Two-loop scalar kinks

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At one loop, quantum kinks are described by a sum of quantum harmonic oscillator Hamiltonians, and the ground state is just the product of the oscillator ground states. Two-loop kink masses are only known in integrable and supersymmetric cases and two-loop states have never been found. We find the two-loop kink mass and explicitly construct the two-loop kink ground state in a scalar field theory with an arbitrary nonderivative potential. We use a coherent state operator that maps the vacuum sector to the kink sector, allowing all states to be treated with a single Hamiltonian that needs to be renormalized only once, eliminating the need for regulator matching conditions. Our calculation is greatly simplified by a recently introduced alternative to collective coordinates, in which the kink momentum is fixed perturbatively.

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I. INTRODUCTION

Quantum solitons at strong coupling are poorly understood, and yet are widely believed to be somehow responsible for confinement in Yang-Mills and QCD. Understanding them is therefore of critical importance. However we would like to suggest that this is premature as solitons at weak coupling are also not understood.

Early papers on quantum solitons produced consistent results. Beginning with the pioneering paper [1], one-loop corrections to kink masses were calculated by introducing a vacuum sector and a kink sector Hamiltonian, regularizing them both, identifying the regulators and renormalizing. In the 1970s, the regulator was a cutoff in the number of modes. In the 1980s, authors instead calculated one-loop corrections to the masses of supersymmetric kinks, regularizing with an energy cutoff. It was only in the following decade that Ref. [2] reported that, when applied to the same kink, these two methods yielded different masses.

The basic problem is as follows. A theory is defined by its Hamitonian together with a regulator and renormalization scheme. One thus expects masses to depend on these three choices. However, once these are fixed, the theory is fixed as are all observables. In particular, nothing may depend on an arbitrary choice of matching conditions for regulators. At most one such inequivalent choice may be correct, but which?

Many responses to this question have since appeared in the literature. The most common interpretation is that some regulator matching conditions give answers which are "bad" [3], and so either different regulators should be used such as in Ref. [1] or different methods, such as that of Ref. [4], which yields the correct soliton mass at one loop. Another response is that the problem is caused by linear divergences, but these may be made logarithmic by taking a derivative with respect to a mass scale and then integrating using a physical principle to fix the constant of integration [5]. This strategy has been successfully employed to reproduce the two-loop mass of the Sine-Gordon soliton. However, as noted by an overlapping collection of authors in [6], this strategy fails with some choices of boundary conditions and, more importantly, it does not shed light on which matching conditions should be allowed. Perhaps the most interesting suggestion, proposed in Ref. [6], is that an ultraviolet cutoff may only be imposed if the nontrivial background itself has no effect above that cutoff. It is an appealing physical principle; however, in practice it does not entirely determine how the density of states is to be corrected. Ultimately the authors chose this correction to reproduce the known answer, leading one to wonder just what prescription works when the answer is not already known. Later it was proposed [7] that instead the matching condition should keep the same mode density in every sector. However the authors note that this proposal is only expected to work at one loop.

This state of affairs has motivated our program to systematically study perturbation theory about quantum solitons in a formalism with no matching conditions. Instead, following [8], we introduce a nonlocal operator

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TABLE I. Summary of notation.

Operator	Description
$\phi(x)$, $\pi(x)$	The real scalar field and its conjugate momentum
$a_p^\dagger$, a_p, $A_p^\dagger$, A_p	Creation and annihilation operators in plane wave basis
$b_k^\dagger$, b_k, $B_k^\dagger$, B_k	Creation and annihilation operators in normal mode basis
ϕ_0, π_0	Zero mode of $\phi(x)$ and $\pi(x)$ in normal mode basis
$::_a$, $::_b$	Normal ordering with respect to a or b operators, respectively
Hamiltonian	**Description**
H	The original Hamiltonian
H'	H with $\phi(x)$ shifted by kink solution $f(x)$
H_n	The ϕ^n term in H'
Symbol	**Description**
$f(x)$	The classical kink solution
$\mathcal{D}_f$	Operator that translates $\phi(x)$ by the classical kink solution
$g_B(x)$	The kink linearized translation mode
$g_k(x)$	Continuum normal mode or breather
γ_i^{mn}	Coefficient of $\phi_0^m B^{\dagger n}\vert 0\rangle_0$ in order i ground state
Γ_i^{mn}	Coefficient of $\phi_0^m B^{\dagger n}\vert 0\rangle_0$ in order i Schrödinger equation
V_{ijk}	Derivative of the potential contracted with various functions
Y_{ijk}	V_{ijk} divided by a sum of frequencies
$\mathcal{I}(x)$	Contraction factor from Wick's theorem
p	Momentum
k_i	The analog of momentum for normal modes
ω_k, ω_p	The frequency corresponding to k or p
Ω_i	Sum of frequencies ω_k
$\tilde{g}$	Inverse Fourier transform of $\sqrt{\lambda}$
Q_n	n-loop correction to kink energy
State	**Description**
$\vert K\rangle$, $\vert\Omega\rangle$	Kink and vacuum sector ground states
$\mathcal{O}\vert\Omega\rangle$	Translation of $\vert K\rangle$ by $\mathcal{D}_f{}^{-1}$
$\mathcal{O}_n\vert\Omega\rangle$	Translation of $\vert K\rangle$ by $\mathcal{D}_f{}^{-1}$ at order n

which maps the vacuum sector to the one soliton sector.[1] This allows all computations involving both sectors to be performed using the original Hamiltonian, with no need to introduce another Hamiltonian for the soliton sector. We thus need to renormalize only once, obviating the need for regulator matching. In Refs. [10,11] this was carried out at one loop in the $1+1$d ϕ^4 and Sine-Gordon models. At one loop these results were known as the theory is free. The first correction to the states was reported in Ref. [12]. The present paper continues to two loops, for a general scalar kink in $1+1$ dimensions. The kink ground state and mass are found.

We begin in Sec. II with a review of our formalism. Then we calculate the two-loop quantum ground states in two steps. Our states are decomposed in a power series in the zero mode ϕ_0 of the scalar field. We refer to the constant terms in this decomposition as ϕ_0 primaries and others as ϕ_0 descendants. In Sec. III we use translation invariance to fix all ϕ_0 descendants in terms of ϕ_0 primaries. Next, in Sec. IV, we use Schrödinger's equation to find the ϕ_0 primaries. As an application, in Sec. V we present a formula for the two-loop mass correction to kinks in $(1+1)$-dimensional scalar theories with an arbitrary potential. In Appendix A we show that the states that we have constructed indeed solve Schrödinger's equation.

[1]For a computationally similar approach without the nonlocal operator, see Ref. [9].

II. REVIEW

In this section we will review the formalism for treating quantum kinks presented in Refs. [10,13,14]. Table I summarizes some of our notation.

Let $\phi(x)$ and $\pi(x)$ be a Schrödinger picture real scalar field and its conjugate in $1+1$ dimensions, whose dynamics are described by the Hamiltonian

$$H = \int dx \mathcal{H}(x),$$
$$\mathcal{H}(x) = \frac{1}{2}:\pi(x)\pi(x):_a + \frac{1}{2}:\partial_x\phi(x)\partial_x\phi(x):_a + \frac{M^2}{\lambda}:\mathcal{V}[\sqrt{\lambda}\phi(x)]:_a. \tag{2.1}$$

Here M and $\sqrt{\lambda}$ have dimensions of mass and action$^{-1/2}$, respectively. We expand in $\lambda\hbar$ and set $\hbar = 1$. Also we will define the dimensionful potential

$$V = M^2\mathcal{V}. \tag{2.2}$$

The normal ordering $::_a$ is defined below. We remind the reader that in $1+1$ dimensions, normal ordering is sufficient to render scalar field theories UV finite.

If V has degenerate minima, then there will be a classical kink solution

$$\phi(x,t) = f(x). \tag{2.3}$$

We normalize M such that $\mathcal{V}''$ evaluated at both minima appearing at the end of the kink is equal to unity, which requires the simplifying assumption that these are equal. In the Schrödinger picture, where we will always work, the displacement operator

$$\mathcal{D}_f = \exp\left(-i\int dx f(x)\pi(x)\right) \tag{2.4}$$

satisfies [10]

$$:F[\pi(x),\phi(x)]:_a\mathcal{D}_f = \mathcal{D}_f :F[\pi(x),\phi(x)+f(x)]:_a, \tag{2.5}$$

where F is an arbitrary functional. This operator takes the vacuum sector to the kink sector. In particular one may relate the ground states $|\Omega\rangle$ and $|K\rangle$ of the two respective sectors

$$|K\rangle = \mathcal{D}_f\mathcal{O}|\Omega\rangle \tag{2.6}$$

using the perturbative operator $\mathcal{O}$. The kink ground state $|K\rangle$ is an eigenstate of the Hamiltonian H and so $\mathcal{O}|\Omega\rangle$ must be an eigenstate of the Hamiltonian

$$H' = \mathcal{D}_f{}^{-1}H\mathcal{D}_f = Q_0 + H_2 + H_I,$$
$$H_2 = \frac{1}{2}\int dx[:\pi^2(x):_a + :(\partial_x\phi(x))^2:_a + V''[\sqrt{\lambda}f(x)]:\phi^2(x):_a]. \tag{2.7}$$

Here Q_0 is the classical mass of the solution $f(x)$ and H_I contains all higher order terms in $\sqrt{\lambda}$.

The free Hamiltonian H_2 leads to classical linear equations of motion whose constant frequency solutions are the normal modes $g(x)$ of the kink

$$\phi(x,t) = e^{-i\omega t}g(x), \qquad V''[\sqrt{\lambda}f(x)]g(x) = \omega^2 g(x) + g''(x). \tag{2.8}$$

There will be continuum solutions $g_k(x)$ labeled by an index k such that[2] $\omega_k = \sqrt{M^2+k^2}$, breathers, and a single Goldstone mode $g_B(x)$

$$g_B(x) = \frac{1}{\sqrt{Q_0}}f'(x), \tag{2.9}$$

with $\omega_B = 0$. For brevity of notation, we will not distinguish between continuum solutions and breathers, and so it will be implicit that integrals over the continuous variable k include a sum over the breathers.

We adopt the normalization conditions

$$\int dx g_{k_1}(x)g^*_{k_2}(x) = 2\pi\delta(k_1-k_2), \qquad \int dx|g_B(x)|^2 = 1, \tag{2.10}$$

and we choose the phases such that

$$g_k(-x) = g^*_k(x) = g_{-k}(x). \tag{2.11}$$

We also define inverse Fourier transforms

$$\tilde{g}(p) = \int dx g(x)e^{ipx}, \tag{2.12}$$

satisfying the completeness relations

$$\tilde{g}_B(p)\tilde{g}_B(q) + \int\frac{dk}{2\pi}\tilde{g}_k(p)\tilde{g}_{-k}(q) = 2\pi\delta(p+q). \tag{2.13}$$

The same quantum field and its conjugate may be expanded in terms of plane waves

$$\phi(x) = \int\frac{dp}{2\pi}\frac{1}{\sqrt{2\omega_p}}(a^\dagger_p + a_{-p})e^{-ipx},$$
$$\omega_p = \sqrt{M^2+p^2},$$
$$\pi(x) = i\int\frac{dp}{2\pi}\sqrt{\frac{\omega_p}{2}}(a^\dagger_p - a_{-p})e^{-ipx} \tag{2.14}$$

or normal modes

[2]The sign of k is chosen to agree with the momentum of the corresponding plane wave at $|x| \gg 0$.

$$\phi(x) = \phi_B(x) + \phi_C(x), \qquad \pi(x) = \pi_B(x) + \pi_C(x),$$
$$\phi_B(x) = \phi_0 g_B(x), \qquad \phi_C(x) = \int \frac{dk}{2\pi} \frac{1}{\sqrt{2\omega_k}} (b_k^\dagger + b_{-k}) g_k(x),$$
$$\pi_B(x) = \pi_0 g_B(x), \qquad \pi_C(x) = i \int \frac{dk}{2\pi} \sqrt{\frac{\omega_k}{2}} (b_k^\dagger - b_{-k}) g_k(x). \tag{2.15}$$

We define the plane wave normal ordering $::_a$ by placing the $a^\dagger$ to the left of the a and normal mode normal ordering $::_b$ by placing $b^\dagger$ and ϕ_0 to the left of b and π_0.

Using the canonical algebra satisfied by $\phi(x)$ and $\pi(x)$ together with the completeness of the solutions [12]

$$g_B(x) g_B(y) + \int \frac{dk}{2\pi} g_k(x) g_{-k}(y) = \delta(x - y), \tag{2.16}$$

one finds

$$[a_p, a_q^\dagger] = 2\pi\delta(p - q), \qquad [\phi_0, \pi_0] = i, \qquad [b_{k_1}, b_{k_2}^\dagger] = 2\pi\delta(k_1 - k_2).$$

These allow the plane wave normal ordered H_2 to be rewritten in terms of a normal mode normal ordered free Hamiltonian plus a constant Q_1, which is the one-loop correction to the kink mass. This can be achieved one term a time

$$:\pi_B^2(x):_a = :\pi_B^2(x):_b + g_B(x)\hat{\hat{g}}_B(x), \qquad \hat{\hat{g}}_B(x) = -\int \frac{dp}{2\pi} e^{-ixp} \frac{\omega_p}{2} \tilde{g}_B(p)$$
$$:\pi_C^2(x):_a = :\pi_C^2(x):_b + \int \frac{dk}{2\pi} g_k(x) \hat{\hat{g}}_{-k}(x), \qquad \hat{\hat{g}}_k(x) = \int \frac{dp}{2\pi} e^{-ixp} \left(\frac{\omega_k - \omega_p}{2}\right) \tilde{g}_k(p)$$
$$:\phi_B^2(x):_a = :\phi_B^2(x):_b + g_B(x)\hat{g}_B(x), \qquad \hat{g}_B(x) = -\int \frac{dp}{2\pi} e^{-ixp} \frac{1}{2\omega_p} \tilde{g}_B(p)$$
$$:\phi_C^2(x):_a = :\phi_C^2(x):_b + \int \frac{dk}{2\pi} g_k(x) \hat{g}_{-k}(x), \qquad \hat{g}_k(x) = \int \frac{dp}{2\pi} e^{-ixp} \left(\frac{1}{2\omega_k} - \frac{1}{2\omega_p}\right) \tilde{g}_k(p). \tag{2.17}$$

Applying the classical equations of motion (2.8) one finds

$$V''[\sqrt{\lambda} f(x)]:\phi_B^2(x):_a = V''[\sqrt{\lambda} f(x)]:\phi_B^2(x):_b + g_B''(x)\hat{g}_B(x),$$
$$V''[\sqrt{\lambda} f(x)]:\phi_C^2(x):_a = V''[\sqrt{\lambda} f(x)]:\phi_C^2(x):_b + \int \frac{dk}{2\pi} (\omega_k^2 g_k(x) + g_k''(x)) \hat{g}_{-k}(x). \tag{2.18}$$

The g'' terms cancel $:\partial\phi(x)\partial\phi(x):_a - :\partial\phi(x)\partial\phi(x):_b$ after an integration by parts, leaving

$$H_2 = Q_1 + \frac{\pi_0^2}{2} + \int \frac{dk}{2\pi} \omega_k b_k^\dagger b_k,$$
$$Q_1 = -\frac{1}{4} \int \frac{dk}{2\pi} \int \frac{dp}{2\pi} \frac{(\omega_p - \omega_k)^2}{\omega_p} \tilde{g}_k^2(p) - \frac{1}{4} \int \frac{dp}{2\pi} \omega_p \tilde{g}_B(p) \tilde{g}_B(p). \tag{2.19}$$

We perform a semiclassical expansion of the kink ground state[3] in powers of $\sqrt{\lambda}$

$$\mathcal{O}|\Omega\rangle = \sum_{i=0}^{\infty} |0\rangle_i. \tag{2.20}$$

The one-loop kink ground state $|0\rangle_0$ is a product of free vacua

[3] The n-loop ground state is the sum up to $i = 2n - 2$. Note that there is no tree-level term in our expansion. In a sense made precise in Ref. [12], the tree-level ground state $|\Omega\rangle$ is automatically included in the one-loop $|0\rangle_0$ by the condition (2.21). Beginning the expansion with a tree-level ground state at a fixed center of mass would lead to an infinite first correction [15].

$$\pi_0|0\rangle_0 = b_k|0\rangle_0 = 0. \tag{2.21}$$

In Ref. [16] we found a general Wick's formula for the conversion of plane wave to normal mode normal ordering. For powers of $\phi(x)$ it reads

$$:\phi^n(x):_a = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{2^m m!(n-2m)!} \mathcal{I}^m(x) :\phi^{n-2m}(x):_b, \tag{2.22}$$

where

$$\mathcal{I}(x) = g_B(x)\hat{g}_B(x) + \int \frac{dk}{2\pi} g_{-k}(x)\hat{g}_k(x),$$
$$\hat{g}_B(x) = -\int \frac{dp}{2\pi} e^{-ipx} \frac{\tilde{g}_B(p)}{2\omega_p}, \qquad \hat{g}_k(x) = \int \frac{dp}{2\pi} e^{-ipx} \tilde{g}_k(p) \left(\frac{1}{2\omega_k} - \frac{1}{2\omega_p}\right). \tag{2.23}$$

Using the completeness relations (2.13) one can show [12,16] that $\mathcal{I}(x)$ is determined by

$$\partial_x \mathcal{I}(x) = \int \frac{dk}{2\pi} \frac{1}{2\omega_k} \partial_x |g_k(x)|^2 \tag{2.24}$$

together with the condition that it vanish at spatial infinity.

III. TRANSLATION INVARIANCE

In this section we will calculate the translation operator that acts on our states $\mathcal{O}|\Omega\rangle$ and will use it to fix all ϕ_0 descendants (components of states that include operators ϕ_0).

A. The translation operator

Let us define the shorthand

$$\Delta_{ij} = \int dx g_i(x) g'_j(x) = i \int \frac{dp}{2\pi} p \tilde{g}_i(p) \tilde{g}_j(-p), \tag{3.1}$$

where i and j may be a bound state or a momentum k. Note that Δ is antisymmetric. We will use reweighted creation and annihilation operators

$$B_k^\dagger = \frac{b_k^\dagger}{\sqrt{2\omega_k}}, \qquad B_k = \sqrt{2\omega_k} b_k, \tag{3.2}$$

which satisfy the same Heisenberg commutation relations as $b^\dagger$ and b.

The identity

$$P\mathcal{D}_f = \mathcal{D}_f (P - \sqrt{Q_0}\pi_0) \tag{3.3}$$

implies that the translation invariance

$$P|K\rangle = P\mathcal{D}_f \sum_i |0\rangle_i = 0 \tag{3.4}$$

is equivalent to

$$P|0\rangle_i = \sqrt{Q_0}\pi_0 |0\rangle_{i+1}. \tag{3.5}$$

Our strategy will be to solve this equation by inverting π_0. Thus translation invariance fixes our states entirely up to an element of the kernel of π_0. We then *only* use the Schrödinger equation to fix the element of the kernel of π_0, thus greatly simplifying the problem. Note that the kernel of π_0 consists precisely of the ϕ_0-primary states.

Let us write the translation operator as

$$\begin{aligned} P &= -\int dx \pi(x) \partial_x \phi(x), \\ &= -\int dx \left[\pi_0 g_B(x) \int \frac{dk}{2\pi} \phi_k g'_k(x) + \left(\int \frac{dk}{2\pi} \pi_k g_k(x)\right) \phi_0 g'_B(x) + \int \frac{d^2k}{(2\pi)^2} \pi_{k_1} \phi_{k_2} g_{k_1}(x) g'_{k_2}(x)\right], \\ &= \int \frac{dk}{2\pi} \Delta_{kB} \left[i\phi_0 \left(-\omega_k B_k^\dagger + \frac{B_{-k}}{2}\right) + \pi_0 \left(B_k^\dagger + \frac{B_{-k}}{2\omega_k}\right)\right] + i \int \frac{d^2k}{(2\pi)^2} \Delta_{k_1 k_2} \left(-\omega_{k_1} B_{k_1}^\dagger B_{k_2}^\dagger + \frac{B_{-k_1} B_{-k_2}}{4\omega_{k_2}} - \frac{1}{2}\left(1 + \frac{\omega_{k_1}}{\omega_{k_2}}\right) B_{k_1}^\dagger B_{-k_2}\right), \end{aligned} \tag{3.6}$$

and expand the ith order kink ground state as

$$|0\rangle_i = \sum_{m,n=0}^{\infty} |0\rangle_i^{mn}, \qquad |0\rangle_i^{mn} = Q_0^{-i/2} \int \frac{d^n k}{(2\pi)^n} \gamma_i^{mn}(k_1 \cdots k_n) \phi_0^m B_{k_1}^\dagger \cdots B_{k_n}^\dagger |0\rangle_0. \tag{3.7}$$

We will refer to $m=0$ states or matrix elements γ_i^{0n} as ϕ_0 primary and $m>0$ states as ϕ_0 descendants. Then the translation invariance (3.5) yields the recursion relation

$$\begin{aligned}\gamma_{i+1}^{mn}(k_1\cdots k_n) &= \Delta_{k_n B}\left(\gamma_i^{m,n-1}(k_1\cdots k_{n-1}) + \frac{\omega_{k_n}}{m}\gamma_i^{m-2,n-1}(k_1\cdots k_{n-1})\right)\\
&\quad + \int \frac{dk'}{2\pi}\Delta_{-k'B}\sum_{j=0}^{n}\left(\frac{\gamma_i^{m,n+1}(k_1\cdots k_j,k',k_{j+1}\cdots k_n)}{2\omega_{k'}} - \frac{\gamma_i^{m-2,n+1}(k_1\cdots k_j,k',k_{j+1}\cdots k_n)}{2m}\right)\\
&\quad + \frac{1}{2m}\sum_{j=1}^{n}\int\frac{dk'}{2\pi}\Delta_{k_n,-k'}\left(1+\frac{\omega_{k_n}}{\omega_{k'}}\right)\gamma_i^{m-1,n}(k_1\cdots k_{j-1},k',k_j\cdots k_{n-1}) + \frac{\omega_{k_{n-1}}\Delta_{k_{n-1}k_n}}{m}\gamma_i^{m-1,n-2}(k_1\cdots k_{n-2})\\
&\quad - \int\frac{d^2k'}{(2\pi)^2}\frac{\Delta_{-k_1',-k_2'}}{2m\omega_{k_2'}}\sum_{j_1=1}^{n+1}\sum_{j_2=j_1+1}^{n+2}\gamma_i^{m-1,n+2}(k_1\cdots k_{j_1-1},k_1',k_{j_1}\cdots k_{j_2-2},k_2',k_{j_2-1}\cdots k_n).\end{aligned} \tag{3.8}$$

This recursion relation determines all ϕ_0 descendants in terms of ϕ_0-primary states plus the free state corresponding to the one-loop initial condition γ_0. It does not determine the ϕ_0 primaries, as it corresponds to a particular solution of Eq. (3.5) and the addition of any element of the kernel of π_0, in other words any ϕ_0-primary state, is another solution.

In general this recursion relation leads to infrared (IR) divergences. In Ref. [17], two kinds of IR divergences are identified. The first results from singularities in $\Delta_{k_1k_2}$ as k_1+k_2 tends to zero [see for example Eq. (5.18)] and, in that case, describes the recoil momentum of a kink when a normal mode is excited in the center of mass frame. The second results from divergences in the initial conditions γ_0, for example if one begins with an excited isolated continuum normal mode. These divergences correspond to interactions that do not involve the excited mode. In both cases the divergences must in general be kept to arrive at the correct final answer. Realistic initial conditions for excited states are wave packets that depend smoothly on the continuum k and we do not know whether such IR divergences are avoided at all orders in that case. In the present paper, we are interested in the ground state and so γ_0 is independent of k and we have no recoil. In this case no IR divergences appear to the order calculated below, although there are terms of the form $(\omega_{k_1}-\omega_{k_2})\delta(k_1-k_2)$ in which divergences are avoided by the structure of the coefficients.

B. Constructing translation-invariant states

At one loop, the quantum kink is described by a series of harmonic oscillators and so its spectrum is known precisely [1]. To find a Hamiltonian eigenstate at higher but finite order, one need only start the recursion (3.8) at $i=0$ with the one-loop avatar of the state of interest.

In this note we will apply this strategy to the ground state, corresponding to the initial condition

$$\gamma_0^{mn} = \delta_{m0}\delta_{n0}\gamma_0^{00}. \tag{3.9}$$

The first recursion is depicted in the left panel of Fig. 1, where it determines the squares in terms of the star, which corresponds to the initial condition. More precisely, it yields

$$\gamma_1^{12}(k_1,k_2) = \omega_{k_1}\Delta_{k_1k_2}\gamma_0^{00}, \qquad \gamma_1^{21}(k_1) = \frac{\omega_{k_1}\Delta_{k_1B}}{2}\gamma_0^{00}. \tag{3.10}$$

We are not interested in calculating the ϕ_0 primaries ($m=0$ terms) because these are in the kernel of π_0, and so they are

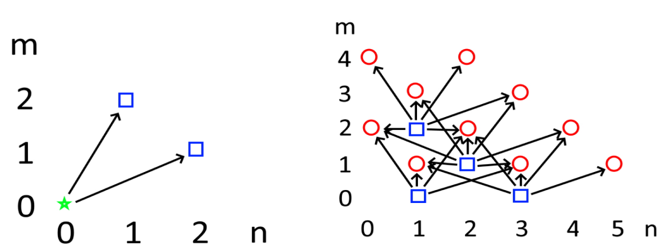

FIG. 1. The γ_i^{mn} generated by the recursion relation at $i=1$ (left) and $i=2$ (right). Green stars, blue squares and red circles represent elements at $i=0$, $i=1$, and $i=2$, respectively. As ϕ_0 primaries ($m=0$ elements) are in the kernel of π_0, they are not fixed by (3.5) and so arrows to such elements are not shown.

not determined by translation invariance. These will be calculated using Schrödinger's equation in Sec. IV.

We may continue by simply plugging in to our recursion relation (3.8). But we can simplify things somewhat by noticing that (3.7) does not completely determine the functions $\gamma_i^{mn}(k_1 \cdots k_n)$. For example, one may add any function that is antisymmetric under the exchange of any k_i and k_j without affecting $|0\rangle$. Therefore we are free to symmetrize each function. As this will simplify our answer, that will be our convention: it will be understood that after calculating each γ using (3.8) it should be symmetrized before the next recursion. This convention allows one to perform all of the sums in our recursion relation (3.8), leaving

$$\begin{aligned}\gamma_{i+1}^{mn}(k_1\cdots k_n) &= \Delta_{k_nB}\left(\gamma_i^{m,n-1}(k_1\cdots k_{n-1}) + \frac{\omega_{k_n}}{m}\gamma_i^{m-2,n-1}(k_1\cdots k_{n-1})\right)\\ &+(n+1)\int\frac{dk'}{2\pi}\Delta_{-k'B}\left(\frac{\gamma_i^{m,n+1}(k_1\cdots k_n,k')}{2\omega_{k'}} - \frac{\gamma_i^{m-2,n+1}(k_1\cdots k_n,k')}{2m}\right)\\ &+\frac{\omega_{k_{n-1}}\Delta_{k_{n-1}k_n}}{m}\gamma_i^{m-1,n-2}(k_1\cdots k_{n-2}) + \frac{n}{2m}\int\frac{dk'}{2\pi}\Delta_{k_n,-k'}\left(1+\frac{\omega_{k_n}}{\omega_{k'}}\right)\gamma_i^{m-1,n}(k_1\cdots k_{n-1},k')\\ &-\frac{(n+2)(n+1)}{2m}\int\frac{d^2k'}{(2\pi)^2}\frac{\Delta_{-k_1',-k_2'}}{2\omega_{k_2'}}\gamma_i^{m-1,n+2}(k_1\cdots k_n,k_1',k_2').\end{aligned}\tag{3.11}$$

In summary, the recursion relation (3.8) always yields a correct γ_{i+1} whereas the simpler (3.11) is also correct if one first symmetrizes each $\gamma_i^{mn}(k_1\cdots k_n)$ with respect to its arguments $k_1\cdots k_n$. Thus to apply (3.11) to derive γ_2 we must first symmetrize all γ_1^{mn} with $n \geq 2$. We only found one such element, which, after symmetrizing using the antisymmetry of Δ, becomes

$$\gamma_1^{12}(k_1,k_2) = \frac{(\omega_{k_1}-\omega_{k_2})\Delta_{k_1k_2}}{2}\gamma_0^{00}. \tag{3.12}$$

What about the ϕ_0 primaries γ_1^{0n}? These are not fixed by translation invariance as they are in the kernel of π_0. Rather they are determined using the Schrödinger equation. In a scalar theory with a canonical kinetic term, ϕ will have dimensions of $[\text{action}]^{1/2}$. As each $|0\rangle_i$ is suppressed by $\hbar^{1/2}$ with respect to $|0\rangle_{i-1}$, it may only depend on terms in the potential up to ϕ^{2+i}. Therefore $|0\rangle_1$ and so γ_1 only depend on ϕ^3 terms. As a result the only nonzero entries resulting from the Schrödinger equation can be γ_1^{01} and γ_1^{03}.

Finally we are ready to apply (3.11) to calculate γ_2. Remember that the recursion relations only determine ϕ_0 descendants ($m > 0$), so over all we expect 3, 4, 5, and 6 contributions from γ_1^{01}, γ_1^{03}, γ_1^{21}, and γ_1^{12}, respectively. These are the circles in the right panel of Fig. 1. The γ_2^{mn} corresponding to a circle at (m,n) is a sum of terms, one for each arrow ending on that circle, which are each proportional to the $\gamma_1^{m'n'}$ found at the beginning of the corresponding arrow.

At $m = 1$ we find

$$\begin{aligned}\gamma_2^{11}(k_1) &= \int\frac{dk'}{2\pi}\Delta_{-k'B}\frac{\gamma_1^{12}(k_1,k')}{\omega_{k'}} - \frac{3}{4}\int\frac{d^2k'}{(2\pi)^2}\frac{\Delta_{-k_1',-k_2'}}{\omega_{k_2'}}\gamma_1^{03}(k_1,k_1',k_2') + \frac{1}{2}\int\frac{dk'}{2\pi}\Delta_{k_1,-k'}\left(1+\frac{\omega_{k_1}}{\omega_{k'}}\right)\gamma_1^{01}(k'),\\ &= \frac{1}{2}\int\frac{dk'}{2\pi}\left(\frac{\omega_{k_1}}{\omega_{k'}}-1\right)\Delta_{k_1k'}\Delta_{-k'B}\gamma_0^{00} - \frac{3}{2}\int\frac{d^2k'}{(2\pi)^2}\frac{\Delta_{-k_1',-k_2'}}{\omega_{k_2'}}\gamma_1^{03}(k_1,k_1',k_2') + \frac{1}{2}\int\frac{dk'}{2\pi}\Delta_{k_1,-k'}\left(1+\frac{\omega_{k_1}}{\omega_{k'}}\right)\gamma_1^{01}(k'),\end{aligned}\tag{3.13}$$

and

$$\begin{aligned}
\gamma_2^{13}(k_1,k_2,k_3) &= \omega_{k_2}\Delta_{k_2k_3}\gamma_1^{01}(k_1) + \Delta_{k_3B}\gamma_1^{12}(k_1,k_2) + \frac{3}{2}\int\frac{dk'}{2\pi}\Delta_{k_3,-k'}\left(1+\frac{\omega_{k_3}}{\omega_{k'}}\right)\gamma_1^{03}(k_1,k_2,k'),\\
&= \omega_{k_2}\Delta_{k_2k_3}\gamma_1^{01}(k_1) + \frac{1}{2}\Delta_{k_3B}(\omega_{k_1}-\omega_{k_2})\Delta_{k_1k_2}\gamma_0^{00} + \frac{3}{2}\int\frac{dk'}{2\pi}\Delta_{k_3,-k'}\left(1+\frac{\omega_{k_3}}{\omega_{k'}}\right)\gamma_1^{03}(k_1,k_2,k'),\\
\gamma_2^{15}(k_1\cdots k_5) &= \omega_{k_4}\Delta_{k_4k_5}\gamma_1^{03}(k_1,k_2,k_3).
\end{aligned} \tag{3.14}$$

Next at $m=2$

$$\begin{aligned}
\gamma_2^{20} &= \int\frac{dk'}{2\pi}\Delta_{-k'B}\left(\frac{\gamma_1^{21}(k')}{2\omega_{k'}}-\frac{\gamma_1^{01}(k')}{4}\right) - \frac{1}{4}\int\frac{d^2k'}{(2\pi)^2}\frac{\Delta_{-k'_1,-k'_2}}{\omega_{k'_2}}\gamma_1^{12}(k'_1,k'_2),\\
&= \frac{1}{4}\int\frac{dk'}{2\pi}\Delta_{-k'B}(\Delta_{k'B}\gamma_0^{00}-\gamma_1^{01}(k')) + \frac{1}{8}\int\frac{d^2k'}{(2\pi)^2}\left(1-\frac{\omega_{k'_1}}{\omega_{k'_2}}\right)\Delta_{k'_1k'_2}\Delta_{-k'_1,-k'_2}\gamma_0^{00},
\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}
\gamma_2^{22}(k_1,k_2) &= \Delta_{k_2B}\left(\gamma_1^{21}(k_1)+\frac{\omega_{k_2}}{2}\gamma_1^{01}(k_1)\right) - \frac{3}{4}\int\frac{dk'}{2\pi}\Delta_{-k'B}\gamma_1^{03}(k_1,k_2,k') + \frac{1}{2}\int\frac{dk'}{2\pi}\Delta_{k_2,-k'}\left(1+\frac{\omega_{k_2}}{\omega_{k'}}\right)\gamma_1^{12}(k_1,k'),\\
&= \frac{\Delta_{k_2B}}{2}(\omega_{k_1}\Delta_{k_1B}\gamma_0^{00}+\omega_{k_2}\gamma_1^{01}(k_1)) - \frac{3}{4}\int\frac{dk'}{2\pi}\Delta_{-k'B}\gamma_1^{03}(k_1,k_2,k')\\
&\quad + \frac{1}{4}\int\frac{dk'}{2\pi}\Delta_{k_2,-k'}\left(1+\frac{\omega_{k_2}}{\omega_{k'}}\right)(\omega_{k_1}-\omega_{k'})\Delta_{k_1k'}\gamma_0^{00},\\
\gamma_2^{24}(k_1\cdots k_4) &= \frac{\omega_{k_3}\Delta_{k_3k_4}}{2}\gamma_1^{12}(k_1,k_2) + \Delta_{k_4B}\frac{\omega_{k_4}}{2}\gamma_1^{03}(k_1\cdots k_3),\\
&= \frac{\omega_{k_1}\omega_{k_3}\Delta_{k_1k_2}\Delta_{k_3k_4}}{2}\gamma_0^{00} + \frac{\omega_{k_4}\Delta_{k_4B}}{2}\gamma_1^{03}(k_1\cdots k_3).
\end{aligned} \tag{3.16}$$

Continuing to $m=3$ we find

$$\begin{aligned}
\gamma_2^{31}(k_1) &= -\frac{1}{3}\int\frac{dk'}{2\pi}\Delta_{-k'B}\gamma_1^{12}(k_1,k') + \frac{1}{6}\int\frac{dk'}{2\pi}\Delta_{k_1,-k'}\left(1+\frac{\omega_{k_1}}{\omega_{k'}}\right)\gamma_1^{21}(k'),\\
&= \frac{\gamma_0^{00}}{6}\int\frac{dk'}{2\pi}\left[(\omega_{k'}-\omega_{k_1})\Delta_{k_1k'}\Delta_{-k'B} + \frac{1}{2}\Delta_{k_1,-k'}(\omega_{k_1}+\omega_{k'})\omega_{k'}\Delta_{k'B}\right],\\
&= \gamma_0^{00}\int\frac{dk'}{2\pi}\left(\frac{\omega_{k'}}{4}-\frac{\omega_{k_1}}{12}\right)\Delta_{k_1k'}\Delta_{-k'B},\\
\gamma_2^{33}(k_1,k_2,k_3) &= \frac{\omega_{k_3}\Delta_{k_3B}}{3}\gamma_1^{12}(k_1,k_2) + \frac{\omega_{k_2}\Delta_{k_2k_3}}{3}\gamma_1^{21}(k_1),\\
&= (\omega_{k_3}\Delta_{k_3B}(\omega_{k_1}-\omega_{k_2})\Delta_{k_1k_2} + \omega_{k_2}\Delta_{k_2k_3}\omega_{k_1}\Delta_{k_1B})\frac{\gamma_0^{00}}{6}.
\end{aligned} \tag{3.17}$$

Note that, since γ_2^{33} is defined by its symmetric contraction with $B_{k_1}^\dagger B_{k_2}^\dagger B_{k_3}^\dagger$, one is free to add any term that is annihilated by the symmetrization of k_1, k_2, and k_3. Thus one may freely redefine

$$\gamma_2^{33}(k_1,k_2,k_3) = \frac{\omega_{k_1}\Delta_{k_1B}\omega_{k_2}\Delta_{k_2k_3}}{2}\gamma_0^{00}. \tag{3.18}$$

In other words, different paths from γ_0^{00} to γ_2^{33} lead to contributions that are proportional. This suggests that to some extent it may be possible to explicitly solve our recursion formula. Finally the $m = 4$ terms are

$$\gamma_2^{40} = -\int \frac{dk'}{2\pi} \Delta_{-k'B} \frac{\gamma_1^{21}(k')}{8} = \frac{\gamma_0^{00}}{16} \int \frac{dk'}{2\pi} \omega_{k'} \Delta_{Bk'} \Delta_{-k'B},$$
$$\gamma_2^{42}(k_1, k_2) = \Delta_{k_2B} \frac{\omega_{k_2}}{4} \gamma_1^{21}(k_1) = \frac{\omega_{k_1} \Delta_{k_1B} \omega_{k_2} \Delta_{k_2B}}{8} \gamma_0^{00}. \tag{3.19}$$

IV. SCHRÖDINGER'S EQUATION

Let us define the symbol Γ by any solution of

$$\sum_{j=0}^{i} \left(H_{i+2-j} - Q_{\frac{i-j}{2}+1} \right) |0\rangle_j = Q_0^{-i/2} \sum_{mn} \int \frac{d^n k}{(2\pi)^n} \Gamma_i^{mn}(k_1 \cdots k_n) \phi_0^m B_{k_1}^\dagger \cdots B_{k_n}^\dagger |0\rangle_0. \tag{4.1}$$

Then the Schrödinger equation

$$(H - Q)|0\rangle = 0 \tag{4.2}$$

is solved if

$$\Gamma_i^{mn} = 0. \tag{4.3}$$

Note that Γ is not uniquely defined by (4.1). A necessary and sufficient condition for a solution to Schrödinger's equations is that Γ_i^{mn} vanishes when summed over all permutations of the k_j. The number of loops can be defined by counting powers of $\hbar$ and is equal to $i/2 + 1$. Note that only integral numbers of loops correct the energy, and so Q vanishes if its subscript is a half-integer. Here Q is defined to be the energy of the ground state. For applications to other states, Q should be replaced with their respective energies.

Let us begin with the one-loop approximation, $i = 0$. Using

$$H_2 - Q_1 = \frac{\pi_0^2}{2} + \int \frac{dk}{2\pi} \omega_k B_k^\dagger B_k \tag{4.4}$$

one finds that the Schrödinger equation is satisfied if

$$\pi_0|0\rangle_0 = B_k|0\rangle_0 = 0. \tag{4.5}$$

These are both satisfied by the initial condition $\gamma_0^{mn} = \delta_{m0}\delta_{n0}$ of our recursion.

A. Leading corrections

At $i = 1$ the Schrödinger equation is

$$H_3|0\rangle_0 + (H_2 - Q_1)|0\rangle_1 = 0. \tag{4.6}$$

Using

$$H_3 = \frac{1}{6} \int dx V^{(3)}[\sqrt{\lambda} f(x)] :\phi^3(x):_a,$$
$$= \frac{1}{6} \int dx V^{(3)}[\sqrt{\lambda} f(x)] :\phi^3(x):_b$$
$$+ \frac{1}{2} \int dx V^{(3)}[\sqrt{\lambda} f(x)] \phi(x) \mathcal{I}(x), \tag{4.7}$$

where we have defined $V^{(n)}[\sqrt{\lambda} f(x)]$ to be the nth derivative of $\lambda^{-1} V[\sqrt{\lambda}\phi(x)]$ with respect to $\phi(x)$, evaluated at $\phi(x) = f(x)$, one finds that the leading correction to the states (3.10) yields

$$\Gamma_1^{21} = \sqrt{Q_0} \frac{V_{BBk_1}}{2} + \frac{\omega_{k_1}^2 \Delta_{k_1B}}{2},$$
$$\Gamma_1^{12} = \sqrt{Q_0} \frac{V_{Bk_1k_2}}{2} + \frac{(\omega_{k_1} - \omega_{k_2})(\omega_{k_1} + \omega_{k_2}) \Delta_{k_1k_2}}{2}, \tag{4.8}$$

where we have introduced the notation

$$V_{\mathcal{I} \overset{m}{\cdots} \mathcal{I}, \alpha_1 \cdots \alpha_n} = \int dx V^{(2m+n)}[\sqrt{\lambda} f(x)] \mathcal{I}^m(x) g_{\alpha_1}(x) \cdots g_{\alpha_n(x)}, \tag{4.9}$$

where α_j can be B or k_j.

Substituting the identities [12]

$$\begin{aligned}
V_{BBk} &= \int dx V^{(3)}[\sqrt{\lambda}f(x)]g_B(x)\frac{f'(x)}{\sqrt{Q_0}}g_k(x) = \frac{1}{\sqrt{Q_0}}\int dx\partial_x(V^{(2)}[\sqrt{\lambda}f(x)])g_B(x)g_k(x),\\
&= -\frac{1}{\sqrt{Q_0}}\int dx V^{(2)}[\sqrt{\lambda}f(x)](g'_B(x)g_k(x)+g_B(x)g'_k(x)),\\
&= -\frac{1}{\sqrt{Q_0}}\int dx(g'_B(x)\omega_k^2 g_k(x)+g'_B(x)g''_k(x)+g''_B(x)g'_k(x)),\\
&= -\frac{\omega_k^2}{\sqrt{Q_0}}\Delta_{kB},\\
V_{Bk_1k_2} &= -\frac{1}{\sqrt{Q_0}}\int dx V^{(2)}[\sqrt{\lambda}f(x)](g'_{k_1}(x)g_{k_2}(x)+g_{k_1}(x)g'_{k_2}(x)),\\
&= -\frac{1}{\sqrt{Q_0}}\int dx(g'_{k_1}(x)\omega_{k_2}^2 g_{k_2}(x)+g'_{k_1}(x)g''_{k_2}(x)+\omega_{k_1}^2 g_{k_1}(x)g'_{k_2}(x)+g''_{k_1}(x)g'_{k_2}(x)),\\
&= \frac{\omega_{k_2}^2-\omega_{k_1}^2}{\sqrt{Q_0}}\Delta_{k_1k_2}
\end{aligned} \tag{4.10}$$

into (4.8) one finds $\Gamma = 0$, and so these matrix elements of Schrödinger's equation are satisfied by the states (3.10), which were derived from translation invariance alone. This is consistent with our claim that all ϕ_0 descendants ($m > 0$ components of states) are determined in terms of ϕ_0 primaries by imposing the eigenvalue of the momentum, in this case zero.

Actually the derivation of the second identity in (4.10) is not quite correct. As the kink is localized by definition, $g_B(x)$ tends to zero at large x, while $V^{(3)}[\sqrt{\lambda}f(x)]$ tends to a constant. As the remaining terms tend to $e^{-i(k_1+k_2)x}$, the integral over x defining the left-hand side converges at large x. The problem is that the integration by parts, used to convert $V^{(3)}[\sqrt{\lambda}f(x)]g_B$ to $V^{(2)}[\sqrt{\lambda}f(x)]$, leads to a boundary term $V^{(2)}[\sqrt{\lambda}f(x)]g_{k_1}(x)g_{k_2}(x)$, which does not vanish at large x. Instead, it becomes $me^{-i(k_1+k_2)x_m}$, where x_m is the upper limit of integration. As one takes the limit $x_m \to \infty$, this vanishes by the Riemann-Lebesgue lemma when integrated over any kernel continuous in $k_1 + k_2$. As the left-hand side is finite at $k_1 + k_2 = 0$ and the dropped boundary term vanishes in the sense of a distribution, the final expression must also vanish in the sense of a distribution. Let us check this for the term

$$\Delta_{k_1k_2} \supset i\pi(k_2-k_1)\delta(k_1+k_2), \tag{4.11}$$

which appears for any potential V. This term indeed does not contribute to the last line of (4.10) as it vanishes in the sense of a distribution when multiplied by $\omega_{k_2}^2 - \omega_{k_1}^2$.

Similarly, in our main result (5.9), the Δ is multiplied by $(\omega_{k_1} - \omega_{k_2})$. The $\delta(k_1 + k_2)$ term in Δ therefore does not contribute to the two-loop energy of the ground state. It does, however, contribute to the two-loop energy of a kink with an excited normal mode [17] where it leads to the recoil kinetic energy of the bulk motion of the kink. We also note that one cannot simply divide (4.10) through by $(\omega_{k_2}^2 - \omega_{k_1}^2)$ to obtain $\Delta_{k_1k_2}$ as a function of $V_{Bk_1k_2}$, as this quantity vanishes on the support of the delta function (4.11).

The other components of the Schrödinger equation at $i = 1$ are

$$\begin{aligned}
\Gamma_1^{01} &= \frac{\sqrt{Q_0}}{2}V_{\mathcal{I}k_1} - \frac{\omega_{k_1}\Delta_{k_1B}}{2} + \omega_{k_1}\gamma_1^{01},\\
\Gamma_1^{03} &= \frac{\sqrt{Q_0}}{6}V_{k_1k_2k_3} + (\omega_{k_1}+\omega_{k_2}+\omega_{k_3})\gamma_1^{03}
\end{aligned} \tag{4.12}$$

and so the state at order $i = 1$ is given by the ϕ_0 descendants in Eqs. (3.10) and (3.12) together with the ϕ_0 primaries

$$\gamma_1^{01} = \frac{\Delta_{k_1B}}{2} - \frac{\sqrt{Q_0}}{2}Y_{\mathcal{I}k_1}, \quad \gamma_1^{03} = -\frac{\sqrt{Q_0}}{6}Y_{k_1k_2k_3}, \tag{4.13}$$

where we have defined the reduced potential

$$Y_{k_1\cdots k_j} = \frac{V_{k_1\cdots k_j}}{\omega_{k_1}+\cdots+\omega_{k_j}}, \qquad Y_{\mathcal{I},k_1\cdots k_j} = \frac{V_{\mathcal{I},k_1\cdots k_j}}{\omega_{k_1}+\cdots+\omega_{k_j}}. \tag{4.14}$$

This is depicted in the top-left panel of Fig. 2. Here one sees that the leading order γ_0^{00} contribution corresponding to the star contributes to four elements γ_1, shown as squares. Two of these are descendants and so were already fixed by translation invariance.

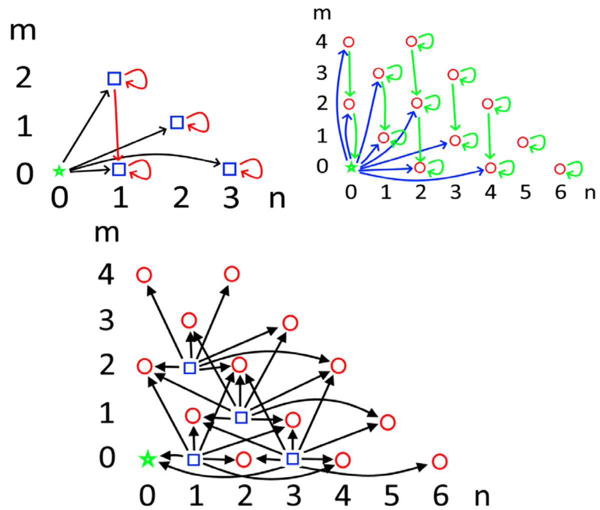

FIG. 2. Terms in the Schrödinger equation: $H_3|0\rangle_0$ in black and $(H_2 - Q_1)|0\rangle_1$ in red (top left), $(H_4 - Q_2)|0\rangle_0$ in blue and $(H_2 - Q_1)|0\rangle_2$ in green (top right), and $H_3|0\rangle_1$ in black (bottom).

Note that in models like the ϕ^4 double well, in which the third derivative of the potential is nonzero at the minima, $V_{k_1k_2k_3}$ will have a divergence of the form $\delta(\sum_i k_i)$. When integrated over k to determine the state, this divergence leads to finite coefficients. However at two loops it leads to an infrared divergence in the energy of the kink state. As we will see in Sec. V B, this infrared divergence also appears in the vacuum energy and so the kink mass, which is the difference between the energies of the two states, is finite.

B. The kink ground state at two loops

Translation invariance fixes all ϕ_0-descendant components γ_i^{mn} in any Hamiltonian eigenstate. The ϕ_0-primary terms γ_i^{0n}, at each order i are fixed by the Schrödinger equation. Interaction terms relate these coefficients to those at lower orders. Thus the other γ_i^{mn} are only related to γ_i^{0n} by the free Hamiltonian (4.4). More specifically, γ_i^{0n} is related to γ_i^{2n} by the $\pi_0^2/2$ term and to γ_i^{0n} by the oscillator term. This allows each ϕ_0 primary γ_i^{0n} to be determined from γ_i^{2n} and the state at orders less than i. In theories, like those considered here, with nonderivative interactions the situation is even simpler because interactions never decrease m. Thus Schrödinger's equation determines ϕ_0 primaries γ_i^{0n} in terms of γ_i^{2n} and ϕ_0 primaries $\gamma_j^{0n'}$ with $j < i$. In other words, only the ϕ_0 descendants at $m = 2$ are needed. Similarly the energy at each order i is determined by γ_i^{20} together with the ϕ_0 primaries γ_j^{0n} at lower orders $j < i$.

This observation in practice leads to a dramatic reduction in the complexity of calculations of states and energies. For example, to compute the two-loop energy of the kink ground state, one only needs to know γ_2^{20}, γ_1^{01}, and γ_1^{03}, which themselves are determined from γ_1^{21} and γ_1^{12}. In this subsection we will complete the calculation of the kink ground state at two loops, corresponding to $i = 2(2-1) = 2$, by finding the ϕ_0 primaries.

At $i = 2$ the Schrödinger equation is

$$(H_4 - Q_2)|0\rangle_0 + H_3|0\rangle_1 + (H_2 - Q_1)|0\rangle_2 = 0. \tag{4.15}$$

The H_4 term and H_3 term are, respectively, shown in the top-right and bottom panels of Fig. 2. In both cases, the goal is to compute the red circles, corresponding to γ_2^{mn}, which lie at the ends of the arrows. Again there is one contribution from each arrow, proportional to the γ at the beginning of the arrow.

1. $(m=0, n=6)$

Let us begin with the simplest element, Γ_2^{06}. The previous argument agrees with Fig. 2 showing that there are two contributions, arising from γ_1^{03}, which was found at the previous order, and from γ_2^{06} which is to be found now. Defining the total energy

$$\Omega_n = \sum_{j=1}^{n} \omega_{k_j}, \tag{4.16}$$

these contributions are

$$H_3|0\rangle_1^{03} \supset -\frac{1}{36}\int \frac{d^6k}{(2\pi)^6} Y_{k_1k_2k_3} V_{k_4k_5k_6} B_{k_1}^\dagger \cdots B_{k_6}^\dagger |0\rangle_0,$$
$$H_2|0\rangle_2^{06} = \frac{1}{Q_0}\int \frac{d^6k}{(2\pi)^6} \Omega_6 \gamma_2^{06} B_{k_1}^\dagger \cdots B_{k_6}^\dagger |0\rangle_0, \tag{4.17}$$

and so one finds the matrix element

$$\gamma_2^{06} = \frac{Q_0}{36} Y_{k_1k_2k_3} \frac{V_{k_4k_5k_6}}{\Omega_6}. \tag{4.18}$$

2. $(m=0, n=4)$

To organize the calculations of the other matrix elements, we note that Γ may be decomposed into contributions that do not mix with one another. In particular contributions with different numbers of dummy momenta k' and with different numbers of powers of the undifferentiated[4] contraction factor $\mathcal{I}(x)$ together with $V^{(3)}$ do not mix. We will include this decomposition in the subscript of Γ. Of course each Γ_i^{0n} determines γ_i^{0n} whose form is not known before Γ_i^{0n} is calculated, so terms resulting from γ_i^{0n} will not be included in this decomposition.

[4]If multiplied by $V^{(4)} g_B(x)$ then an integration by parts leads to a differentiated $\mathcal{I}(x)$, which can be evaluated using (2.24) and this argument does not apply. This situation does not arise in the calculation of γ_2^{0n} but does arise when verifying that the Schrödinger equation is satisfied in Appendix A.

Let us begin with all contributions $\Gamma_{2\mathcal{I}}^{04}$ containing a single power of the contraction factor $\mathcal{I}(x)$. These contributions arise from two terms

$$H_3|0\rangle_1^{03} \supset -\frac{1}{12}\int \frac{d^4k}{(2\pi)^4} V_{\mathcal{I}k_4} Y_{k_1k_2k_3} B_{k_1}^\dagger \cdots B_{k_4}^\dagger |0\rangle_0,$$
$$H_3|0\rangle_1^{01} \supset -\frac{1}{12}\int \frac{d^4k}{(2\pi)^4} Y_{\mathcal{I}k_4} V_{k_1k_2k_3} B_{k_1}^\dagger \cdots B_{k_4}^\dagger |0\rangle_0, \tag{4.19}$$

whose sum yields

$$\Gamma_{2\mathcal{I}}^{04} = -\frac{Q_0}{12} Y_{\mathcal{I}k_4} Y_{k_1k_2k_3} \Omega_4. \tag{4.20}$$

Next let us consider the contributions with one contracted momentum k'. There is only one

$$H_3|0\rangle_1^{03} \supset -\frac{1}{8}\int \frac{d^4k}{(2\pi)^4} \int \frac{dk'}{2\pi} Y_{k_1k_2-k'} \frac{V_{k_3k_4-k'}}{\omega_{k'}} B_{k_1}^\dagger \cdots B_{k_4}^\dagger |0\rangle_0 \tag{4.21}$$

yielding

$$\Gamma_{2k'}^{04} = -\frac{Q_0}{8}\int \frac{dk'}{2\pi} Y_{k_1k_2-k'} \frac{V_{k_3k_4-k'}}{\omega_{k'}}. \tag{4.22}$$

Finally there are three contributions with no k'

$$H_3|0\rangle_1^{01} \supset \frac{1}{12\sqrt{Q_0}}\int \frac{d^4k}{(2\pi)^4} \Delta_{k_1B} V_{k_2k_3k_4} B_{k_1}^\dagger \cdots B_{k_4}^\dagger |0\rangle_0,$$
$$\frac{\pi_0^2}{2}|0\rangle_2^{24} = \frac{1}{12Q_0}\int \frac{d^4k}{(2\pi)^4} [-6\omega_{k_1}\omega_{k_3}\Delta_{k_1k_2}\Delta_{k_3k_4} + \sqrt{Q_0} Y_{k_1k_2k_3}\omega_{k_4}\Delta_{k_4B}] B_{k_1}^\dagger \cdots B_{k_4}^\dagger |0\rangle_0,$$
$$H_4|0\rangle_0 = \frac{1}{24}\int \frac{d^4k}{(2\pi)^4} V_{k_1k_2k_3k_4} B_{k_1}^\dagger \cdots B_{k_4}^\dagger |0\rangle_0, \tag{4.23}$$

which sum to

$$\Gamma_{2k'0}^{04} = \frac{\sqrt{Q_0}}{12} Y_{k_1k_2k_3}\Delta_{k_4B}\Omega_4 - \frac{\omega_{k_1}\omega_{k_3}}{2}\Delta_{k_1k_2}\Delta_{k_3k_4} + \frac{Q_0}{24} V_{k_1k_2k_3k_4}. \tag{4.24}$$

The final contribution to Γ_2^{04} arises from

$$\int \frac{dk}{2\pi}\omega_k B_k^\dagger B_{-k}|0\rangle_2^{04} = \frac{1}{Q_0}\int \frac{d^4k}{(2\pi)^4} \Omega_4 \gamma_2^{04} B_{k_1}^\dagger \cdots B_{k_4}^\dagger |0\rangle_0 \tag{4.25}$$

and is

$$\Gamma_{2f}^{04} = \Omega_4 \gamma_2^{04}. \tag{4.26}$$

The Schrödinger equation

$$0 = \Gamma_2^{04} = \Gamma_{2f}^{04} + \Gamma_{2\mathcal{I}}^{04} + \Gamma_{2k'0}^{04} + \Gamma_{2k'}^{04}, \tag{4.27}$$

then yields the matrix element

$$\gamma_2^{04} = -\frac{\Gamma_{2\mathcal{I}}^{04} + \Gamma_{2k'}^{04} + \Gamma_{2k'0}^{04}}{\Omega_4},$$
$$= \frac{Q_0}{12} Y_{\mathcal{I}k_4} Y_{k_1k_2k_3} - \frac{\sqrt{Q_0}}{12} Y_{k_1k_2k_3} \Delta_{k_4B} + \frac{\omega_{k_1}\omega_{k_3}}{2\Omega_4} \Delta_{k_1k_2} \Delta_{k_3k_4} - \frac{Q_0}{24} Y_{k_1k_2k_3k_4} + \frac{Q_0}{8\Omega_4} \int \frac{dk'}{2\pi} Y_{k_1k_2-k'} \frac{V_{k_3k_4-k'}}{\omega_{k'}}. \tag{4.28}$$

Note that in models like the Sine-Gordon model, in which the fourth derivative of the potential is nonzero at the minima, $Y_{k_1k_2k_3k_4}$ will have a divergence of the form $\delta(\sum_i k_i)$. When integrated over k to determine the state, this divergence leads to finite coefficients. However at three loops it leads to an infrared divergence in the energy of the kink state. As in the two-loop divergence in the ϕ^4 kink energy, this divergence also appears in the vacuum energy and so the kink mass remains finite. We expect such cancellations at all loops, as the infrared divergences arise from a regime in x where $f(x)$ is equal to a vacuum value, and so the energy contribution from the kink and vacuum sector should agree.

3. $(m=0, n=2)$

The last matrix element needed to fix the ground state at two loops is $(m=0, n=2)$. There is one contribution with two powers of the contraction factor $\mathcal{I}$

$$H_3|0\rangle_1^{01} \supset -\frac{1}{4} \int \frac{d^2k}{(2\pi)^2} Y_{\mathcal{I}k_1} V_{\mathcal{I}k_2} B_{k_1}^\dagger B_{k_2}^\dagger |0\rangle_0, \tag{4.29}$$

which, after adding an antisymmetric term which does not affect the sum, yields

$$\Gamma_{2\mathcal{I}^2}^{02} = -\frac{Q_0}{8} Y_{\mathcal{I}k_1} Y_{\mathcal{I}k_2} \Omega_2. \tag{4.30}$$

There are four contributions with a single power of $\mathcal{I}$

$$\frac{\pi_0^2}{2}|0\rangle_2^{22} \supset \frac{1}{4\sqrt{Q_0}} \int \frac{d^2k}{(2\pi)^2} Y_{\mathcal{I}k_1} \omega_{k_2} \Delta_{k_2B} B_{k_1}^\dagger B_{k_2}^\dagger |0\rangle_0,$$
$$H_3|0\rangle_1^{01} \supset \int \frac{d^2k}{(2\pi)^2} \left(\frac{1}{4\sqrt{Q_0}} V_{\mathcal{I}k_2} \Delta_{k_1B} - \frac{1}{8} \int \frac{dk'}{2\pi} Y_{\mathcal{I}k'} \frac{V_{k_1k_2-k'}}{\omega_{k'}} \right) B_{k_1}^\dagger B_{k_2}^\dagger |0\rangle_0,$$
$$H_4|0\rangle_0 \supset \frac{1}{4} \int \frac{d^2k}{(2\pi)^2} V_{\mathcal{I}k_1k_2} B_{k_1}^\dagger B_{k_2}^\dagger |0\rangle_0,$$
$$H_3|0\rangle_1^{03} \supset -\frac{1}{8} \int \frac{d^2k}{(2\pi)^2} \int \frac{dk'}{2\pi} Y_{\mathcal{I}k'} Y_{k_1k_2-k'} B_{k_1}^\dagger B_{k_2}^\dagger |0\rangle_0, \tag{4.31}$$

which together contribute

$$\Gamma_{2\mathcal{I}}^{02} = \frac{\sqrt{Q_0}}{4} Y_{\mathcal{I}k_1} \Delta_{k_2B} \Omega_2 + \frac{Q_0 V_{\mathcal{I}k_1k_2}}{4} - \frac{Q_0}{8} \int \frac{dk'}{2\pi} Y_{\mathcal{I}k'} Y_{k_1k_2-k'} \left(2 + \frac{\Omega_2}{\omega_{k'}}\right). \tag{4.32}$$

Now we will organize the terms with no powers of $\mathcal{I}$ by the number of contracted momenta k'. There is one term with two contracted momenta

$$H_3|0\rangle_1^{03} \supset -\frac{1}{8} \int \frac{d^2k}{(2\pi)^2} \int \frac{d^2k'}{(2\pi)^2} Y_{k_1k_1'k_2'} \frac{V_{k_2-k_1'-k_2'}}{\omega_{k_1'}\omega_{k_2'}} B_{k_1}^\dagger B_{k_2}^\dagger |0\rangle_0, \tag{4.33}$$

yielding

$$\Gamma_{2k'^2}^{02} = -\frac{Q_0}{8} \int \frac{d^2k'}{(2\pi)^2} Y_{k_1k_1'k_2'} \frac{V_{k_2-k_1'-k_2'}}{\omega_{k_1'}\omega_{k_2'}}. \tag{4.34}$$

There are two sources of terms with no $\mathcal{I}$ and a single k'

$$H_3|0\rangle_1^{01} \supset \frac{1}{8\sqrt{Q_0}}\int\frac{d^2k}{(2\pi)^2}\int\frac{dk'}{2\pi}\frac{V_{k_1k_2k'}}{\omega_{k'}}\Delta_{-k'B}B^\dagger_{k_1}B^\dagger_{k_2}|0\rangle_0,$$
$$\frac{\pi_0^2}{2}|0\rangle_2^{22} \supset \int\frac{d^2k}{(2\pi)^2}\int\frac{dk'}{2\pi}\left[-\frac{1}{8\sqrt{Q_0}}Y_{k_1k_2k'}\Delta_{-k'B}+\frac{1}{4Q_0}\Delta_{k_1k'}\Delta_{-k'k_2}\left(1+\frac{\omega_{k_2}}{\omega_{k'}}\right)(\omega_{k_1}-\omega_{k'})\right]B^\dagger_{k_1}B^\dagger_{k_2}|0\rangle_0, \tag{4.35}$$

which contribute

$$\Gamma^{02}_{2k'} = \int\frac{dk'}{2\pi}\left[\frac{\sqrt{Q_0}}{8}\frac{\Omega_2}{\omega_{k'}}Y_{k_1k_2k'}\Delta_{-k'B}+\frac{1}{4}\Delta_{k_1k'}\Delta_{-k'k_2}\left(\frac{\omega_{k_1}\omega_{k_2}}{\omega_{k'}}-\omega_{k'}\right)\right]. \tag{4.36}$$

Finally the terms with neither $\mathcal{I}$ nor k' are

$$\frac{\pi_0^2}{2}|0\rangle_2^{22} \supset -\frac{3}{8Q_0}\int\frac{d^2k}{(2\pi)^2}\Omega_2\Delta_{k_1B}\Delta_{k_2B}B^\dagger_{k_1}B^\dagger_{k_2}|0\rangle_0, \tag{4.37}$$

and so

$$\Gamma^{02}_{2k'0} = -\frac{3}{8}\Omega_2\Delta_{k_1B}\Delta_{k_2B}. \tag{4.38}$$

As in the previous cases,

$$\Gamma^{02}_{2f} = \Omega_2\gamma^{02}_2, \tag{4.39}$$

and so the Schrödinger equation

$$0 = \Gamma^{02}_2 = \Gamma^{02}_{2f}+\Gamma^{02}_{2\mathcal{I}^2}+\Gamma^{02}_{2\mathcal{I}}+\Gamma^{02}_{2k'^2}+\Gamma^{02}_{2k'}+\Gamma^{02}_{2k'0} \tag{4.40}$$

fixes the last matrix element

$$\begin{aligned}\gamma^{02}_2 &= -\frac{\Gamma^{02}_{2\mathcal{I}^2}+\Gamma^{02}_{2\mathcal{I}}+\Gamma^{02}_{2k'^2}+\Gamma^{02}_{2k'}+\Gamma^{02}_{2k'0}}{\Omega_2},\\ &= \frac{Q_0}{8}Y_{\mathcal{I}k_1}Y_{\mathcal{I}k_2}-\frac{\sqrt{Q_0}}{4}Y_{\mathcal{I}k_1}\Delta_{k_2B}-\frac{Q_0V_{\mathcal{I}k_1k_2}}{4\Omega_2}+\frac{3}{8}\Delta_{k_1B}\Delta_{k_2B}+\frac{Q_0}{8\Omega_2}\int\frac{d^2k'}{(2\pi)^2}Y_{k_1k'_1k'_2}\frac{V_{k_2-k'_1-k'_2}}{\omega_{k'_1}\omega_{k'_2}}\\ &\quad+\int\frac{dk'}{2\pi}\left[-\frac{\sqrt{Q_0}}{8}\frac{1}{\omega_{k'}}Y_{k_1k_2k'}\Delta_{-k'B}+\frac{1}{4\Omega_2}\left(Q_0Y_{\mathcal{I}k'}Y_{k_1k_2-k'}\left(1+\frac{\Omega_2}{2\omega_{k'}}\right)+\Delta_{k_1k'}\Delta_{-k'k_2}\left(\omega_{k'}-\frac{\omega_{k_1}\omega_{k_2}}{\omega_{k'}}\right)\right)\right].\end{aligned} \tag{4.41}$$

V. THE KINK MASS

A. The energy of the kink ground state

The last Schrödinger equation is $\Gamma^{00}_2 = 0$. This does not fix γ^{00}_2 because Γ^{00}_2 does not depend on γ^{00}_2. This is reasonable because any value of γ^{00}_2 can be absorbed into the normalization of the state. Thus one may normalize the ground state so that

$$\gamma^{00}_i = \delta_{i0}. \tag{5.1}$$

Let us now solve this last Schrödinger equation. There are two terms with two powers of the contraction factor $\mathcal{I}$

$$H_4|0\rangle_0 \supset \frac{V_{\mathcal{I}\mathcal{I}}}{8}|0\rangle_0,$$
$$H_3|0\rangle_1^{01} \supset -\frac{1}{8}\int\frac{dk'}{2\pi}Y_{\mathcal{I}k'}Y_{\mathcal{I}-k'}|0\rangle_0, \quad (5.2)$$

yielding

$$\Gamma^{00}_{2\mathcal{I}^2} \supset \frac{Q_0}{8}\left(V_{\mathcal{I}\mathcal{I}} - \int\frac{dk'}{2\pi}Y_{\mathcal{I}k'}Y_{\mathcal{I}-k'}\right). \quad (5.3)$$

There are also two terms with a single factor of $\mathcal{I}$

$$H_3|0\rangle_1^{01} \supset \frac{1}{8\sqrt{Q_0}}\int\frac{dk'}{2\pi}Y_{\mathcal{I}k'}\Delta_{-k'B}|0\rangle_0,$$
$$\frac{\pi_0^2}{2}|0\rangle_2^{20} \supset -\frac{1}{8\sqrt{Q_0}}\int\frac{dk'}{2\pi}Y_{\mathcal{I}k'}\Delta_{-k'B}|0\rangle_0, \quad (5.4)$$

which precisely cancel. The terms with no factors of $\mathcal{I}$ can be organized by the number of contracted momenta k'. There is one term with 3, 2, and 1 momenta, respectively, which for brevity we summarize together

$$H_3|0\rangle_1^{03} \supset -\frac{1}{48}\int\frac{d^3k'}{(2\pi)^3}Y_{k'_1k'_2k'_3}\frac{V_{-k'_1-k'_2-k'_3}}{\omega_{k'_1}\omega_{k'_2}\omega_{k'_3}},$$
$$\frac{\pi_0^2}{2}|0\rangle_2^{20} \supset \frac{1}{16Q_0}\int\frac{d^2k'}{(2\pi)^2}\frac{(\omega_{k'_1}-\omega_{k'_2})^2}{\omega_{k'_1}\omega_{k'_2}}\Delta_{k'_1k'_2}\Delta_{-k'_1-k'_2},$$
$$\frac{\pi_0^2}{2}|0\rangle_2^{20} \supset -\frac{1}{8Q_0}\int\frac{dk'}{2\pi}\Delta_{Bk'}\Delta_{B-k'}. \quad (5.5)$$

As

$$g_k^*(x) = g_{-k}(x) \quad (5.6)$$

the symbols Δ, V, and Y are all complex conjugated when all of their k arguments are negated. Therefore these contributions can each be rewritten as norms squared and so are real. The corresponding Γ can therefore be written

$$\Gamma^{00}_{2k'^3} = -\frac{Q_0}{48}\int\frac{d^3k'}{(2\pi)^3}\frac{|V_{k'_1k'_2k'_3}|^2}{\omega_{k'_1}\omega_{k'_2}\omega_{k'_3}(\omega_{k'_1}+\omega_{k'_2}+\omega_{k'_3})},$$
$$\Gamma^{00}_{2k'^2} = \frac{1}{16}\int\frac{d^2k'}{(2\pi)^2}\frac{|(\omega_{k'_1}-\omega_{k'_2})\Delta_{k'_1k'_2}|^2}{\omega_{k'_1}\omega_{k'_2}},$$
$$\Gamma^{00}_{2k'} = -\frac{1}{8}\int\frac{dk'}{2\pi}|\Delta_{k'B}|^2. \quad (5.7)$$

The last term may be written in a more convenient form using the completeness relation (2.16)

$$\int\frac{dk}{2\pi}\Delta_{kB}\Delta_{-kB} = \frac{1}{Q_0}\int dx\int dy\int\frac{dk}{2\pi}g_k(x)g_{-k}(y)f''(x)f''(y),$$
$$= \frac{1}{Q_0}\int dx\int dy(\delta(x-y) - g_B(x)g_B(y))f''(x)f''(y) = \frac{1}{Q_0}\int dx|f''(x)|^2, \quad (5.8)$$

where the $g_B(x)f''(x)$ integrals vanish because they are proportional to the total derivative of $g_B^2(x)$.

The Schrödinger equation then gives the two-loop energy

$$Q_2 = \frac{1}{Q_0}(\Gamma^{00}_{2\mathcal{I}^2} + \Gamma^{00}_{2k'^3} + \Gamma^{00}_{2k'^2} + \Gamma^{00}_{2k'}),$$
$$= \frac{V_{\mathcal{I}\mathcal{I}}}{8} - \frac{1}{8}\int\frac{dk'}{2\pi}|Y_{\mathcal{I}k'}|^2 - \frac{1}{48}\int\frac{d^3k'}{(2\pi)^3}\frac{|V_{k'_1k'_2k'_3}|^2}{\omega_{k'_1}\omega_{k'_2}\omega_{k'_3}(\omega_{k'_1}+\omega_{k'_2}+\omega_{k'_3})}$$
$$+ \frac{1}{16Q_0}\int\frac{d^2k'}{(2\pi)^2}\frac{|(\omega_{k'_1}-\omega_{k'_2})\Delta_{k'_1k'_2}|^2}{\omega_{k'_1}\omega_{k'_2}} - \frac{1}{8Q_0^2}\int dx|f''(x)|^2. \quad (5.9)$$

To our knowledge, this is the first time that the two-loop energy has been calculated for kinks that need be neither integrable nor supersymmetric. The explicit calculation of Refs. [18,19], in the case of the Sine-Gordon model, did not require integrability and so could be repeated in this general setting. However in that case we stress that the energy was found by summing 13 divergent Feynman diagrams, and carefully regulating and subtracting the divergences. Here instead we find five terms, each of which is already UV finite. Let us identify each of these terms.

When changing from plane wave to normal mode normal ordering, so that H_2 annihilates the one-loop kink ground state, the interaction Hamiltonian acquired constant and tadpole terms. The first two terms in (5.9) are just the corresponding leading shifts to the energy, equal to the constant $V_{\mathcal{I}\mathcal{I}}$ plus the first perturbative contribution from the tadpole $V_{\mathcal{I}k'}$. The next term is the usual one-loop perturbation theory correction to an energy arising from a cubic interaction, and is given by the same expression as in the vacuum sector (5.16) with plane waves replaced by normal modes. The fourth term is a correction to the third term arising from the fact that derivative operators mix the normal modes, which is not the case for plane waves. The last term was found long ago [18,20] using the collective coordinate approach, where it appeared as the leading term in an expansion of the denominator of an effective Hamiltonian, which came from a canonical transformation that separated a nonlinear extension of $\phi_B(x)$. The manipulations that led to its appearance here are very different from those in the collective coordinate approach, but it is reassuring to see this agreement in the result.

The only trace of renormalization can be found in the first two terms, in the function $\mathcal{I}(x)$ which is the expected difference between two divergent sums weighted by $1/\omega_k$ and $1/\omega_p$, respectively. In models such as the ϕ^4 double well, in which the potential has a nonvanishing third derivative at the minima, the third term will be IR divergent. This divergence arises from the region far from the kink, and so its contribution to the kink mass will be canceled by the same IR divergence in the vacuum energy, which we will now calculate.

B. Vacuum sector energy

The kink mass is generally not Q_2. It is $Q_2 - E_1$ where E_1 is the one-loop correction to the vacuum sector energy, as this contributes at the same order. It is easily computed in perturbation theory. Decompose the field in terms of plane waves as

$$\phi(x) = \int \frac{dp}{2\pi}\left(A_p^\dagger + \frac{A_{-p}}{2\omega_p}\right)e^{-ipx} \tag{5.10}$$

and the free and interaction Hamiltonians can be written

$$H_2 = \int \frac{dp}{2\pi}\omega_p A_p^\dagger A_p,$$
$$H_{n>2} = \frac{1}{n!}\int dx V^{(n)}[\phi_0]:\phi^n(x):_a, \tag{5.11}$$

where ϕ_0 is the minimum of V corresponding to the vacuum. Then the first order of perturbation theory

$$H_3|\Omega\rangle_0 + H_2|\Omega\rangle_1 = 0 \tag{5.12}$$

yields the first order correction $|\Omega\rangle_1$ to the vacuum state $|\Omega\rangle$

$$|\Omega\rangle_1 = -\frac{V^{(3)}[\phi_0]}{6}\int \frac{d^3p}{(2\pi)^3}\frac{2\pi\delta(p_1+p_2+p_3)}{\omega_{p_1}+\omega_{p_2}+\omega_{p_3}} \times A_{p_1}^\dagger A_{p_2}^\dagger A_{p_3}^\dagger|\Omega\rangle_0. \tag{5.13}$$

Acting again with H_3, the $|\Omega\rangle_0$ term yields the one loop correction to the energy

$$H_3|\Omega\rangle_1 \supset -\frac{(V^{(3)}[\phi_0])^2}{48}\int dx \int \frac{d^3p'}{(2\pi)^3}e^{-ix(p_1'+p_2'+p_3')}\frac{2\pi\delta(p_1+p_2+p_3)}{\omega_{p_1'}\omega_{p_2'}\omega_{p_3'}(\omega_{p_1'}+\omega_{p_2'}+\omega_{p_3'})}|\Omega\rangle_0,$$
$$= -\frac{(V^{(3)}[\phi_0])^2}{48}L\int \frac{d^3p'}{(2\pi)^3}\frac{2\pi\delta(p_1'+p_2'+p_3')}{\omega_{p_1'}\omega_{p_2'}\omega_{p_3'}(\omega_{p_1'}+\omega_{p_2'}+\omega_{p_3'})}|\Omega\rangle_0, \tag{5.14}$$

where L is the length of the spatial direction,[5] which serves as an infrared cutoff.

The subleading correction to the Schrödinger equation is

$$(H_4 - E_1)|\Omega\rangle_0 + H_3|\Omega\rangle_1 + H_2|\Omega\rangle_2 = 0. \tag{5.15}$$

[5]Here we are cavalier with boundary conditions, as the theory contains only scalar fields. In practice, we simply subtract the kink and vacuum energy densities before performing the x integration, in which case the integral converges. In a theory with fermions a more careful approach may be warranted, for example adding a distant antikink to each kink to allow identical boundary conditions in each sector.

As H_4 is normal ordered, $H_4|\Omega\rangle_0$ is orthogonal to $|\Omega\rangle_0$ and so does not contribute to E_1. We will chose $|\Omega\rangle_2$ to be orthogonal to $|\Omega\rangle_0$ so that the last term does not contribute to E_1. Then E_1 can be read off of (5.14). Evaluating the delta function, this is

$$E_1 = -\frac{(V^{(3)}[\phi_0])^2}{48} \times L\int \frac{d^2p'}{(2\pi)^2}\frac{1}{\omega_{p_1'}\omega_{p_2'}\omega_{p_1'+p_2'}(\omega_{p_1'}+\omega_{p_2'}+\omega_{p_1'+p_2'})}. \tag{5.16}$$

The dependence on the infrared cutoff L implies that we have calculated an energy density, and not an energy. When this energy density is nonvanishing, it must be subtracted from the kink ground state energy to obtain the kink mass. The kink mass will be finite only if these divergences cancel. This procedure depends on the matching of the infrared divergences, which can be achieved for example if the energy densities are subtracted before they are integrated. If the potential is symmetric about the minimum ϕ_0, as it is in the case of the Sine-Gordon model but not the ϕ^4 double well, $V^{(3)}[\phi_0]$ vanishes, and so $E_1 = 0$ and this complication is avoided. This procedure is performed in the case of the ϕ^4 model in Ref. [21].

C. The Sine-Gordon model

In the case of the Sine-Gordon model, the two-loop mass has been conjectured in [22] and calculated in [5,18,19,23]. It is of course dependent upon the renormalization scheme [22] although in some schemes there is a renormalization group flow invariant coupling that provides a universal relation between the kink and meson mass. No such relation may be expected to hold in general as there are other schemes in which the coupling may be shifted by any finite amount at any scale.

Using the well-known Sine-Gordon normal modes [12]

$$g_k(x) = \frac{e^{-ikx}\mathrm{sign}(k)}{\omega_k}(k - im\tanh(mx)), \qquad g_B(x) = \sqrt{\frac{m}{2}}\mathrm{sech}(mx), \tag{5.17}$$

a contour integration yields

$$\Delta_{kB} = \frac{i\pi\omega_k}{\sqrt{8M}}\mathrm{sech}\left(\frac{k\pi}{2M}\right)\mathrm{sign}(k),$$
$$\Delta_{k_1k_2} = -i(k_1 - k_2)\pi\delta(k_1 + k_2) + \frac{i\pi}{2}\frac{(k_2^2 - k_1^2)}{\omega_{k_1}\omega_{k_2}}\mathrm{csch}\left(\frac{\pi(k_1 + k_2)}{2M}\right)\mathrm{sign}(k_1k_2). \tag{5.18}$$

Using

$$V^{(3)}[\sqrt{\lambda}f(x)] = 2M^2 g\tanh(Mx)\mathrm{sech}(Mx), \tag{5.19}$$

we obtain

$$V_{k_1k_2k_3} = \frac{\pi i}{4}g\mathrm{sign}(k_1k_2k_3)\mathrm{sech}\left(\frac{\pi(k_1 + k_2 + k_3)}{2M}\right)\left[-\left(\frac{\omega_{k_1}^3}{\omega_{k_2}\omega_{k_3}} + \frac{\omega_{k_2}^3}{\omega_{k_1}\omega_{k_3}} + \frac{\omega_{k_3}^3}{\omega_{k_1}\omega_{k_2}}\right) + 2\left(\frac{\omega_{k_1}\omega_{k_2}}{\omega_{k_3}} + \frac{\omega_{k_1}\omega_{k_3}}{\omega_{k_2}} + \frac{\omega_{k_2}\omega_{k_3}}{\omega_{k_1}}\right)\right]. \tag{5.20}$$

We have evaluated the energy (5.9) term by term.

In the first two terms, $\mathcal{I}(x)$ appears. It was calculated in Ref. [12] by integrating the general identity (2.24). Using the present conventions

$$\mathcal{I}(x) = -\frac{\mathrm{sech}^2(Mx)}{2\pi} \tag{5.21}$$

and so

$$V_{\mathcal{I}k} = \frac{i}{8M^2}g\omega_k^3\mathrm{sign}(k)\mathrm{sech}\left(\frac{\pi k}{2M}\right). \tag{5.22}$$

Using

$$V^{(4)}[\sqrt{\lambda}f(x)] = M^2\lambda(-1 + 2\mathrm{sech}^2(Mx)) \tag{5.23}$$

one finds $M\lambda/(40\pi^2)$ and $-M\lambda/(120\pi^2)$ for the first and second terms of (5.9). In the fourth term, the delta function in (5.18) is multiplied by a zero, which leads to a vanishing contribution, as can be checked directly by considering the case $k_1 = k_2$ separately from the beginning. Fixing the mass M and coupling $\sqrt{\lambda}$ to unity, the third, fourth, and fifth terms are equal to terms which may be found in Ref. [18] and they were evaluated analytically by Verwaest who found that the sum of the third and fourth is $-M\lambda/(60\pi^2)$ while the fifth is $-M\lambda/192$. Altogether we find that the two-loop correction to the kink mass is

$$Q_2 = -\frac{M\lambda}{192} \tag{5.24}$$

in agreement with the literature. As shown in Appendix B our normal ordering prescription yields the same meson mass as Ref. [18] and so the soliton to meson mass ratio agrees, in accordance with Refs. [22,23].

VI. REMARKS

Calculations of masses of quantum kinks have been an industry from Ref. [1] to Refs. [24,25]. So far these calculations have been largely at one loop, where they are described by a free theory, with the exception of integrable and supersymmetric models. In this paper, we have calculated the two-loop masses of scalar kinks in theories with arbitrary potentials. We have also explicitly constructed their states, with the ϕ_0 descendants calculated in Sec. III using translation invariance and the ϕ_0-primary components in Sec. IV using the Schrödinger equation. These constructions we feel are even more interesting than the masses, as they allow one to compute matrix elements and so open the door to understanding the phenomenology [26], such as scattering [27–29] and acceleration [30,31] of quantum kinks beyond the harmonic oscillator approximation. For example, one may calculate form factors [32,33].

While we only calculated the ground state, starting our recursion with a superposition of normal modes would have allowed us to apply the same strategy to an arbitrary state in the one-kink sector.

The key step in our calculation was perturbatively imposing the translation invariance conditions, which fixed most matrix elements of the state, the ϕ_0 descendants, in terms of a few coefficients, the ϕ_0 primaries. The ϕ_0-primary components needed to be fixed using ordinary perturbation theory. More generally, in the case of any translation-invariant Hamiltonian, as the Hamiltonian and momentum operators commute, a basis of all Hamiltonian eigenstates may be obtained by first fixing the momentum to obtain the ϕ_0 descendant matrix elements in terms of ϕ_0-primary matrix elements γ_i^{0n} and then using the Schrödinger equation to fix the ϕ_0-primary matrix elements.

In the case of a BPS[6] state in a supersymmetric model one may first impose both translation invariance and also that the state be invariant under the preserved supersymmetries. Presumably this will strongly constrain the state. The big question is whether, in a sufficiently supersymmetric model, this may constrain the state sufficiently that perturbation theory is no longer required. In this case, one would have finally opened the door to a truly quantum understanding of nonperturbative solitons. More precisely, one could understand the physical mechanisms at work behind the nonrenormalization theorems. This of course is a prerequisite for applying lessons from supersymmetric theories to Yang-Mills.

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APPENDIX A: CHECKING SCHRÖDINGER'S EQUATION

We have derived the two-loop ground state using translation invariance together with Schrödinger's equation. We restricted Schrödinger's equation to the ϕ_0 primaries, the subspace of the Fock space with no ϕ_0 acting on $|0\rangle_0$, but we argued that, since the Hamiltonian and momentum operators commute, we expect our solutions to solve the Schrödinger equation in the full Fock space. By imposing a condition on the momentum it is not possible that we lose the ground state solution, since it indeed must have zero momentum. Furthermore, since the solution that we find, given the one-loop contribution, is unique, it must be the ground state.

In this appendix we explicitly check this claim by inserting our two-loop state into the Schrödinger equation and showing that it vanishes on the full Fock space. More precisely, we compute the various ϕ_0-descendant components Γ_2^{mn} at $m > 0$ and show that they each vanish as claimed. Recall that in Sec. IV B we found the ϕ_0 primaries γ_2^{0n} by imposing that Γ_2^{0n} vanishes, and so we already know that the $m = 0$ Schrödinger equation is satisfied.

1. $m = 5$, $n = 1$

The only contribution

$$H_3|0\rangle_1^{21} \supset \frac{1}{6}\int dx V^{(3)}[\sqrt{\lambda}f(x)]g_B^3(x)\phi_0^3|0\rangle_1^{21} = 0 \tag{A1}$$

vanishes because

$$\begin{aligned} V_{BBB} &= \int dx V^{(3)}[\sqrt{\lambda}f(x)]g_B^3(x) = \frac{1}{\sqrt{Q_0}}\int dx(\partial_x V^{(2)}[\sqrt{\lambda}f(x)])g_B^2(x), \\ &= -\frac{2}{\sqrt{Q_0}}\int dx V^{(2)}[\sqrt{\lambda}f(x)]g_B(x)g_B'(x) = -\frac{2}{\sqrt{Q_0}}\int dx g_B''(x)g_B'(x) = 0 \end{aligned} \tag{A2}$$

is a total derivative.

[6] A state is BPS if it is annihilated by any nonzero element of the supersymmetry algebra. In this case, the operator is annihilated by half of the fermionic generators.

2. $m=4$, $n=2$

$$H_3|0\rangle_1^{21} \supset \frac{1}{4\sqrt{Q_0}}\int\frac{d^2k}{(2\pi)^2}\omega_{k_1}\Delta_{k_1B}V_{k_2BB}\phi_0^4B_{k_1}^\dagger B_{k_2}^\dagger|0\rangle_0 \tag{A3}$$

exactly cancels

$$\int\frac{dk'}{2\pi}\omega_{k'}B_{k'}^\dagger B_{k'}|0\rangle_2^{42} = \frac{1}{4Q_0}\int\frac{d^2k}{(2\pi)^2}\omega_{k_1}^2\omega_{k_2}\Delta_{k_1B}\Delta_{k_2B}\phi_0^4B_{k_1}^\dagger B_{k_2}^\dagger|0\rangle_0 \tag{A4}$$

as a result of (4.10).

3. $m=4$, $n=0$

$$H_3|0\rangle_1^{21} \supset -\frac{1}{8}\int\frac{d^1k}{(2\pi)^1}Y_{kBB}Y_{-kBB}\phi_0^4|0\rangle_0 \tag{A5}$$

exactly cancels

$$H_4|0\rangle_0 \supset \frac{V_{BBBB}}{24}\phi_0^4|0\rangle_0 \tag{A6}$$

as

$$\begin{aligned}
V_{BBBB} &= \int dxV^{(4)}[\sqrt{\lambda}f(x)]\lambda_B^2(x) = \int dxV^{(4)}[\sqrt{\lambda}f(x)]g_B^3(x)\frac{f'(x)}{\sqrt{Q_0}},\\
&= \frac{1}{\sqrt{Q_0}}\int dx\partial_x(V^{(3)}[\sqrt{\lambda}f(x)])g_B^3(x) = -\frac{3}{\sqrt{Q_0}}\int dxV^{(3)}[\sqrt{\lambda}f(x)]g_B^2(x)g_B'(x),\\
&= -\frac{3}{\sqrt{Q_0}}\int dxV^{(3)}[\sqrt{\lambda}f(x)]g_B^2(x)\int dy\delta(x-y)g_B'(y),\\
&= -\frac{3}{\sqrt{Q_0}}\int dxV^{(3)}[\sqrt{\lambda}f(x)]g_B^2(x)\int dy\left[g_B(x)g_B(y)+\int\frac{dk}{2\pi}g_k(x)g_{-k}(y)\right]g_B'(y),\\
&= -\frac{3}{\sqrt{Q_0}}\int\frac{dk}{2\pi}V_{kBB}\Delta_{-kB} = 3\int\frac{dk}{2\pi}Y_{kBB}Y_{-kBB}.
\end{aligned} \tag{A7}$$

4. $m=3$, $n=3$

The two terms in

$$\int\frac{dk'}{2\pi}\omega_{k'}B_{k'}^\dagger B_{k'}|0\rangle_2^{33} = -\frac{1}{4\sqrt{Q_0}}\int\frac{d^3k}{(2\pi)^3}[V_{k_1k_2B}\omega_{k_3}\Delta_{k_3B}+V_{k_3BB}(\omega_{k_1}-\omega_{k_2})\Delta_{k_1k_2}]\phi_0^3B_{k_1}^\dagger B_{k_2}^\dagger B_{k_3}^\dagger|0\rangle_0 \tag{A8}$$

are, respectively, canceled by $H_3|0\rangle_1^{21}$ and $H_3|0\rangle_1^{12}$.

5. $m = 3$, $n = 1$

We will need the identity

$$\begin{aligned} V_{\mathcal{I}B} &= \int dx V^{(3)}[\sqrt{\lambda} f(x)] g_B(x)\mathcal{I}(x) = \frac{1}{\sqrt{Q_0}}\int dx (\partial_x V^{(2)}[\sqrt{\lambda}f(x)])\mathcal{I}(x), \\ &= -\frac{1}{\sqrt{Q_0}}\int dx V^{(2)}[\sqrt{\lambda}f(x)]\mathcal{I}'(x) = -\frac{1}{\sqrt{Q_0}}\int dx V^{(2)}[\sqrt{\lambda}f(x)]\int \frac{dk}{2\pi}\frac{g_k(x)g'_{-k}(x)}{\omega_k}, \\ &= -\frac{1}{\sqrt{Q_0}}\int\frac{dk}{2\pi}\int dx\left(\omega_k g_k(x) + \frac{g''_k(x)}{\omega_k}\right)g'_{-k}(x), \\ &= -\frac{1}{2\sqrt{Q_0}}\int\frac{dk}{2\pi}\int dx \partial_x\left(\omega_k|g_k(x)|^2 + \frac{|g'_k(x)|^2}{\omega_k}\right) = 0. \end{aligned} \tag{A9}$$

Note that although this is the integral of a total derivative, the differentiated function does not vanish at infinity. The integral vanishes because the differentiated function is even in x. This is true at each k if the potential is symmetric under an inversion that exchanges the two minima responsible for the kink, as it is in the Sine-Gordon model and the ϕ^4 model. More generally, in the large x region which fixes this integral by the fundamental theorem of calculus, the functions $g_k(x)$ are plane waves and their norm is constant and independent of the potential. In the case of a reflectionless potential, the norm is equal in both asymptotic regimes and so this integral vanishes for each value of k. More generally, the integral vanishes when summed over k and $-k$ as the summed norms squared are equal in the two asymptotic regions.

Using this identity, one evaluates the contribution of $|0\rangle_1^{21}$ to be

$$H_3|0\rangle_1^{21} \supset \frac{1}{4\sqrt{Q_0}}\int\frac{dk}{2\pi}\int\frac{dk'}{2\pi}V_{Bk-k'}\Delta_{k'B}\phi_0^3 B_k^\dagger|0\rangle_0. \tag{A10}$$

Similarly

$$\begin{aligned} H_3|0\rangle_1^{12} &\supset \frac{1}{4\sqrt{Q_0}}\int\frac{dk}{2\pi}\int\frac{dk'}{2\pi}\left(\frac{\omega_k}{\omega_{k'}} - 1\right)V_{BB-k'}\Delta_{kk'}\phi_0^3 B_k^\dagger|0\rangle_0, \\ \int\frac{dk'}{2\pi}\omega_{k'}B_{k'}^\dagger B_{k'}|0\rangle_2^{31} &= \frac{1}{12Q_0}\int\frac{dk}{2\pi}\int\frac{dk'}{2\pi}(3\omega_{k'} - \omega_k)\omega_k\Delta_{-k'B}\Delta_{kk'}\phi_0^3 B_k^\dagger|0\rangle_0. \end{aligned} \tag{A11}$$

The final contribution is

$$H_4|0\rangle_0 = \frac{1}{6}\int\frac{dk}{2\pi}V_{BBBk}\phi_0^3 B_k^\dagger|0\rangle_0. \tag{A12}$$

The Vs may all be traded for Δsusing (4.10) and, as may be derived similarly to (A7),

$$\begin{aligned} V_{BBBk} &= \frac{1}{\sqrt{Q_0}}\int dx \partial_x(V^{(3)}[\sqrt{\lambda}f(x)])g_B(x)^2 g_k(x), \\ &= -\frac{1}{\sqrt{Q_0}}\int dx V^{(3)}[\sqrt{\lambda}f(x)]\left[2g_B(x)g_k(x)\int dy\left(g_B(x)g_B(y) + \int\frac{dk'}{2\pi}g_{k'}(x)g_{-k'}(y)\right)g'_B(y)\right. \\ &\quad \left. + g_B^2(x)\int dy\left(g_B(x)g_B(y) + \int\frac{dk'}{2\pi}g_{k'}(x)g_{-k'}(y)\right)g'_k(y)\right], \\ &= -\frac{1}{\sqrt{Q_0}}\int\frac{dk'}{2\pi}(2V_{Bkk'}\Delta_{-k'B} + V_{BBk'}\Delta_{-k'k}) = \frac{1}{Q_0}\int\frac{dk'}{2\pi}(2\omega_k^2 - 3\omega_{k'}^2)\Delta_{kk'}\Delta_{-k'B}. \end{aligned} \tag{A13}$$

Combining these contributions

$$\Gamma_2^{31}(k) = \int \frac{dk'}{2\pi} \Delta_{kk'} \Delta_{-k'B} \left(\frac{\omega_{k'}^2 - \omega_k^2}{4} - \frac{\omega_{k'}^2}{4} \left(\frac{\omega_k}{\omega_{k'}} - 1 \right) + \frac{3\omega_{k'}\omega_k - \omega_k^2}{12} + \frac{2\omega_k^2 - 3\omega_{k'}^2}{6} \right), \\ = 0. \tag{A14}$$

6. $m = 2$, $n = 4$

The contributions are

$$H_3|0\rangle_1^{21} \supset \frac{1}{12\sqrt{Q_0}} \int \frac{d^4k}{(2\pi)^4} \omega_{k_1} \Delta_{k_1 B} V_{k_2k_3k_4} \phi_0^2 B_{k_1}^\dagger B_{k_2}^\dagger B_{k_3}^\dagger B_{k_4}^\dagger |0\rangle_0, \\ H_3|0\rangle_1^{12} \supset \frac{1}{4\sqrt{Q_0}} \int \frac{d^4k}{(2\pi)^4} (\omega_{k_1} - \omega_{k_2}) \Delta_{k_1k_2} V_{Bk_3k_4} \phi_0^2 B_{k_1}^\dagger B_{k_2}^\dagger B_{k_3}^\dagger B_{k_4}^\dagger |0\rangle_0, \\ H_3|0\rangle_1^{03} \supset -\frac{1}{12} \int \frac{d^4k}{(2\pi)^4} V_{BBk_1} Y_{k_2k_3k_4} \phi_0^2 B_{k_1}^\dagger B_{k_2}^\dagger B_{k_3}^\dagger B_{k_4}^\dagger |0\rangle_0, \tag{A15}$$

and

$$\int \frac{dk'}{2\pi} \omega_{k'} B_{k'}^\dagger B_{k'} |0\rangle_2^{24} = \int \frac{d^4k}{(2\pi)^4} \left(\frac{Y_{Bk_1k_2} V_{Bk_3k_4}}{4} - \Omega_4 \frac{\omega_{k_1} \Delta_{k_1B} Y_{k_2k_3k_4}}{12\sqrt{Q_0}} \right) \phi_0^2 B_{k_1}^\dagger B_{k_2}^\dagger B_{k_3}^\dagger B_{k_4}^\dagger |0\rangle_0. \tag{A16}$$

Therefore

$$\Gamma_2^{24}(k_1 \cdots k_4) = \frac{\sqrt{Q_0} \Delta_{k_1B} V_{k_2k_3k_4}}{12} \left[\omega_{k_1} + \frac{\omega_{k_1}^2}{\omega_{k_2} + \omega_{k_3} + \omega_{k_4}} - \frac{\omega_{k_1} \Omega_4}{\omega_{k_2} + \omega_{k_3} + \omega_{k_4}} \right] \\ + \frac{\sqrt{Q_0} \Delta_{k_1k_2} V_{Bk_3k_4}}{4} \left[(\omega_{k_1} - \omega_{k_2}) + \frac{\omega_{k_2}^2 - \omega_{k_1}^2}{\omega_{k_1} + \omega_{k_2}} \right] = 0 \tag{A17}$$

as the terms in each square bracket vanish.

7. $m = 2$, $n = 2$

From here on there will be many more contributions to each Γ_2^{mn}, and so we will decompose them into pieces that are not expected to mix as was done in Sec. IV B. First let us consider contributions that depend on $\mathcal{I}(x)$ and so on our renormalization scheme. As we have seen that $V_{\mathcal{I}B}$ vanishes, there are three contributions

$$H_3|0\rangle_1^{21} \supset \frac{1}{4\sqrt{Q_0}} \int \frac{d^2k}{(2\pi)^2} V_{\mathcal{I}k_1} \omega_{k_2} \Delta_{k_2B} \phi_0^2 B_{k_1}^\dagger B_{k_2}^\dagger |0\rangle_0, \\ H_3|0\rangle_1^{01} \supset \frac{1}{4\sqrt{Q_0}} \int \frac{d^2k}{(2\pi)^2} Y_{\mathcal{I}k_1} \omega_{k_2}^2 \Delta_{k_2B} \phi_0^2 B_{k_1}^\dagger B_{k_2}^\dagger |0\rangle_0, \\ \int \frac{dk'}{2\pi} \omega_{k'} B_{k'}^\dagger B_{k'} |0\rangle_2^{22} \supset -\frac{1}{4\sqrt{Q_0}} \int \frac{d^2k}{(2\pi)^2} (\omega_{k_1} + \omega_{k_2}) Y_{\mathcal{I}k_1} \omega_{k_2} \omega_{k_2}^2 \Delta_{k_2B} \phi_0^2 B_{k_1}^\dagger B_{k_2}^\dagger |0\rangle_0, \tag{A18}$$

whose sum is easily seen to vanish.

Similarly to (A13) one may derive

$$V_{BBk_1k_2} = \frac{1}{Q_0} \left[-(\omega_{k_1}^2 + \omega_{k_2}^2) \Delta_{k_1B} \Delta_{k_2B} + \int \frac{dk'}{2\pi} [-\sqrt{Q_0} V_{k_1k_2k'} \Delta_{-k'B} + (\omega_{k_1}^2 + \omega_{k_2}^2 - 2\omega_{k'}^2) \Delta_{k_2k'} \Delta_{-k'k_1}] \right]. \tag{A19}$$

There are four terms that contain $V_{k_1k_2k_3}$

$$\begin{aligned}
H_4|0\rangle_0 &\supset -\frac{1}{4\sqrt{Q_0}}\int\frac{d^2k}{(2\pi)^2}\int\frac{dk'}{2\pi}V_{k_1k_2k'}\Delta_{-k'B}\phi_0^2B_{k_1}^\dagger B_{k_2}^\dagger|0\rangle_0,\\
H_3|0\rangle_1^{03} &\supset \frac{1}{8\sqrt{Q_0}}\int\frac{d^2k}{(2\pi)^2}\int\frac{dk'}{2\pi}Y_{k_1k_2k'}\omega_{k'}\Delta_{-k'B}\phi_0^2B_{k_1}^\dagger B_{k_2}^\dagger|0\rangle_0,\\
H_3|0\rangle_1^{21} &\supset \frac{1}{8\sqrt{Q_0}}\int\frac{d^2k}{(2\pi)^2}\int\frac{dk'}{2\pi}V_{k_1k_2k'}\Delta_{-k'B}\phi_0^2B_{k_1}^\dagger B_{k_2}^\dagger|0\rangle_0,\\
\int\frac{dk'}{2\pi}\omega_{k'}B_{k'}^\dagger B_{k'}|0\rangle_2^{22} &\supset \frac{1}{8\sqrt{Q_0}}\int\frac{d^2k}{(2\pi)^2}\int\frac{dk'}{2\pi}(\omega_{k_1}+\omega_{k_2})Y_{k_1k_2k'}\Delta_{-k'B}\phi_0^2B_{k_1}^\dagger B_{k_2}^\dagger|0\rangle_0,
\end{aligned}\tag{A20}$$

which again sum to zero, as the second plus the fourth and also the third are equal to minus one half of the first. The four terms with no k' integral are

$$\begin{aligned}
H_4|0\rangle_0 &\supset -\frac{1}{4Q_0}\int\frac{d^2k}{(2\pi)^2}(\omega_{k_1}^2+\omega_{k_2}^2)\Delta_{k_1B}\Delta_{k_2B}\phi_0^2B_{k_1}^\dagger B_{k_2}^\dagger|0\rangle_0,\\
H_3|0\rangle_1^{01} &\supset -\frac{1}{8Q_0}\int\frac{d^2k}{(2\pi)^2}(\omega_{k_1}^2+\omega_{k_2}^2)\Delta_{k_1B}\Delta_{k_2B}\phi_0^2B_{k_1}^\dagger B_{k_2}^\dagger|0\rangle_0,\\
\frac{\pi_0^2}{2}|0\rangle_2^{42} &= -\frac{3}{4Q_0}\int\frac{d^2k}{(2\pi)^2}\omega_{k_1}\omega_{k_2}\Delta_{k_1B}\Delta_{k_2B}\phi_0^2B_{k_1}^\dagger B_{k_2}^\dagger|0\rangle_0,\\
\int\frac{dk'}{2\pi}\omega_{k'}B_{k'}^\dagger B_{k'}|0\rangle_2^{22} &\supset \frac{3}{8Q_0}\int\frac{d^2k}{(2\pi)^2}\Omega_2^2\Delta_{k_1B}\Delta_{k_2B}\phi_0^2B_{k_1}^\dagger B_{k_2}^\dagger|0\rangle_0,
\end{aligned}\tag{A21}$$

which sum easily to zero as well. Finally the three terms with k' but no $V_{k_1k_2k_3}$ are

$$\begin{aligned}
H_4|0\rangle_0 &\supset \frac{1}{4Q_0}\int\frac{d^2k}{(2\pi)^2}\int\frac{dk'}{2\pi}(\omega_{k_1}^2+\omega_{k_2}^2-2\omega_{k'}^2)\Delta_{k_1k'}\Delta_{-k'k_2}\phi_0^2B_{k_1}^\dagger B_{k_2}^\dagger|0\rangle_0,\\
H_3|0\rangle_1^{12} &\supset \frac{1}{2Q_0}\int\frac{d^2k}{(2\pi)^2}\int\frac{dk'}{2\pi}\left(\frac{\omega_{k_2}^2}{\omega_{k'}}-\omega_{k'}\right)(\omega_{k_1}-\omega_{k'})\Delta_{k_1k'}\Delta_{-k'k_2}\phi_0^2B_{k_1}^\dagger B_{k_2}^\dagger|0\rangle_0,\\
\int\frac{dk'}{2\pi}\omega_{k'}B_{k'}^\dagger B_{k'}|0\rangle_2^{22} &\supset -\frac{1}{4Q_0}\int\frac{d^2k}{(2\pi)^2}\int\frac{dk'}{2\pi}\Omega_2\left(\frac{\omega_{k_1}\omega_{k_2}}{\omega_{k'}}-\omega_{k'}\right)\Delta_{k_1k'}\Delta_{-k'k_2}\phi_0^2B_{k_1}^\dagger B_{k_2}^\dagger|0\rangle_0.
\end{aligned}\tag{A22}$$

Notice that in the second term, all factors are symmetric with respect to $k_1 \leftrightarrow k_2$ except for the factors of ω. Therefore these may be symmetrized to

$$\frac{1}{2}\left(\frac{\omega_{k_1}\omega_{k_2}\Omega_2}{\omega_{k'}}-\omega_{k_1}^2-\omega_{k_2}^2-\omega_{k'}\Omega_2\right)+\omega_{k'}^2,\tag{A23}$$

which exactly cancels the corresponding contributions from the first and third terms. We thus conclude that $\Gamma_2^{22}=0$.

8. $m=2$, $n=0$

We will see that this is the most interesting case so far, because it is the first that strongly depends on the form of $\mathcal{I}(x)$. To see this, let us try to proceed as above. The terms that depend on $\mathcal{I}(x)$ are

$$\begin{aligned}
H_4|0\rangle_0 &\supset \frac{1}{4}V_{\mathcal{I}BB}\phi_0^2|0\rangle_0,\\
H_3|0\rangle_1^{21} &\supset \frac{1}{8\sqrt{Q_0}}\int\frac{dk'}{2\pi}V_{\mathcal{I}k'}\Delta_{-k'k_2}\phi_0^2|0\rangle_0,\\
H_3|0\rangle_1^{01} &\supset \frac{1}{8\sqrt{Q_0}}\int\frac{dk'}{2\pi}V_{\mathcal{I}k'}\Delta_{-k'k_2}\phi_0^2|0\rangle_0.
\end{aligned}\tag{A24}$$

As above, we may eliminate $V^{(4)}$ using integration by parts and then inserting the completeness relation (2.16)

$$\begin{aligned} V_{\mathcal{I}BB} &= \int dx V^{(4)}[\sqrt{\lambda} f(x)]\mathcal{I}(x) g_B^2(x) = \frac{1}{\sqrt{Q_0}} \int dx (\partial_x V^{(3)}[\sqrt{\lambda} f(x)])\mathcal{I}(x) g_B(x), \\ &= -\frac{1}{\sqrt{Q_0}} \int dx V^{(3)}[\sqrt{\lambda} f(x)](\mathcal{I}'(x) g_B(x) + \mathcal{I}(x) g_B'(x)), \\ &= -\frac{1}{\sqrt{Q_0}} \left(V_{\mathcal{I}'B} + \int \frac{dk'}{2\pi} V_{\mathcal{I}k'} \Delta_{-k'B} \right). \end{aligned} \tag{A25}$$

The second term cancels the contributions from $H_3|0\rangle_1^{21}$ and $H_3|0\rangle_1^{01}$, leaving

$$\Gamma_{2,\mathcal{I}}^{20} = -\frac{\sqrt{Q_0}}{4} V_{\mathcal{I}'B}. \tag{A26}$$

Let us rewrite $V_{\mathcal{I}'B}$ in terms of quantities that we expect to find in other contributions.

Writing the identity (2.24) as

$$\partial_x \mathcal{I}(x) = \int \frac{dk}{2\pi} \frac{1}{2\omega_k} (g_k(x) g'_{-k}(x) + g'_k(x) g_{-k}(x)), \tag{A27}$$

one finds

$$\begin{aligned} V_{\mathcal{I}'B} &= \int \frac{dk}{2\pi} \frac{1}{2\omega_k} \int dx V^{(3)}[\sqrt{\lambda} f(x)] g_B(x) (g_k(x) g'_{-k}(x) + g'_k(x) g_{-k}(x)), \\ &= \int \frac{dk}{2\pi} \frac{1}{\omega_k} \left(V_{BBk} \Delta_{Bk} + \int \frac{dk'}{2\pi} V_{Bkk'} \Delta_{-k'k} \right), \\ &= -\frac{1}{\sqrt{Q_0}} \int \frac{d^2k'}{(2\pi)^2} \frac{\omega_{k'_1}^2 - \omega_{k'_2}^2}{\omega_{k'_1}} \Delta_{k'_1 k'_2} \Delta_{-k'_1 -k'_2} - \frac{1}{\sqrt{Q_0}} \int \frac{dk'}{2\pi} \omega_{k'} \Delta_{Bk'} \Delta_{-k'B}. \end{aligned} \tag{A28}$$

Inserting this into (A26) and symmetrizing dummy indices one finds

$$\Gamma_{2,\mathcal{I}}^{20} = \frac{1}{8} \int \frac{d^2k'}{(2\pi)^2} \left(-\omega_{k'_1} - \omega_{k'_2} + \frac{\omega_{k'_1}^2}{\omega_{k'_2}} + \frac{\omega_{k'_2}^2}{\omega_{k'_1}} \right) \Delta_{k'_1 k'_2} \Delta_{-k'_1 -k'_2} + \frac{1}{4} \int \frac{dk'}{2\pi} \omega_{k'} \Delta_{Bk'} \Delta_{-k'B}. \tag{A29}$$

Schrödinger's equation will only be satisfied if these terms are canceled by contributions with no $\mathcal{I}(x)$.

There is only one contribution with no $\mathcal{I}(x)$ that has two contracted momenta k'

$$H_3|0\rangle_1^{12} \supset \frac{1}{8Q_0} \int \frac{d^2k'}{(2\pi)^2} \left(\omega_{k'_1} + \omega_{k'_2} - \frac{\omega_{k'_1}^2}{\omega_{k'_2}} - \frac{\omega_{k'_2}^2}{\omega_{k'_1}} \right) \Delta_{k'_1 k'_2} \Delta_{-k'_1 -k'_2} \phi_0^2 |0\rangle_0, \tag{A30}$$

which indeed cancels the first term in (A29). There are two contributions with no $\mathcal{I}(x)$ and a single k'

$$\begin{aligned} H_3|0\rangle_1^{01} &\supset \frac{1}{8Q_0} \int \frac{dk'}{2\pi} \omega_{k'} \Delta_{Bk'} \Delta_{-k'B} \phi_0^2 |0\rangle_0, \\ \frac{\pi_0^2}{2} |0\rangle_2^{40} &= -\frac{3}{8Q_0} \int \frac{dk'}{2\pi} \omega_{k'} \Delta_{Bk'} \Delta_{-k'B} \phi_0^2 |0\rangle_0, \end{aligned} \tag{A31}$$

which are equal to $1/2$ and $-3/2$ of the second term in (A29), and so altogether they cancel, leaving $\Gamma_2^{20} = 0$.

9. $m = 1$, $n = 5$

There are only three contributions to this term

$$H_3|0\rangle_1^{03} \supset \frac{1}{12\sqrt{Q_0}} \int \frac{d^5k}{(2\pi)^5} (\omega_{k_4}^2 - \omega_{k_5}^2) Y_{k_1k_2k_3} \Delta_{k_4k_5} \phi_0 B_{k_1}^\dagger \cdots B_{k_5}^\dagger |0\rangle_0,$$
$$H_3|0\rangle_1^{12} \supset \frac{1}{12\sqrt{Q_0}} \int \frac{d^5k}{(2\pi)^5} (\omega_{k_4} - \omega_{k_5}) V_{k_1k_2k_3} \Delta_{k_4k_5} \phi_0 B_{k_1}^\dagger \cdots B_{k_5}^\dagger |0\rangle_0,$$
$$\int \frac{dk'}{2\pi} \omega_{k'} B_{k'}^\dagger B_{k'} |0\rangle_2^{15} = -\frac{1}{12\sqrt{Q_0}} \int \frac{d^5k}{(2\pi)^5} \left(\omega_{k_4} - \omega_{k_5} + \frac{\omega_{k_4}^2 - \omega_{k_5}^2}{\Omega_3}\right) V_{k_1k_2k_3} \Delta_{k_4k_5} \phi_0 B_{k_1}^\dagger \cdots B_{k_5}^\dagger |0\rangle_0, \tag{A32}$$

whose sum is readily seen to vanish.

10. $m = 1$, $n = 3$

Again let us divide the 11 contributions to Γ_2^{13} into three subsets that are expected to cancel separately. First, terms involving $\mathcal{I}(x)$ are

$$H_3|0\rangle_1^{12} \supset \frac{1}{4\sqrt{Q_0}} \int \frac{d^3k}{(2\pi)^3} (\omega_{k_2} - \omega_{k_3}) V_{\mathcal{I}k_1} \Delta_{k_2k_3} \phi_0 B_{k_1}^\dagger B_{k_2}^\dagger B_{k_3}^\dagger |0\rangle_0,$$
$$H_3|0\rangle_1^{01} \supset \frac{1}{4\sqrt{Q_0}} \int \frac{d^3k}{(2\pi)^3} (\omega_{k_2}^2 - \omega_{k_3}^2) Y_{\mathcal{I}k_1} \Delta_{k_2k_3} \phi_0 B_{k_1}^\dagger B_{k_2}^\dagger B_{k_3}^\dagger |0\rangle_0,$$
$$\int \frac{dk'}{2\pi} \omega_{k'} B_{k'}^\dagger B_{k'} |0\rangle_2^{13} \supset -\frac{1}{4\sqrt{Q_0}} \int \frac{d^3k}{(2\pi)^3} (\omega_{k_2} - \omega_{k_3}) \Omega_3 Y_{\mathcal{I}k_1} \Delta_{k_2k_3} \phi_0 B_{k_1}^\dagger B_{k_2}^\dagger B_{k_3}^\dagger |0\rangle_0, \tag{A33}$$

which sum to zero.

We will now need

$$V_{Bk_1k_2k_3} = \frac{1}{Q_0}[(\omega_{k_2}^2 - \omega_{k_3}^2)\Delta_{Bk_1}\Delta_{k_2k_3} + (\omega_{k_1}^2 - \omega_{k_3}^2)\Delta_{Bk_2}\Delta_{k_1k_3} + (\omega_{k_1}^2 - \omega_{k_2}^2)\Delta_{Bk_3}\Delta_{k_1k_2}]$$
$$- \frac{1}{\sqrt{Q_0}} \int \frac{dk'}{2\pi} [V_{k_2k_3k'}\Delta_{-k'k_1} + V_{k_1k_3k'}\Delta_{-k'k_2} + V_{k_1k_2k'}\Delta_{-k'k_3}]. \tag{A34}$$

The terms which have a contracted index k' are

$$H_4|0\rangle_0 \supset -\frac{1}{2\sqrt{Q_0}} \int \frac{d^3k}{(2\pi)^3} \int \frac{dk'}{2\pi} V_{k_1k_2k'} \Delta_{-k'k_3} \phi_0 B_{k_1}^\dagger B_{k_2}^\dagger B_{k_3}^\dagger |0\rangle_0,$$
$$H_3|0\rangle_1^{03} \supset \frac{1}{4\sqrt{Q_0}} \int \frac{d^3k}{(2\pi)^3} \int \frac{dk'}{2\pi} \frac{\omega_{k'}^2 - \omega_{k_3}^2}{\omega_{k'}} Y_{k_1k_2k'} \Delta_{-k'k_3} \phi_0 B_{k_1}^\dagger B_{k_2}^\dagger B_{k_3}^\dagger |0\rangle_0,$$
$$H_3|0\rangle_1^{12} \supset \frac{1}{4\sqrt{Q_0}} \int \frac{d^3k}{(2\pi)^3} \int \frac{dk'}{2\pi} \frac{\omega_{k'} - \omega_{k_3}}{\omega_{k'}} V_{k_1k_2k'} \Delta_{-k'k_3} \phi_0 B_{k_1}^\dagger B_{k_2}^\dagger B_{k_3}^\dagger |0\rangle_0,$$
$$\int \frac{dk'}{2\pi} \omega_{k'} B_{k'}^\dagger B_{k'} |0\rangle_2^{13} \supset \frac{1}{4\sqrt{Q_0}} \int \frac{d^3k}{(2\pi)^3} \int \frac{dk'}{2\pi} \Omega_3 \frac{\omega_{k'} + \omega_{k_3}}{\omega_{k'}} Y_{k_1k_2k'} \Delta_{-k'k_3} \phi_0 B_{k_1}^\dagger B_{k_2}^\dagger B_{k_3}^\dagger |0\rangle_0, \tag{A35}$$

whose sum also vanishes. Finally the terms with neither $\mathcal{I}(x)$ nor k' are

$$H_4|0\rangle_0 \supset \frac{1}{2Q_0}\int\frac{d^3k}{(2\pi)^3}(\omega_{k_2}^2-\omega_{k_3}^2)\Delta_{Bk_1}\Delta_{k_2k_3}\phi_0 B_{k_1}^\dagger B_{k_2}^\dagger B_{k_3}^\dagger|0\rangle_0,$$
$$H_3|0\rangle_1^{01} \supset \frac{1}{4Q_0}\int\frac{d^3k}{(2\pi)^3}(\omega_{k_2}^2-\omega_{k_3}^2)\Delta_{Bk_1}\Delta_{k_2k_3}\phi_0 B_{k_1}^\dagger B_{k_2}^\dagger B_{k_3}^\dagger|0\rangle_0,$$
$$\frac{\pi_0^2}{2}|0\rangle_2^{33} \supset \frac{3}{4Q_0}\int\frac{d^3k}{(2\pi)^3}\omega_{k_1}(\omega_{k_2}-\omega_{k_3})\Delta_{Bk_1}\Delta_{k_2k_3}\phi_0 B_{k_1}^\dagger B_{k_2}^\dagger B_{k_3}^\dagger|0\rangle_0,$$
$$\int\frac{dk'}{2\pi}\omega_{k'}B_{k'}^\dagger B_{k'}|0\rangle_2^{13} \supset -\frac{3}{4Q_0}\int\frac{d^3k}{(2\pi)^3}[\omega_{k_1}(\omega_{k_2}-\omega_{k_2})+\omega_{k_2}^2-\omega_{k_3}^2]\Delta_{Bk_1}\Delta_{k_2k_3}\phi_0 B_{k_1}^\dagger B_{k_2}^\dagger B_{k_3}^\dagger|0\rangle_0, \tag{A36}$$

which again trivially cancel, leaving $\Gamma_2^{13}=0$.

11. $m=1$, $n=1$

Finally we turn our attention to Γ_2^{11}. Like Γ_2^{20} we will see that it only vanishes if $\mathcal{I}(x)$ satisfies (2.24). The terms involving $\mathcal{I}(x)$ are

$$H_4|0\rangle_0 \supset \frac{1}{2}\int\frac{dk}{2\pi}V_{\mathcal{I}Bk}\phi_0 B_k^\dagger|0\rangle_0,$$
$$H_3|0\rangle_1^{12} \supset \frac{1}{4\sqrt{Q_0}}\int\frac{dk}{2\pi}\int\frac{dk'}{2\pi}(\omega_{k'}-\omega_k)Y_{\mathcal{I}k'}\Delta_{-k'k}\phi_0 B_k^\dagger|0\rangle_0,$$
$$H_3|0\rangle_1^{01} \supset \frac{1}{4\sqrt{Q_0}}\int\frac{dk}{2\pi}\int\frac{dk'}{2\pi}\left(\frac{\omega_{k'}^2-\omega_k^2}{\omega_{k'}}\right)Y_{\mathcal{I}k'}\Delta_{-k'k}\phi_0 B_k^\dagger|0\rangle_0,$$
$$\int\frac{dk'}{2\pi}\omega_{k'}B_{k'}^\dagger B_{k'}|0\rangle_2^{11} \supset \frac{1}{4\sqrt{Q_0}}\int\frac{dk}{2\pi}\int\frac{dk'}{2\pi}\omega_k\left(\frac{\omega_{k'}+\omega_k}{\omega_{k'}}\right)Y_{\mathcal{I}k'}\Delta_{-k'k}\phi_0 B_k^\dagger|0\rangle_0. \tag{A37}$$

Again, as in the case of Γ_2^{20} integration by parts allows us to remove the fourth derivative

$$V_{\mathcal{I}Bk} = -\frac{1}{\sqrt{Q_0}}\int dx V^{(3)}[\sqrt{\lambda}f(x)](\mathcal{I}'(x)g_k(x)+\mathcal{I}(x)g_k'(x)), \tag{A38}$$

and the two terms can be simplified using completeness (2.16) and the formula (2.24) for $\mathcal{I}'(x)$

$$-\frac{1}{\sqrt{Q_0}}\int dx V^{(3)}[\sqrt{\lambda}f(x)]\mathcal{I}'(x)g_k(x) = \frac{1}{Q_0}\int\frac{dk'}{2\pi}\left(\frac{\omega_{k'}^2-\omega_k^2}{\omega_{k'}}\right)\Delta_{kk'}\Delta_{-k'B} - \frac{1}{2\sqrt{Q_0}}\int\frac{d^2k'}{(2\pi)^2}\left(\frac{\omega_{k_1'}-\omega_{k_2'}}{\omega_{k_1'}\omega_{k_2'}}\right)V_{kk_1'k_2'}\Delta_{-k_1'-k_2'},$$
$$-\frac{1}{\sqrt{Q_0}}\int dx V^{(3)}[\sqrt{\lambda}f(x)]\mathcal{I}(x)g_k'(x) = -\frac{1}{\sqrt{Q_0}}V_{\mathcal{I}k'}\Delta_{-k'k}. \tag{A39}$$

The second equation in (A39) substituted into the first term in (A37) cancels the second, third, and fourth terms. This leaves only the first equation in (A39), which when substituted into (A37) yields

$$\Gamma_{2,\mathcal{I}}^{11} = \frac{1}{2}\int\frac{dk'}{2\pi}\left(\frac{\omega_{k'}^2-\omega_k^2}{\omega_{k'}}\right)\Delta_{kk'}\Delta_{-k'B} - \frac{\sqrt{Q_0}}{4}\int\frac{d^2k'}{(2\pi)^2}\left(\frac{\omega_{k_1'}-\omega_{k_2'}}{\omega_{k_1'}\omega_{k_2'}}\right)V_{kk_1'k_2'}\Delta_{-k_1'-k_2'}. \tag{A40}$$

Summing the three contributions with integrals over k'_1 and k'_2

$$H_3|0\rangle_1^{03} \supset \frac{1}{8\sqrt{Q_0}}\int\frac{dk}{2\pi}\int\frac{d^2k'}{(2\pi)^2}\left(\frac{\omega_{k'_1}^2-\omega_{k'_2}^2}{\omega_{k'_1}\omega_{k'_2}}\right)Y_{kk'_1k'_2}\Delta_{-k'_1-k'_2}\phi_0B_k^\dagger|0\rangle_0,$$

$$H_3|0\rangle_1^{12} \supset \frac{1}{8\sqrt{Q_0}}\int\frac{dk}{2\pi}\int\frac{d^2k'}{(2\pi)^2}\left(\frac{\omega_{k'_1}-\omega_{k'_2}}{\omega_{k'_1}\omega_{k'_2}}\right)V_{kk'_1k'_2}\Delta_{-k'_1-k'_2}\phi_0B_k^\dagger|0\rangle_0,$$

$$\int\frac{dk'}{2\pi}\omega_{k'}B_{k'}^\dagger B_{k'}|0\rangle_2^{11} \supset \frac{1}{8\sqrt{Q_0}}\int\frac{dk}{2\pi}\int\frac{d^2k'}{(2\pi)^2}\omega_{k_1}\left(\frac{\omega_{k'_1}-\omega_{k'_2}}{\omega_{k'_1}\omega_{k'_2}}\right)Y_{kk'_1k'_2}\Delta_{-k'_1-k'_2}\phi_0B_k^\dagger|0\rangle_0,$$

one obtains

$$\Gamma_{2,2k'}^{11} = \frac{\sqrt{Q_0}}{4}\int\frac{d^2k'}{(2\pi)^2}\left(\frac{\omega_{k'_1}-\omega_{k'_2}}{\omega_{k'_1}\omega_{k'_2}}\right)V_{kk'_1k'_2}\Delta_{-k'_1-k'_2}, \tag{A41}$$

which cancels the second term in $\Gamma_{2,\mathcal{I}}^{11}$.

Finally the terms with no $\mathcal{I}(x)$ and one k' are

$$H_3|0\rangle_1^{01} \supset \frac{1}{4Q_0}\int\frac{dk}{2\pi}\int\frac{dk'}{2\pi}\left(\omega_{k'}-\frac{\omega_k^2}{\omega_{k'}}\right)\Delta_{kk'}\Delta_{-k'B}\phi_0B_k^\dagger|0\rangle_0,$$

$$\frac{\pi_0^2}{2}|0\rangle_2^{31} \supset \frac{1}{4Q_0}\int\frac{dk}{2\pi}\int\frac{dk'}{2\pi}(\omega_k-3\omega_{k'})\Delta_{kk'}\Delta_{-k'B}\phi_0B_k^\dagger|0\rangle_0,$$

$$\int\frac{dk'}{2\pi}\omega_{k'}B_{k'}^\dagger B_{k'}|0\rangle_2^{11} \supset \frac{1}{4Q_0}\int\frac{dk}{2\pi}\int\frac{dk'}{2\pi}\left(-\omega_k+3\frac{\omega_k^2}{\omega_{k'}}\right)\Delta_{kk'}\Delta_{-k'B}\phi_0B_k^\dagger|0\rangle_0, \tag{A42}$$

which sum to

$$\Gamma_{2,1k'}^{11} = \frac{1}{2}\int\frac{dk'}{2\pi}\left(\frac{\omega_k^2-\omega_{k'}^2}{\omega_{k'}}\right)\Delta_{kk'}\Delta_{-k'B}, \tag{A43}$$

canceling the first term in $\Gamma_{2,\mathcal{I}}^{11}$. Summarizing, we have verified that

$$\Gamma_2^{11} = \Gamma_{2,\mathcal{I}}^{11}+\Gamma_{2,1k'}^{11}+\Gamma_{2,2k'}^{11} = 0, \tag{A44}$$

and so the state $|0\rangle_2$ that we have found indeed solves Schrödinger's equation at two loops.

APPENDIX B: THE MESON MASS IN THE SINE-GORDON MODEL

In this appendix we briefly review the Schrödinger picture derivation of the two-loop meson mass in the normal-ordered Sine-Gordon model. First one expands the scalar field in terms of Heisenberg operators

$$\phi(x) = \int\frac{dp}{2\pi}\left(A_p^\dagger+\frac{A_{-p}}{2\omega_p}\right)e^{-ipx}. \tag{B1}$$

The Sine-Gordon potential

$$V(x) = \frac{M^2}{\lambda}(1-:\cos(\sqrt{\lambda}\phi(x)):_a) \tag{B2}$$

at fourth order is the interaction

$$H_4 = -\frac{M^2\lambda}{24}\int dx:\phi^4(x):_a, \tag{B3}$$

while the free Hamiltonian is

$$H_2 = \int \frac{dp}{2\pi}\omega_p A_p^\dagger A_p. \tag{B4}$$

Let the meson state $|p\rangle$ be an eigenstate of the full Hamiltonian. Expand it in powers of $\sqrt{\lambda}$

$$|p\rangle = \sum_{n=0}^{\infty} |p\rangle_n, \tag{B5}$$

where

$$|p\rangle_0 = A_p^\dagger|\Omega\rangle. \tag{B6}$$

The tree-level Schrödinger equation

$$H_2|p\rangle_0 = E_0|p\rangle_0 \tag{B7}$$

is solved by $E_0 = \omega_p$. At one loop

$$0 = (H_4 - E_1)|p\rangle_0 + (H_2 - E_0)|p\rangle_1 \tag{B8}$$

together with the convention[7]

$${}_0\langle p|p\rangle_i = \delta_{0i} \tag{B9}$$

are solved by $E_1 = 0$ and

$$|p\rangle_1 = \frac{M^2\lambda}{24}\int dx \int \frac{d^4q}{(2\pi)^4}\frac{e^{-ix\sum_j^4 q_j}}{\sum_j^4 \omega_{q_j}} A_{q_1}^\dagger \cdots A_{q_4}^\dagger A_p^\dagger|\Omega\rangle + \frac{M^2\lambda}{12}\int dx \int \frac{d^3q}{(2\pi)^3}\frac{e^{-ix(-p+\sum_j^3 q_j)}}{\omega_p(-\omega_p + \sum_j^3 \omega_{q_j})} \times A_{q_1}^\dagger \cdots A_{q_3}^\dagger|\Omega\rangle. \tag{B10}$$

At two loops, the Schrödinger equation is

$$0 = (H_6 - E_2)|p\rangle_0 + (H_4 - E_1)|p\rangle_1 + (H_2 - E_0)|p\rangle_2. \tag{B11}$$

Let us left multiply ${}_0\langle p|$ and use the orthogonality condition (B9). As H_6 is normal ordered, its matrix element vanishes and one finds

$$E_2 = {}_0\langle p|H_4|p\rangle_1 = A + B, \tag{B12}$$

where

$$A = -\frac{M^4\lambda^2}{48\omega_p}\int \frac{d^2q}{(2\pi)^2}\frac{\omega_{q_1} + \omega_{q_2} + \omega_{p-q_1-q_2}}{\omega_{q_1}\omega_{q_2}\omega_{p-q_1-q_2}[(\omega_{q_1} + \omega_{q_2} + \omega_{p-q_1-q_2})^2 - \omega_p^2]},$$
$$B = -\frac{M^4\lambda^2}{384}\int \frac{d^3q}{(2\pi)^3}\frac{1}{\omega_{q_1}\omega_{q_2}\omega_{q_3}\omega_{q_1+q_2+q_3}(\omega_{q_1+q_2+q_3} + \sum_i^3 \omega_{q_i})}. \tag{B13}$$

The infrared divergent term B is equal to the two-loop energy of the vacuum state $|\Omega\rangle$ and so it does not contribute to the meson mass. Therefore the two-loop meson mass correction M_2 is equal to A evaluated at $p = 0$

$$M_2 = -\frac{M^3\lambda^2}{48}\int \frac{d^2q}{(2\pi)^2}\frac{\omega_{q_1} + \omega_{q_2} + \omega_{q_1+q_2}}{\omega_{q_1}\omega_{q_2}\omega_{q_1+q_2}[(\omega_{q_1} + \omega_{q_2} + \omega_{q_1+q_2})^2 - M^2]} = -\frac{M\lambda^2}{768} \tag{B14}$$

in agreement with the pole mass in Ref. [18].

[7]For simplicity we have fixed p and applied a p-dependent normalization condition on $|\Omega\rangle$.

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