

Extended Nappi-Witten geometry for the fractional quantum Hall effectPatricio Salgado-Rebolledo¹ and Giandomenico Palumbo²¹*Université Libre de Bruxelles and International Solvay Institutes,
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Motivated by the recent progresses in the formulation of geometric theories for the fractional quantum Hall states, we propose a novel nonrelativistic geometric model for the Laughlin states based on an extension of the Nappi-Witten geometry. We show that the $U(1)$ gauge sector responsible for the fractional Hall conductance, the gravitational Chern-Simons action and Wen-Zee term associated to the Hall viscosity can be derived from a single Chern-Simons theory with a gauge connection that takes values in the extended Nappi-Witten algebra. We then provide a new derivation of the chiral boson associated to the gapless edge states from the Wess-Zumino-Witten model that is induced by the Chern-Simons theory on the boundary.

DOI: [10.1103/PhysRevD.103.125006](https://doi.org/10.1103/PhysRevD.103.125006)**I. INTRODUCTION**

The fractional quantum Hall effect (FQHE) is recognized as one of the most important physical phenomena in condensed matter physics [1–5]. Being its microscopic origin a central research topic in topological phases of matter, it has given rise to an enormous amount of developments in quantum mesoscopic physics. In the low energy regime, the fractional quantum Hall states (FQHs) can be described by Abelian and non-Abelian Chern-Simons theories, while the corresponding chiral edge states by rational conformal field theories (CFTs) [5–9]. In the case of Laughlin states, the topological field theory depends on an emergent $U(1)$ gauge field and the electromagnetic field, while the edge states are described by chiral bosons. Recently, there have been an intense research on the geometric aspects of the FQHE. On one hand, the incompressibility of the FQHs is due to the presence of the strong external magnetic field and many-body interactions, which is encoded in the Girvin-Macdonald-Platzman (GMP) mode [8,10,11]. Notice that the underlying GMP algebra is related to the W_∞ algebra, which encodes the area-preserving diffeomorphisms. Thus, the GMP mode can be naturally understood as a propagating nonrelativistic spin-2 boson related to an emergent quantum geometry [12–14]. On the other hand, the background (ambient) geometry plays a central role in the Hall viscosity, which is a linear response effect of the Hall fluid in the bulk [15–19]. For all these reasons, several geometric models for the FQHE have been recently proposed [20–37]. These $2 + 1$ -dimensional effective field theories are based on nonrelativistic geometry. In other words, the models are spatially covariant, and their formulation rely on

Newton-Cartan geometry [33]. This nonrelativistic theory is a geometric reformulation of Newton's gravity, that imitates the geometric formulation of general relativity. It has been shown that Newton-Cartan gravity is based on the gauging of the Bargmann algebra (namely, centrally extended Galilei algebra) [38–40]. Importantly, there have been many developments and generalizations of this theory by replacing the Bargmann algebra with the Newton-Hooke algebra, the Maxwell algebra, etc. Behind these extensions of the Newton-Cartan theory, there appears the Nappi-Witten algebra [41]. This special algebra plays also an important role in certain Wess-Zumino-Witten (WZW) models and pp-wave spacetime [42], and string-inspired $1 + 1$ -dimensional gravity [43]. Thus, generalized non-relativistic geometries can provide a novel scenario where to characterize the geometric features of FQHs. Notice, this research line follows in spirit the recent studies of the geometric aspects of topological insulators and topological superconductors where relativistic non-Riemannian geometries have been employed [44–48]. In this work, we will present a novel geometric model for the Laughlin states given by a Chern-Simons (CS) theory with the gauge connection that takes values in an extended Nappi-Witten algebra. We will show that this CS action naturally contains not only the Wen-Zee [49] and nonrelativistic gravitational CS [20] terms but also the $U(1)$ topological terms responsible for the fractional Hall conductance [5]. From this topological effective field theory we will derive the chiral WZW model on the boundary of the system. This CFT will allow us to describe the gapless edge states in terms of a chiral boson that takes contribution from both the charge and gravitational sectors. Finally, we will show that the

extended Nappi-Witten geometry can be naturally embedded in a full nonrelativistic space-time AdS-Lorentz geometry. Our work paves the way for the characterization of FQHE from the point of view of generalized nonrelativistic geometries, where both the charge and gravitational sectors are encoded into a unified geometric formalism.

II. EXTENDED NAPPI-WITTEN ALGEBRA

The Nappi-Witten algebra is the central extension of the Euclidean algebra in two dimensions [42,50]

$$[P_a, P_b] = -\epsilon_{ab}T, \quad [J, P_a] = \epsilon_a{}^b P_b, \quad (1)$$

where $a = \{1, 2\}$, P_a stands for the translations in the two-dimensional plane, J is the generator of rotations, and T is central. Its Lorentzian version is isomorphic to the Newton-Hook algebra [51] as well as to the Maxwell algebra in two dimensions [52]. The Maxwell algebra describes particle systems in the presence of a constant electromagnetic field [53,54]. Thus, the Nappi-Witten algebra has been found in the description of the integer QHE [55,56] because it naturally contains the magnetic translation algebra in two dimensions. Furthermore it has been shown that the Nappi-Witten geometry properly describes the momentum-space cigar geometry of a certain kind of two-dimensional topological phases [45].

Based in the previous discussion, it seems natural to expect the Nappi-Witten algebra to be relevant in the description of two-dimensional interacting systems where a constant external electromagnetic field plays a central role. The quantum field theory of the Laughlin states is characterized by an extra field content given by an emergent gauge field a [5]. Thus, as a first attempt to describe these quantum states, we add an Abelian generator Y to Eq. (1) and consider the direct product Nappi-Witten $\times \mathfrak{u}(1)$. The central extension of the translations T will be associated to the external electromagnetic field A and the Abelian generator Y , to the emergent gauge field a . The Nappi-Witten $\times \mathfrak{u}(1)$ algebra admits a nondegenerate invariant bilinear form given by

$$\begin{aligned} \langle JJ \rangle &= \mu_0, & \langle P_a P_b \rangle &= \mu_1 \delta_{ab}, & \langle JT \rangle &= -\mu_1, \\ \langle YJ \rangle &= \rho_0, & \langle YY \rangle &= \rho_1. \end{aligned} \quad (2)$$

The relevant gauge fields of the system are to be encoded in a connection one-form $\mathbb{A} = \mathbb{A}_\mu^A T_A dx^\mu$ where $\mu = 0, 1, 2$ is a space-time index and $T_A = \{J, P_a, Z, Y\}$ collectively denotes the generators of the Nappi-Witten $\times \mathfrak{u}(1)$ algebra. Explicitly, the gauge connection has the form

$$\mathbb{A} = \omega J + \frac{1}{\ell} e^a P_a + AT + aY. \quad (3)$$

where ω is the connection of rotations (spin connection), e^a the spatial dreibein and we have introduced a parameter ℓ

with dimensions of length in such a way that the Lie algebra generators are dimensionless. We now introduce the following shift in the connection one-form in Eq. (3)

$$\omega \rightarrow \omega + \beta a, \quad (4)$$

which can be translated into the definition of a new Lie algebra generator

$$Z = Y + \beta J. \quad (5)$$

This leads to the following commutation relations

$$[P_a, P_b] = -\epsilon_{ab}T, \quad [J, P_a] = \epsilon_a{}^b P_b = \frac{1}{\beta} [Z, P_a], \quad (6)$$

which we will refer to as *extended Nappi-Witten algebra*. This algebra admits a nondegenerate invariant bilinear form given by

$$\begin{aligned} \langle JJ \rangle &= \mu_0, & \langle P_a P_b \rangle &= \mu_1 \delta_{ab}, & \langle JT \rangle &= -\mu_1, & \langle ZJ \rangle &= \mu_2, \\ \langle ZT \rangle &= -\beta \mu_1, & \langle ZZ \rangle &= \mu_3, \end{aligned} \quad (7)$$

where μ_i are real constants. One can show that implementing the shift in Eq. (4) in the connection in Eq. (3) and using the Nappi-Witten $\times \mathfrak{u}(1)$ algebra leads to the gauge connection for the extended Nappi-Witten algebra

$$\mathbb{A} = \omega J + \frac{1}{\ell} e^a P_a + AT + aZ. \quad (8)$$

The corresponding curvature $\mathbb{F} = (1/2)\mathbb{F}_{\mu\nu}^A T_A dx^\mu \wedge dx^\nu$ has the form

$$\mathbb{F} = d\omega J + \frac{1}{\ell} R^a P_a + daZ + RT, \quad (9)$$

where we have defined the one-forms

$$\begin{aligned} R^a &= de^a + \epsilon^a{}_b e^b (\omega + \beta a), \\ R &= dA + \frac{1}{2} \epsilon_{ab} e^a e^b. \end{aligned} \quad (10)$$

A gauge transformation $\delta_\xi \mathbb{A} = d\xi + [\mathbb{A}, \xi]$ leads to the following transformation laws for the gauge fields

$$\begin{aligned} \delta_\xi \omega &= d\xi^J, & \delta_\xi a &= d\xi^Z, & \delta_\xi A &= d\xi^T + \frac{1}{\ell^2} \epsilon_{ab} \xi^P e^a e^b, \\ \delta_\xi e^a &= d\xi^P e^a + \epsilon^a{}_b (\xi^J e^b - \omega \xi^P e^b + \beta (\xi^Z e^b - a \xi^P e^b)). \end{aligned} \quad (11)$$

Diffeomorphisms with parameter χ^μ act on the connection as

$$\mathcal{L}_\chi \mathbb{A}_\mu = \mathbb{F}_{\mu\nu} \chi^\nu + \delta_{\xi_a \mathbb{A}^a} \mathbb{A}_\mu \quad (12)$$

and thus they are on-shell equivalent to gauge transformations with parameter $\xi_a^{\mathbb{A}\mu}$.

III. TOPOLOGICAL HALL RESPONSE FROM THE CHERN-SIMONS THEORY

Because we are willing to derive the topological response of the fractional Hall states, in this section we consider the Chern-Simons action for the connection in Eq. (8), given by

$$S = -\frac{k}{4\pi} \int \left\langle \mathbb{A} \wedge d\mathbb{A} + \frac{2}{3} \mathbb{A} \wedge \mathbb{A} \wedge \mathbb{A} \right\rangle, \quad (13)$$

with k an integer level. Importantly, this effective action can be derived from a microscopic theory by integrating out the fermionic fields associated to gapped spinful matter minimally coupled to the extended Nappi-Witten geometry (notice, spinful matter is compatible with a torsionful geometric background). By employing Eq. (7), the above action can be rewritten as follows (from now on the wedge product between forms will be omitted for simplicity)

$$S = -\frac{k}{4\pi} \int \left[\mu_0 \omega d\omega + \frac{\mu_1}{\ell^2} e^a R_a - 2\mu_1 A d\omega - 2\mu_1 \beta A da + 2\mu_2 a d\omega + \mu_3 a da \right]. \quad (14)$$

Variation with respect to the spatial dreibein e^a leads to the field equation $R^a = 0$, which in turn yields the following equation for torsion

$$T^a \equiv de^a + \epsilon^a_b e^b \omega = -\beta \epsilon^a_b e^b a. \quad (15)$$

This equation allows us to formally express the dreibein in terms of the spin connection and the field a . Now, by varying the action with respect to the field a , we obtain

$$a = \frac{\beta\mu_1}{\mu_3} A - \frac{\mu_2}{\mu_3} \omega. \quad (16)$$

By replacing this expression in Eq. (14) and by taking the following identifications of the parameters [57]

$$\begin{aligned} k = \mu_3 = 1, \quad \mu_1 = 2\bar{\nu}, \quad \mu_0 = \frac{c}{12}, \\ \mu_2 = \bar{\nu} \sqrt{\nu}, \quad \beta = \frac{1}{2\bar{\nu} \sqrt{\nu}}, \end{aligned} \quad (17)$$

we finally obtain

$$S = \int \left[\left(\frac{\nu\bar{\nu}^2}{4\pi} - \frac{c}{48\pi} \right) \omega d\omega + \frac{\nu}{4\pi} A dA + \frac{\nu\bar{\nu}}{2\pi} A d\omega \right], \quad (18)$$

which is compatible with the effective geometric action for the FQHE analyzed in Refs. [20–28] although it is

important to bear in mind that our spin connection is torsionful. Here, c is the chiral central charge, ν is the fractional filling and $\bar{\nu}$ is the average orbital spin. For the Laughlin states we have: $c = 1$, $\nu = 1/(2p + 1)$ and $\bar{\nu} = (2p + 1)/2$, with p an integer. The first and third terms in the action are usually referred as the gravitational Chern-Simons [20] and Wen-Zee term [49], respectively. Here, the coefficient in front of the U(1) CS term is associated to the Hall conductance, while the coefficient in front of the Wen-Zee term is related to the Hall viscosity [16–19,58]. These are the two main topological responses in the Abelian FQH states on manifolds with genus 0 and 1 [31]. The electron Hall density and current derived from the above action are respectively given by

$$\begin{aligned} \rho &= \frac{\nu}{2\pi} B + \frac{\nu\bar{\nu}}{4\pi} R, \\ J^i &= \frac{\nu}{2\pi} \epsilon^{ij} E_j + \frac{\nu\bar{\nu}}{2\pi} \epsilon^{ij} \mathcal{E}_j, \end{aligned} \quad (19)$$

where B and E_j are the magnetic and electric fields, respectively while $R = (2/\sqrt{|e|}) \epsilon^{ij} \partial_i \omega_j$ and $\mathcal{E}_j = \partial_j \omega_0 - \partial_0 \omega_j$ are the Abelian Ricci scalar and the gravi-electric field, respectively. The Hall viscosity comes from the response of the system to shear or strain [15]. Because our spin connection is torsionful, we follow here the approach developed in Refs. [22,59] that allows us to define the symmetric Cauchy stress-mass tensor from the spin current s^μ . This current is given by the variation of our topological action with respect to ω

$$s^\mu = \frac{\nu\bar{\nu}}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda + \left(\frac{\nu\bar{\nu}^2}{2\pi} - \frac{c}{24\pi} \right) \epsilon^{\mu\nu\lambda} \partial_\nu \omega_\lambda, \quad (20)$$

where the first term can be rewritten as follows

$$\frac{\nu\bar{\nu}}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda = \eta_H u^\mu, \quad (21)$$

with $u^\mu = (1, (1/B)\epsilon^{ij} E_j)$ the covariant drift velocity and $\eta_H = \nu\bar{\nu}B/4\pi$ the Hall viscosity [59]. Importantly, on a manifold with boundary both the Abelian and the gravitational CS terms contribute to the boundary gapless modes. These edge states are usually identified by chiral bosons. In the next section, we will derive the action for the chiral boson for the Laughlin states starting from the WZW action induced on the boundary by our CS theory in Eq. (13).

IV. CHIRAL BOSON FROM THE WZW MODEL

It is well known that Chern-Simons theories are gauge invariant only on compact manifolds. For this reason, in this section we consider a manifold with boundary such that the CS action gives rise to a chiral WZW model on the boundary [60]. This CFT represents the natural effective theory for the gapless edge excitations of the FQH states.

In the specific case of Laughlin states, the edge states are given by a chiral boson [5–9] that is usually described by the Floreanini-Jackiw action in flat space [61]. In our framework, we now derive the chiral boson from a WZW model. In fact, the CS action can be reduced to a boundary theory by solving the corresponding field equations for the spatial connection

$$\mathbb{F}_{ij} = 0, \quad (22)$$

and then plugging the solution back in the action. A local solution is given by

$$\mathbb{A}_i = g^{-1} \partial_i g. \quad (23)$$

Imposing the condition

$$\mathbb{A}_t + v \mathbb{A}_\phi = 0 \quad (24)$$

at the boundary and taking care of the surface integrals in (13), the action reduces to the following chiral WZW model [62]

$$S_{\text{WZW}} = \frac{k}{4\pi} \int_0^{2\pi} d\phi \int dt \langle g^{-1} \dot{g} g^{-1} g' + v (g^{-1} g')^2 \rangle + \frac{k}{12\pi} \int_M \langle (g^{-1} dg)^3 \rangle, \quad (25)$$

where $\dot{g} = \partial_t g$, $g' = \partial_\phi g$ and we have chosen for simplicity a 2 + 1-dimensional manifold $M = D \times \mathbb{R}$, where D is topologically equivalent to a disk and ϕ is the angular variable associated to the unitary circle that represents the spatial boundary $\mathbb{S}^1 \equiv \partial D$. Because we are mainly interested in the kinematics of the chiral boson, we look for the left-invariant Maurer-Cartan form and neglect the explicit derivation of the topological winding number of the WZW action. One way to find the left-invariant Maurer-Cartan form [60] is to solve the corresponding Maurer-Cartan equation

$$d\Omega + \Omega \wedge \Omega = 0, \quad (26)$$

where

$$\Omega = g^{-1} dg = \Omega_J J + \Omega_Z Z + \Omega_P^a P_a + \Omega_T T. \quad (27)$$

By employing the commutations of the extended Nappi-Witten algebra, the Maurer-Cartan equations can be shown to be equivalent to the following systems of equations

$$\begin{aligned} d\Omega_J &= 0, & d\Omega_Z &= 0, \\ d\Omega_P^a + \epsilon^a_b \Omega_P^b (\Omega_J + \beta \Omega_Z) &= 0, \\ d\Omega_T - \frac{1}{2} \epsilon_{ab} \Omega_P^a \Omega_P^b &= 0. \end{aligned} \quad (28)$$

The first two equations imply

$$\Omega_J = d\theta, \quad \Omega_Z = d\varphi, \quad (29)$$

while for the others we have

$$\begin{aligned} \Omega_P^a &= d\sigma^a - \epsilon^a_b \sigma^b (d\theta + \beta d\varphi), \\ \Omega_T &= d\vartheta + \frac{1}{2} \epsilon_{ab} \sigma^a d\sigma^b + \frac{1}{2} \sigma^a \sigma_a (d\theta + \beta d\varphi), \end{aligned} \quad (30)$$

with θ , φ and ϑ three real scalar fields and σ^a a vector field. By defining the following coordinates

$$x^\pm = \frac{1}{2} \left(t + \frac{1}{v} \phi \right), \quad \partial_\pm = \partial_t + v \partial_\phi, \quad (31)$$

with v the Fermi velocity, we find the explicit form of the WZW action

$$S_{\text{WZW}} = \frac{k}{4\pi} \int dt d\phi [\mu_0 \partial_+ \theta \theta' + 2\mu_2 \partial_+ \theta \varphi' + \mu_3 \partial_+ \varphi \varphi' + \mu_1 (\partial_+ \sigma^a \sigma'_a - (2\partial_+ \vartheta + \epsilon_{ab} \sigma^a \partial_+ \sigma^b) (\theta' + \beta \varphi'))]. \quad (32)$$

We see that ϑ is a Lagrange multiplier which gives rise to the following constraint

$$\partial_+ \theta' + \beta \partial_+ \varphi' = 0, \quad (33)$$

which implies

$$\theta = -\beta \varphi + a(t) + b(x^-), \quad (34)$$

with $a(t)$ and $b(x^-)$ arbitrary functions of their arguments. On the other hand the field equation for σ^a is given by

$$\partial_+ \sigma'_a + \epsilon_{ab} \partial_+ \sigma^b (\theta' + \beta \varphi') = 0. \quad (35)$$

By replacing these expressions back in the action lead to an action for a single chiral boson, given by

$$S_{\text{WZW}} = \frac{c}{192\pi} \left(\frac{1}{\nu \bar{s}} \right) \int d\phi dt (\dot{\varphi} \varphi' + v \varphi' \varphi'), \quad (36)$$

which agrees with the boundary theory previously derived in literature [26,27] for the Laughlin states where $1/(\nu \bar{s}^2) = 4\nu$ and $c = 1$. Importantly, the above chiral boson action can be naturally employed to describe the edge states of spin- j Laughlin states [30–32] where

$$\bar{s} = \frac{1}{2\nu} - j. \quad (37)$$

V. ADS-LORENTZ ALGEBRA AND EXTENDED NEWTON-CARTAN GEOMETRY

The extended Nappi-Witten algebra can be naturally embedded in a full nonrelativistic space-time algebra. This allows to interpret the Nappi-Witten geometry as a submanifold of a particular extended Newton-Cartan geometry [63]. In $2 + 1$ dimensions, the Galilei algebra admits two central extensions. This leads to the extended Bargmann symmetry [64–67]

$$\begin{aligned} [J, G_a] &= \epsilon_a{}^b G_b, & [J, P_a] &= [H, G_a] = \epsilon_a{}^b P_b, \\ [G_a, G_b] &= -\epsilon_{ab} S, & [G_a, P_b] &= -\epsilon_{ab} M. \end{aligned} \quad (38)$$

One can consider the following an extension of the Bargmann algebra by including a new set of generators $\{Z, Z_a, T\}$ and the commutation relations

$$\begin{aligned} [J, Z_a] &= [H, P_a] = [Z, G_a] = \epsilon_a{}^b Z_b, \\ [P_a, P_b] &= [G_a, Z_b] = -\epsilon_{ab} T. \end{aligned} \quad (39)$$

This is the Maxwellian Exotic Bargmann algebra [68] and defines a central extension of the “electric” nonrelativistic Maxwell algebra [69,70]. One can further extend the Bargmann algebra by introducing a parameter ℓ with dimension of length and the commutation relations [41]

$$\begin{aligned} [Z, Z_a] &= \epsilon_a{}^b Z_b, & [H, Z_a] &= [Z, P_a] = \epsilon_a{}^b P_b, \\ [P_a, Z_b] &= -\epsilon_{ab} M, & [Z_a, Z_b] &= -\epsilon_{ab} T. \end{aligned} \quad (40)$$

This algebra can be obtained as a nonrelativistic limit of the AdS-Lorentz algebra in $2 + 1$ dimensions [71], which in turn is a semisimple extension of the Maxwell algebra [72]. As it happens in the relativistic case, the Maxwell and the AdS-Lorentz symmetries are related by an Inönü-Wigner contraction. Indeed, one can use a length parameter ℓ to reinsert dimensions in the Lie algebra generators as

$$\begin{aligned} H &\rightarrow \ell H, & P_a &\rightarrow \ell P_a, & M &\rightarrow \ell M, \\ Z &\rightarrow \ell^2 Z, & Z_a &\rightarrow \ell^2 Z_a, & T &\rightarrow \ell^2 T, \end{aligned} \quad (41)$$

Thus, it is clear that in the limit $\ell \rightarrow \infty$ the nonrelativistic AdS-Lorentz algebra reduces to the electric Maxwell symmetry. One can see that the extended Nappi-Witten algebra (6) is the subalgebra of Eqs. (38)–(40) spanned by the generators $\{J, P_a, Z, T\}$, where one has to use the redefinition $Z \rightarrow \beta Z$.

At the relativistic level, the Maxwell algebra has been previously used to construct a geometric model for the gapped boundary of three-dimensional topological insulators [44]. However, in order to go beyond the integer QHE, a Chern-Simons term for the emergent U(1) gauge field is necessary. This term can be obtained by either extending the Maxwell to include its dual space, or by going to the

AdS-Lorentz extension [48]. Since the FQHE is intrinsically nonrelativistic, it is natural to expect that the nonrelativistic AdS-Lorentz is a good candidate to construct an effective geometric description. Here, we have shown that this is indeed the case. A Chern-Simons action (13) invariant under the nonrelativistic AdS-Lorentz algebra is constructed by means of the connection one-form

$$\begin{aligned} \mathbb{A} &= \omega J + \tau H + aZ + \omega^a G_a + e^a P_a \\ &+ k^a Z_a + mM + sS + AT. \end{aligned} \quad (42)$$

We can chose an absolute time and fix the reference frame by imposing the conditions

$$\tau_\mu = \delta_\mu^0, \quad \omega_\mu^a = 0 \quad (43)$$

Furthermore, we consider the particular case where

$$k_\mu^a = 0, \quad m_\mu = 0 = s_\mu. \quad (44)$$

One can show that in this case the Chern-Simons action invariant under the nonrelativistic AdS-Lorentz algebra reduces to the effective model in Eq. (14). It is important to note, however, that nondegenerate invariant bilinear form for the nonrelativistic AdS-Lorentz symmetry that generalized (7) is given by

$$\begin{aligned} \langle G_a G_b \rangle &= \lambda_0 \delta_{ab}, & \langle G_a P_b \rangle &= \langle P_a Z_b \rangle = \lambda_1 \delta_{ab} \\ \langle P_a P_b \rangle &= \langle G_a Z_b \rangle = \langle Z_a Z_b \rangle = \mu_1 \delta_{ab} \\ \langle JM \rangle &= \langle HS \rangle = \langle HT \rangle = \langle ZM \rangle = -\lambda_1, \\ \langle JT \rangle &= \langle ZS \rangle = \langle HM \rangle = \langle ZT \rangle = -\mu_1 \\ \langle JS \rangle &= -\lambda_0, & \langle JJ \rangle &= \mu_0, & \langle JH \rangle &= \lambda_4, & \langle JZ \rangle &= \mu_2, \\ \langle HH \rangle &= \lambda_2, & \langle HZ \rangle &= \lambda_3, & \langle ZZ \rangle &= \mu_3. \end{aligned} \quad (45)$$

This expression is more general than the invariant bilinear form derived in [41], which is the one that comes from the relativistic AdS-Lorentz symmetry upon contraction. This indicates that the model here presented is purely non-relativistic and does not have a relativistic counterpart.

It is known that the gauging of the Bargmann algebra leads to Newton-Cartan geometry [38–40]. Similarly, since the AdS-Lorentz algebra is an extension of the Bargmann symmetry, its gauging leads to an extended Newton-Cartan geometry. This is in complete analogy with the extended relativistic geometry that follows from Maxwell algebra [73] (see also [74]), which is an extension of the Poincaré symmetry that underlies Minkowski space.

VI. CONCLUSIONS AND OUTLOOK

In this paper, we have proposed a novel geometric model for the Laughlin states. We have shown that the U(1) and gravitational CS terms together with the Wen-Zee term can

be derived from a single CS action where the gauge connection takes values in the extended Nappi-Witten algebra. Besides the topological response in the bulk given by the Hall conductance and Hall viscosity, we have provided a novel way to derive the effective field theory for the chiral boson that lives on the edge of the system. We have shown that the extended Nappi-Witten symmetry can be naturally embedded in the nonrelativistic AdS-Lorentz algebra, which is a particular extension of the Bargmann algebra in $2 + 1$ dimensions. In this way, the geometry behind our model can be thought as part of a generalized Newton-Cartan geometry. Several directions will be considered in future work. In particular, we will extend our formalism by including multicharged emergent gauge fields to describe FQH states beyond the Laughlin states [5,75] and a second emergent metric to properly encode the

GMP mode and nematic states [35,36]. Finally, we will define a novel nonrelativistic higher-spin Nappi-Witten algebra to properly describe the higher-spin modes in the FQHE [76,77]. Our work paves the way for the description of the geometric and topological features of interacting topological fluids through generalized nonrelativistic geometries where both the charge and gravitational sectors are dealt in a unified framework.

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