

$\mathbb{Z}_N$  symmetries, anomalies, and the modular bootstrapYing-Hsuan Lin<sup>1,2,\*</sup> and Shu-Heng Shao<sup>3,†</sup><sup>1</sup>*Jefferson Physical Laboratory, Harvard University, Cambridge, Massachusetts 02138, USA*<sup>2</sup>*Walter Burke Institute for Theoretical Physics, California Institute of Technology, Pasadena, California 91125, USA*<sup>3</sup>*School of Natural Sciences, Institute for Advanced Study, Princeton, New Jersey 08540, USA*

(Received 3 February 2021; accepted 29 April 2021; published 1 June 2021)

We explore constraints on  $(1+1)d$  unitary conformal field theory with an internal  $\mathbb{Z}_N$  global symmetry, by bounding the lightest symmetry-preserving scalar primary operator using the modular bootstrap. Among the other constraints we have found, we prove the existence of a  $\mathbb{Z}_N$ -symmetric relevant/marginal operator if  $N-1 \leq c \leq 9-N$  for  $N \leq 4$ , with the end points saturated by various Wess-Zumino-Witten models that can be embedded into  $(e_8)_1$ . Its existence implies that robust gapless fixed points are not possible in this range of  $c$  if only a  $\mathbb{Z}_N$  symmetry is imposed microscopically. We also obtain stronger, more refined bounds that depend on the 't Hooft anomaly of the  $\mathbb{Z}_N$  symmetry.

DOI: [10.1103/PhysRevD.103.125001](https://doi.org/10.1103/PhysRevD.103.125001)

## I. INTRODUCTION

Global symmetries and their 't Hooft anomalies (i.e., obstruction to gauging) are central tools in analyzing strongly coupled quantum systems. In this paper, we continue our exploration in [1] of universal constraints imposed by the symmetries and anomalies in conformal field theory (CFT). We will apply the techniques of the conformal bootstrap, which exploits the internal consistency of CFT, to derive general constraints on  $(1+1)d$  unitary bosonic CFT with  $\mathbb{Z}_N$  global symmetry.

Of special importance are constraints on the robust (or stable), gapless fixed points of renormalization group (RG) flows. Typically, we start from a short-distanced quantum system, which can, for example, be another quantum field theory or a lattice model, and impose a global symmetry  $G_{UV}$  on the system. The global symmetry  $G_{UV}$  generally carries certain 't Hooft anomalies. We would like to ask whether this system can flow to a gapless fixed point without fine-tuning the parameters at short distances. Typically the global symmetry in the infrared (IR) CFT is enhanced to a bigger group  $G_{IR}$  with an embedding  $G_{UV} \rightarrow G_{IR}$ , in which case the IR CFT has to match the 't Hooft anomalies of the UV system.

A generic local perturbation of the UV system becomes a local operator in the low-energy quantum field theory. For the candidate low-energy CFT to be a robust fixed point, there cannot be any  $G_{UV}$ -invariant relevant local operator.<sup>1</sup> The absence of such an operator at low energy guarantees that, if the flow approaches close to the fixed point, it will be attracted to it. See, for example, [3–5] for applications of this type of argument and [6] for a recent discussion on the robustness of quantum field theory.

Our approach to this general question is to exploit the internal consistency of CFT in  $(1+1)$  dimensions. In  $(1+1)d$  CFT, the spectrum of local operators is highly constrained by the modular invariance of the torus partition function. Most famously, Cardy derived a universal formula for the density of heavy local operators in terms of the central charge [7]. More generally, when the CFT has a certain global symmetry, the spectra of charged and uncharged operators are constrained by the modular covariance of the torus partition function dressed with symmetry defects. The modern modular bootstrap program, which exploits the above consistency conditions, provides an ideal platform to map out the space of CFTs with global symmetry. See, for example, [1,8–39] for various exciting developments in the modular bootstrap.

In the special case of a  $\mathbb{Z}_N$  global symmetry with small  $N$ , we will derive a universal upper bound  $\Delta_{\text{scalar}}^{Q=0}$  on the scaling dimension of the lightest symmetry-preserving scalar operator. This generalizes the previous works when there is no symmetry [17] and when the symmetry is  $\mathbb{Z}_2$  [1].

<sup>1</sup>However, see [2] for an interesting example of a  $(1+1)d$  spin chain where the low-energy gapless fixed point is stable even in the presence of symmetry-preserving relevant local operators.

\*yhlin@fas.harvard.edu  
†shao@ias.edu

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International license](https://creativecommons.org/licenses/by/4.0/). Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP<sup>3</sup>.

This upper bound  $\Delta_{\text{scalar}}^{Q=0}(N, k, c)$  depends on the central charge  $c$  and the 't Hooft anomaly, denoted as  $k$ , of the  $\mathbb{Z}_N$  global symmetry.<sup>2</sup>

For small  $N$  and a given anomaly  $k$ , our bound further implies that there must be a symmetry-preserving relevant scalar operator for any CFT within a certain range of the central charge. In other words, robust gapless fixed points are impossible within this range of the central charge if we only impose a  $\mathbb{Z}_N$  symmetry microscopically.<sup>3</sup>

In  $(1+1)d$ , the anomaly of a global symmetry is encoded in the  $F$ -symbols of the fusion category characterizing the symmetry defects (see, for example, [44,45]). For each  $N$ , there is a special value of the anomaly  $k$  that is natural from the point of view of  $(2+1)d$  topological quantum field theory, which is described by unitary modular tensor category (UMTC). Specifically, for  $k=0$  when  $N$  is odd and for  $k=N/2$  when  $N$  is even, the corresponding  $\mathbb{Z}_N$  fusion category with anomaly  $k$  can be lifted to a UMTC with  $\mathbb{Z}_N$  fusion rule [46] (see also [47–49]).

For this special anomaly and small  $N$ , we find that the bounds at  $c=N-1$  and  $c=9-N$  are saturated by the center symmetries of a pair of Wess-Zumino-Witten (WZW) models  $(\mathfrak{g}_1, \mathfrak{h}_1)$  at level 1. In these cases, the marginal operators are the current bilinear operators  $J\bar{J}$ , which are neutral under the center symmetry. Moreover, there must be a  $\mathbb{Z}_N$ -symmetric relevant/marginal operator in between,

$$N-1 \leq c \leq 9-N, \quad N \leq 4. \quad (1.1)$$

See Table I.

For nonanomalous  $\mathbb{Z}_5$  at  $c=4$ , which is the value of the central charge extrapolated from (1.1) to  $N=5$ , the bound on the scalar primary operator is  $\Delta_{\text{scalar}}^{Q=0} < 2.045$ . This bound is almost saturated by the marginal current bilinear operator in the  $\mathfrak{su}(5)_1$  WZW model. For  $\mathbb{Z}_6$  with the  $k=3$  anomaly, the range of  $c$  is  $3.98 < c < 4.85$ , and the bound at  $c=5$  is  $\Delta_{\text{scalar}}^{Q=0} < 2.022$ , which is almost saturated by the  $\mathfrak{su}(6)_1$  WZW model. For  $\mathbb{Z}_6$  with other anomalies and for  $\mathbb{Z}_7$  with any anomaly, we could not establish any range of  $c$  for which the bound drops to or below 2. This presents a natural stopping point for our numerical exploration.

<sup>2</sup>For simplicity, we assume that the left and right central charges are equal in this paper, i.e.,  $c \equiv c_L = c_R$ .

<sup>3</sup>The prototypical example of a robust gapless phase with a nonanomalous  $\mathbb{Z}_N$  global symmetry is the  $\mathbb{Z}_N$  clock model [40]. For  $N \geq 5$ , it is known that there is a region in the parameter space where the low-energy phase is gapless [41–43]. Curiously, our numerical bootstrap bounds for the nonanomalous  $\mathbb{Z}_N$  symmetry only give a nontrivial range of  $c$  when  $N \leq 4$  and therefore do not constrain the phase diagram of the clock model. We thank Pranay Gorantla, Ho Tat Lam, and Nathan Seiberg for related discussions.

TABLE I. For  $N \leq 4$ , we show that any CFT with a  $\mathbb{Z}_N$  global symmetry must have a  $\mathbb{Z}_N$ -symmetric relevant/marginal operator in a range of  $c$  that is numerically very close to that above. For each range of  $c$ , the bounds at the two ends are saturated by a pair of WZW models that can be embedded into  $(\mathfrak{e}_8)_1$ .

Symmetry	$c$	CFTs saturating the bounds
None	$0 \leq c \leq 8$	(Trivial, $(\mathfrak{e}_8)_1$ )
$\mathbb{Z}_2$	$1 \leq c \leq 7$	$(\mathfrak{su}(2)_1, (\mathfrak{e}_7)_1)$
$\mathbb{Z}_3$	$2 \leq c \leq 6$	$(\mathfrak{su}(3)_1, (\mathfrak{e}_6)_1)$
$\mathbb{Z}_4$	$3 \leq c \leq 5$	$(\mathfrak{su}(4)_1 = \mathfrak{so}(6)_1, \mathfrak{so}(10)_1)$

Surprisingly, our numerical bootstrap bounds have an interesting connection to the  $(\mathfrak{e}_8)_1$  vertex operator algebra (VOA). For each  $N \leq 4$ , the pair  $(\mathfrak{g}_1, \mathfrak{h}_1)$  are both sub-VOAs of  $(\mathfrak{e}_8)_1$  and they are the commutants of each other. Furthermore, the decomposition of  $(\mathfrak{e}_8)_1$  into this pair of mutual commutants preserves precisely a  $\mathbb{Z}_N$  global symmetry with the anomaly described above. The analogous decompositions for  $N=5, 6$  are into  $\mathfrak{su}(5)_1 \times \mathfrak{su}(5)_1$  and  $\mathfrak{su}(2)_1 \times \mathfrak{su}(3)_1 \times \mathfrak{su}(6)_1$ , respectively.<sup>4</sup> Interestingly, our bootstrap bounds for nonanomalous  $\mathbb{Z}_5$  at  $c=4$  and that for  $\mathbb{Z}_6$  with the  $k=3$  anomaly at  $c=5$  appear to be (almost) saturated by  $\mathfrak{su}(5)_1$  and  $\mathfrak{su}(6)_1$ .<sup>5</sup> We leave these curious observations and an analytic derivation of (1.1) for future investigations. Some of these commutant pairs of  $(\mathfrak{e}_8)_1$  have recently been discussed in [50,51].

The ranges of  $c$  from the other anomalies are wider than (1.1). Therefore, among the other constraints we have derived, we conclude that any CFT with a  $\mathbb{Z}_N$  symmetry in the range (1.1) must have a  $\mathbb{Z}_N$ -symmetric relevant/marginal operator.

To check whether our bounds are correct, we will compute the  $\mathbb{Z}_N$  anomalies in a variety of  $(1+1)d$  CFTs. An explicit formula for the 't Hooft anomaly can be derived from the following input data<sup>6</sup>:

- (i) the Hilbert space  $\mathcal{H}$  of local operators and their conformal weights  $(h, \bar{h})$ ,
- (ii) the action of the  $\mathbb{Z}_N$  global symmetry on the local operators.

First, from the input data one can readily compute  $Z^\eta(\tau, \bar{\tau})$ , the torus partition function with a  $\mathbb{Z}_N$  symmetry operator  $\hat{\eta}$  inserted at a fixed time. The  $\mathbb{Z}_N$  anomaly  $k$ , which is an element of  $H^3(\mathbb{Z}_N, \text{U}(1)) = \mathbb{Z}_N$ , can then be expressed as

$$e^{\frac{2\pi i k}{N}} = \frac{Z^\eta(-\frac{1}{\tau+N}, -\frac{1}{\bar{\tau}+N})}{Z^\eta(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}})}. \quad (1.2)$$

<sup>4</sup>We thank Theo Johnson-Freyd for an illuminating discussion on this point.

<sup>5</sup>However, the bound for  $\mathbb{Z}_6$  with the  $k=3$  anomaly at  $c=3$  is significantly above 2 and is not saturated by the  $\mathfrak{su}(2)_1 \times \mathfrak{su}(3)_1$  WZW model.

<sup>6</sup>See [52] for a related discussion on the lattice. We thank Michael Levin for discussions on this point.

Despite appearance, the right-hand side is actually independent of  $\tau, \bar{\tau}$ . It is important to stress that this formula for the anomaly holds true for general  $(1+1)d$  bosonic CFT with a unique vacuum and is not restricted to rational or unitary CFT. We will apply this formula to the  $c = 1$  compact boson, the WZW models, and the parafermion CFT in Sec. III.

This paper is organized as follows. Section II reviews the modern perspective on global symmetries and their anomalies in terms of the topological defects. In Sec. III, we compute the 't Hooft anomaly for various CFTs with  $\mathbb{Z}_N$  global symmetry. In Sec. IV, we set up the modular bootstrap system and present universal upper bounds on the lightest symmetry-preserving scalar operators. In Appendix B, we collect further details on the modular bootstrap system. In Appendix B, we present more general upper bounds on charged local operators.

## II. SYMMETRIES AND ANOMALIES IN $(1+1)d$

In this section, we start with a general characterization of global symmetries and their 't Hooft anomalies in  $(1+1)d$  in terms of the topological defect lines. See [1,44,45,53–56] and references therein for recent discussions on this topic.

### A. Topological defect lines

The 0-form global symmetry in a general quantum field theory in  $d$  spacetime dimensions is implemented by a codimension-one topological defect [57,58]. When the global symmetry group is  $U(1)$ , the topological defect is nothing but  $\exp[i\theta \int_{\Sigma} \star j]$ , where  $j_{\mu}(x)$  is the Noether current. Here  $\Sigma$  is a codimension-one closed manifold in spacetime (sometimes taken to be the whole space at a fixed time) where the topological defect has support. In this example, the topological nature of the defect follows from the conservation equation of the Noether current,  $\partial^{\mu} j_{\mu} = 0$ , in flat spacetime with trivial background.

For a discrete global symmetry where there is no local Noether current, there is still a codimension-one topological defect  $\mathcal{L}_g$  associated with every group element  $g$ . In the special case when the manifold of support  $\Sigma$  is taken to be the whole space at a fixed time,  $\mathcal{L}_g$  is the charge operator of the symmetry. In Euclidean signature, we can encircle a local operator  $\phi(x)$  by a topological defect  $\mathcal{L}_g$  and then contract the defect without changing the correlation function. This process produces another local operator  $\phi'(x) = g \cdot \phi(x)$ .<sup>7</sup>

<sup>7</sup>Strictly speaking,  $\phi'(x) = \langle \mathcal{L}_g \rangle_{\mathbb{R}^d} \times g \cdot \phi(x)$ , where  $\langle \mathcal{L}_g \rangle_{\mathbb{R}^d}$  is the vacuum expectation value of the topological defect in  $\mathbb{R}^d$ , which might not always be +1. For example, an anomalous  $\mathbb{Z}_2$  topological defect line in  $(1+1)d$  has  $\langle \mathcal{L} \rangle_{\mathbb{R}^2} = -1$ . See Sec. II. D of [45,59].

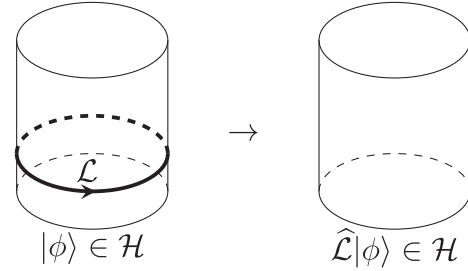


FIG. 1. A topological defect line  $\mathcal{L}$  wrapped around the spatial circle of a cylinder leads to an action  $\hat{\mathcal{L}}$  on the Hilbert space  $\mathcal{H}$ .

In  $(1+1)d$ , such codimension-one topological defects are lines. In this paper, we will focus on the topological defect line (TDL)  $\mathcal{L}$  associated with an internal, unitary  $\mathbb{Z}_N$  symmetry in a bosonic  $(1+1)d$  CFT. The  $\mathbb{Z}_N$  TDL implements a  $\mathbb{Z}_N$  action on the Hilbert space  $\mathcal{H}$  quantized on a circle  $S^1$ . This action can be realized on the cylinder  $S^1 \times \mathbb{R}$  by wrapping the  $\mathbb{Z}_N$  TDL around the compact (spatial)  $S^1$  direction, to act on a state  $|\phi\rangle \in \mathcal{H}$  prepared at an earlier time (see Fig. 1). We will denote this  $\mathbb{Z}_N$  unitary operator as

$$\hat{\mathcal{L}}: \mathcal{H} \rightarrow \mathcal{H}. \tag{2.1}$$

We can grade the Hilbert space  $\mathcal{H}$  using the  $\mathbb{Z}_N$  charge  $Q = 0, 1, \dots, N-1$ ,

$$\mathcal{H} = \bigoplus_{Q=0}^{N-1} \mathcal{H}^{(Q)}. \tag{2.2}$$

Via the operator-state correspondence, the TDL also implements a  $\mathbb{Z}_N$  action on local operators: as a  $\mathbb{Z}_N$  TDL is swept past a local operator  $\phi(x)$ , the correlation function changes by a phase  $e^{\frac{2\pi i Q}{N}}$  (see Fig. 2).

The fusion of topological symmetry lines obeys the group multiplication law. In particular, as we bring  $N$  parallel  $\mathbb{Z}_N$  loops together, they fuse to a trivial line. Thus  $\hat{\mathcal{L}}^N = 1$ .

Consider the theory on a cylinder  $S^1 \times \mathbb{R}$  with  $\mathcal{L}$  running along the time  $\mathbb{R}$  direction (see Fig. 3). The  $\mathcal{L}$  TDL intersects with the spatial  $S^1$  and therefore implements a twisted periodic boundary condition in the quantization.

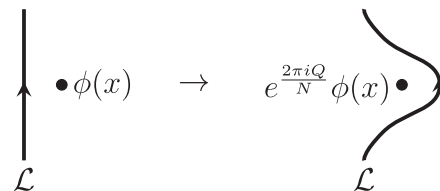


FIG. 2. As a  $\mathbb{Z}_N$  TDL is swept past a local operator  $\phi(x)$ , the correlation function changes by a phase  $e^{\frac{2\pi i Q}{N}}$  where  $Q = 0, 1, \dots, N-1$  is the  $\mathbb{Z}_N$  charge of  $\phi$ .

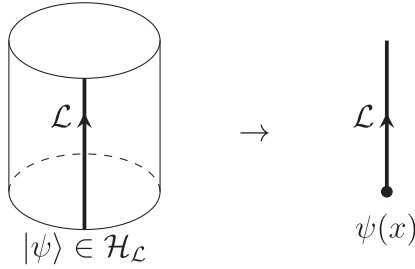


FIG. 3. By quantizing the system on a spatial circle with a topological defect line  $\mathcal{L}$  inserted at a point in space, we define a defect Hilbert space  $\mathcal{H}_\mathcal{L}$ . Via the operator-state correspondence, the states in  $\mathcal{H}_\mathcal{L}$  are mapped to operators living at the end of  $\mathcal{L}$ .

This defines a ‘‘defect Hilbert space’’ denoted by  $\mathcal{H}_\mathcal{L}$ . Via the operator-state correspondence, a defect Hilbert space state  $|\psi\rangle \in \mathcal{H}_\mathcal{L}$  is mapped to an operator living at the end of the  $\mathbb{Z}_N$  TDL.

Since a TDL commutes with the stress tensor, the states in the defect Hilbert space  $\mathcal{H}_\mathcal{L}$  can be organized into representations of the left and right Virasoro algebras. In particular, the defect Hilbert space can be diagonalized to have definite conformal weights  $(h, \bar{h})$ .

### B. Explicit formula for $\mathbb{Z}_N$ anomalies

The ‘t Hooft anomaly of a global symmetry is characterized by certain splitting-and-joining relations of their TDLs. More specifically, it is captured by the  $F$ -symbols of the fusion category formed by these TDLs. We refer the readers to [1,44,45,54] for physicists’ expositions on this subject.

Here we focus on the case of a  $\mathbb{Z}_N$  global symmetry in  $(1+1)d$  bosonic CFT. An explicit formula for the  $\mathbb{Z}_N$  anomaly can be derived from the input data of the conformal weights  $(h, \bar{h})$  and the  $\mathbb{Z}_N$  charges of all the local operators in the Hilbert space  $\mathcal{H}$ . (The relation between modular invariance/covariance and anomalies has been discussed extensively in [60–64].)

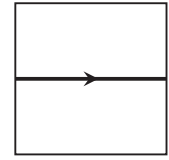
The  $\mathbb{Z}_N$  anomaly is classified by the group cohomology  $H^3(\mathbb{Z}_N, \text{U}(1)) = \mathbb{Z}_N$  [60,65,66]. Let  $\eta$  be a generator of the  $\mathbb{Z}_N$  global symmetry with ‘t Hooft anomaly  $k$ , which is an integer modulo  $N$ . The cocycle of this group cohomology can be chosen to be (see, for example, [46])

$$\alpha(a, b, c) = \exp \left[ \frac{2\pi i k}{N^2} a(b + c - \langle b + c \rangle) \right], \quad (2.3)$$

where  $a, b, c = \{0, 1, \dots, N-1\}$  and  $\langle a \rangle$  is the mod  $N$  function to  $\{0, 1, \dots, N-1\}$ .

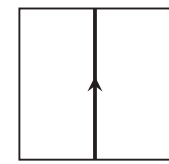
Consider the partition function  $Z^\eta(\tau, \bar{\tau})$  with the symmetry line  $\eta$  wrapped around the spatial circle (a twist in the time direction) of a torus with complex modulus  $\tau$ . This partition function admits the following interpretation as a

trace over the Hilbert space  $\mathcal{H}$  of local operators with the  $\mathbb{Z}_N$  symmetry operator  $\hat{\eta}$  inserted:



$$Z^\eta(\tau, \bar{\tau}) = \text{Tr}_{\mathcal{H}} \hat{\eta} q^{h - \frac{c}{24}} \bar{q}^{\bar{h} - \frac{\bar{c}}{24}}, \quad (2.4)$$

where  $q = \exp(2\pi i \tau)$ ,  $\bar{q} = \exp(-2\pi i \bar{\tau})$ . Using the trace interpretation of the right-hand side, the partition function  $Z^\eta(\tau, \bar{\tau})$  can be computed by the input data above. Under a modular S transform  $\tau \rightarrow -1/\tau$ , we obtain the torus partition function with an  $\eta$  line running in the time direction,



$$\begin{aligned} Z_\eta(\tau, \bar{\tau}) &= \text{Tr}_{\mathcal{H}_\eta} q^{h - \frac{c}{24}} \bar{q}^{\bar{h} - \frac{\bar{c}}{24}} \\ &= Z^\eta(-1/\tau, -1/\bar{\tau}), \end{aligned} \quad (2.5)$$

where  $\mathcal{H}_\eta$  is the defect Hilbert space introduced in Sec. II A.

It was explained in [45] (see also [67]) that the spins  $h - \bar{h}$  of the states in the defect Hilbert space  $\mathcal{H}_\eta$  of such an anomalous  $\mathbb{Z}_N$  symmetry are constrained as

$$\mathcal{H}_\eta: s \equiv h - \bar{h} \in \frac{k}{N^2} + \frac{\mathbb{Z}}{N}. \quad (2.6)$$

We will refer to this constraint as a spin selection rule.<sup>8</sup>

The spin selection rule (2.6) implies that under the modular  $T^N$  transformation, the partition function  $Z_\eta$  of  $\mathcal{H}_\eta$  is shifted by a phase determined by the anomaly

$$Z_\eta(\tau + N, \bar{\tau} + N) = e^{2\pi i k} Z_\eta(\tau, \bar{\tau}). \quad (2.7)$$

Written in terms of  $Z^\eta$ , we find the following formula for the ‘t Hooft anomaly  $k$ :

$$e^{\frac{2\pi i k}{N}} = \frac{Z^\eta(-\frac{1}{\tau+N}, -\frac{1}{\bar{\tau}+N})}{Z^\eta(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}})}. \quad (2.8)$$

In dividing the partition function, we have assumed that our CFT has a unique vacuum (i.e., there is a unique local operator with  $h = \bar{h} = 0$ ), so that  $Z^\eta(\tau, \bar{\tau})$  is not a zero function. We stress that  $Z^\eta(\tau, \bar{\tau})$  is completely and

<sup>8</sup>This spin selection rule for a (possibly anomalous)  $\mathbb{Z}_N$  symmetry in  $(1+1)d$  CFT is related to the spins of the anyons in the  $(2+1)d$   $\mathbb{Z}_N$  gauge theory (possibly with a Dijkgraaf-Witten twist [68]). See [1,69] for the case of  $\mathbb{Z}_2$  symmetry.

straightforwardly determined by the input data specified above. Note that, despite appearance, the right-hand side is actually independent of  $\tau, \bar{\tau}$ .

Starting from a  $\mathbb{Z}_N$  symmetry and its anomaly, we now discuss the anomaly of its group extension and its subgroup. A  $\mathbb{Z}_N$  symmetry can be viewed as an unfaithful  $\mathbb{Z}_{pN}$  symmetry, for some positive integer  $p$ . Mathematically, this means that the  $\mathbb{Z}_N$  symmetry is *extended* to  $\mathbb{Z}_{pN}$ , where  $\eta$  is now regarded as an order  $pN$  element,  $\eta^{pN} = 1$ . By viewing the spin selection rule (2.6) as that for a  $\mathbb{Z}_{pN}$  defect Hilbert space, we find that the anomaly of this  $\mathbb{Z}_{pN}$  symmetry generated by  $\eta$  is  $p^2k \bmod pN$ . In particular, by choosing  $p = N/\gcd(k, N)$ , the  $\mathbb{Z}_N$  anomaly is trivialized when extended to  $\mathbb{Z}_{pN}$ . See [53,70] for a more general discussion on how anomalies can be trivialized via extension.

Next, consider the  $\mathbb{Z}_{N/\gcd(r, N)}$  subgroup of  $\mathbb{Z}_N$  generated by  $\eta^r$  for some positive integer  $r$ . The spin selection rule for the defect Hilbert space of  $\eta^r$  is

$$\mathcal{H}_{\eta^r}: s \in \frac{r^2k}{N^2} + \frac{\mathbb{Z}}{N/\gcd(r, N)}. \quad (2.9)$$

Hence the anomaly of the  $\mathbb{Z}_{N/\gcd(r, N)}$  subgroup generated by  $\eta^r$  is  $(\frac{r}{\gcd(r, N)})^2k \bmod \frac{N}{\gcd(r, N)}$ .

The torus partition functions  $Z^{\eta^r}$  and  $Z_{\eta^r}$  form a system closed under the modular S transform. The consistency of the modular S transform can then be exploited to study and constrain different charge sectors. This is the subject of Sec. IV.

### III. EXAMPLES OF $\mathbb{Z}_N$ SYMMETRIES AND ANOMALIES

In this section we review and compute the  $\mathbb{Z}_N$  anomalies in various different CFTs.

#### A. Anomalies in diagonal rational CFTs

In rational CFT with diagonal modular invariance, there is a natural class of TDLs known as the ‘‘Verlinde lines’’ [71,72]. They have the distinguished property that they commute not only with the Virasoro algebra, but also with the entire left and right chiral algebras. Modular covariance constrains the Verlinde lines to be in one-to-one correspondence with the primaries of the chiral algebra.

Let us review the action of the Verlinde lines on the Hilbert space. Let  $|\phi_i\rangle$  be the chiral algebra primary of a rational CFT, with the index  $i$  labels different primaries. The Verlinde line  $\mathcal{L}_k$  associated with the primary  $\phi_k$  acts on the primary  $|\phi_i\rangle$  as

$$\hat{\mathcal{L}}_k|\phi_i\rangle = \frac{S_{ki}}{S_{0i}}|\phi_i\rangle, \quad (3.1)$$

where  $S_{ki}$  is the modular S matrix and 0 stands for the identity operator. In particular, the Verlinde lines in a diagonal rational conformal field theory (RCFT) commutes with parity, which exchanges  $h$  with  $\bar{h}$ .

The Verlinde lines, such as the Kramers-Wannier duality line in the Ising CFT [73,74], are generally noninvertible. We will refer to the invertible ones as the ‘‘Verlinde symmetry lines.’’ Simple examples of symmetries that are realized by Verlinde lines are the  $\mathbb{Z}_2$  symmetry in the Ising CFT and the center symmetry in the WZW model that commutes with the left and the right current algebras.

Using the spin selection rule, we will now show that certain ‘t Hooft anomalies cannot be realized by Verlinde symmetry lines. Since the Verlinde symmetry lines commute with parity, their defect Hilbert spaces should be invariant under flipping the sign of the spin  $s = h - \bar{h} \rightarrow -s$ . We conclude that an anomaly can be realized by the Verlinde symmetry lines only if the spin selection rules of its defect Hilbert spaces are invariant under  $s \rightarrow -s$ .

Consider the special case of a  $\mathbb{Z}_N$  global symmetry generated by  $\eta$  with anomaly  $k$ . Let  $\mathcal{S}$  be the set of spins in the defect Hilbert space  $\mathcal{H}_{\eta}$ . Every element  $s$  of  $\mathcal{S}$  is constrained by the spin selection rule to satisfy (2.6). It follows that a necessary condition for the set  $\mathcal{S}$  to be invariant under  $s \rightarrow -s$  is

$$2k \in N\mathbb{Z}. \quad (3.2)$$

This is equivalent to

$$\begin{aligned} k &= 0 \pmod N, \quad \text{for odd } N, \\ k &= 0, \quad \frac{N}{2} \pmod N, \quad \text{for even } N. \end{aligned} \quad (3.3)$$

We conclude that a  $\mathbb{Z}_N$  symmetry with anomaly  $k$  can be realized by Verlinde symmetry lines only if  $2k \in N\mathbb{Z}$ . For small  $N$ , we tabulate the possible  $\mathbb{Z}_N$  anomalies that can be realized by the Verlinde symmetry lines in certain diagonal RCFTs in Table II.

TABLE II.  $\mathbb{Z}_N$  anomalies that can be realized as Verlinde symmetry lines in diagonal RCFTs for  $N = 2, \dots, 5$ . On the third column we tabulate some diagonal RCFTs that realize these anomalies. Here  $n$  is any positive integer.

$G$	Anomaly $k$	RCFTs
$\mathbb{Z}_2$	0	Ising, $\mathfrak{su}(2)_{2n}$
	1	$\mathfrak{su}(2)_{2n+1}$
$\mathbb{Z}_3$	0	$\mathbb{Z}_3$ parafermion, $\mathfrak{su}(3)_n$
$\mathbb{Z}_4$	0	$\mathbb{Z}_4$ parafermion, $\mathfrak{su}(4)_{2n}$
	2	$\mathfrak{su}(4)_{2n+1}, \mathfrak{so}(4n+2)_1$
$\mathbb{Z}_5$	0	$\mathbb{Z}_5$ parafermion, $\mathfrak{su}(5)_n$

In any diagonal RCFT, the set of Verlinde lines can be lifted into a UMTC.<sup>9</sup> In particular, the Verlinde symmetry lines, which are the subset of invertible Verlinde lines, can be lifted to a unitary braided tensor category (UBTC). The constraints (3.2) imply that not every ‘t Hooft anomaly, which is encoded in the  $F$ -symbols (2.3) of the TDLs, is compatible with a UBTC. It is indeed known that only those  $F$ -symbols obeying (3.3) can be lifted to a UBTC [46]. Above we provide a derivation of this well-known fact from a different perspective.

For instance, the anomalous  $\mathbb{Z}_3$  symmetry cannot be lifted to a UBTC and therefore cannot be realized by the Verlinde lines in a diagonal RCFT. As another example, the  $\mathbb{Z}_4$  symmetries with  $k = 1, 3$  anomalies in Sec. III D also cannot be lifted to a UBTC.

### B. $c = 1$ compact boson

The  $c = 1$  compact boson at radius  $R$  is described by a target space field  $X(z, \bar{z})$  with periodicity  $X(z, \bar{z}) \sim X(z, \bar{z}) + 2\pi R$  and the operator product expansion (OPE)<sup>10</sup>

$$X(z, \bar{z})X(0, 0) \sim -\frac{1}{2} \log |z|^2. \quad (3.4)$$

The left and right current algebras are generated by the holomorphic and antiholomorphic currents  $i\partial X(z)$  and  $i\bar{\partial}X(\bar{z})$ , which have the OPE

$$i\partial X(z)i\partial X(0) \sim \frac{1}{2z}, \quad i\bar{\partial}X(\bar{z})i\bar{\partial}X(0) \sim \frac{1}{2\bar{z}}. \quad (3.5)$$

The current algebra primaries are exponential operators labeled by a momentum number  $n \in \mathbb{Z}$  and a winding number  $w \in \mathbb{Z}$ ,

$$\mathcal{O}_{n,w}(z, \bar{z}) = \exp \left[ i \left( \frac{n}{R} + wR \right) X_L(z) + i \left( \frac{n}{R} - wR \right) X_R(\bar{z}) \right]; \quad (3.6)$$

with conformal weights

$$h = \frac{1}{4} \left( \frac{n}{R} + wR \right)^2, \quad \bar{h} = \frac{1}{4} \left( \frac{n}{R} - wR \right)^2. \quad (3.7)$$

Here  $::$  stands for normal ordering of the operator. The scaling dimension and the spin are

$$\Delta = h + \bar{h} = \frac{1}{2} \left( \frac{n^2}{R^2} + w^2 R^2 \right), \quad s = h - \bar{h} = nw. \quad (3.8)$$

<sup>9</sup>The relative ‘t Hooft anomaly in RCFT has recently been analyzed from this perspective in [75].

<sup>10</sup>Our convention for the radius  $R$  is such that  $R = 1$  is the self-dual point with  $\mathfrak{su}(2)_1$  current algebra.

The OPE between the (anti)holomorphic currents and the primaries is

$$\begin{aligned} i\partial X(z)\mathcal{O}_{n,w}(0, 0) &\sim \frac{\frac{n}{R} + wR}{2z} \mathcal{O}_{n,w}(0, 0), \\ i\bar{\partial}X(\bar{z})\mathcal{O}_{n,w}(0, 0) &\sim \frac{\frac{n}{R} - wR}{2\bar{z}} \mathcal{O}_{n,w}(0, 0). \end{aligned} \quad (3.9)$$

The Ward identity then implies that, at an irrational  $R^2$ , the exponential operators have mutually irrational charges under each current, i.e.,  $i\partial X(z)$  and  $i\bar{\partial}X(\bar{z})$  each generate a noncompact  $\mathbb{R}$  instead of a compact  $U(1)$  global symmetry. Nonetheless, the symmetry group is a compact  $U(1) \times U(1)$  torus. The compact cycles are generated by the momentum and winding currents that have holomorphic and antiholomorphic components

$$\begin{aligned} \text{momentum } U(1)_n: J_n(z) &= iR\partial X(z), \quad \bar{J}_n(z) = iR\bar{\partial}X(\bar{z}), \\ \text{winding } U(1)_w: J_w(z) &= \frac{i}{R}\partial X(z), \quad \bar{J}_w(z) = -\frac{i}{R}\bar{\partial}X(\bar{z}), \end{aligned} \quad (3.10)$$

under which the exponential operators  $\mathcal{O}_{n,w}(z, \bar{z})$  have integer charges  $n$  and  $m$ , respectively.

The  $U(1)_n$  and  $U(1)_w$  symmetries are each nonanomalous by themselves, but they have a mixed anomaly. For a pair of coprime integers  $p, q$ , consider the compact  $U(1)_{p,q}$  subgroup generated by a Noether current with holomorphic and antiholomorphic components

$$\begin{aligned} J_{p,q}(z) &= pJ_n(z) + qJ_w(z) = \left( pR + \frac{q}{R} \right) i\partial X(z), \\ \bar{J}_{p,q}(\bar{z}) &= p\bar{J}_n(\bar{z}) + q\bar{J}_w(\bar{z}) = \left( pR - \frac{q}{R} \right) i\bar{\partial}X(\bar{z}). \end{aligned} \quad (3.11)$$

Its anomaly is given by  $k_{U(1)} = \frac{1}{2} \left[ \frac{(pR + \frac{q}{R})^2}{2} - \frac{(pR - \frac{q}{R})^2}{2} \right] = pq$ .<sup>11</sup> Generally, given a  $U(1)$  symmetry with  $k_{U(1)}$  anomaly, its  $\mathbb{Z}_N$  subgroup has  $k_N \equiv k_{U(1)} \pmod{N}$  anomaly. To summarize, the  $U(1)$  anomaly and the anomaly of its  $\mathbb{Z}_N$  subgroup are given by

$$k_{U(1)} = pq, \quad k_N \equiv pq \pmod{N}. \quad (3.13)$$

<sup>11</sup>We follow the convention that the  $U(1)$  anomaly  $k_{U(1)}$  in a bosonic quantum field theory is an integer and is related to the two-point function of the Noether currents by

$$\langle J(z)J(0) \rangle = \frac{k}{z^2}, \quad \langle \bar{J}(\bar{z})\bar{J}(0) \rangle = \frac{\bar{k}}{\bar{z}^2}, \quad k_{U(1)} = \frac{k - \bar{k}}{2} \in \mathbb{Z}. \quad (3.12)$$

Therefore, the  $c = 1$  compact boson at any definite radius  $R$  realizes all possible U(1) and cyclic anomalies.<sup>12</sup>

### C. Center symmetries in WZW models

We now consider the center symmetries in the  $\mathfrak{su}(N)_1$ ,  $\mathfrak{so}(4N+2)_1$ , and  $(\mathfrak{e}_6)_1$  WZW models. The centers of the other WZW models with simple Lie algebras are either trivial or products of  $\mathbb{Z}_2$ , which have already been considered in [1]. We focus on WZW models at level 1 because some of them turn out to saturate our bounds. The anomaly of the center symmetry for general WZW models have also been considered in [64].

For the above WZW models at level 1, the center symmetry lines constitute all the Verlinde lines. Therefore, the fusion category of the  $\mathbb{Z}_N$  center symmetry can be lifted to a UMTC. It follows that the anomaly  $k$  of the  $\mathbb{Z}_N$  center symmetry, which is encoded in the  $F$ -symbols, is determined to be  $k = 0$  if  $N$  is odd, and  $k = N/2$  if  $N$  is even [46]. Below we confirm this by a  $(1+1)$ -dimensional computation using the algorithm in Sec. II B.

#### 1. $\mathfrak{su}(N)_1$

The  $\mathfrak{su}(N)_1$  WZW model has central charge  $c = N - 1$  and center symmetry  $\mathbb{Z}_N$ . It contains  $N$  current algebra primaries with  $\mathbb{Z}_N$  charges  $Q = 0, 1, \dots, N - 1$  and weights

$$h_Q = \bar{h}_Q = \frac{Q(N-Q)}{2N}. \quad (3.15)$$

The modular  $S$  matrix is

$$S_{QQ'} = \frac{\omega^{QQ'}}{\sqrt{N}}, \quad \omega = e^{\frac{2\pi i}{N}}. \quad (3.16)$$

In the spectrum of *Virasoro* primaries, the gap in each charge  $Q$  sector is

$$\Delta^Q = \begin{cases} 2 & Q = 0, \\ \frac{Q(N-Q)}{N} & Q \neq 0. \end{cases} \quad (3.17)$$

<sup>12</sup>The exponential operators  $\mathcal{O}_{n,w}(z, \bar{z})$  have charge  $Q_{p,q} = pn + qw$  under  $U(1)_{p,q}$ . With respect to  $\mathbb{Z}_N < U(1)_{p,q}$ , the uncharged sector consists of the exponential operators with  $Q_{p,q} = pn + qw \equiv 0 \pmod{N}$ , and the uncharged scalar gap in the spectrum of exponential operators (hence excluding  $J\bar{J}$  which has scaling dimension 2 and all other current algebra descendants) is given by

$$\Delta_{\text{scalar,exp}}^{Q=0} = \frac{1}{2} \min_R \left[ \frac{N^2}{\text{gcd}(p, N)^2 R^2}, \frac{\text{gcd}(q, N)^2 R^2}{N^2} \right]. \quad (3.14)$$

Whenever  $\Delta_{\text{scalar,exp}}^{Q=0}$  is greater than 2, there is a range of  $R$  such that the  $c = 1$  compact boson is robust against perturbation preserving this  $\mathbb{Z}_N$  symmetry with anomaly  $k_N = pq \pmod{N}$ .

Here the lightest  $\mathbb{Z}_N$ -symmetric Virasoro primary with  $\Delta^{Q=0} = 2$  is the current bilinear operator.

Denote the  $\mathfrak{su}(N)_1$  characters (without flavor fugacity) by  $\chi_Q(\tau)$ . For later convenience, we define the subscript modulo  $N$ , i.e.,  $\chi_{Q+N}(\tau) = \chi_Q(\tau)$ . The torus partition function with no twist is

$$Z(\tau, \bar{\tau}) = \sum_{Q=0}^{N-1} |\chi_Q(\tau)|^2. \quad (3.18)$$

Let  $\eta$  be a generator of  $\mathbb{Z}_N$ . The torus partition function with a  $\mathbb{Z}_N$  twist in the time direction is

$$Z^\eta(\tau, \bar{\tau}) = \sum_{Q=0}^{N-1} \omega^Q |\chi_Q(\tau)|^2. \quad (3.19)$$

Performing a modular  $S$  transform on the latter gives the torus partition function with a  $\mathbb{Z}_N$  twist in the spatial direction

$$Z_\eta(\tau, \bar{\tau}) = \sum_{Q=0}^{N-1} \chi_Q(\tau) \overline{\chi_{Q+1}(\tau)}. \quad (3.20)$$

Therefore, the spins of the operators in the defect Hilbert space  $\mathcal{H}_\eta$  are

$$s = \frac{Q}{N} - \frac{N-1}{2N}, \quad (3.21)$$

with  $0 \leq Q \leq N - 1$ . Comparing these spins with the spin selection rule (2.6), we have found the anomaly of the  $\mathbb{Z}_N$  center symmetry in the  $\mathfrak{su}(N)_1$  WZW model,

$$k = \begin{cases} 0 & N \text{ odd} \\ \frac{N}{2} & N \text{ even} \end{cases}. \quad (3.22)$$

#### 2. $\mathfrak{so}(4N+2)_1$

The  $\mathfrak{so}(4N+2)_1$  WZW model has central charge  $c = 2N + 1$  and a  $\mathbb{Z}_4$  center symmetry  $\mathbb{Z}_4$ .<sup>13</sup> It is the bosonization of  $4N + 2$  free Majorana fermions. There are four current algebra primaries with  $\mathbb{Z}_4$  charges  $Q = 0, 1, 2, 3$  and weights

$$\begin{aligned} h_0 = \bar{h}_0 = 0, \quad h_1 = \bar{h}_1 = h_3 = \bar{h}_3 = \frac{2N+1}{8}, \\ h_2 = \bar{h}_2 = \frac{1}{2}. \end{aligned} \quad (3.23)$$

<sup>13</sup>The  $\mathfrak{so}(4N)_1$  WZW model, by contrast, has a  $\mathbb{Z}_2 \times \mathbb{Z}_2$  center symmetry.

The modular  $S$  matrix is

$$S = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \quad \text{for odd } N,$$

$$S = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \quad \text{for even } N. \quad (3.24)$$

In the spectrum of Virasoro primaries, the gaps in each charge  $Q$  sector are

$$\Delta^Q = \begin{cases} 2 & Q = 0, \\ \frac{2N+1}{4} & Q = 1, 3, \\ 1 & Q = 2. \end{cases} \quad (3.25)$$

The torus partition function with no twist is

$$Z(\tau, \bar{\tau}) = \sum_{Q=0,1,2,3} |\chi_Q(\tau)|^2. \quad (3.26)$$

The torus partition function with a  $\mathbb{Z}_4$  twist in the time direction is ( $\eta$  is a generator of  $\mathbb{Z}_4$ )

$$Z^\eta(\tau, \bar{\tau}) = |\chi_0(\tau)|^2 - |\chi_2(\tau)|^2. \quad (3.27)$$

Under a modular  $S$  transformation, we find

$$Z_\eta(\tau, \bar{\tau}) = [\chi_0(\tau) + \chi_1(\tau)]\overline{[\chi_2(\tau) + \chi_3(\tau)]} + \chi_2(\tau)\overline{[\chi_0(\tau) + \chi_1(\tau)]}, \quad (3.28)$$

which has spin content

$$h - \bar{h} = \left\{ \pm \frac{1}{2}, \pm \frac{2N-3}{8} \right\}. \quad (3.29)$$

Comparing with (2.6), we see that the  $\mathbb{Z}_4$  center has anomaly

$$k = 2. \quad (3.30)$$

### 3. $(\mathfrak{e}_6)_1$

The  $(\mathfrak{e}_6)_1$  WZW model has central charge  $c = 6$  and a  $\mathbb{Z}_3$  center symmetry. It has three current algebra primaries with  $\mathbb{Z}_3$  charges  $Q = 0, 1, 2$  and weights

$$h_0 = \bar{h}_0 = 0, \quad h_1 = \bar{h}_1 = h_2 = \bar{h}_2 = \frac{2}{3}. \quad (3.31)$$

In the spectrum of Virasoro primaries, the gaps in each charge sector are

$$\Delta^Q = \begin{cases} 2 & Q = 0, \\ \frac{4}{3} & Q = 1, 2. \end{cases} \quad (3.32)$$

The  $S$  matrix is the complex conjugate of  $\mathfrak{su}(3)_1$ ,

$$S = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}, \quad \omega = e^{\frac{2\pi i}{3}}. \quad (3.33)$$

Hence, the determination of the spin content proceeds in a similar fashion to the case of  $\mathfrak{su}(3)_1$ ,

$$Z^\eta(\tau, \bar{\tau}) = \sum_{Q=0}^2 \omega^Q |\chi_Q(\tau)|^2,$$

$$Z_\eta(\tau, \bar{\tau}) = \sum_{Q=0}^2 \chi_Q(\tau) \overline{\chi_{Q-1}(\tau)}, \quad s = \frac{1}{3}, \frac{2}{3}, 0. \quad (3.34)$$

Comparing with (2.6), we conclude that the  $\mathbb{Z}_3$  center of  $(\mathfrak{e}_6)_1$  is nonanomalous.

### D. T-duality $\mathbb{Z}_4$ in $\mathfrak{su}(2)_1$

In this subsection we discuss examples of anomalous  $\mathbb{Z}_4$  symmetries in the  $\mathfrak{su}(2)_1$  WZW model. These  $\mathbb{Z}_4$  symmetries have the  $k = 1$  or  $k = 3$  anomalies, which according to (3.2), cannot be realized as Verlinde symmetry lines, i.e., they do not commute with the maximally extended chiral algebra.

At a generic radius of the  $c = 1$  compact boson, T-duality is not a global symmetry, but a map of one description of the theory with radius  $R$  to a different description of the same theory with radius  $1/R$ . See [76] for discussions on T-duality and topological interfaces in the compact boson theory.

However, at the self-dual point  $R = 1$ , i.e., the  $\mathfrak{su}(2)_1$  WZW model, the T-duality becomes a global symmetry of the theory. The right T-duality is the chiral  $\pi$  rotation of the right  $\mathfrak{su}(2)_1$  current algebra and is an order 4 action [77]. It is not a Verlinde symmetry because it acts nontrivially on the right  $\mathfrak{su}(2)_1$  algebra.

Similarly, there is another order 4 action corresponding to the chiral  $\pi$  rotation of the left  $\mathfrak{su}(2)_1$  current algebra, which we call the left T-duality. The square of either  $\mathbb{Z}_4$  is the anomalous  $\mathbb{Z}_2$  center symmetry.

Let us describe the right T-duality  $\mathbb{Z}_4$  in more detail and compute its anomaly. The  $\mathfrak{u}(1)$  current algebra primaries are labeled by a momentum number and a winding number,



$$\begin{aligned} \mathcal{O}_{n,w}(z, \bar{z}) &= \exp [i(n+w)X_L(z) + i(n-w)X_R(\bar{z})], \\ h &= \frac{1}{4}(n+w)^2, \quad \bar{h} = \frac{1}{4}(n-w)^2. \end{aligned} \quad (3.35)$$

The left and the right  $\mathfrak{su}(2)_1$  current algebra generators are

$$\begin{aligned} J_L^{\pm}(z) &= i\partial X_L(z), \quad J_L^{\pm}(\bar{z}) = \mathcal{O}_{\pm 1, \pm 1} = e^{\pm 2iX_L(z)}, \\ J_R^{\pm}(z) &= i\partial X_R(\bar{z}), \quad J_R^{\pm}(\bar{z}) = \mathcal{O}_{\pm 1, \mp 1} = e^{\pm 2iX_R(\bar{z})}. \end{aligned} \quad (3.36)$$

There are two current algebra primaries, the vacuum primary  $|h=0, \bar{h}=0\rangle$  and the spin- $\frac{1}{2}$  primary  $|h=\frac{1}{4}, \bar{h}=\frac{1}{4}\rangle$ . Those  $\mathcal{O}_{n,w}$  with  $n=w \pmod 2$  belong to the vacuum module, while those with  $n=w+1 \pmod 2$  belong to the spin- $\frac{1}{2}$  module.

The right T-duality  $\mathbb{Z}_4$  acts on  $X_L$ ,  $X_R$ , and  $\mathcal{O}_{n,w}$  as [77]

$$\begin{aligned} X_L(z) &\rightarrow X_L(z), \quad X_R(\bar{z}) \rightarrow -X_R(\bar{z}), \\ \mathcal{O}_{n,w} &\rightarrow e^{\frac{i\pi}{2}(n+w)^2} \mathcal{O}_{w,n}. \end{aligned} \quad (3.37)$$

Note that this  $\mathbb{Z}_4$  does not commute with the right-moving  $\mathfrak{u}(1)$  current algebra generated by  $\bar{\partial}X_R(\bar{z})$ . The square of the generator of this  $\mathbb{Z}_4$  acts on  $\mathcal{O}_{n,w}$  as

$$\mathbb{Z}_2: \mathcal{O}_{n,w} \rightarrow (-1)^{n+w} \mathcal{O}_{n,w}, \quad (3.38)$$

which is the anomalous  $\mathbb{Z}_2$  center symmetry of the  $\mathfrak{su}(2)_1$  WZW model (see, for example, [1]).

The torus partition function of the  $\mathfrak{su}(2)_1$  WZW model without any defect is

$$\begin{aligned} Z(\tau, \bar{\tau}) &= \text{Tr}_{\mathcal{H}} [q^{h-\frac{1}{24}} \bar{q}^{\bar{h}-\frac{1}{24}}] = \sum_{n,w \in \mathbb{Z}} \frac{q^{\frac{1}{4}(n+w)^2 - \frac{1}{24}} \bar{q}^{\frac{1}{4}(n-w)^2 - \frac{1}{24}}}{\prod_{m=1}^{\infty} (1-q^m)(1-\bar{q}^m)} \\ &= \left| \frac{\theta_3(2\tau)}{\eta(\tau)} \right|^2 + \left| \frac{\theta_2(2\tau)}{\eta(\tau)} \right|^2. \end{aligned} \quad (3.39)$$

We now compute the partition function  $Z^\eta$  with a right T-duality  $\mathbb{Z}_4$  line  $\eta$  extended along the spatial direction. Since  $\mathbb{Z}_4$  swaps  $\mathcal{O}_{n,w}$  with  $\mathcal{O}_{w,n}$ , only those terms with  $n=w$  contribute. Furthermore, the right-moving current algebra oscillators contribute with a minus sign to the partition function. Hence we have

$$\begin{aligned} Z^\eta(\tau, \bar{\tau}) &= \text{Tr}_{\mathcal{H}} [\hat{\eta} q^{h-\frac{1}{24}} \bar{q}^{\bar{h}-\frac{1}{24}}] = \sum_{n \in \mathbb{Z}} \frac{q^{n^2 - \frac{1}{24}} \bar{q}^{-\frac{1}{24}}}{\prod_{m=1}^{\infty} (1-q^m)(1+\bar{q}^m)} \\ &= \frac{\theta_3(2\tau)}{\eta(\tau)} \sqrt{\frac{2\eta(\tau)}{\theta_2(\tau)}}. \end{aligned} \quad (3.40)$$

Its modular S transform is the partition function of the  $\mathbb{Z}_4$  defect Hilbert space,

$$\begin{aligned} Z_\eta(\tau, \bar{\tau}) &= \text{Tr}_{\mathcal{H}_\eta} [q^{h-\frac{1}{24}} \bar{q}^{\bar{h}-\frac{1}{24}}] = \left( \frac{\theta_3(2\tau) + \theta_2(2\tau)}{\eta(\tau)} \right) \sqrt{\frac{\eta(\tau)}{\theta_4(\tau)}} \\ &= \frac{1}{|\eta(\tau)|^2} \left( \bar{q}^{\frac{1}{16}} + 2q^{\frac{1}{4}} \bar{q}^{\frac{1}{16}} + \bar{q}^{\frac{9}{16}} + 2q^{\frac{1}{4}} \bar{q}^{\frac{9}{16}} + 2q \bar{q}^{\frac{1}{16}} + \dots \right). \end{aligned} \quad (3.41)$$

From the spins of these operators, we conclude that the T-duality  $\mathbb{Z}_4$  has the  $k=3$  anomaly. Similarly, the left T-duality  $\mathbb{Z}_4$ , which acts by the chiral  $\pi$  rotation of the left  $\mathfrak{su}(2)_1$  current algebra, has the  $k=1$  anomaly.

### E. $\mathbb{Z}_N$ parafermion CFT

The  $\mathbb{Z}_N$  parafermion CFT, also known as the  $\frac{\mathfrak{su}(2)_N}{\mathfrak{u}(1)}$  coset CFT, is a  $c = \frac{2(N-1)}{N+2}$  diagonal RCFT [78]. It consists of  $\frac{N(N+1)}{2}$  primaries labeled by two integers  $(\ell, m)$  in the range

$$0 \leq \ell \leq N, \quad -\ell + 2 \leq m \leq \ell, \quad \ell - m \in 2\mathbb{Z}, \quad (3.42)$$

with conformal weights

$$h = \bar{h} = \frac{\ell(\ell+2)}{4(N+2)} - \frac{m^2}{4N}. \quad (3.43)$$

There is a nonanomalous  $\mathbb{Z}_N$  symmetry that commutes with the parafermion algebra, and the primary labeled by  $(\ell, m)$  has  $\mathbb{Z}_N$  charge  $m \pmod N$ . The uncharged sector consists of the identity  $(\ell = m = N)$  and all the primaries with  $m = 0$ . The lightest nontrivial (scalar) operator in the uncharged sector is the  $\ell = 2, m = 0$  primary, giving

$$\Delta_{\text{scalar}}^{Q=0} = \frac{4}{N+2}. \quad (3.44)$$

The first few parafermion CFTs with  $N = 2, 3, 4$  are the familiar  $c = \frac{1}{2}$  Ising CFT, the  $c = \frac{4}{3}$  three-state Potts CFT, and the  $c = 1$  free boson orbifold  $S^1/\mathbb{Z}_2$  at radius  $R = \sqrt{3}$ .

Let us explicitly verify that the  $\mathbb{Z}_N$  symmetry is non-anomalous by analyzing the spin contents in  $Z_\eta$ . We start with

$$Z^\eta(\tau, \bar{\tau}) = \sum_{\ell, m} e^{\frac{2\pi i m}{N}} \chi_{\ell, m}(\tau) \overline{\chi_{\ell, -m}(\tau)}. \quad (3.45)$$

Using the modular S matrix [79]

$$S_{\ell, m}^{\ell', m'} = \frac{2}{\sqrt{N(N+2)}} \sin \frac{\pi(\ell+1)(\ell'+1)}{N+2} e^{\frac{2\pi i m m'}{N}} \quad (3.46)$$

we obtain the partition function in the defect Hilbert space

$$\begin{aligned}
 Z_\eta(\tau, \bar{\tau}) &= \sum_{\ell, m, \bar{\ell}, \bar{m}} \left[ \sum_{\ell', m'} S_{\ell', m'}^{\ell, m} e^{\frac{2\pi i m'}{N}} S_{\ell', -m'}^* \right] \chi_{\ell, m}(\tau) \overline{\chi_{\bar{\ell}, \bar{m}}(\bar{\tau})} \\
 &= \sum_{\ell, m, \bar{\ell}, \bar{m}} n_{\ell, m; \bar{\ell}, \bar{m}} \chi_{\ell, m}(\tau) \overline{\chi_{\bar{\ell}, \bar{m}}(\bar{\tau})}, \quad (3.47)
 \end{aligned}$$

where

$$\begin{aligned}
 n_{\ell, m; \bar{\ell}, \bar{m}} &= \frac{4}{N(N+2)} \sum_{\ell', m'} \sin \frac{\pi(\ell+1)(\ell'+1)}{N+2} \\
 &\times \sin \frac{\pi(\bar{\ell}+1)(\bar{\ell}'+1)}{N+2} e^{\frac{\pi i m'(m-\bar{m}+2)}{N}}. \quad (3.48)
 \end{aligned}$$

Using the fact that the summand is invariant under the simultaneous shift of  $\ell' \rightarrow N - \ell'$ ,  $m' \rightarrow m' + N$ , we can rewrite the sum as ( $m' = 2p'$ )

$$\sum_{\ell', m'} \rightarrow \frac{1}{2} \sum_{\ell'=0}^N \left( \sum_{p'=-\frac{\ell'}{2}+1}^{\frac{\ell'}{2}} + \sum_{p'=-\frac{N-\ell'}{2}+1+\frac{N}{2}}^{\frac{N-\ell'}{2}+\frac{N}{2}} \right) \rightarrow \frac{1}{2} \sum_{\ell'=0}^N \sum_{p'=-\frac{\ell'}{2}+1}^{\frac{N-\ell'}{2}}, \quad (3.49)$$

and, furthermore,

$$\begin{aligned}
 &\sum_{p'=-\frac{\ell'}{2}+1}^{\frac{N-\ell'}{2}} e^{\frac{2\pi i p'(m-\bar{m}+2)}{N}} \\
 &= \begin{cases} N & \bar{m} = m + 2 \pmod{2N}, \\ -(-)^{N-\ell'} N & \bar{m} = m + 2 + N \pmod{2N}, \\ 0 & \text{otherwise.} \end{cases} \quad (3.50)
 \end{aligned}$$

With this, one can easily carry out the sum over  $\ell'$  to arrive at

$$\begin{aligned}
 n_{\ell, m; \bar{\ell}, \bar{m}} &= \begin{cases} 1 & \bar{\ell} = \ell \text{ and } \bar{m} \equiv m + 2 \pmod{2N}, \\ 1 & \bar{\ell} = N - \ell \text{ and } \bar{m} \equiv m + 2 + N \pmod{2N}, \\ 0 & \text{otherwise.} \end{cases} \quad (3.51)
 \end{aligned}$$

It follows that the spins in  $Z_\eta$  are always  $\frac{m+1}{N} + \mathbb{Z}$ , in agreement with the spin selection rule for a nonanomalous  $\mathbb{Z}_N$  symmetry.

#### IV. MODULAR CONSTRAINTS ON SYMMETRY-PRESERVING SCALAR OPERATORS

##### A. Modular bootstrap

To apply the bootstrap algorithm, we need to identify a set of observables  $\mathbf{Z}$  that (a) has an expansion on a certain

basis of functions with non-negative coefficients, and (b) obeys a certain crossing equation. For our application, the observables  $\mathbf{Z}$  will be linear combinations of the torus partition functions twisted by different symmetry lines. We will pick a basis for these partition functions so that they have non-negative expansions in the Virasoro characters. The crossing equation will come from the modular S transformation. Below we discuss in detail these partition functions and the crossing equation they satisfy.

Focusing on  $c > 1$ , the Virasoro character  $\chi_h(\tau)$  is nondegenerate when  $h > 0$ , and degenerate when  $h = 0$ ,

$$\chi_h(\tau) = \begin{cases} \frac{q^{\frac{h-c-1}{24}}}{\eta(\tau)} & h > 0, \\ \frac{q^{\frac{h-c-1}{24}}}{\eta(\tau)} (1-q) & h = 0, \end{cases} \quad (4.1)$$

where as before  $q = \exp(2\pi i\tau)$ ,  $\bar{q} = \exp(-2\pi i\bar{\tau})$ .

Let  $\eta$  be a generator of  $\mathbb{Z}_N$ . First consider the torus partition function  $Z^{\eta^r}$  with the insertion of the  $\eta^r$  line along the spatial direction,

$$\begin{aligned}
 Z^{\eta^r}(\tau, \bar{\tau}) &= \sum_{h, \bar{h}} \left[ n_{h, \bar{h}}^0 + \sum_{Q=1}^{N-1} \omega^{rQ} n_{h, \bar{h}}^Q \right] \chi_h(\tau) \overline{\chi_{\bar{h}}(\bar{\tau})}, \\
 \omega &= e^{\frac{2\pi i}{N}}, \quad (4.2)
 \end{aligned}$$

where  $n_{h, \bar{h}}^Q \geq 0$  is the number of Virasoro primaries with conformal weight  $(h, \bar{h})$  and  $\mathbb{Z}_N$  charge  $Q \pmod{N}$ .

The modular S transform of  $Z^{\eta^r}(\tau, \bar{\tau})$  is a partition function with the insertion of the  $\eta^r$  line along the time direction,

$$Z_{\eta^r}(\tau, \bar{\tau}) = \sum_{h, \bar{h}} (n_{\eta^r})_{h, \bar{h}} \chi_h(\tau) \overline{\chi_{\bar{h}}(\bar{\tau})}, \quad (4.3)$$

where  $(n_{\eta^r})_{h, \bar{h}}$  is the number of Virasoro primaries with conformal weight  $(h, \bar{h})$  in the defect Hilbert space  $\mathcal{H}_{\eta^r}$ .

To perform bootstrap, it is important to work in a basis where each partition function has an expansion in Virasoro characters with non-negative coefficients. The twist basis (4.2) does not have this property, so we consider instead the charge basis

$$Z^Q(\tau, \bar{\tau}) = \text{Tr}_{\substack{|\phi\rangle \in \mathcal{H} \\ \eta|\phi\rangle = \omega^Q |\phi\rangle}} q^{h-c/24} \bar{q}^{\bar{h}-\bar{c}/24} = \sum_{h, \bar{h}} n_{h, \bar{h}}^Q \chi_h(\tau) \overline{\chi_{\bar{h}}(\bar{\tau})}. \quad (4.4)$$

The twist and charge bases are related by a discrete Fourier transform

$$Z^{\eta^r}(\tau, \bar{\tau}) = \sum_{Q=0}^{N-1} \omega^{rQ} Z^Q(\tau, \bar{\tau}). \quad (4.5)$$

**1. Bootstrap system**

To facilitate the presentation, define the  $(2N - 1)$ -dimensional modular vector  $\mathbf{Z}$  in the charge basis with

$$\mathbf{Z}_i = \begin{cases} Z^{i-1} & i = 1, \dots, N, \\ Z_{\eta^{i-N}} & i = N + 1, \dots, 2N - 1, \end{cases} \quad (4.6)$$

and that in the twist basis

$$\tilde{\mathbf{Z}}_i = \begin{cases} Z^{\eta^{i-1}} & i = 1, \dots, N, \\ Z_{\eta^{i-N}} & i = N + 1, \dots, 2N - 1. \end{cases} \quad (4.7)$$

The two are related by the Fourier transform

$$\tilde{\mathbf{Z}} = F\mathbf{Z}, \quad (4.8)$$

where the matrix  $F$  can be read off from (4.5).

Since our primary interest is to bound the  $\mathbb{Z}_N$ -invariant scalar primary operators, we should work with the partition functions in the charge basis  $\mathbf{Z}$ . On the other hand, the modular  $S$  matrix is simpler in the twist basis  $\tilde{\mathbf{Z}}$ . Let  $\tilde{S}$  be the modular  $S$  matrix in the twist basis, i.e.,  $\tilde{\mathbf{Z}}(-1/\tau, -1/\bar{\tau}) = \tilde{S}\tilde{\mathbf{Z}}(\tau, \bar{\tau})$ . It can be easily written down by noting that  $Z^{\eta^i} \leftrightarrow Z_{\eta^i}$  under the  $S$  transformation. In other words,  $\tilde{S}$  is the permutation matrix representing the cycle  $(2, N + 1)(3, N + 2)\dots(N, 2N - 1)$ .

The modular  $S$  matrix in the charge basis is then

$$S = F^{-1}\tilde{S}F. \quad (4.9)$$

The bootstrap equation is

$$\mathbf{Z}(-1/\tau, -1/\bar{\tau}) = S\mathbf{Z}(\tau, \bar{\tau}). \quad (4.10)$$

Depending on the symmetry and anomaly, this bootstrap equation can be reduced to one of smaller dimensionality. In Appendix A, we explain this reduction and present the explicit reduced modular  $S$  matrices for small  $N$ .

The linear functional method for ruling out putative spectra  $\mathcal{P}$  proceeds as in [1]. The idea is to expand the bootstrap equation (4.10) into a sum of characters  $\chi_h(\tau)\chi_{\bar{h}}(\bar{\tau})$  over the putative spectrum  $(h, \bar{h}) \in \mathcal{P}$ , subject to the appropriate spin selection rule (2.9), and search for a linear functional whose action on the combination  $I\chi_h(-1/\tau)\chi_{\bar{h}}(-1/\bar{\tau}) - S\chi_h(\tau)\chi_{\bar{h}}(\bar{\tau})$  ( $I$  is the identity matrix) is of definite sign on the entire  $\mathcal{P}$ . If such a functional exists, then the putative spectrum is ruled out. Iterating this procedure produces various constraints, such as bounds on the gap in the spectrum of  $\mathbb{Z}_N$ -invariant scalar primaries.

More specifically, we expand a general linear function  $\alpha$  in the derivative basis, up to a certain derivative order  $\Lambda$ , such that on a test vector-valued function  $f(\tau, \bar{\tau})$ ,

$$\alpha[f] = \sum_{\substack{m=n=0 \\ m+n \leq \Lambda}} \partial_\tau^m \partial_{\bar{\tau}}^n f(\tau, \bar{\tau})|_{\tau=i, \bar{\tau}=-i}. \quad (4.11)$$

The numerical bounds in this paper are all obtained at derivative order  $\Lambda = 25$  and with spin truncation  $s_{\max} = 50$ .<sup>14</sup> The search for a linear function utilizes the semidefinite programming solver SDPB [80,81].<sup>15</sup>

**B. Bounds on symmetry-preserving scalar primaries**

In the following, we present the numerical bounds on the gap in the spectrum of scalar Virasoro primaries for  $\mathbb{Z}_N$  with small  $N$  and every possible anomaly. We will find saturation by several WZW models. The  $c = 1$  numerical bounds have to be interpreted differently because there are additional degenerate Virasoro modules at  $h = \frac{n^2}{4}$  with  $n \in \mathbb{Z}$ , which are not incorporated into the bootstrap system.<sup>16</sup>

Note that bootstrap is blind to unfaithfulness: a  $\mathbb{Z}_N$  global symmetry can be extended to a  $\mathbb{Z}_{pN}$  symmetry realized unfaithfully for some positive integer  $p$ . The discussion in Sec. II B implies that our numerical bounds for  $\mathbb{Z}_{pN}$  symmetry with anomaly  $k' \bmod pN$  can be applied to CFTs with a  $\mathbb{Z}_N$  global symmetry that has anomaly  $k \bmod N$  if  $k' = p^2 k \bmod pN$ .

**1. No symmetry**

The upper bound on the scalar primaries without assuming any global symmetry was done in [17]. It was found that relevant operators must be present when the central charge lies within  $0 < c < 8$ . The two ends of this range is saturated by the trivial theory and the  $(e_8)_1$  WZW model, in which the current bilinear operator is a marginal operator.

**2.  $\mathbb{Z}_2$  symmetry**

Modular bootstrap with  $\mathbb{Z}_2$  global symmetry was studied by the present authors in [1]. It was found that  $\mathbb{Z}_2$ -even relevant/marginal operators must be present when the

<sup>14</sup>The positivity of the linear functional acting on  $I\chi_h(-1/\tau)\chi_{\bar{h}}(-1/\bar{\tau}) - S\chi_h(\tau)\chi_{\bar{h}}(\bar{\tau})$  is enforced for  $|h - \bar{h}| \leq 50$ .  
<sup>15</sup>We use the following SDPB parameter settings: (binary) precision = 768, initialMatrixScalePrimal = 1e-10, initialMatrixScaleDual = 1e-10, maxComplementarity = 1e-30, feasibleCenteringParameter = 0.1, infeasibleCenteringParameter = 0.3, stepLengthReduction = 0.7.

<sup>16</sup>We cannot compare the bounds with (3.14) since the latter are uncharged scalar gaps for  $\mathfrak{u}(1)$  current algebra primaries. It is a coincidence that, for  $\mathbb{Z}_3$  and  $\mathbb{Z}_4$ , our bounds agree with the values given by (3.14).

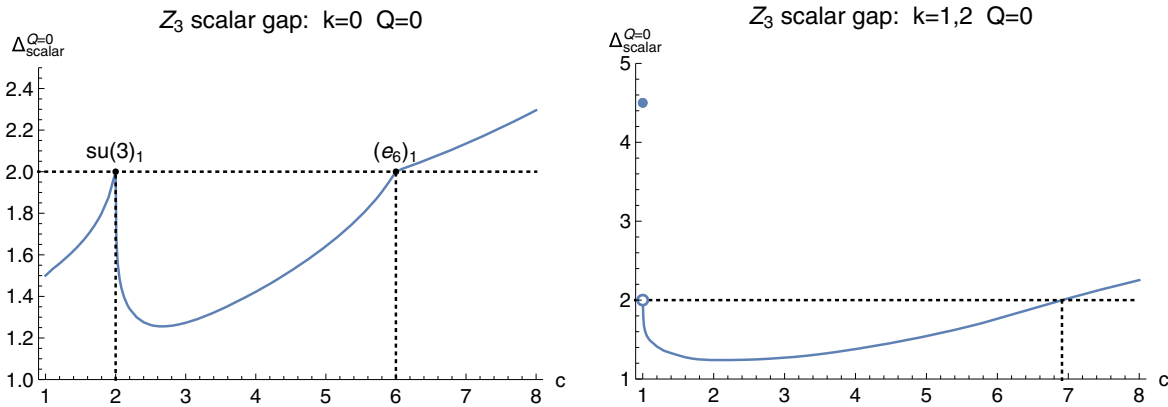


FIG. 4. Upper bounds  $\Delta_{\text{scalar}}^{Q=0}$  on the lightest  $\mathbb{Z}_3$ -symmetric scalar operator in the spectrum of Virasoro primaries with anomaly  $k$ .

central charge lies within  $1 \leq c \leq 7$ . Moreover, for anomalous  $\mathbb{Z}_2$  ( $k = 1$ ), the bounds  $\Delta_{\text{scalar}}^{Q=0} = 2$  at  $c = 1$  and  $c = 7$  are saturated by the center symmetries of the  $\mathfrak{su}(2)_1$  and  $(e_7)_1$  WZW models, respectively.

### 3. $\mathbb{Z}_3$ symmetry

The upper bounds on the lightest  $\mathbb{Z}_3$ -symmetric scalar operator for different anomalies are shown in Fig. 4, and other types of bounds can be found in Appendix B. Depending on the anomaly, we find that there must exist a  $\mathbb{Z}_3$ -symmetric relevant/marginal scalar primary in the following ranges of the central charge:

$$\begin{aligned} k = 0: & \quad 1 < c \leq 6, & \quad \Delta_{\text{scalar}}^{Q=0} = 2 & \quad \text{at } c = 2, 6, \\ k = 1, 2: & \quad 1 < c < 6.91, \\ \text{any } k: & \quad 1 < c \leq 6. \end{aligned} \quad (4.12)$$

For  $k = 0$ , the bounds  $\Delta_{\text{scalar}}^{Q=0} = 2$  at  $c = 2$  and  $c = 6$  are saturated by the center symmetries in the  $\mathfrak{su}(3)_1$  and  $(e_6)_1$  WZW models, respectively.<sup>17</sup>

### 4. $\mathbb{Z}_4$ symmetry

The upper bounds on the lightest  $\mathbb{Z}_4$ -symmetric scalar operator for different anomalies are shown in Fig. 5, and other types of bounds can be found in Appendix B. We find that there must exist a  $\mathbb{Z}_4$ -symmetric relevant/marginal scalar primary in the following ranges of the central charge:

<sup>17</sup>For  $\mathbb{Z}_3$  symmetry with arbitrary anomaly, the bootstrap bound  $\Delta_{\text{scalar}}^{Q=0}$  at  $c = 1$  coincidentally has the same value as the uncharged scalar gap given in (3.14) for the  $c = 1$  compact boson. However, (3.14) lists the uncharged scalar gap in the spectrum of exponential operators [i.e.,  $\mathfrak{u}(1)$  current algebra primaries] and not that in the spectrum of Virasoro primaries. The same holds true for  $\mathbb{Z}_4$  with arbitrary anomaly.

$$\begin{aligned} k = 0: & \quad 2 \leq c \leq 6, & \quad \Delta_{\text{scalar}}^{Q=0} = 2 & \quad \text{at } c = 2, 4, 6, \\ k = 1, 3: & \quad 2.19 < c < 5.90, \\ k = 2: & \quad 1 < c < 5.74, & \quad \Delta_{\text{scalar}}^{Q=0} = 2 & \quad \text{at } c = 3, 5, \\ \text{any } k: & \quad 2.19 < c < 5.74. \end{aligned} \quad (4.13)$$

For  $\mathbb{Z}_4$  with the  $k = 2$  anomaly, the bounds at  $c = 1, 3, 5$  are saturated by the  $\mathbb{Z}_4$  center symmetry of the  $\mathfrak{so}(2)_1$ ,  $\mathfrak{so}(6)_1$ ,  $\mathfrak{so}(10)_1$  WZW models (see Sec. III C 2).

Next, we move on to the bounds for a nonanomalous  $\mathbb{Z}_4$  ( $k = 0$ ). As discussed in Sec. II B, a  $\mathbb{Z}_2$  symmetry, anomalous or not, can be extended to an unfaithful nonanomalous  $\mathbb{Z}_4$  symmetry (see also [53,70]). Hence the corresponding  $\mathbb{Z}_2$  twisted partition functions also satisfy or saturate our  $\mathbb{Z}_4$   $k = 0$  bootstrap bounds.

The bounds  $\Delta_{\text{scalar}}^{Q=0} = 2$  at  $c = 2, 4, 6$  are saturated by a nonanomalous  $\mathbb{Z}_2$  symmetry in the  $\mathfrak{so}(4)_1$ ,  $\mathfrak{so}(8)_1$ ,  $\mathfrak{so}(12)_1$  WZW models. This  $\mathbb{Z}_2$  symmetry is a center symmetry discussed in (B.28) of [1]. At  $c = 1$ , the bound for nonanomalous  $\mathbb{Z}_4$  is saturated by the  $\mathbb{Z}_4$  subgroup of the momentum  $U(1)_n$  of the compact boson at radius  $R = 2$ . The  $\mathbb{Z}_4$ -symmetric operators that saturate the bound are  $\mathcal{O}_{4,0}$  and  $\mathcal{O}_{0,1}$  with  $\Delta = 2$ . It is also saturated by the anomalous center  $\mathbb{Z}_2$  symmetry, realized unfaithfully as a nonanomalous  $\mathbb{Z}_4$ , of the  $\mathfrak{su}(2)_1$  WZW model.

The  $c_L = c_R = \frac{N}{2}$  bosonic  $\mathfrak{so}(N)_1$  WZW model [sometimes known as the  $spin(N)_1$  WZW model] is the bosonization of  $N$  free (nonchiral) Majorana fermions. Indeed, the chiral fermion parity  $(-1)^{F_L}$  that flips the signs of all left-moving Majorana fermions under bosonization becomes a  $\mathbb{Z}_2$  symmetry if  $N = 0$  or  $4 \pmod{8}$  and is extended to an anomalous  $\mathbb{Z}_4$  with  $k = 2$  anomaly if  $N = 2$  or  $6 \pmod{8}$  [82,83].

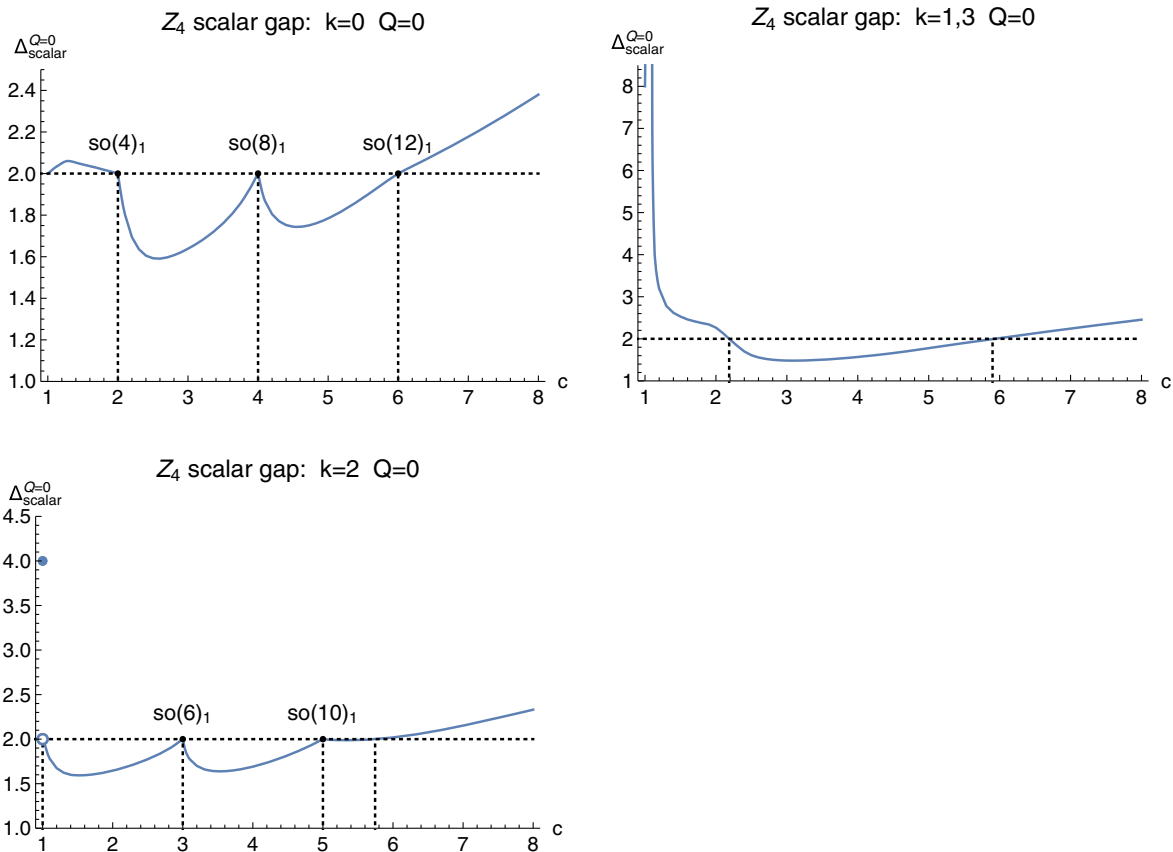


FIG. 5. Upper bounds  $\Delta_{\text{scalar}}^{Q=0}$  on the lightest  $\mathbb{Z}_4$ -symmetric scalar operator in the spectrum of Virasoro primaries with anomaly  $k$ .

### 5. $\mathbb{Z}_5$ symmetry

For a  $\mathbb{Z}_5$  symmetry with any given ‘t Hooft anomaly, we do not find any range of the central charge where there must be a  $\mathbb{Z}_5$ -symmetric relevant/marginal scalar operator (see Fig. 6). For nonanomalous  $\mathbb{Z}_5$  at  $c = 4$ , which is the value of the central charge extrapolated from (1.1) to  $N = 5$ , the bound on the scalar primary operator  $\Delta_{\text{scalar}}^{Q=0} < 2.045$  is almost saturated by the marginal current bilinear operator in the  $\mathfrak{su}(5)_1$  WZW model.

Nevertheless, the dependence of the bound on the derivative order  $\Lambda$ , as shown in the upper-right corner of Fig. 6, suggests that the extrapolating to infinite  $\Lambda$  does not bring the bound much closer to  $\Delta_{\text{scalar}}^{Q=0} = 2$ . Such “so close yet so far” instances concerning bound saturation occur often in numerical bootstrap studies, for instance, in the classic  $(2 + 1)d$  Ising bootstrap[84,85]. It is often believed that exact saturation will be attained once the full set of bootstrap constraints—beyond modular invariance/covariance and four-point crossing—are taken into account. While the shrinking of islands [86] in the  $(2 + 1)d$  Ising case provides some evidence for such belief, it is generally difficult to make such improvements. Thus we leave open the interpretation of the close to saturation by the  $\mathfrak{su}(5)_1$  WZW model.

### 6. $\mathbb{Z}_6$ symmetry

The upper bound on the  $\mathbb{Z}_6$ -symmetric scalar operator for the  $k = 3$  anomaly is shown in Fig. 7. We find that there must exist a  $\mathbb{Z}_6$ -symmetric relevant/marginal scalar primary in the range of the central charge

$$k = 3: 3.98 < c < 4.85. \tag{4.14}$$

The bound  $\Delta_{\text{scalar}}^{Q=0} < 2.022$  at  $c = 5$  is almost saturated by the center symmetry in the  $\mathfrak{su}(6)_1$  WZW model. For other anomalies  $k \neq 3$ , the bounds  $\Delta_{\text{scalar}}^{Q=0}$  do not drop to or below 2. Thus if we are blind to the ‘t Hooft anomaly, then we do not find any range of the central charge where there must be a  $\mathbb{Z}_6$ -symmetric relevant/marginal scalar operator.

### 7. $\mathbb{Z}_7$ symmetry

For a  $\mathbb{Z}_7$  symmetry with any given ‘t Hooft anomaly, we do not find any range of the central charge where there must be a  $\mathbb{Z}_7$ -symmetric relevant/marginal scalar operator. This presents a natural stopping point for our numerical exploration of constraints on robust gapless fixed points with  $\mathbb{Z}_N$  symmetry.

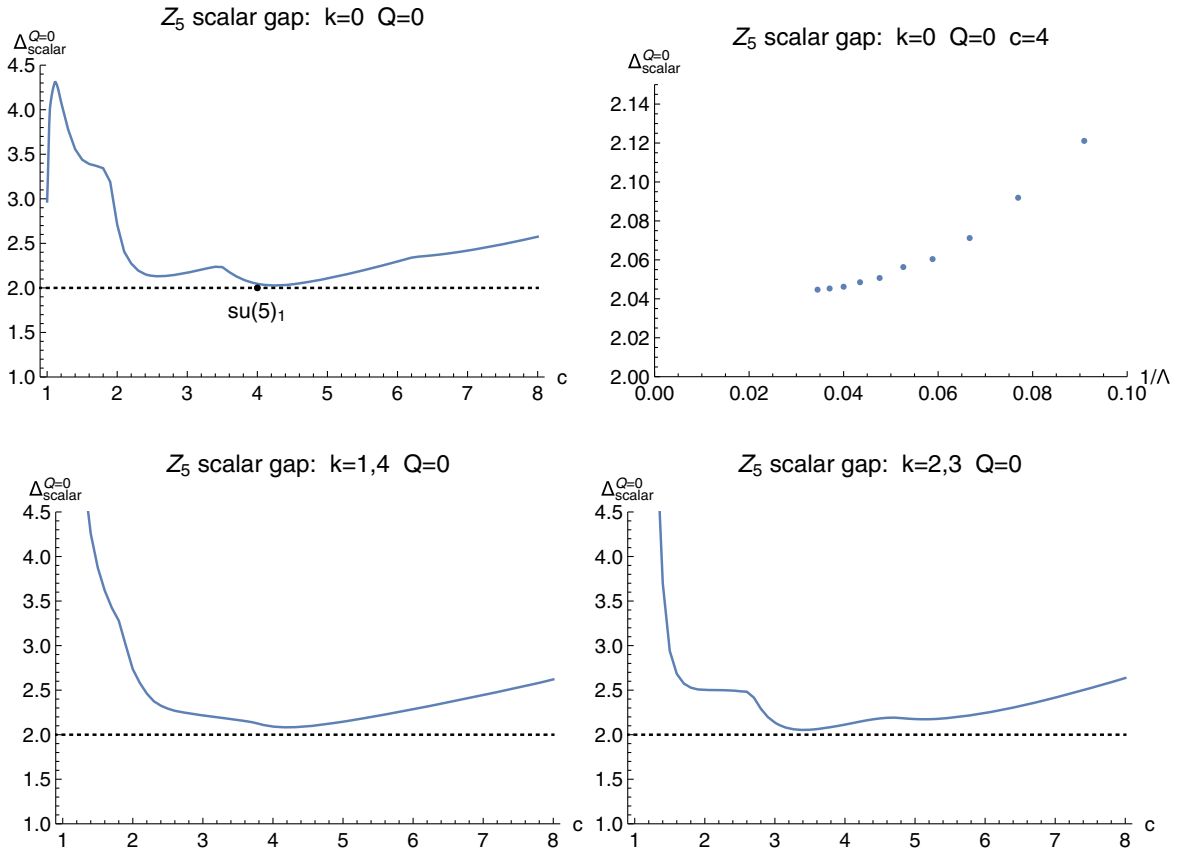


FIG. 6. Upper bounds  $\Delta_{\text{scalar}}^{Q=0}$  on the lightest  $\mathbb{Z}_5$ -symmetric scalar operator in the spectrum of Virasoro primaries with anomaly  $k$ . The upper-right figure shows the bound as a function of the inverse derivative order  $1/\Lambda$  for nonanomalous  $\mathbb{Z}_5$  at  $c = 4$ .

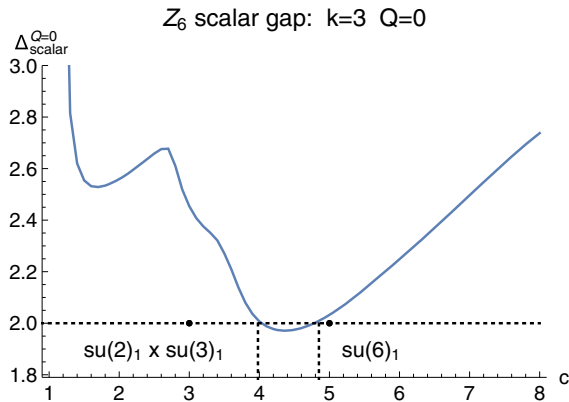


FIG. 7. Upper bounds  $\Delta_{\text{scalar}}^{Q=0}$  on the lightest  $\mathbb{Z}_6$ -symmetric scalar operator in the spectrum of Virasoro primaries with anomaly  $k = 3$ .

## ACKNOWLEDGMENTS

We thank Luca Delacrétaz, Meng Cheng, Pranay Gorantla, Theo Johnson-Freyd, Zohar Komargodski, Ho Tat Lam, Michael Levin, Kantaro Ohmori, and Nathan Seiberg for helpful discussions. We are grateful to Meng

Cheng, Theo Johnson-Freyd, and Justin Kulp for comments on the first draft. Y.L. is supported by the Sherman Fairchild Foundation, by the U.S. Department of Energy, Office of Science, Office of High Energy Physics, under Award No. DE-SC0011632, and by the Simons Collaboration Grant on the Nonperturbative Bootstrap. S.H.S. is supported by the Simons Collaboration on Ultra-Quantum Matter, which is a grant from the Simons Foundation (651440, NS). This research was supported in part by the National Science Foundation under Grant No. NSF PHY-1748958.

## APPENDIX A: MORE ON THE MODULAR BOOTSTRAP EQUATIONS

### A. Reduction of the bootstrap system

We can consider a reduced bootstrap system by combining basis partition functions together. The reduction takes the schematic form

$$\mathbf{Z}_{\text{red}} \equiv R\mathbf{Z}, \quad (\text{A1})$$

where  $R$  is the *reduction* matrix. For instance, for  $\mathbb{Z}_N$ , if we combine all charge sectors together and all defect partition functions together, then

$$R = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 1 & \dots & 1 \end{pmatrix}. \quad (\text{A2})$$

The reduced partition vector obeys the crossing equation

$$\begin{aligned} \mathbf{Z}_{\text{red}}(-1/\tau, -1/\bar{\tau}) &= S_{\text{red}} \mathbf{Z}_{\text{red}}(\tau, \bar{\tau}), \\ S_{\text{red}} &= RSR^t(RR^t)^{-1}. \end{aligned} \quad (\text{A3})$$

Given any consistent solution  $\mathbf{Z}$  to the original bootstrap system (4.10),  $\mathbf{Z}_{\text{red}} = R\mathbf{Z}$  solves the reduced bootstrap system (A3). Hence the constraints from the former cannot be weaker than those from the latter.

Can the constraints from the reduced bootstrap system be as strong as the original one? If there exists a *lifting* matrix  $L$  satisfying  $LS_{\text{red}} = SL$ , then any solution  $\mathbf{Z}_{\text{red}}$  to the reduced bootstrap system can be lifted to a solution  $\mathbf{Z} = L\mathbf{Z}_{\text{red}}$  of the original bootstrap system

$$LRSR^t(RR^t)^{-1} = SL \quad (\text{A4})$$

In the example (A2), a consistent lifting matrix is given by

$$L = R^t(RR^t)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{N-1} & 0 \\ 0 & \vdots & 0 \\ 0 & \frac{1}{N-1} & 0 \\ 0 & 0 & \frac{1}{N-1} \\ 0 & \vdots & 0 \\ 0 & 0 & \frac{1}{N-1} \end{pmatrix}. \quad (\text{A5})$$

However, an important caveat is that the lift may not be compatible with the spin selection rule. To avoid such incompatibility, we should restrict the reduction  $R$  to only combine defect partition functions with the same spin selection rule and require the lift  $L$  to respect the spin selection rule. The example (A2) satisfies this restriction for nonanomalous  $\mathbb{Z}_N$  and anomalous  $\mathbb{Z}_{N \leq 3}$ , but not for anomalous  $\mathbb{Z}_{N \geq 4}$ . If a lift satisfying this restriction exists, then the reduced bootstrap system loses no generality, i.e., it produces the same bounds as the original; otherwise, it produces (possibly) weaker bounds.

In the following we present the reduced bootstrap systems that lose no generality for  $N = 3, 4, 5, 6$ .

### B. $\mathbb{Z}_3$ bootstrap system

Define the partition vector

$$\mathbf{Z}_{\mathbb{Z}_3} \equiv \begin{pmatrix} Z^0 \\ Z^1 + Z^2 \\ Z_\eta + Z_{\eta^2} \end{pmatrix}, \quad (\text{A6})$$

which obeys the crossing equation

$$\begin{aligned} \mathbf{Z}_{\mathbb{Z}_3}(-1/\tau, -1/\bar{\tau}) &= S_{\mathbb{Z}_3} \mathbf{Z}_{\mathbb{Z}_3}(\tau, \bar{\tau}), \\ S_{\mathbb{Z}_3} &= \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ 2 & -1 & 0 \end{pmatrix}. \end{aligned} \quad (\text{A7})$$

The spin selection rule for  $Z_\eta + Z_{\eta^2}$  is

$$h - \bar{h} \in \frac{k}{9} + \frac{\mathbb{Z}}{3}. \quad (\text{A8})$$

### C. $\mathbb{Z}_4$ bootstrap system

Define the partition vector

$$\mathbf{Z}_{\mathbb{Z}_4} \equiv \begin{pmatrix} Z^0 \\ Z^1 + Z^3 \\ Z^2 \\ Z_\eta + Z_{\eta^3} \\ Z_{\eta^2} \end{pmatrix}, \quad (\text{A9})$$

which obeys the crossing equation

$$\begin{aligned} \mathbf{Z}_{\mathbb{Z}_4}(-1/\tau, -1/\bar{\tau}) &= S_{\mathbb{Z}_4} \mathbf{Z}_{\mathbb{Z}_4}(\tau, \bar{\tau}), \\ S_{\mathbb{Z}_4} &= \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ 2 & 0 & -2 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (\text{A10})$$

The spin selection rule for  $Z_\eta + Z_{\eta^3}$  is

$$h - \bar{h} \in \frac{k}{16} + \frac{\mathbb{Z}}{4}, \quad (\text{A11})$$

and that for  $Z_{\eta^2}$  is

$$h - \bar{h} \in \frac{k}{4} + \frac{\mathbb{Z}}{2}. \quad (\text{A12})$$

**D.  $\mathbb{Z}_5$  bootstrap system**

Define the partition vector

$$\mathbf{Z}_{\mathbb{Z}_5} \equiv \begin{pmatrix} Z^0 \\ Z^1 + Z^4 \\ Z^2 + Z^3 \\ Z_\eta + Z_{\eta^4} \\ Z_{\eta^2} + Z_{\eta^3} \end{pmatrix}, \quad (\text{A13})$$

which obeys the crossing equation

$$\begin{aligned} \mathbf{Z}_{\mathbb{Z}_5}(-1/\tau, -1/\bar{\tau}) &= S_{\mathbb{Z}_5} \mathbf{Z}_{\mathbb{Z}_5}(\tau, \bar{\tau}), \\ &= \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{2}{5} & \frac{\sqrt{5}-1}{10} & \frac{-1-\sqrt{5}}{10} \\ \frac{2}{5} & \frac{2}{5} & \frac{2}{5} & \frac{-1-\sqrt{5}}{10} & \frac{\sqrt{5}-1}{10} \\ 2 & \frac{\sqrt{5}-1}{2} & \frac{-1-\sqrt{5}}{2} & 0 & 0 \\ 2 & \frac{-1-\sqrt{5}}{2} & \frac{\sqrt{5}-1}{2} & 0 & 0 \end{pmatrix}. \end{aligned} \quad (\text{A14})$$

The spin selection rule for  $Z_\eta + Z_{\eta^4}$  is

$$h - \bar{h} \in \frac{k}{25} + \frac{\mathbb{Z}}{5}, \quad (\text{A15})$$

and that for  $Z_{\eta^2} + Z_{\eta^3}$  is

$$h - \bar{h} \in -\frac{k}{25} + \frac{\mathbb{Z}}{5}. \quad (\text{A16})$$

**E.  $\mathbb{Z}_6$  bootstrap system**

Define the partition vector

$$\mathbf{Z}_{\mathbb{Z}_6} \equiv \begin{pmatrix} Z^0 \\ Z^1 + Z^5 \\ Z^2 + Z^4 \\ Z^3 \\ Z_\eta + Z_{\eta^5} \\ Z_{\eta^2} + Z_{\eta^4} \\ Z_{\eta^3} \end{pmatrix}, \quad (\text{A17})$$

which obeys the crossing equation

$$\mathbf{Z}_{\mathbb{Z}_6}(-1/\tau, -1/\bar{\tau}) = S_{\mathbb{Z}_6} \mathbf{Z}_{\mathbb{Z}_6}(\tau, \bar{\tau}),$$

$$S_{\mathbb{Z}_6} = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} & -\frac{1}{6} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & -\frac{1}{6} & \frac{1}{6} & -\frac{1}{6} \\ 2 & 1 & -1 & -2 & 0 & 0 & 0 \\ 2 & -1 & -1 & 2 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 \end{pmatrix}. \quad (\text{A18})$$

The spin selection rule for  $Z_\eta + Z_{\eta^5}$  is

$$h - \bar{h} \in \frac{k}{36} + \frac{\mathbb{Z}}{6}, \quad (\text{A19})$$

that for  $Z_{\eta^2} + Z_{\eta^4}$  is

$$h - \bar{h} \in \frac{k}{9} + \frac{\mathbb{Z}}{3}, \quad (\text{A20})$$

and that for  $Z_{\eta^3}$  is

$$h - \bar{h} \in \frac{k}{4} + \frac{\mathbb{Z}}{2}. \quad (\text{A21})$$

**APPENDIX B: CONSTRAINTS ON THE GAP OF ALL PRIMARIES**

In the main text, we focused on upper bounds on the gap of  $\mathbb{Z}_N$ -symmetric scalar primaries since they are relevant for the discussion of robust, symmetry-preserving gapless phases. However, the general framework of modular bootstrap with symmetry lines can produce more general bounds on primary operators (not just scalars) with different symmetry charges. We present these bounds in Figs. 8–10.

In the case of a  $\mathbb{Z}_2$  symmetry [1], at a fixed  $c$ , an upper bound on the scaling dimension of charged operators only exists if the symmetry is anomalous. Below we see the same phenomenon for more general  $\mathbb{Z}_N$  symmetries. Note that there is also no bound for the  $\mathbb{Z}_4$  symmetry with the  $k = 2$  anomaly in the  $Q = 1$  sector, since this is equivalent to bounding operators charged (odd) under its nonanomalous  $\mathbb{Z}_2$  subgroup.



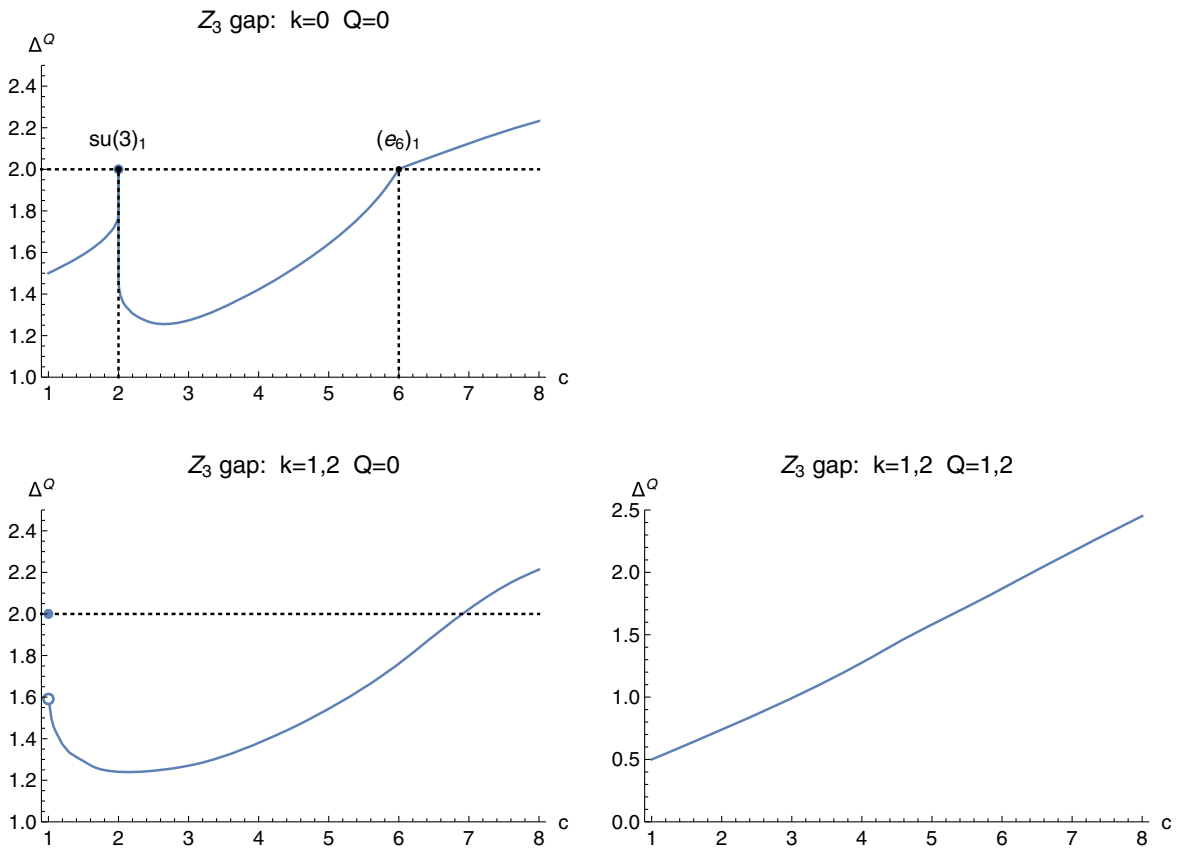


FIG. 8. Upper bounds on the gap  $\Delta^Q$  in the spectrum of Virasoro primaries, for  $\mathbb{Z}_3$  anomaly  $k = 0, 1, 2$  and  $\mathbb{Z}_3$  charge  $Q = 0, 1, 2$ . For  $k = 0, Q = 0$ , note the discontinuous jump at  $c = 2$  to  $\Delta_{\text{scalar}}^{Q=0} = 2$ .

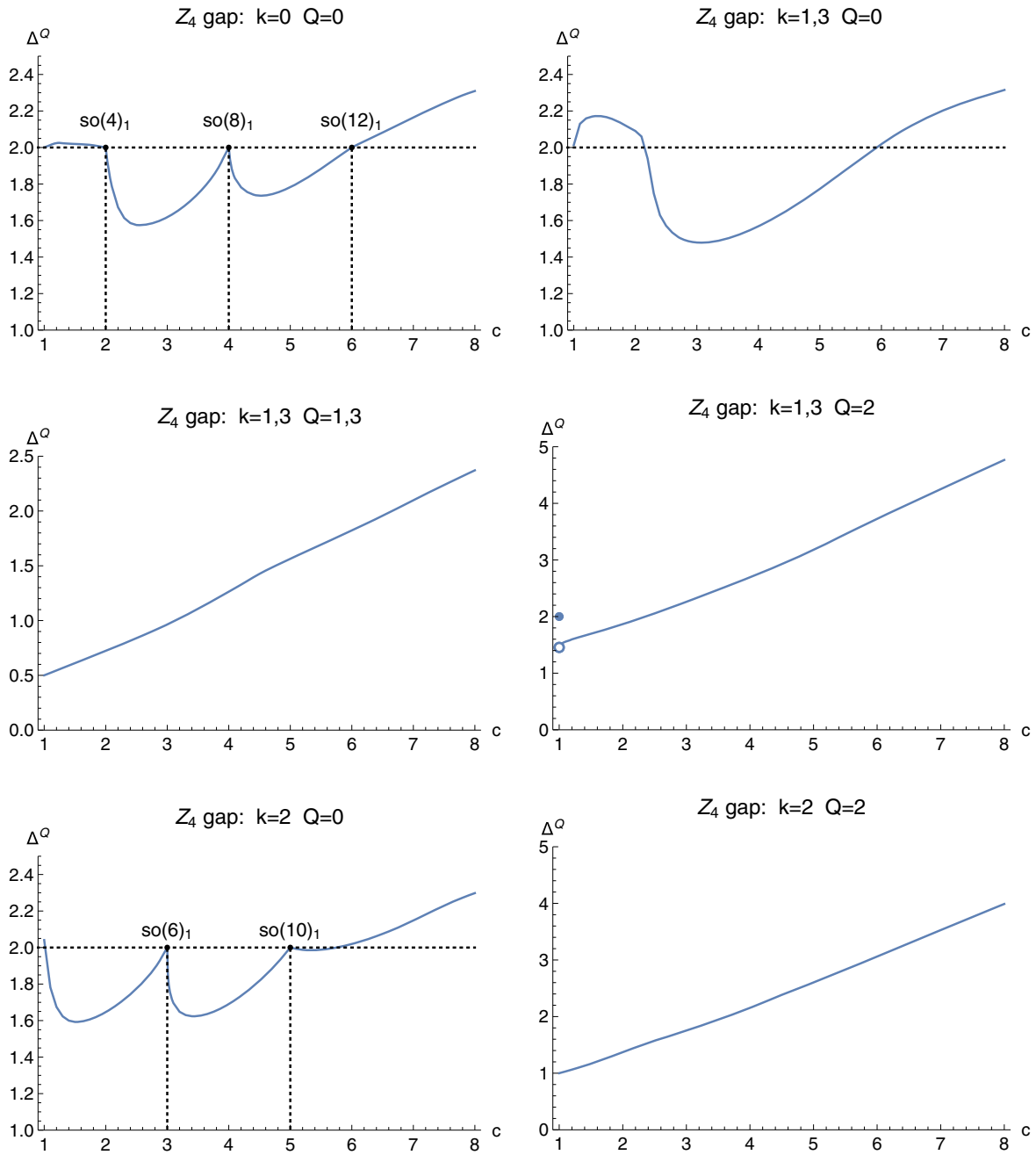


FIG. 9. Upper bounds on the gap  $\Delta^Q$  in the spectrum of Virasoro primaries, for  $Z_4$  anomaly  $k = 0, 1, 2, 3$  and  $Z_4$  charge  $Q = 0, 1, 2, 3$ .

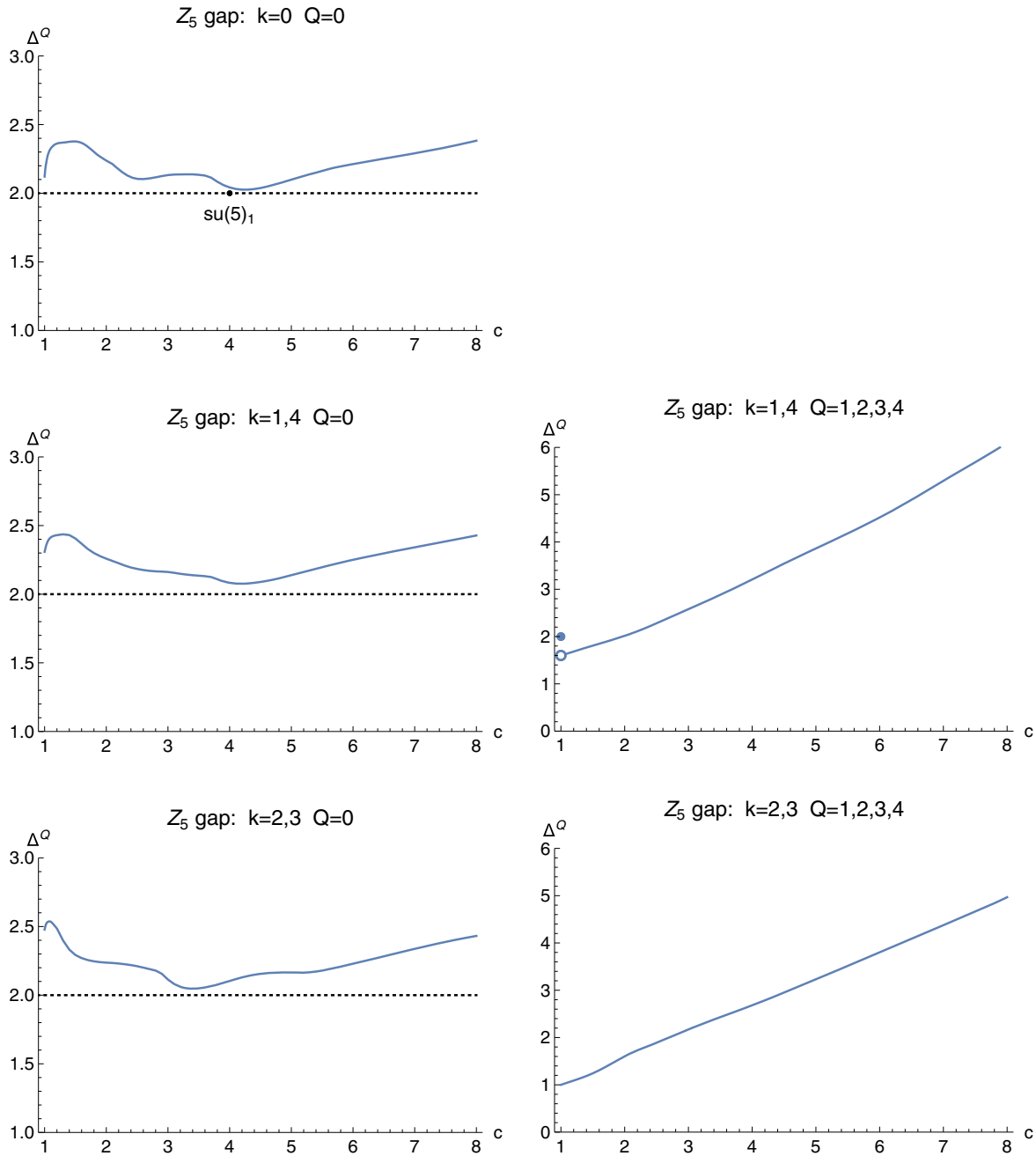


FIG. 10. Upper bounds on the gap  $\Delta^Q$  in the spectrum of Virasoro primaries, for  $\mathbb{Z}_5$  anomaly  $k = 0, 1, 2, 3, 4$  and  $\mathbb{Z}_5$  charge  $Q = 0, 1, 2, 3, 4$ .

[1] Y.-H. Lin and S.-H. Shao, Anomalies and bounds on charged operators, *Phys. Rev. D* **100**, 025013 (2019).  
 [2] Y. Alavirad and M. Barkeshli, Anomalies and unnatural stability of multi-component Luttinger liquids in  $\mathbb{Z}_n \times \mathbb{Z}_n$  spin chains, [arXiv:1910.00589](https://arxiv.org/abs/1910.00589).  
 [3] I. Affleck, Field theory methods and quantum critical phenomena, in *Les Houches Summer School in Theoretical*

*Physics: Fields, Strings, Critical Phenomena Les Houches, France* (1988), pp. 563–640, <https://inspirehep.net/literature/265102>.  
 [4] I. Affleck, Critical behavior of  $SU(n)$  quantum chains and topological nonlinear  $\sigma$  models, *Nucl. Phys.* **B305**, 582 (1988).  
 [5] K. Ohmori, N. Seiberg, and S.-H. Shao, Sigma models on flags, *SciPost Phys.* **6**, 017 (2019).

- [6] N. Seiberg and S.-H. Shao, Exotic symmetries, duality, and fractons in  $2 + 1$ -dimensional quantum field theory, *SciPost Phys.* **10**, 027 (2021).
- [7] J. L. Cardy, Operator content of two-dimensional conformally invariant theories, *Nucl. Phys.* **B270**, 186 (1986).
- [8] S. Hellerman, A universal inequality for CFT and quantum gravity, *J. High Energy Phys.* **08** (2011) 130.
- [9] S. Hellerman and C. Schmidt-Colinet, Bounds for state degeneracies in 2d conformal field theory, *J. High Energy Phys.* **08** (2011) 127.
- [10] C. A. Keller and H. Ooguri, Modular constraints on Calabi-Yau compactifications, *Commun. Math. Phys.* **324**, 107 (2013).
- [11] D. Friedan and C. A. Keller, Constraints on 2d CFT partition functions, *J. High Energy Phys.* **10** (2013) 180.
- [12] J. D. Qualls and A. D. Shapere, Bounds on operator dimensions in 2D conformal field theories, *J. High Energy Phys.* **05** (2014) 091.
- [13] T. Hartman, C. A. Keller, and B. Stoica, Universal spectrum of 2d conformal field theory in the large  $c$  limit, *J. High Energy Phys.* **09** (2014) 118.
- [14] J. D. Qualls, Universal bounds in even-spin CFTs, *J. High Energy Phys.* **12** (2015) 001.
- [15] H. Kim, P. Kravchuk, and H. Ooguri, Reflections on conformal spectra, *J. High Energy Phys.* **04** (2016) 184.
- [16] N. Benjamin, E. Dyer, A. L. Fitzpatrick, and S. Kachru, Universal bounds on charged states in 2d CFT and 3d gravity, *J. High Energy Phys.* **08** (2016) 041.
- [17] S. Collier, Y.-H. Lin, and X. Yin, Modular bootstrap revisited, *J. High Energy Phys.* **09** (2018) 061.
- [18] S. Collier, P. Kravchuk, Y.-H. Lin, and X. Yin, Bootstrapping the spectral function: On the uniqueness of Liouville and the universality of BTZ, *J. High Energy Phys.* **09** (2018) 150.
- [19] M. Cho, S. Collier, and X. Yin, Genus two modular bootstrap, *J. High Energy Phys.* **04** (2019) 022.
- [20] C. A. Keller, G. Mathys, and I. G. Zadeh, Bootstrapping chiral CFTs at genus two, *Adv. Theor. Math. Phys.* **22**, 1447 (2018).
- [21] J. Cardy, A. Maloney, and H. Maxfield, A new handle on three-point coefficients: OPE asymptotics from genus two modular invariance, *J. High Energy Phys.* **10** (2017) 136.
- [22] J.-B. Bae, S. Lee, and J. Song, Modular constraints on conformal field theories with currents, *J. High Energy Phys.* **12** (2017) 045.
- [23] E. Dyer, A. L. Fitzpatrick, and Y. Xin, Constraints on flavored 2d CFT partition functions, *J. High Energy Phys.* **02** (2018) 148.
- [24] T. Anous, R. Mahajan, and E. Shaghoulian, Parity and the modular bootstrap, *SciPost Phys.* **5**, 022 (2018).
- [25] J.-B. Bae, S. Lee, and J. Song, Modular constraints on superconformal field theories, *J. High Energy Phys.* **01** (2019) 209.
- [26] N. Afkhami-Jeddi, T. Hartman, and A. Tajdini, Fast conformal bootstrap and constraints on 3d gravity, *J. High Energy Phys.* **05** (2019) 087.
- [27] B. Mukhametzhanov and A. Zhiboedov, Modular invariance, Tauberian theorems and microcanonical entropy, *J. High Energy Phys.* **10** (2019) 261.
- [28] T. Hartman, D. Mazac, and L. Rastelli, Sphere packing and quantum gravity, *J. High Energy Phys.* **12** (2019) 048.
- [29] S. Ganguly and S. Pal, Bounds on the density of states and the spectral gap in  $\text{CFT}_2$ , *Phys. Rev. D* **101**, 106022 (2020).
- [30] N. Benjamin, H. Ooguri, S.-H. Shao, and Y. Wang, Light-cone modular bootstrap and pure gravity, *Phys. Rev. D* **100**, 066029 (2019).
- [31] S. Pal and Z. Sun, Tauberian-Cardy formula with spin, *J. High Energy Phys.* **01** (2020) 135.
- [32] N. Benjamin, H. Ooguri, S.-H. Shao, and Y. Wang, Twist gap and global symmetry in two dimensions, *Phys. Rev. D* **101**, 106026 (2020).
- [33] B. Mukhametzhanov and S. Pal, Beurling-Selberg Extremization and modular bootstrap at high energies, *SciPost Phys.* **8**, 088 (2020).
- [34] S. Pal and Z. Sun, High energy modular bootstrap, global symmetries and defects, *J. High Energy Phys.* **08** (2020) 064.
- [35] N. Benjamin and Y.-H. Lin, Lessons from the Ramond sector, *SciPost Phys.* **9**, 065 (2020).
- [36] N. Afkhami-Jeddi, H. Cohn, T. Hartman, D. de Laat, and A. Tajdini, High-dimensional sphere packing and the modular bootstrap, *J. High Energy Phys.* **12** (2020) 066.
- [37] L. F. Alday, J.-B. Bae, N. Benjamin, and C. Jorge-Diaz, On the spectrum of pure higher spin gravity, *J. High Energy Phys.* **12** (2020) 001.
- [38] A. Dymarsky and A. Shapere, Solutions of Modular Bootstrap Constraints from Quantum Codes, *Phys. Rev. Lett.* **126**, 161602 (2021).
- [39] A. Dymarsky and A. Shapere, Quantum stabilizer codes, lattices, and CFTs, *J. High Energy Phys.* **03** (2021) 160.
- [40] J. V. José, L. P. Kadanoff, S. Kirkpatrick, and D. R. Nelson, Renormalization, vortices, and symmetry-breaking perturbations in the two-dimensional planar model, *Phys. Rev. B* **16**, 1217 (1977).
- [41] S. Elitzur, R. B. Pearson, and J. Shigemitsu, Phase structure of discrete Abelian spin and gauge systems, *Phys. Rev. D* **19**, 3698 (1979).
- [42] J. L. Cardy, General discrete planar models in two dimensions: Duality properties and phase diagrams, *J. Phys. A* **13**, 1507 (1980).
- [43] F. C. Alcaraz and R. Koberle, Duality and the phases of  $Z(n)$  spin systems, *J. Phys. A* **13**, L153 (1980).
- [44] L. Bhardwaj and Y. Tachikawa, On finite symmetries and their gauging in two dimensions, *J. High Energy Phys.* **03** (2018) 189.
- [45] C.-M. Chang, Y.-H. Lin, S.-H. Shao, Y. Wang, and X. Yin, Topological defect lines and renormalization group flows in two dimensions, *J. High Energy Phys.* **01** (2019) 026.
- [46] G. W. Moore and N. Seiberg, Classical and quantum conformal field theory, *Commun. Math. Phys.* **123**, 177 (1989).
- [47] P. Bonderson, K. Shtengel, and J. Slingerland, Interferometry of non-Abelian anyons, *Ann. Phys. (Amsterdam)* **323**, 2709 (2008).
- [48] M. Barkeshli, P. Bonderson, M. Cheng, and Z. Wang, Symmetry fractionalization, defects, and gauging of topological phases, *Phys. Rev. B* **100**, 115147 (2019).

- [49] P.-S. Hsin, H. T. Lam, and N. Seiberg, Comments on one-form global symmetries and their gauging in 3d and 4d, *SciPost Phys.* **6**, 039 (2019).
- [50] S. Mukhi and R. Poddar, Universal correlators and novel cosets in 2d RCFT, *J. High Energy Phys.* **02** (2021) 158.
- [51] S. Hegde and D. P. Jatkar, Defect partition function from TDLs in commutant pairs, [arXiv:2101.12189](https://arxiv.org/abs/2101.12189).
- [52] K. Kawagoe and M. Levin, Computing anomalies in SPT edge theories, [arXiv:2105.02909](https://arxiv.org/abs/2105.02909).
- [53] Y. Tachikawa, On gauging finite subgroups, *SciPost Phys.* **8**, 015 (2020).
- [54] R. Thorngren and Y. Wang, Fusion category symmetry I: Anomaly in-flow and gapped phases, [arXiv:1912.02817](https://arxiv.org/abs/1912.02817).
- [55] D. Gaiotto and J. Kulp, Orbifold groupoids, *J. High Energy Phys.* **02** (2021) 132.
- [56] Z. Komargodski, K. Ohmori, K. Roumpedakis, and S. Seifnashri, Symmetries and strings of adjoint QCD<sub>2</sub>, *J. High Energy Phys.* **03** (2021) 103.
- [57] A. Kapustin and N. Seiberg, Coupling a QFT to a TQFT and duality, *J. High Energy Phys.* **04** (2014) 001.
- [58] D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett, Generalized global symmetries, *J. High Energy Phys.* **02** (2015) 172.
- [59] C. Córdova, K. Ohmori, S.-H. Shao, and F. Yan, Decorated  $\mathbb{Z}_2$  symmetry defects and their time-reversal anomalies, *Phys. Rev. D* **102**, 045019 (2020).
- [60] D. S. Freed and C. Vafa, Global anomalies on orbifolds, *Commun. Math. Phys.* **110**, 349 (1987); Erratum, *Commun. Math. Phys.* **117**, A349 (1988).
- [61] G. Felder, K. Gawedzki, and A. Kupiainen, Spectra of Wess-Zumino-Witten models with arbitrary simple groups, *Commun. Math. Phys.* **117**, 127 (1988).
- [62] D. Gaiotto, Monster symmetry and extremal CFTs, *J. High Energy Phys.* **11** (2012) 149.
- [63] O. M. Sule, X. Chen, and S. Ryu, Symmetry-protected topological phases and orbifolds: Generalized Laughlin's argument, *Phys. Rev. B* **88**, 075125 (2013).
- [64] T. Numasawa and S. Yamaguchi, Mixed global anomalies and boundary conformal field theories, *J. High Energy Phys.* **11** (2018) 202.
- [65] X. Chen, Z.-X. Liu, and X.-G. Wen, Two-dimensional symmetry-protected topological orders and their protected gapless edge excitations, *Phys. Rev. B* **84**, 235141 (2011).
- [66] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Symmetry protected topological orders and the group cohomology of their symmetry group, *Phys. Rev. B* **87**, 155114 (2013).
- [67] L.-Y. Hung and X.-G. Wen, Universal symmetry-protected topological invariants for symmetry-protected topological states, *Phys. Rev. B* **89**, 075121 (2014).
- [68] R. Dijkgraaf and E. Witten, Topological gauge theories and group cohomology, *Commun. Math. Phys.* **129**, 393 (1990).
- [69] W. Ji and X.-G. Wen, Non-invertible anomalies and mapping-class-group transformation of anomalous partition functions, *Phys. Rev. Research* **1**, 033054 (2019).
- [70] J. Wang, X.-G. Wen, and E. Witten, Symmetric Gapped Interfaces of SPT and SET States: Systematic Constructions, *Phys. Rev. X* **8**, 031048 (2018).
- [71] E. P. Verlinde, Fusion rules and modular transformations in 2D conformal field theory, *Nucl. Phys.* **B300**, 360 (1988).
- [72] V. B. Petkova and J. B. Zuber, Generalized twisted partition functions, *Phys. Lett. B* **504**, 157 (2001).
- [73] J. Frohlich, J. Fuchs, I. Runkel, and C. Schweigert, Kramers-Wannier Duality from Conformal Defects, *Phys. Rev. Lett.* **93**, 070601 (2004).
- [74] H. A. Kramers and G. H. Wannier, Statistics of the two-dimensional ferromagnet. Part I, *Phys. Rev.* **60**, 252 (1941).
- [75] M. Cheng and D. J. Williamson, Relative anomaly in (1+1)d rational conformal field theory, *Phys. Rev. Research* **2**, 043044 (2020).
- [76] J. Fuchs, M. R. Gaberdiel, I. Runkel, and C. Schweigert, Topological defects for the free boson CFT, *J. Phys. A* **40**, 11403 (2007).
- [77] J. A. Harvey and G. W. Moore, An uplifting discussion of T-duality, *J. High Energy Phys.* **05** (2018) 145.
- [78] V. A. Fateev and A. B. Zamolodchikov, Parafermionic currents in the two-dimensional conformal quantum field theory and self-dual critical points in  $Z(n)$  invariant statistical systems, *Zh. Eksp. Teor. Fiz.* **89**, 380 (1985) [*Sov. Phys. JETP* **62**, 215 (1985)].
- [79] D. Gepner and Z.-a. Qiu, Modular invariant partition functions for parafermionic field theories, *Nucl. Phys.* **B285**, 423 (1987).
- [80] D. Simmons-Duffin, A semidefinite program solver for the conformal bootstrap, *J. High Energy Phys.* **06** (2015) 174.
- [81] W. Landry and D. Simmons-Duffin, Scaling the semidefinite program solver SDPB, [arXiv:1909.09745](https://arxiv.org/abs/1909.09745).
- [82] R. Thorngren, Anomalies and bosonization, *Commun. Math. Phys.* **378**, 1775 (2020).
- [83] W. Ji, S.-H. Shao, and X.-G. Wen, Topological transition on the conformal manifold, *Phys. Rev. Research* **2**, 033317 (2020).
- [84] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, and A. Vichi, Solving the 3D Ising model with the conformal bootstrap, *Phys. Rev. D* **86**, 025022 (2012).
- [85] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, and A. Vichi, Solving the 3d Ising model with the conformal bootstrap II. c-minimization and precise critical exponents, *J. Stat. Phys.* **157**, 869 (2014).
- [86] F. Kos, D. Poland, and D. Simmons-Duffin, Bootstrapping mixed correlators in the 3D Ising model, *J. High Energy Phys.* **11** (2014) 109.