Nonexistence and existence of shock profiles in the Bemfica-Disconzi-Noronha model

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This paper studies a four-field hyperbolic PDE model that was recently introduced by Bemfica, Disconzi, and Noronha for the pure radiation fluid with viscosity, and asks whether shock waves admit continuous profiles in this description. The model containing two free parameters μ, ν and being causal whenever one chooses (μ, ν) from a certain range $\mathcal{C} \subset \mathbb{R}^2$, this paper shows that for any choice of (μ, ν) in the interior of $\mathcal{C}$, there is a dichotomy in so far as (i) shocks of sufficiently small amplitude admit profiles and (ii) certain other, thus necessarily nonsmall, shocks do not. This finding does not preclude the possibility that if one chooses (μ, ν) from a specific part $\mathcal{S}$ of the boundary of $\mathcal{C}$, the dichotomy disappears and *all* shocks have profiles; the parameter set $\mathcal{S}$ corresponds to the "sharply causal" case, in which one of the characteristic speeds of the dissipation operator is the speed of light.

I. INTRODUCTION

In their admirable 2018 paper [1], Bemfica, Disconzi, and Noronha have proposed a four-field PDE formulation

$$\partial_\beta(T^{\alpha\beta} + \Delta T^{\alpha\beta}) = 0 \qquad (1.1)$$

for the dynamics of the pure radiation fluid, ideally

$$T^{\alpha\beta} = \frac{\partial(p(\theta)\psi^\alpha)}{\partial\psi_\beta} = \theta^3 p'(\theta)\psi^\alpha\psi^\beta + p(\theta)g^{\alpha\beta},$$

$$\psi^\gamma = \frac{u^\gamma}{\theta}, \qquad p(\theta) = \frac{1}{3}\theta^4, \qquad (1.2)$$

where $g^{\alpha\beta} = \text{diag}(-1, 1, 1, 1)$ is the Minkowski metric and θ, p, u^γ denote the local temperature, pressure and 4-velocity of the fluid in units with respect to which the speed of light is 1. In their dissipation tensor

$$\Delta T^{\alpha\beta} = -B^{\alpha\beta\gamma\delta}(\psi)\frac{\partial\psi_\gamma}{\partial x^\delta}, \qquad (1.3)$$

the classical Eckart viscosity ansatz [2]

$$B_{\text{visc}}^{\alpha\beta\gamma\delta} = \Pi^{\alpha\gamma}\Pi^{\beta\delta} + \Pi^{\alpha\delta}\Pi^{\beta\gamma} - \frac{2}{3}\Pi^{\alpha\beta}\Pi^{\gamma\delta} \quad \text{with}$$

$$\Pi^{\alpha\gamma} = u^\alpha u^\beta + g^{\alpha\beta} \qquad (1.4)$$

is augmented as[1]

$$B^{\alpha\beta\gamma\delta} = \eta B_{\text{visc}}^{\alpha\beta\gamma\delta} - \mu B_{\text{ther}}^{\alpha\beta\gamma\delta} - \nu B_{\text{velo}}^{\alpha\beta\gamma\delta} \qquad (1.5)$$

with

$$B_{\text{ther}}^{\alpha\beta\gamma\delta} = (3u^\alpha u^\beta + \Pi^{\alpha\beta})(3u^\gamma u^\delta + \Pi'^\delta),$$

$$B_{\text{velo}}^{\alpha\beta\gamma\delta} = (u^\alpha\Pi^\beta{}_\epsilon + u^\beta\Pi^\alpha{}_\epsilon)(u^\gamma\Pi^{\delta\epsilon} + u^\delta\Pi'^\epsilon). \quad (1.6)$$

Like that of a different augmentation introduced by B. Temple and the author in [3,5], the purpose of the combination (1.5) is to make the dissipation causal and thus remedy a well-known deficiency of the Eckart theory.[2] It was shown in [1] that this formulation, which we will here briefly refer to as "the BDN model," is indeed causal if and only if, relative to the classical coefficient η of viscosity, the coefficients μ and ν of the "regulators" $B_{\text{ther}}^{\alpha\beta\gamma\delta}, B_{\text{velo}}^{\alpha\beta\gamma\delta}$ satisfy

$$\mu \geq \frac{4}{3}\eta \quad \text{and} \quad \nu \leq \left(\frac{1}{3\eta} - \frac{1}{9\mu}\right)^{-1}. \qquad (1.7)$$

The present note focusses on shock waves, whose ideal version is given by discontinuous solutions to the corresponding inviscid, Euler, equations

$$\partial_\beta T^{\alpha\beta} = 0 \qquad (1.8)$$

of the (prototypical) form

$$\psi(x) = \begin{cases} \psi_-, & x^\beta \xi_\beta < 0, \\ \psi_+, & x^\beta \xi_\beta > 0, \end{cases} \quad \text{with spacelike normal } \xi \colon \xi^\beta \xi_\beta = 1,$$

$$(1.9)$$

and asks whether they can be properly represented in the viscous setting (1.1). A standard way to achieve such representation of a "viscous shock wave" is a "dissipation profile," i.e., a regular solution of (1.1) that depends also only on $x^\beta \xi_\beta$ and connects the two states forming the shock, in other words, a solution of the ODE

$$\xi_\beta \xi_\delta B^{\alpha\beta\gamma\delta}(\psi)\psi'_\gamma = \xi_\beta T^{\alpha\beta}(\psi) - q^\alpha, \qquad q^\alpha := \xi_\beta T^{\alpha\beta}(\psi_\pm),$$

$$(1.10)$$

on $\mathbb{R}$ which is heteroclinic to them,

$$\hat{\psi}(-\infty) = \psi_-, \qquad \hat{\psi}(+\infty) = \psi_+. \qquad (1.11)$$

A solution (1.9) of (1.8) is called a Lax shock if the characteristic speeds of (1.8), i.e., the eigenvalues of the Jacobian

$$\left(\frac{\partial T^{\alpha\beta}}{\partial \psi_\gamma} \xi_\beta\right)_{\alpha\gamma},$$

at the two states ψ_- and ψ_+ satisfy Lax's ordering condition ([8], Sec. VII); physically speaking, this means that the discontinuity wave is supersonic with respect to the downstream and subsonic with respect to the upstream flow.

The technical main purpose of this paper is to show the following.

Theorem 1. For any choice of the coefficients $\eta, \mu, \nu > 0$,

(i) every Lax shock of sufficiently small amplitude possesses a dissipation profile, and

(ii) if the coefficients satisfy the strict causality condition

$$\mu \geq \frac{4}{3}\eta \quad \text{and} \quad \nu < \left(\frac{1}{3\eta} - \frac{1}{9\mu}\right)^{-1}, \qquad (1.12)$$

then there always exist other shock waves that do not admit a dissipation profile.

Besides the dichotomy that Theorem 1 states so distinctly, we draw the reader's attention to the slight difference between the two causality conditions (1.7) and (1.12). To understand its meaning note that all characteristic speeds σ (cf. the Appendix) of the operator

$$B^{\alpha\beta\gamma\delta} \frac{\partial^2 \psi_\gamma}{\partial x^\beta \partial x^\delta}$$

satisfy $0 \leq \sigma^2 \leq 1$ if and only if both inequalities in (1.7) hold, and one of the speeds is luminal, $\sigma^2 = 1$, if and only if

$$\mu \geq \frac{4}{3}\eta \quad \text{and} \quad \nu = \left(\frac{1}{3\eta} - \frac{1}{9\mu}\right)^{-1}, \qquad (1.13)$$

a condition that we therefore call *sharp causality*.

Obviously, Theorem 1 has the following two implications:

(A) In the strictly causal case (1.12), upon variation from small to large amplitudes, necessarily some kind of transition occurs in the phase portrait of the dynamical system (1.10).

(B) As it is well possible (though not formally proved in this paper, cf. Remark 1 at the end of Sec. III) that *all* shocks have profiles when

$$\nu = \nu_*(\eta, \mu) = \left(\frac{1}{3\eta} - \frac{1}{9\mu}\right)^{-1}, \qquad (1.14)$$

it seems that for the modeling one might prefer this sharply causal tuning over others.

Both (A)—cf. Fig. 1—and (B) are under further investigation [9].

Dissipation profiles for fluid dynamical shock waves have been widely studied. For a selection of aspects, including the interesting (though quite different) heteroclinic bifurcation in standard magnetohydrodynamics, the interested reader is referred to [5,10–15].

As regards to the present paper, we parametrize the ideal shock waves of the pure radiation fluid in Sec. II, Sec. III contains the proof of the theorem, and a brief Appendix concisely reconsiders the causality conditions (1.7), (1.13).

II. RANKINE-HUGONIOT CONDITIONS OF PURE RADIATION

Due to Lorentz invariance, we can restrict attention to shocks of speed $s = 0$, and because of the system's natural isotropy, we may assume that the spatial direction of propagation is (1,0,0). Correspondingly, we fix the spatio-temporal normal to the shock as

$$(\xi_0, \xi_1, \xi_2, \xi_3) = (0, 1, 0, 0),$$

henceforth consider only states ψ with $\psi_2 = \psi_3 = 0$, and let the indices run over 0 and 1 instead from 0 to 3; thus, for instance, the metric $g^{\alpha\gamma}$ and the projection $\Pi^{\alpha\gamma} = g^{\alpha\gamma} + u^\alpha u^\gamma$ on the orthogonal complement of the velocity u^α are given by the matrices

$$(g^{\alpha\gamma}) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad (\Pi^{\alpha\gamma}) = \begin{pmatrix} u_1^2 & u^0 u^1 \\ u^0 u^1 & u_0^2 \end{pmatrix}.$$

On the state space $\Psi \equiv \{\psi = (\psi^0, \psi^1) \in \mathbb{R}^2 : \psi^0 > |\psi^1|\}$, we use the temperature and the velocity,
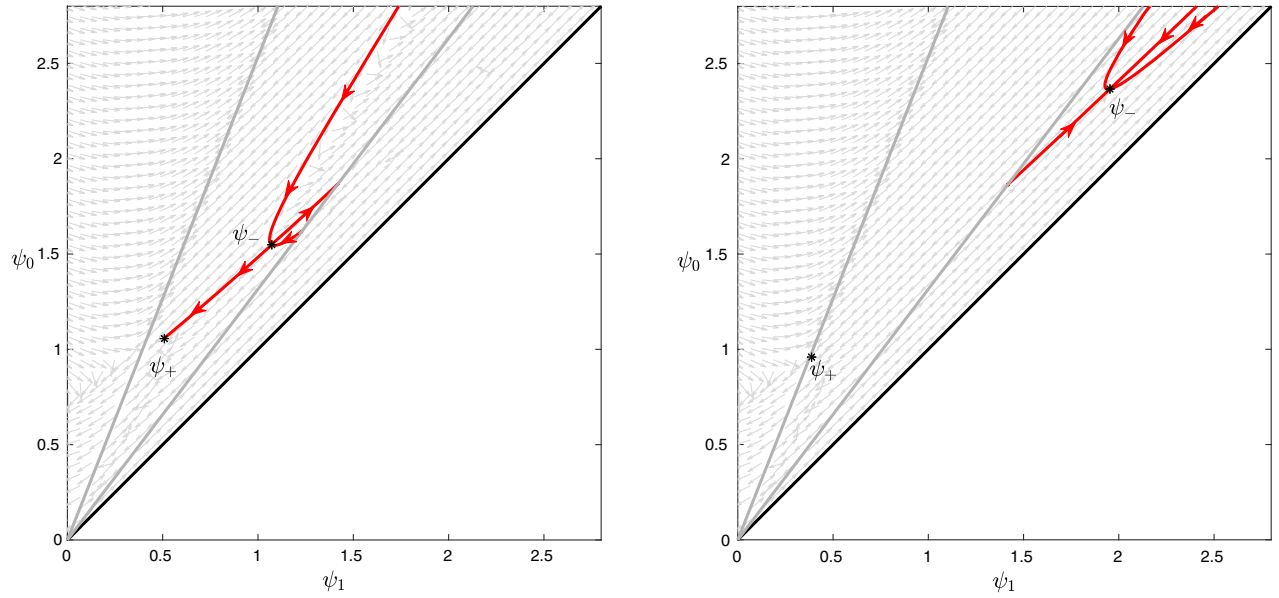
FIG. 1. Phase portrait of (1.10) for strictly causal parameter values $(\eta, \mu, \nu) = (1, 7, 20)$. Left: $\tilde{q} = 31/40$. Right: $\tilde{q} = 34/40$. Gray lines indicate det $B = 0$. Plots by V. Pellhammer.

$$\theta = (-\psi_\gamma \psi^\gamma)^{-1/2}, \qquad (u, v) := (u^0, u^1) = \theta(\psi^0, \psi^1)$$
$$\text{with} \quad u = (1 + v^2)^{1/2},$$

as coordinates. All possible shocks can be identified by screening the preimage set of the ξ_β component of the ideal stress $T^{\alpha\beta}$ [16,17]:

Lemma 1. On Ψ, the equation

$$T^{\alpha 1}(\psi) = q^\alpha \tag{2.1}$$

has more than one solution if and only if

$$q^1 > 0 \quad \text{and} \quad (q^1)^2 < (q^0)^2 < \frac{2}{\sqrt{3}}(q^1)^2.$$

In that case, it has exactly two, and these two states form a standing Lax shock in right-moving or left-moving flow if $q^0 > 0$ or $q^0 < 0$, respectively.

Proof.—As the relation $T^{11}(\psi) = q^1$ is equivalent to

$$\frac{4}{3}v^2 + \frac{1}{3} = q^1\theta^{-4},$$

its solution set is empty if $q^1 \leq 0$. If $q^1 > 0$, it is the curve

$$H^1(q^1) = \{h^1(v, q^1) : v \in \mathbb{R}\}$$

with

$$h^1(v, q^1) = \frac{1}{\theta^1(v, q^1)}(\sqrt{1 + v^2}, v), \quad \text{where}$$

$$\theta^1(v, q^1) := \left(\frac{1}{q^1}\left(\frac{4}{3}v^2 + \frac{1}{3}\right)\right)^{-1/4}.$$

Since the relation $T^{01}(\psi) = q^0$ is equivalent to

$$\left(\frac{4}{3}\right)^2(1 + v^2)v^2 = (q^0)^2\theta^{-8} \quad \text{and} \quad q^0 v \geq 0,$$

its solution set is $H^0(0) = (0, \infty) \times \{0\}$ if $q^0 = 0$, but otherwise it is the curve

$$H^0(q^0) = \{h^0(v, q^0) : q^0 v > 0\}$$

with

$$h^0(v, q^0) = \frac{1}{\theta^0(v, q^0)}(\sqrt{1 + v^2}, v), \quad \text{where}$$

$$\theta^0(v, q^0) := \left(\left(\frac{4}{3}\right)^2\frac{(1 + v^2)v^2}{(q^0)^2}\right)^{-1/8}.$$

Being the intersection of the two curves $H^0(q^0)$ and $H^1(q^1)$, the solution set $H(q)$ of (2.1) can have more than one element only if we have $q^0 \neq 0$. As (2.1) is invariant under the reflection $q^0 \mapsto -q^0, u^1 \mapsto -u^1$, it is without loss of generality that we henceforth assume

$$q^1 > 0 \quad \text{and} \quad q^0 > 0. \tag{2.2}$$

Observing that

$$h^0(v, q^0) = h^1(v, q^1) \Leftrightarrow \theta^0(v, q^0) = \theta^1(v, q^1)$$
$$\Leftrightarrow (4v^2 + 1)^2 = 16\tilde{q}^2 v^2 (1 + v^2)$$
$$\Leftrightarrow (16(1 - \tilde{q})v^4 + 8(1 - 2\tilde{q})v^2 + 1$$

with

$$\tilde{q} = (q^1/q^0)^2,$$

we find that the equation $h^0(v, q^0) = h^1(v, q^1)$ has two different positive solutions $v_- > v_+$ if and only if

$$3/4 < \tilde{q} < 1$$

and in that case, these solutions are given through

$$v_\pm^2(\tilde{q}) = \frac{(2\tilde{q} - 1) \mp \sqrt{\tilde{q}(4\tilde{q} - 3)}}{4(1 - \tilde{q})}.$$

We note that

$$\lim_{\tilde{q} \searrow 3/4} v_\pm(\tilde{q}) =: v_* = \frac{1}{\sqrt{2}}$$

while

$$\lim_{\tilde{q} \nearrow 1} v_+(\tilde{q}) = \frac{1}{2\sqrt{2}} \quad \text{and} \quad \lim_{\tilde{q} \nearrow 1} v_-(\tilde{q}) = \infty.$$

I. e., the case $\tilde{q} \searrow 3/4$ corresponds to the zero amplitude and $\tilde{q} \nearrow 1$ to the infinite amplitude limit.

As (2.1) is also invariant under homotheties $(\psi, q) \mapsto (a\psi, a^{-2}q), a > 0$, it again causes no loss of generality when we from now on also assume that $q^1 = 1$. This means that we consider a $\tilde{q}$-dependent pair of states

$$(\psi_-(\tilde{q}), \psi_+(\tilde{q})) \quad \text{with}$$
$$\psi_\pm = (((4/3)v^2 + 1/3)^{1/4}((1 + v^2)^{1/2}, v))|_{v=v_\pm(\tilde{q})},$$
$$3/4 < \tilde{q} < 1,$$

that originates from the bifurcation point

$$\psi_* = \frac{1}{\sqrt{2}}(\sqrt{3}, 1)$$

when $\tilde{q}$ grows starting from its lower limiting value $3/4$. Since $v_- > v_+ > 0$, ψ_- as a left-hand (upstream) state and ψ_+ as a right-hand (downstream) state form a shock that stands in a right-moving decelerating flow.

To confirm that it is a Lax shock, we observe that up to a positive scalar factor, the Jacobian

$$\frac{\partial T^{\alpha 1}}{\partial \psi_\gamma} = \theta^2 p_\theta (g^{\alpha 1} u^\gamma + g^{\alpha \gamma} u^1 + g^{1\gamma} u^\alpha) + (\theta^3 p_\theta)_\theta u^\alpha u^1 u^\gamma$$
$$= \frac{4}{3} \theta^5 ((g^{\alpha 1} u^\gamma + g^{\alpha \gamma} u^1 + g^{1\gamma} u^\alpha) + 6u^\alpha u^1 u^\gamma)$$

is given by the matrix

$$\begin{pmatrix} -v + 6u^2 v & u + 6uv^2 \\ u + 6uv^2 & 3v + 6v^3 \end{pmatrix}$$
$$= \begin{pmatrix} v(6v^2 + 5) & \sqrt{1 + v^2}(6v^2 + 1) \\ \sqrt{1 + v^2}(6v^2 + 1) & v(6v^2 + 3) \end{pmatrix}, \quad (2.3)$$

whose determinant

$$v^2(6v^2 + 5)(6v^2 + 3) - (1 + v^2)(6v^2 + 1)^2 = 2v^2 - 1 \quad (2.4)$$

is positive for $v^2 > 1/2$ and negative for $v < 1/2$. Thus both characteristic speeds—here corresponding to right- and left-moving sound waves—are positive at ψ_-, while for ψ_+ one of them is positive and the other negative: this means that ψ_- is subsonic, ψ^+ supersonic and, in the terminology of [8] defines a 1-shock. ∎

III. SHOCK PROFILES IN THE BDN PICTURE

We write the profile equation

$$B^{\alpha 1 \gamma 1}(\psi)\psi_\gamma' = T^{\alpha 1}(\psi) - q^\alpha.$$

and a *scaled* version of its linearization at a given state $\bar{\psi}$,

$$B^{\alpha 1 \gamma 1}(\bar{\psi})\psi_\gamma' = \frac{\partial T^{\alpha 1}}{\partial \psi_\gamma}(\bar{\psi})\psi_\gamma,$$

as

$$B(\psi)\psi' = F(\psi, \tilde{q}) \quad (3.1)$$

and

$$B(\bar{\psi})\psi' = A(\bar{\psi})\psi, \quad (3.2)$$

respectively, with

$$B = \tilde{\eta}\tilde{B}_{\text{visc}} - \mu B_{\text{ther}} - \nu B_{\text{velo}} \quad (3.3)$$

where

$$\tilde{B}_{\text{visc}}(\psi) = \begin{pmatrix} u^2 v^2 & -u^3 v \\ -u^3 v & u^4 \end{pmatrix}, \qquad \tilde{\eta} = \frac{4}{3}\eta, \quad (3.4)$$

$$B_{\text{ther}}(\psi) = \begin{pmatrix} 16u^2v^2 & -4uv(4v^2+1) \\ -4uv(4v^2+1) & (4v^2+1)^2 \end{pmatrix}, \quad (3.5)$$

$$B_{\text{velo}}(\psi) = \begin{pmatrix} (u^2+v^2)^2 & -2(u^2+v^2)uv \\ -2(u^2+v^2)uv & 4u^2v^2 \end{pmatrix}, \quad (3.6)$$

and

$$A(\psi) = \begin{pmatrix} v(6u^2-1) & -u(6v^2+1) \\ -u(6v^2+1) & v(6v^2+3) \end{pmatrix};$$

according to (1.4), (1.6), and (2.3). (The minus signs in the offdiagonal entries of the above matrices are induced by the distinction between contra- and covariant indices, $\psi^\epsilon = g^{\epsilon\gamma}\psi_\gamma$.)

For any $\tilde{q} \in (3/4, 1)$, the two states $\psi_-(\tilde{q}), \psi_+(\tilde{q})$ we have identified above, obviously are (the only) rest points of the ODE system (3.1).

We first consider the small-amplitude case. At the bifurcation point ψ_*, all four matrices

$$A(\psi_*) = \frac{1}{\sqrt{2}}\begin{pmatrix} 8 & -4\sqrt{3} \\ -4\sqrt{3} & 6 \end{pmatrix}, \qquad \tilde{B}_{\text{visc}}(\psi_*) = \begin{pmatrix} u^2v^2 & -u^3v \\ -u^3v & u^4 \end{pmatrix} = \frac{3}{4}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix},$$

$$B_{\text{ther}}(\psi) = \begin{pmatrix} 16u^2v^2 & -4uv(4v^2+1) \\ -4uv(4v^2+1) & (4v^2+1)^2 \end{pmatrix} = \begin{pmatrix} 12 & -6\sqrt{3} \\ -6\sqrt{3} & 9 \end{pmatrix},$$

$$B_{\text{velo}}(\psi) = \begin{pmatrix} (u^2+v^2)^2 & -2(u^2+v^2)uv \\ -2(u^2+v^2)uv & 4u^2v^2 \end{pmatrix} = \begin{pmatrix} 4 & -2\sqrt{3} \\ -2\sqrt{3} & 3 \end{pmatrix}.$$

are positive multiples of orthogonal projectors, with

$$\ker A(\psi_*) = \ker B_{\text{ther}}(\psi_*) = \ker B_{\text{velo}}(\psi_*) \text{ spanned by } r = \begin{pmatrix} 3 \\ 2\sqrt{3} \end{pmatrix}$$

and

$$\ker \tilde{B}_{\text{visc}}(\psi_*) \text{ spanned by } \begin{pmatrix} 3 \\ \sqrt{3} \end{pmatrix} \not\parallel r.$$

As thus both $\eta\tilde{B}_{\text{visc}}(\psi_*)$ and $\mu B_{\text{ther}}(\psi_*) + \nu B_{\text{velo}}(\psi_*)$ are nontrivial multiples of orthogonal projectors with different one-dimensional images, the matrix $B(\psi_*)$, and so $B(\psi)$ for any ψ near ψ_*, is invertible. Consequently, the augmented profile equation

$$\psi' = (B(\psi))^{-1}F(\psi, \tilde{q}), \qquad \tilde{q}' = 0 \qquad (3.7)$$

has a $1+1$-dimensional center manifold $\mathcal{C}$ at $(\psi_*, \tilde{q}_* = \frac{3}{4})$. For any value of $\tilde{q}$ above and sufficiently close to $\tilde{q}_*$, the 1-dimensional fiber

$$\mathcal{C}_{\tilde{q}} = \{\psi \in \Psi : (\psi, \tilde{q}) \in \mathcal{C}\}$$

contains (cf. [14], p. 242) the nearby rest points $\psi_-(\tilde{q}), \psi_+(\tilde{q})$, and as these are the only rest points, the segment of the curve $\mathcal{C}_{\tilde{q}}$ between them is a single orbit $\hat{\psi}(\mathbb{R})$ which is heteroclinic to them—this orbit is the sought after traveling wave. A straightforward center manifold reduction (cf. [14], pp. 245, 246) confirms that it has $\hat{\psi}(-\infty) = \psi_-(\tilde{q}), \hat{\psi}(+\infty) = \psi_+(\tilde{q})$ and not vice versa. This finishes the proof of Assertion (i).

Regarding arbitrary shocks, we first show the following properties of B.

Lemma 2. Assume μ and ν satisfy the causality condition (1.7). Then

(i) $\psi_-(\tilde{q})$ is an attractor at least for certain values of $\tilde{q} \in (3/4, 1)$, unless (1.14) holds.

(ii) If (1.14) holds, $\psi_-(\tilde{q})$ is a hyperbolic saddle for all values of $\tilde{q} \in (3/4, 1)$.

Proof.—Relations (3.3)–(3.6) readily yield

$$B(\psi) = \begin{pmatrix} \tilde{\eta}u^2v^2 - 16\mu u^2v^2 - \nu(u^2+v^2)^2 & -uv[\tilde{\eta}u^2 - 4\mu(4v^2+1) - 2\nu(u^2+v^2)] \\ uv[\tilde{\eta}u^2 - 4\mu(4v^2+1) - 2\nu(u^2+v^2)] & -\tilde{\eta}u^4 - \mu(4v^2+1)^2 - 4\nu u^2v^2 \end{pmatrix},$$

which entails

$$\det B(\psi) = -9\tilde{\eta}\mu(1 + v^2)v^2 - \tilde{\eta}\nu(1 + v^2)^2 + \mu\nu(2v^2 - 1)^2$$
$$= c_4(\tilde{\eta}, \mu, \nu)v^4 + c_2(\tilde{\eta}, \mu, \nu)v^2 + c_0(\tilde{\eta}, \mu, \nu)$$

(3.8)

with

$$c_4(\tilde{\eta}, \mu, \nu) = -9\tilde{\eta}\mu - \tilde{\eta}\nu + 4\mu\nu$$
$$c_2(\tilde{\eta}, \mu, \nu) = -9\tilde{\eta}\mu - 2\tilde{\eta}\nu - 4\mu\nu$$
$$c_0(\tilde{\eta}, \mu, \nu) = -\tilde{\eta}\nu + \mu\nu.$$

Assume first that (1.14) holds. In this case,

$$c_4(\tilde{\eta}, \mu, \nu) = 0$$
$$c_2(\tilde{\eta}, \mu, \nu) = -\nu(\tilde{\eta} + 8\mu)$$
$$c_0(\tilde{\eta}, \mu, \nu) = \nu(\mu - \tilde{\eta}),$$

thus $\det B(\psi^\pm) = \nu(-(\tilde{\eta} + 8\mu)v^2 + (\mu - \tilde{\eta})) < 0$ and, as according to (2.4) $\det A(\psi_-) > 0$, also

$$\det(B(\psi_-)^{-1}A(\psi_-)) < 0,$$

which proves (ii).

As $c_4(\tilde{\eta}, \mu, \nu_*(\eta, \mu)) = 0$ and $\partial c_4(\tilde{\eta}, \mu, \nu)/\partial\nu = 4\mu - \tilde{\eta} > 0$, the leading-order coefficient in (3.8) is positive whenever

$$\nu > \nu_*(\tilde{\eta}, \mu). \tag{3.9}$$

Assuming (3.9), we thus have $\det B(\psi_-) > 0$ for sufficiently large v, which implies that

$$\det B(\psi_-) > 0 \quad \text{for any } \tilde{q} < 1 \text{ sufficiently close to 1.}$$

(3.10)

Since $B(\psi)$ has negative trace and $A(\psi_-) > 0$, this shows that the eigenvalues of $B(\psi_-)^{-1}$ and thus those of $B(\psi_-)^{-1}A(\psi_-)$ are negative and thus

$$\psi_- \text{ is an attractor for these same } \tilde{q}; \tag{3.11}$$

this verifies (i). ∎

Proof of Assertion (ii) of Theorem 1:—This is a direct corollary of Assertion (i) of Lemma 2, as an attractor cannot be the α-limit of any other state; in particular, no heteroclinic orbit can originate in ψ_- for any of the $\tilde{q}$ in (3.10), (3.11). Cf. the second plot in Fig. 1. ∎

Remark 1. Assertion (ii) of Lemma 2 leaves the possibility that in the sharply causal case (1.13), $\psi_-(\tilde{q})$ is connected to $\psi_+(\tilde{q})$ by a heteroclinic orbit for *all* choices of $\tilde{q} \in (3/4, 1)$.

Remark 2. We conjecture that the phenomena established in this note have counterparts in generalizations of the original BDN model to other fluids as have meanwhile been considered in [4,18,19].

APPENDIX: STRICT AND SHARP CAUSALITY

For the reader's convenience, we briefly discuss the causality condition (1.7) following [1].[3] Fix $\tilde{\eta}, \mu, \nu > 0$. The dispersion relation of B being[4]

$$0 = \det\left(\lambda^2 B^{\alpha0\gamma0} + i\xi\lambda(B^{\alpha0\gamma1} + B^{\alpha1\gamma0}) - \xi^2 B^{\alpha1\gamma1}\right)$$
$$= \begin{vmatrix} 9\mu\lambda^2 - \nu\xi^2 & i\xi\lambda(3\mu + \nu) \\ i\xi\lambda(3\mu + \nu) & \nu\lambda^2 + (\tilde{\eta} - \mu)\xi^2 \end{vmatrix},$$

the speeds $\sigma = -i\lambda/\xi$ satisfy

$$0 = \pi(\sigma^2) = 9\mu\nu\sigma^4 - 3\mu(3\tilde{\eta} + 2\nu)\sigma^2 - \nu(\tilde{\eta} - \mu).$$

Looking at the signs of the coefficients and the discriminant of the polynomial π,

$$\Delta = \tilde{\eta}\{81\mu^2\tilde{\eta} + 36\mu\nu(3\mu + \nu)\},$$

one sees that the speeds have $\sigma^2 \geq 0$ if and only if $\mu \geq \tilde{\eta}$; we assume this now.

Observing that

$$\pi(1) = 4\mu\nu - (9\mu + \nu)\tilde{\eta},$$

one easily concludes that the speeds satisfy $\sigma^2 \leq 1$ if and only if

$$\nu \leq \left(\frac{1}{3\eta} - \frac{1}{9\mu}\right)^{-1}$$

holds, with $\sigma^2 = 1$ occurring exactly in case the last inequality holds with "=".

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[3]In their notation, (1.7) says that coefficients χ and λ, which correspond to what here is 3μ and ν, must satisfy $\chi = a_1\eta$, $\lambda \geq 3a_1\eta/(a_1 - 1)$ for some $a_1 \geq 4$. See [1], p. 10, and [4].

[4]We evaluate it in the fluid's rest frame. This means no loss of generality as these modes are neutral, cf. the proof of Proposition 1 in [20].

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