

Extension of Kodama vector and quasilocal quantities in three-dimensional axisymmetric spacetimes

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Spherically symmetric spacetimes admit the so-called Kodama vector, which provides a locally conserved current and a preferred time even for dynamical spacetime without any time translation symmetry. A charge associated with this conserved current leads to a quasilocal mass which agrees with the Misner-Sharp mass. In three dimensions, spherically symmetric spacetimes correspond to axisymmetric ones, while axisymmetry allows spacetimes to be rotating with angular momentum. We extend the notion of the Kodama vector to axisymmetric rotating spacetimes in three dimensions. We also define a quasilocal mass taking into account angular momentum in three-dimensional axisymmetric spacetimes.

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I. INTRODUCTION

Conservation laws of energy and momentum have played significant roles in physics. In gravitational theories, however, diffeomorphism invariance makes local notion of energy and momentum for gravitating system ambiguous. Thus, gravitational energy-momentum should be defined nonlocally in finite spacetime domain. Such various definitions of quasi-local mass and angular momentum have been proposed (see [1,2] and references therein).

On the other hand, it is well-known that if a spacetime admits a Killing vector field generating some symmetry, a locally conserved current vector can be obtained by contracting divergence-free, symmetric tensor such as energy-momentum tensor, or Einstein tensor, with the Killing vector. In particular, if a Killing vector is timelike, one can obtain a locally conserved energy current in terms of the Killing vector. In this case the spacetime is static or stationary with time translation symmetry, so that this conserved current cannot be applied to dynamics of gravitating systems.

Remarkably, in spherically symmetric spacetimes we can construct a local conserved energy current by using the so-called Kodama vector, even though the spacetime is dynamical without any time translation Killing vector [3]. This conserved current gives us quasilocal mass as Misner-Sharp mass [4,5]. Generalizations to higher-dimensional spacetime or other gravitational theories such as Gauss-Bonnet and $f(R)$ gravity were studied in Refs. [6–9].

In three dimensions, the counterparts of spherically symmetric spacetimes are axisymmetric (or circularly symmetric) ones. However, since axisymmetry does not rule out rotating systems, which are of physical interest significantly, we can take account of angular momentum

unlike spherical symmetry of higher-dimensional spacetimes. Such a rotating spacetime, in general, cannot be described by warped product metric. It turns out that in spacetimes with angular momentum the conventional Kodama vector does not always yield a conserved current. (The Kodama vector in warped product spacetimes was discussed in Ref. [10].)

In this paper we will extend the notion of the Kodama vector to spacetimes with nonzero angular momentum, which are not described by warped product metric. The organization of the paper is as follows. In Sec. II we consider general axisymmetric spacetimes in three dimensions, and explore a new vector that satisfies similar conditions to the Kodama vector in spherically symmetric cases. This new vector provides a locally conserved current. In Sec. III we define quasilocal quantities as charges associated with locally conserved currents.

While this paper was in preparation, Ref. [11] appeared where the authors have proposed the same quasilocal mass.

II. EXTENSION OF KODAMA VECTOR

Let us consider a three-dimensional axisymmetric spacetime with an axial Killing vector field given by

$$\psi^\mu = \left(\frac{\partial}{\partial \phi} \right)^\mu. \quad (1)$$

The spacetime metric can be generally written as

$$g_{\mu\nu} dx^\mu dx^\nu = h_{ij}(y) dy^i dy^j + r^2(y) [d\phi + a_i(y) dy^i]^2, \quad (2)$$

where h_{ij} is a two-dimensional metric on the orbit space of the Killing vector. Note that $a_i(y)$ has a gauge freedom as $a_i(y) \rightarrow a_i(y) + \partial_i \lambda(y)$, where $\lambda(y)$ is an arbitrary function

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on the orbit space. If we assume that the axial Killing vector field has closed orbits with $\phi \sim \phi + 2\pi$, the norm of the Killing vector provides the circumferential radius $r(y) = (\psi^\mu \psi_\mu)^{1/2}$. Greek indices indicate components of total three-dimensional spacetime, and Latin indices indicate two-dimensional orbit space components.

When ψ^μ is a hypersurface orthogonal Killing vector $\psi_{[\mu} \nabla_\nu \psi_{\lambda]} = 0$, that is, nontrivial $a_i(y)$ vanishes, the spacetime becomes simply a warped product. In this case we obtain the Kodama vector by conventional definition [3] as $\tilde{K}^i = -e^{ij} \nabla_j r$, where e^{ij} is the two-dimensional volume form associated with h_{ij} . This definition can be naturally adapted to a vector on the full three-dimensional spacetime as

$$\tilde{K}^\mu = -\frac{1}{r} \epsilon^{\mu\nu\alpha} \psi_\alpha \nabla_\nu r, \quad (3)$$

where $\epsilon_{\mu\nu\alpha}$ is the totally antisymmetric 3-tensor corresponding to a three-dimensional volume form. The metric of the two-dimensional orbit space is written as

$$h_{\mu\nu} = g_{\mu\nu} - \frac{1}{r^2} \psi_\mu \psi_\nu. \quad (4)$$

Even though the spacetime is not a warped product, \tilde{K}^μ can still satisfy the following properties: $\tilde{K}^\mu \nabla_\mu r = 0$ and $\nabla_\mu \tilde{K}^\mu = 0$ in a similar manner as the Kodama vector. However, it turns out that $G^{\mu\nu} \nabla_\mu \tilde{K}_\nu \neq 0$ for the three-dimensional Einstein tensor $G_{\mu\nu}$ of generic axisymmetric spacetimes.¹ Thus, $G^{\mu\nu} \tilde{K}_\nu$ is not always a locally conserved current unless the axisymmetric spacetime becomes a warped product such as nonrotating spacetimes.

Now, we define a new vector associated with the axial Killing vector as

$$K^\mu \equiv -\frac{1}{2} \epsilon^{\mu\alpha\beta} \nabla_\alpha \psi_\beta. \quad (5)$$

It is worth noting that this vector can be decomposed into normal and tangential components to ψ^μ as

$$K^\mu = -\frac{1}{r} \epsilon^{\mu\nu\alpha} \psi_\alpha \nabla_\nu r - \frac{\beta}{2r^2} \psi^\mu, \quad \beta = \epsilon^{\lambda\alpha\beta} \psi_\lambda \nabla_\alpha \psi_\beta. \quad (6)$$

Thus, two vectors K^μ and \tilde{K}^μ are related as $K^\mu = \tilde{K}^\mu - \beta \psi^\mu / (2r^2)$. If $\beta = 0$, which means that the axial Killing vector ψ^μ is hypersurface orthogonal and the spacetime becomes warped product, then both vectors are identical to just the Kodama vector.

¹For a specific spacetime, it is possible that $G^{\mu\nu} \nabla_\mu \tilde{K}_\nu = 0$ also holds while the spacetime is not warped product. We can see an example of such spacetimes in Ref. [12], which is a solution of a three dimensional higher derivative gravity.

In what follows we will examine some properties of K^μ . By definition, K^μ itself is (i) divergence-free: $\nabla_\mu K^\mu = 0$. We can see that (ii) K^μ is tangent to $r = \text{const.}$ surfaces, because

$$\begin{aligned} K^\mu \nabla_\mu r &= K^\mu \nabla_\mu (\psi^\nu \psi_\nu)^{1/2} = \frac{1}{r} \psi^\nu K^\mu \nabla_\mu \psi_\nu \\ &= \frac{1}{r} \psi^\nu K^\mu \epsilon_{\alpha\mu\nu} K^\alpha = 0. \end{aligned} \quad (7)$$

The Killing equation for the Killing vector ψ^μ gives us

$$\nabla_\mu \nabla_\alpha \psi_\beta = -R_{\alpha\beta\mu}{}^\nu \psi_\nu, \quad (8)$$

and, in three dimensions the Riemann tensor $R_{\alpha\beta\mu\nu}$ can be expressed as²

$$R_{\alpha\beta\mu\nu} = \epsilon_{\alpha\beta\gamma} \epsilon_{\mu\nu\lambda} G^{\gamma\lambda}. \quad (9)$$

Using the above equations, we find (iii) $G^{\mu\nu} \nabla_\mu K_\nu = 0$ as follows:

$$\begin{aligned} G^{\mu\nu} \nabla_\mu K_\nu &= -\frac{1}{2} \epsilon_\nu{}^{\alpha\beta} G^{\mu\nu} \nabla_\mu \nabla_\alpha \psi_\beta \\ &= \frac{1}{2} \epsilon_\nu{}^{\alpha\beta} G^{\mu\nu} R_{\alpha\beta\mu\lambda} \psi^\lambda \\ &= G^{\mu\nu} (\epsilon_{\nu\alpha\lambda} G_\mu{}^\alpha - \epsilon_{\nu\alpha\mu} R_\lambda{}^\alpha) \psi^\lambda \\ &= 0. \end{aligned} \quad (10)$$

As a result, we confirm that K^μ can always satisfy the same conditions (i) $\nabla_\mu K^\mu = 0$, (ii) $K^\mu \nabla_\mu r = 0$, and (iii) $G^{\mu\nu} \nabla_\mu K_\nu = 0$ as the Kodama vector in spherically symmetric spacetimes. This implies that the vector K^μ defined by (5) is natural extension of the Kodama vector to axisymmetric spacetime with angular momentum in three dimensions.

III. QUASILOCAL QUANTITIES

In spherically symmetric cases, a locally conserved current constructed from the Kodama vector yields a quasilocal mass as an associated charge [3,5]. In this

²In three-dimensional spacetime, we can directly show

$$\begin{aligned} \epsilon_{\alpha\beta\gamma} \epsilon^{\mu\nu\lambda} G^\gamma{}_\lambda &= -3! g_{[\alpha}{}^\mu g_\beta{}^\nu g_{\gamma]}{}^\lambda G^\gamma{}_\lambda \\ &= -(g_\alpha{}^\mu g_\beta{}^\nu g_\gamma{}^\lambda + g_\beta{}^\mu g_\gamma{}^\nu g_\alpha{}^\lambda + g_\gamma{}^\mu g_\alpha{}^\nu g_\beta{}^\lambda - g_\alpha{}^\mu g_\gamma{}^\nu g_\beta{}^\lambda \\ &\quad - g_\gamma{}^\mu g_\beta{}^\nu g_\alpha{}^\lambda - g_\beta{}^\mu g_\alpha{}^\nu g_\gamma{}^\lambda) G^\gamma{}_\lambda \\ &= g_\alpha{}^\mu R_\beta{}^\nu - g_\beta{}^\mu R_\alpha{}^\nu + R_\alpha{}^\mu g_\beta{}^\nu - R_\beta{}^\mu g_\alpha{}^\nu \\ &\quad - \frac{R}{2} (g_\alpha{}^\mu g_\beta{}^\nu - g_\beta{}^\mu g_\alpha{}^\nu) \\ &= R_{\alpha\beta}{}^{\mu\nu}. \end{aligned}$$

section, we will construct quasilocal quantities related to the extended Kodama vector.

In general, if a conserved current \mathcal{J}^μ satisfying $\nabla_\mu \mathcal{J}^\mu = 0$ is axisymmetric, we obtain

$$0 = \xi_\psi \mathcal{J}^\mu = \psi^\nu \nabla_\nu \mathcal{J}^\mu - \mathcal{J}^\nu \nabla_\nu \psi^\mu = 2\nabla_\nu (\psi^{[\nu} \mathcal{J}^{\mu]}), \quad (11)$$

where we have used the fact that ψ^μ and \mathcal{J}^μ are divergence-free. This implies that there exists a scalar function \mathcal{Q} such that

$$\nabla_\mu \mathcal{Q} = 2\pi \epsilon_{\mu\alpha\beta} \psi^\alpha \mathcal{J}^\beta. \quad (12)$$

By the Stokes theorem this scalar function is written in the integral form

$$\mathcal{Q}[C] \equiv \frac{1}{2\pi} \oint_C \mathcal{Q} d\phi = \int_S \mathcal{J}^\mu dS_\mu, \quad (13)$$

where C is a closed Killing orbit of the axial Killing vector and S is an arbitrary spacelike surface whose boundary is C . It turns out that scalar function \mathcal{Q} is a charge associated with each closed Killing orbit.

Since K^μ and $G^{\mu\nu} K_\nu$ are local conserved currents as we mentioned in the previous section, we obtain scalar functions $\psi^\mu \psi_\mu$ and $K^\mu K_\mu$ associated with each current, respectively. Indeed, we can explicitly confirm that

$$\begin{aligned} \nabla_\mu (\psi^\nu \psi_\nu) &= 2\psi^\nu \nabla_\mu \psi_\nu \\ &= 2K^\alpha \epsilon_{\alpha\mu\nu} \psi^\nu, \end{aligned} \quad (14)$$

and

$$\begin{aligned} \nabla_\mu (K^\nu K_\nu) &= -\nabla^\alpha \psi^\beta \nabla_\mu \nabla_\alpha \psi_\beta \\ &= \nabla^\alpha \psi^\beta R_{\alpha\beta\mu\nu} \psi^\nu \\ &= \nabla^\alpha \psi^\beta \epsilon_{\alpha\beta\gamma} \epsilon_{\mu\nu\lambda} G^{\gamma\lambda} \psi^\nu \\ &= -2K_\gamma G^{\gamma\lambda} \epsilon_{\lambda\mu\nu} \psi^\nu. \end{aligned} \quad (15)$$

Because any linear combination of local conserved currents is also locally conserved, we should adopt a charge associated with energy current as quasilocal mass. Now, we define a mass function m as

$$m \equiv \frac{1}{8G_3} (-\Lambda \psi^\nu \psi_\nu + K^\nu K_\nu), \quad (16)$$

where G_3 and Λ denote a three-dimensional gravitational constant and a cosmological constant, respectively. By using Eqs. (14) and (15), we obtain

$$\begin{aligned} \nabla_\mu m &= \frac{1}{4G_3} \epsilon_{\mu\nu\alpha} \psi^\alpha (G^{\nu\beta} + \Lambda g^{\nu\beta}) K_\beta \\ &= 2\pi \epsilon_{\mu\nu\alpha} \psi^\alpha T^{\nu\beta} K_\beta, \end{aligned} \quad (17)$$

where we have assumed the Einstein equation $G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_3 T_{\mu\nu}$ in the last line. Thus, m is a charge associated with the conserved energy current, $-T^{\mu\nu} K_\nu$, in Einstein gravity.

In addition, we have an angular-momentum function as

$$j \equiv \frac{1}{8G_3} \epsilon^{\alpha\mu\nu} \psi_\alpha \nabla_\mu \psi_\nu = -\frac{1}{4G_3} \psi_\mu K^\mu, \quad (18)$$

which satisfies

$$\begin{aligned} \nabla_\mu j &= -\frac{1}{4G_3} \epsilon_{\mu\nu\alpha} \psi^\alpha G^{\nu\beta} \psi_\beta \\ &= -2\pi \epsilon_{\mu\nu\alpha} \psi^\alpha T^{\nu\beta} \psi_\beta. \end{aligned} \quad (19)$$

This also implies that j is a charge associated with the conserved current, $T^{\mu\nu} \psi_\nu$, which is nothing but a well-known conserved current with respect to the axial Killing vector ψ^μ . Note that formula (18) of angular-momentum function agrees with that of the Komar angular momentum.

The above mass and angular-momentum functions are rewritten as

$$8G_3 m = -\Lambda r^2 - \nabla_\mu r \nabla^\mu r + \frac{(4G_3 j)^2}{r^2}. \quad (20)$$

It turns out that the mass function consists of the conventional Misner-Sharp mass in spherically symmetric cases and the angular-momentum term.³ In the Bañados-Teitelboim-Zanelli (BTZ) solution [13], which is stationary, axisymmetric vacuum solution with a negative cosmological constant ($\Lambda < 0$), the spacetime metric is given by

$$\begin{aligned} ds^2 &= -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 \left(d\phi - \frac{4G_3 J}{r^2} dt \right)^2, \\ f(r) &= -\Lambda r^2 - 8G_3 M + \frac{16G_3^2 J^2}{r^2}, \end{aligned} \quad (21)$$

where M and J denote mass and angular-momentum parameters, respectively. In this case quasilocal mass m and quasilocal angular momentum j defined by Eqs. (16) and (18) become constants equal to the mass and angular-momentum parameters everywhere as $m = M$ and $j = J$. Moreover, the extended Kodama vector coincides with usual time-translational Killing vector, $K^\mu = (\partial/\partial t)^\mu$.

IV. CONCLUSION

In this paper we have proposed extending the notion of the Kodama vector to axisymmetric, three-dimensional

³Precisely speaking, we can add an arbitrary constant term to definition of quasilocal mass (16). This constant is related to a value of mass in vacuum. Here, we have set $m = -1/8G_3$ in pure AdS_3 .

spacetimes with nonzero angular momentum. The extended Kodama vector $K^\mu = -\frac{1}{2}e^{\mu\alpha\beta}\nabla_\alpha\psi_\beta$, which is associated with an axial Killing vector ψ^v , satisfies the same conditions as the conventional Kodama vector in spherically symmetric cases. In particular, satisfying condition $G^{\mu\nu}\nabla_\mu K_\nu = 0$ for the Einstein tensor makes $G^{\mu\nu}K_\nu$ divergence free for generic axisymmetric spacetimes. This provides a locally conserved energy current assuming the Einstein equation. We have shown that quasilocal mass like the Misner-Sharp mass can be defined as a charge associated with the locally conserved energy current. This quasilocal mass contains angular momentum of rotating spacetime.

These quasilocal quantities and conservation laws were also pointed out in Ref. [11], where they were useful for reducing the Einstein equations in numerical simulations of axisymmetric Einstein-perfect fluid systems. What we should emphasize in the present paper is that the extended Kodama vector guarantees the conserved currents and provides the quasilocal quantities as conserved charges. The above properties originate from purely geometrical identities for three-dimensional axisymmetric spacetimes independent of field equations of a specific gravitational theory. This means that any axisymmetric, three-dimensional spacetime can admit a conserved current associated with the extended Kodama vector even in gravitational theories other

than the Einstein gravity. The field equations will give physical meaning of the conserved current by connecting with energy-momentum tensor of matter in each gravitational theory.

It is fascinating to generalize the extended Kodama vector to higher-dimensional axisymmetric spacetimes, or spacetimes different from warped product, other than three dimensions. It does not seem to be so easy, because geometrical properties in three-dimensional spacetimes are necessary in order to explore the extended Kodama vector.⁴ However, once one can reduce to a three-dimensional axisymmetric spacetime, it is expected that there are certain extended Kodama vector and locally conserved current associated with it. For example, if a cylindrical spacetime in four dimensions are rotating, there is expected to be a similar conserved current.

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⁴It was shown that, if a three-dimensional spacetime admits a non-null Killing vector field, the spacetime metric and the Ricci tensors are determined by two scalars characterizing the Killing vector in Ref. [14].

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