General relativistic effects in neutron star electrodynamics

Bobur Turimov,[1,2,3,4,*] Zdeněk Stuchlík,[1,†] Javlon Rayimbayev,[2,3,5,‡] and Ahmadjon Abdujabbarov[6,3,5,§]

[1]*Research Centre for Theoretical Physics and Astrophysics, Institute of Physics in Opava, Silesian University in Opava, Bezručovo náměstf 13, CZ-74601 Opava, Czech Republic*
[2]*Akfa University, Kichik Halqa Yuli Street 17, Tashkent 100095, Uzbekistan*
[3]*Ulugh Beg Astronomical Institute, Astronomicheskaya str. 33, Tashkent 100052, Uzbekistan*
[4]*Webster University in Tashkent, Alisher Navoiy 13, Tashkent 100011, Uzbekistan*
[5]*National University of Uzbekistan, Tashkent 100174, Uzbekistan*
[6]*Shanghai Astronomical Observatory, 80 Nandan Road, Shanghai 200030, People's Republic of China*

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The paper explores general relativistic (GR) effects in electromagnetic fields of the rotating neutron star. The star has been assumed as a perfect conductor with infinity electric conductivity, i.e., $\sigma \to \infty$. The analytical form of general relativistic Maxwell's equations for the electromagnetic fields has been derived in the presence of gravity. It is shown that six components of the electromagnetic fields can be expressed in terms of two profile functions. It has been shown that the Lense-Thirring term plays an important role in the generation of the multipole electromagnetic fields. We obtain that the rotation of the quadrupole magnetic field can create the dipole electric field. Moreover, we have also shown that GR effects are reasonably large for the highest order of electromagnetic multipole. Finally, as a test of our results, we investigate the effect of the Lense-Thirring term on the luminosity of magnetodipolar radiations.

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I. INTRODUCTION

Recent observation shows that general relativistic (GR) effects are very important in the electromagnetic field of radio pulsars [1]. That is why in both theoretical and astrophysical points of view, it is interesting to study the electromagnetic fields in this vicinity of gravitational compact objects such as neutron stars and black holes in a strong gravity regime. Extensive observations of radio pulsars and soft gamma-ray repeaters have been shown where the surface magnetic field about 10^{12} G for a typical neutron star, while it may reach 10^{15} G for magnetars observed as soft gamma-ray repeaters and anomalous x-ray pulsars [2–5]. Therefore, the comparison of the evolution of magnetic fields and the rotation spin-down observed in neutron stars with those modeled and theoretically predicted provides a great challenge and powerful tool to get the constraints on the neutron star properties in the extreme physics regime and conditions.

The first time solution for the exterior electromagnetic fields of a rotating magnetized sphere in the Newtonian framework has been obtained by Deutsch [6] and interior fields are studied by many authors, for example, in the paper of [7]. The general-relativistic correction to electromagnetic fields of outside magnetized compact gravitational objects has been investigated by Ginzburg and Ozernoy [8] and later has been extended by several authors [9–17], while the effects from the alternative theory on the electromagnetic fields of relativistic neutron stars and black holes have been studied in [18–25]. A semianalytical estimation for the magnetodipole radiation [26] and the oscillations of a relativistic magnetized star including damping due to heating have also been studied [27]. The time evaluation dipole magnetic field [28,29] and multipole magnetic field [30–32] at the surface of the magnetized neutron star has been studied. Decaying of the magnetic field through the Hall drift in stellar crusts has been studied in [33]. In Ref. [34], the effect of the magnetic field in deformation of relativistic stars due to the magnetic stress has been investigated.

In Ref. [35], the general relativistic form of the Grad-Shafranov (GS) equation for an ideal magnetohydrodynamics system in stationary axially symmetric spacetimes has been discussed. It has been investigated in the toroidal and poloidal magnetic fields in stationary axially symmetric configurations of magnetized stars in the framework of GR ideal magnetohydrodynamics in [36,37].

The electromagnetic signal detected from radio pulsars is mainly due to the magnetodipolar radiation from the rotating magnetized compact star. The energy loss due to the electromagnetic radiation causes the spin-down of the rotating relativistic star [38,39]. The structure of the pulsar magnetosphere and related astrophysical processes

*bturimov@astrin.uz
†zdenek.stuchlik@fpf.slu.cz
‡javlon@astrin.uz
§ahmadjon@astrin.uz

in it have been widely studied; see e.g., [12,24,40–45]. In Ref. [46] the x-ray light curve of the misaligned pulsar form realistic magnetospheric model has been studied. The influence of external magnetic fields and cosmic repulsion on accretion disks rotating around rotating black holes has been studied in [47].

The paper is organized as follows: In Sec. III we construct the stellar model of the slowly rotating relativistic magnetized neutron star. In Sec. IV we study general relativistic Maxwell equations for electromagnetic fields in the background metric of the rotating magnetized relativistic star. We expand solutions for electromagnetic fields in terms of spherical harmonics and write coupled radial equations for profile functions of the electric field and the magnetic fields. Section V is devoted to the solution of Maxwell equations for the electromagnetic fields in the case of a slowly rotating limit (i.e., $\omega^2 \to 0$ and $\Omega^2 \to 0$) and presents analytical expressions for the components of the electric and the magnetic fields. In Sec. VI we present an approximate solution of the Maxwell equation in a wave zone (far from the source) for in the power of compactness of the star. In Sec. VII we investigate in the interior solutions of the general relativistic Maxwell equation the electric and magnetic fields for interior Schwarzschild spacetime. Finally in Sec. VIII we summarize and discuss our results. We use in this paper a system of units in which $c = 1 = G$, a spacelike signature $(-,+,+,+)$, a spherical coordinate system (t,r,θ,ϕ), and Greek letters (running from 0 to 3) for four-dimensional spacetime tensor components, while Latin letters (running from 1 to 3) will be employed for three-dimensional spatial tensor components.

II. FORMALISM

In this section we give guidelines to investigate Maxwell equations in arbitrary axially symmetric spacetime, which can be given by the following metric: $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$ in spherical coordinates $x^\alpha = (t,r,\theta,\phi)$, where the metric tensor is represented as

$$g_{\alpha\beta} = \begin{pmatrix} g_{tt} & 0 & 0 & g_{t\phi} \\ 0 & g_{rr} & 0 & 0 \\ 0 & 0 & g_{\theta\theta} & 0 \\ g_{t\phi} & 0 & 0 & g_{\phi\phi} \end{pmatrix}, \tag{1}$$

$$g^{\alpha\beta} = \begin{pmatrix} g_{\phi\phi}/\tilde{g} & 0 & 0 & g_{t\phi}/\tilde{g} \\ 0 & 1/g_{rr} & 0 & 0 \\ 0 & 0 & 1/g_{\theta\theta} & 0 \\ g_{t\phi}/\tilde{g} & 0 & 0 & g_{tt}/\tilde{g} \end{pmatrix}, \tag{2}$$

where $\tilde{g} = g_{tt}g_{\phi\phi} - g_{t\phi}^2$ and the determinant of the metric tensor is given by $g = \tilde{g}g_{rr}g_{\theta\theta}$. Notice that the spacetime metric tensor (1) to be taken as a function of r and θ, i.e., $g_{\alpha\beta} = g_{\alpha\beta}(r,\theta)$, which indicates the spacetime is stationary and axially symmetric.

The general form of Maxwell equations in a curved space are

$$\frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}F^{\alpha\beta}) = -4\pi J^\beta, \tag{3}$$

$$\frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g})^*F^{\alpha\beta} = 0, \tag{4}$$

$$^*F^{\alpha\beta} = \frac{1}{2\sqrt{-g}}\epsilon_{\alpha\beta\mu\nu}F^{\mu\nu}, \tag{5}$$

where J^α is the source of the electromagnetic fields. $F_{\alpha\beta}$ and $^*F_{\alpha\beta}$ are, respectively, the electromagnetic field tensor and its dual partner which can be expressed in terms of the electric and the magnetic fields in the form

$$F_{\alpha\beta} = u_\alpha E_\beta - u_\beta E_\alpha + \frac{1}{\sqrt{-g}}\epsilon_{\alpha\beta\mu\nu}u^\mu B^\nu, \tag{6}$$

$$^*F_{\alpha\beta} = u_\alpha B_\beta - u_\beta B_\alpha + \frac{1}{\sqrt{-g}}\epsilon_{\alpha\beta\mu\nu}u^\mu E^\nu, \tag{7}$$

where u^α is the four-velocity in the zero angular momentum observer (ZAMO) frame of reference. Notice that the four-velocity satisfies the following normalization condition $u_\alpha u^\alpha = -1$ which allows one to write the components of the electromagnetic field in the form

$$E_\alpha = F_{\alpha\beta}u^\beta = \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}u^{\beta *}F^{\mu\nu}, \tag{8}$$

$$B_\alpha = {}^*F_{\alpha\beta}u^\beta = \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}u^\beta F^{\mu\nu}, \tag{9}$$

where $\epsilon_{\alpha\beta\mu\nu}$ is the Levi-Civita tensor in the four-dimensional space.

In the present paper, we present Maxwell equations (3) in terms of the electromagnetic fields. To do this we use the fundamental invariants in classical electrodynamics such as $F_{\alpha\beta}F^{\alpha\beta}$ and $F_{\alpha\beta}{}^*F^{\alpha\beta}$. Recalling the equation (6) and taking into account normalization of the four-velocity, i.e., $u_\alpha u^\alpha = -1$, the fundamental invariants of the electrodynamics can be expressed as

$$F_{\alpha\beta}F^{\alpha\beta} = 2(B_iB^i - E_iE^i) = 2(\mathbf{B}^2 - \mathbf{E}^2), \tag{10}$$

$$F_{\alpha\beta}{}^*F^{\alpha\beta} = -4E_iB^i = -4\mathbf{E}\cdot\mathbf{B}, \tag{11}$$

where $\mathbf{E} = (E^{\hat{r}}, E^{\hat{\theta}}, E^{\hat{\phi}})$ and $\mathbf{B} = (B^{\hat{r}}, B^{\hat{\theta}}, B^{\hat{\phi}})$ are, respectively, the electric and magnetic fields observed in the ZAMO frame and $E^{\hat{i}}$ and $B^{\hat{i}}$ in brackets are orthogonal components of each field. Using the relation (10) we can express the components of the electromagnetic fields in terms of the electromagnetic field tensor through the metric tensor in the form

$$B^{\hat{i}} = \frac{1}{2}\epsilon_{ijk}\sqrt{g_{jj}g_{kk}}F^{jk} = \frac{1}{2}\epsilon_{ijk}\sqrt{g^{jj}g^{kk}}F_{jk}, \tag{12}$$

$$\begin{aligned} E^{\hat{i}} &= \sqrt{-g_{ii}F^{it}(g_{tt}F^{it} + g_{t\phi}F^{i\phi})} \\ &= \sqrt{-g^{ii}F_{it}(g^{tt}F_{it} + g^{t\phi}F_{i\phi})}, \end{aligned} \tag{13}$$

where ϵ_{ijk} is the Levi-Civita tensor in three-dimensional space. Here one has to emphasize that very similar definitions as shown in (12) and (13) can be found in Ref. [8] in the case of spherically symmetric and static spacetime. As we mentioned before, the main idea of introducing the relations in (12) and (13) is to express Maxwell equations in terms of the components of the electromagnetic fields in the background of an arbitrary axially symmetric spacetime. As we see from the expression (12), the components magnetic fields are linearly proportional to the electromagnetic field tensor while Eq. (13) shows that due to nondiagonal components of the metric tensor (or due to frame dragging) the components of the electric field will be completed, in particular, radial $E^{\hat{r}}$ and tangential $E^{\hat{\theta}}$ components; however, azimuthal component $E^{\hat{\phi}}$ is linearly proportional to the electromagnetic field tensor. Taking into account Eq. (12), Eq. (13) can be written as

$$F^{rt2} - \frac{g_{t\phi}B^{\hat{\theta}}}{g_{tt}\sqrt{g_{rr}g_{\phi\phi}}}F^{rt} = -\frac{E^{\hat{r}2}}{g_{tt}g_{rr}}, \tag{14}$$

$$F^{\theta t2} + \frac{g_{t\phi}B^{\hat{r}}}{g_{tt}\sqrt{g_{rr}g_{\phi\phi}}}F^{\theta t} = -\frac{E^{\hat{\theta}2}}{g_{tt}g_{\theta\theta}}, \tag{15}$$

and

$$F^2_{rt} - \frac{g^{t\phi}B^{\hat{\theta}}}{g^{tt}\sqrt{g^{rr}g^{\phi\phi}}}F_{rt} = -\frac{E^{\hat{r}2}}{g^{tt}g^{rr}}, \tag{16}$$

$$F^2_{\theta t} + \frac{g^{t\phi}B^{\hat{r}}}{g^{tt}\sqrt{g^{rr}g^{\phi\phi}}}F_{\theta t} = -\frac{E^{\hat{\theta}2}}{g^{tt}g^{\theta\theta}}, \tag{17}$$

which are quadratic equations with respect to F_{rt}, F^{rt} and $F_{\theta t}$, $F^{\theta t}$. Hereafter performing simple algebraic calculations, we can easily show the following relation:

$$\frac{g^{t\phi}}{\sqrt{-g^{tt}g^{\phi\phi}}} = \frac{g_{t\phi}}{\sqrt{-g_{tt}g_{\phi\phi}}}, \tag{18}$$

in arbitrary axially symmetric spactime.

Finally, taking all facts above the components of the electromagnetic field tensor are expressed as

$$F_{\alpha\beta} = \begin{pmatrix} 0 & -\frac{E^{\hat{r}}\zeta_r}{\sqrt{-g^{tt}g^{rr}}} & -\frac{E^{\hat{\theta}}\zeta_\theta}{\sqrt{-g^{tt}g^{\theta\theta}}} & -\frac{E^{\hat{\phi}}}{\sqrt{-g^{tt}g^{\phi\phi}}} \\ \frac{E^{\hat{r}}\zeta_r}{\sqrt{-g^{tt}g^{rr}}} & 0 & \frac{B^{\hat{\phi}}}{\sqrt{g^{rr}g^{\theta\theta}}} & -\frac{B^{\hat{\theta}}}{\sqrt{g^{rr}g^{\phi\phi}}} \\ \frac{E^{\hat{\theta}}\zeta_\theta}{\sqrt{-g^{tt}g^{\theta\theta}}} & -\frac{B^{\hat{\phi}}}{\sqrt{g^{rr}g^{\theta\theta}}} & 0 & \frac{B^{\hat{r}}}{\sqrt{g^{\theta\theta}g^{\phi\phi}}} \\ \frac{E^{\hat{\phi}}}{\sqrt{-g^{tt}g^{\phi\phi}}} & \frac{B^{\hat{\theta}}}{\sqrt{g^{rr}g^{\phi\phi}}} & -\frac{B^{\hat{r}}}{\sqrt{g^{\theta\theta}g^{\phi\phi}}} & 0 \end{pmatrix}, \tag{19}$$

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -\frac{E^{\hat{r}}\zeta_r}{\sqrt{-g_{tt}g_{rr}}} & -\frac{E^{\hat{\theta}}\zeta_\theta}{\sqrt{-g_{tt}g_{\theta\theta}}} & -\frac{E^{\hat{\phi}}}{\sqrt{-g_{tt}g_{\phi\phi}}} \\ \frac{E^{\hat{r}}\zeta_r}{\sqrt{-g_{tt}g_{rr}}} & 0 & \frac{B^{\hat{\phi}}}{\sqrt{g_{rr}g_{\theta\theta}}} & -\frac{B^{\hat{\theta}}}{\sqrt{g_{rr}g_{\phi\phi}}} \\ \frac{E^{\hat{\theta}}\zeta_\theta}{\sqrt{-g_{tt}g_{\theta\theta}}} & -\frac{B^{\hat{\phi}}}{\sqrt{g_{rr}g_{\theta\theta}}} & 0 & \frac{B^{\hat{r}}}{\sqrt{g_{\theta\theta}g_{\phi\phi}}} \\ \frac{E^{\hat{\phi}}}{\sqrt{-g_{tt}g_{\phi\phi}}} & \frac{B^{\hat{\theta}}}{\sqrt{g_{rr}g_{\phi\phi}}} & -\frac{B^{\hat{r}}}{\sqrt{g_{\theta\theta}g_{\phi\phi}}} & 0 \end{pmatrix}, \tag{20}$$

where dimensionless functions ζ_r and ζ_θ are defined as

$$\zeta_r = \sqrt{1 + \left(\frac{g_{t\phi}}{2\sqrt{-g_{tt}g_{\phi\phi}}}\right)^2\left(\frac{B^{\hat{\theta}}}{E^{\hat{r}}}\right)^2} + \frac{g_{t\phi}}{2\sqrt{-g_{tt}g_{\phi\phi}}}\frac{B^{\hat{\theta}}}{E^{\hat{r}}}, \tag{21}$$

$$\zeta_\theta = \sqrt{1 - \left(\frac{g_{t\phi}}{2\sqrt{-g_{tt}g_{\phi\phi}}}\right)^2\left(\frac{B^{\hat{r}}}{E^{\hat{\theta}}}\right)^2} - \frac{g_{t\phi}}{2\sqrt{-g_{tt}g_{\phi\phi}}}\frac{B^{\hat{r}}}{E^{\hat{\theta}}}. \tag{22}$$

In the case of nonrotating spacetime, i.e., $g_{t\phi} = 0$, one can obtain $\zeta_r = \zeta_\theta = 1$, while for the slow rotation limit of the spacetime, i.e., $g^2_{t\phi} \to 0$, expressions (21) can be reduced as

$$\zeta_r = 1 + \frac{g_{t\phi}}{\sqrt{-g_{tt}g_{\phi\phi}}}\frac{B^{\hat{\theta}}}{E^{\hat{r}}}, \quad \zeta_\theta = 1 - \frac{g_{t\phi}}{\sqrt{-g_{tt}g_{\phi\phi}}}\frac{B^{\hat{r}}}{E^{\hat{\theta}}}. \tag{23}$$

Taking into account all facts above the Maxwell equations in (3) can be expressed in terms of the electromagnetic fields in the ZAMO reference frame as

$$\partial_r\left(\frac{E^{\hat{r}}\zeta_r}{\sqrt{g^{\theta\theta}g^{\phi\phi}}}\right) + \partial_\theta\left(\frac{E^{\hat{\theta}}\zeta_\theta}{\sqrt{g^{rr}g^{\phi\phi}}}\right) + \partial_\phi\left(\frac{E^{\hat{\phi}}}{\sqrt{g^{rr}g^{\theta\theta}}}\right) = 4\pi\sqrt{-g}J^t, \tag{24}$$

$$\partial_t\left(\frac{E^{\hat{r}}\zeta_r}{\sqrt{g^{\theta\theta}g^{\phi\phi}}}\right) = \partial_\theta\left(\frac{B^{\hat{\phi}}}{\sqrt{-g^{tt}g^{\phi\phi}}}\right) - \partial_\phi\left(\frac{B^{\hat{\theta}}}{\sqrt{-g^{tt}g^{\theta\theta}}}\right) + 4\pi\sqrt{-g}J^r, \tag{25}$$

$$\partial_t\left(\frac{E^{\hat{\theta}}\zeta_\theta}{\sqrt{g^{rr}g^{\phi\phi}}}\right) = \partial_\phi\left(\frac{B^{\hat{r}}}{\sqrt{-g^{tt}g^{rr}}}\right) - \partial_r\left(\frac{B^{\hat{\phi}}}{\sqrt{-g^{tt}g^{\phi\phi}}}\right) + 4\pi\sqrt{-g}J^\theta, \tag{26}$$

$$\partial_t\left(\frac{E^{\hat{\phi}}}{\sqrt{g^{rr}g^{\theta\theta}}}\right) = \partial_r\left(\frac{B^{\hat{\theta}}}{\sqrt{-g^{tt}g^{\theta\theta}}}\right) - \partial_\theta\left(\frac{B^{\hat{r}}}{\sqrt{-g^{tt}g^{rr}}}\right) + 4\pi\sqrt{-g}J^\phi, \tag{27}$$

and

$$\partial_r\left(\frac{B^{\hat{r}}}{\sqrt{g^{\theta\theta}g^{\phi\phi}}}\right) + \partial_\theta\left(\frac{B^{\hat{\theta}}}{\sqrt{g^{rr}g^{\phi\phi}}}\right) + \partial_\phi\left(\frac{B^{\hat{\phi}}}{\sqrt{g^{rr}g^{\theta\theta}}}\right) = 0, \tag{28}$$

$$\partial_t\left(\frac{B^{\hat{r}}}{\sqrt{g^{\theta\theta}g^{\phi\phi}}}\right) = \partial_\phi\left(\frac{E^{\hat{\theta}}\zeta_\theta}{\sqrt{-g^{tt}g^{\theta\theta}}}\right) - \partial_\theta\left(\frac{E^{\hat{\phi}}}{\sqrt{-g^{tt}g^{\phi\phi}}}\right), \tag{29}$$

$$\partial_t\left(\frac{B^{\hat{\theta}}}{\sqrt{g^{rr}g^{\phi\phi}}}\right) = \partial_r\left(\frac{E^{\hat{\phi}}}{\sqrt{-g^{tt}g^{\phi\phi}}}\right) - \partial_\phi\left(\frac{E^{\hat{r}}\zeta_r}{\sqrt{-g^{tt}g^{rr}}}\right), \tag{30}$$

$$\partial_t\left(\frac{B^{\hat{\phi}}}{\sqrt{g^{rr}g^{\theta\theta}}}\right) = \partial_\theta\left(\frac{E^{\hat{r}}\zeta_r}{\sqrt{-g^{tt}g^{rr}}}\right) - \partial_r\left(\frac{E^{\hat{\theta}}\zeta_\theta}{\sqrt{-g^{tt}g^{\theta\theta}}}\right). \tag{31}$$

As we can see from Eqs. (24)–(31), the general relativistic Maxwell equations can be expressed in terms of measurable components of the electromagnetic fields and the covariant form of the metric tensor. So far we have derived the general form of relativistic Maxwell equations in curved spacetime described by arbitrary metric coefficients. We now focus on investigating the source terms in Eqs. (24)–(27). First of all, one has to mention that the four-current satisfies the continuity equation $\nabla_\alpha J^\alpha = 0$, and the square of the four-current vector reads

$$\mathbf{J}^2 = J^{\hat{\alpha}2} = g_{\alpha\beta}J^\alpha J^\beta, \tag{32}$$

which is useful to decompose the four-current vector into measurable quantities. Hereafter making simple algebra, one can easily obtain the following expressions:

$$J^r = J^{\hat{r}}\frac{1}{\sqrt{g_{rr}}}, \qquad J^\theta = J^{\hat{\theta}}\frac{1}{\sqrt{g_{\theta\theta}}}, \tag{33}$$

$$J^\phi = \frac{\sqrt{g_{tt}}J^{\hat{\phi}2}}{\sqrt{\mathcal{J} + \frac{1}{2}g_{t\phi}^2(J^{\hat{t}2} - J^{\hat{\phi}2}) + g_{tt}g_{\phi\phi}J^{\hat{\phi}^2}}}, \tag{34}$$

$$J^t = \frac{\sqrt{g_{\phi\phi}}J^{\hat{t}2}}{\sqrt{\mathcal{J} + \frac{1}{2}g_{t\phi}^2(J^{\hat{\phi}2} - J^{\hat{t}2}) + g_{tt}g_{\phi\phi}J^{\hat{t}2}}}, \tag{35}$$

where $\mathcal{J}$ is defined as

$$\mathcal{J} = g_{t\phi}\sqrt{g_{tt}g_{\phi\phi}(J^{\hat{t}}J^{\hat{\phi}})^2 + \frac{1}{2}g_{t\phi}^2(J^{\hat{t}2} - J^{\hat{\phi}2})^2}. \tag{36}$$

In the case of the nonrotating spacetime, i.e., $g_{t\phi} = 0$, we obtain

$$J^\alpha = \left(\frac{J^{\hat{t}}}{\sqrt{-g_{tt}}}, \frac{J^{\hat{r}}}{\sqrt{g_{rr}}}, \frac{J^{\hat{\theta}}}{\sqrt{g_{\theta\theta}}}, \frac{J^{\hat{\phi}}}{\sqrt{g_{\phi\phi}}}\right), \tag{37}$$

while in the slowly rotating limit, one has

$$J^t = \frac{J^{\hat{t}}}{\sqrt{-g_{tt}}} + \frac{g_{t\phi}}{2g_{tt}}\frac{J^{\hat{\phi}}}{\sqrt{g_{\phi\phi}}},$$
$$J^r = \frac{J^{\hat{r}}}{\sqrt{g_{rr}}}, \qquad J^\theta = \frac{J^{\hat{\theta}}}{\sqrt{g_{\theta\theta}}},$$
$$J^\phi = \frac{J^{\hat{\phi}}}{\sqrt{g_{\phi\phi}}} - \frac{g_{t\phi}}{2g_{\phi\phi}}\frac{J^{\hat{t}}}{\sqrt{-g_{tt}}}. \tag{38}$$

III. STELLAR MODEL

In this section, we focus on constructing the stellar model, which will be very useful in further calculations. We first assume that the energy of the electromagnetic field is much smaller than the gravitational one so that the electromagnetic field does not change the spacetime of the relativistic star. The fractional energies of the electromagnetic and the gravitational fields of highly magnetized magnetars are roughly estimated as [26,48]

$$\frac{B^2}{8\pi\bar{\rho}c^2} \simeq 2.2\times 10^{-7}\left(\frac{B}{10^{15}\ \mathrm{G}}\right)^2\left(\frac{M}{1.4\ M_\odot}\right)^{-1}\left(\frac{R}{15\ \mathrm{km}}\right)^{-3}, \tag{39}$$

where M and R are, respectively, the total mass and radii of the NS, B is the surface magnetic field strength, and $\bar{\rho}$ is an averaged mass density of NS. From Eq. (39) one can see that indeed effects of electromagnetic fields are very small in order to change the spacetime geometry.

In fact, most NS have their own spin, and the question arises of how the spin of NS affects spacetime. To check this argument, the ratio of spin parameter a and mass M of NS can be estimated as

$$\frac{a}{M} = \frac{I\Omega}{M^2} \simeq 0.15\left(\frac{P}{1\ \mathrm{ms}}\right)^{-1}\left(\frac{M}{1.4\,M_\odot}\right)^{-1}\left(\frac{R}{15\,\mathrm{km}}\right)^2, \tag{40}$$

which means $a^2/M^2 \simeq 0.026$ for typical NS, so it gives $\sim 3\%$ contribution even for rapidly rotating NS which is negligibly small. In Eq. (40) the quantities J and I are, respectively, the angular momentum and moment of inertia of NS, P is the period of the star which can be related to its angular velocity as $\Omega = 2\pi/P$. That is why in the slowly rotating limit, the spacetime metric for NS, in a coordinate system $x^\alpha = (t, r, \theta, \phi)$, can be expressed as [14,49–51]

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) - 2\omega(r)r^2\sin^2\theta dt d\phi, \quad (41)$$

where $\Phi(r)$ and $\Lambda(r)$ are lapse functions which can be found by solving Einstein field equations:

$$8\pi\rho(r) = e^{-2\Lambda(r)}\left(\frac{2\Lambda'(r)}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2}, \quad (42)$$

$$8\pi p(r) = e^{-2\Lambda(r)}\left(\frac{2\Phi'(r)}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2}, \quad (43)$$

$$8\pi p(r) = e^{-2\Lambda(r)}\left(\Phi''(r) + \Phi'^2(r) - \Phi'(r)\Lambda'(r) + \frac{\Phi'^2(r) - \Lambda'^2(r)}{r}\right), \quad (44)$$

where $\rho(r)$ and $p(r)$ are mass density and pressure, respectively. The function $\omega(r)$ in Eq. (41) is the Lense-Thirring angular velocity, which represents the angular velocity of the dragging of inertial frames. The radial dependence of $\omega(r)$ has to be found as the solution of the differential equation (see, for example, [50])

$$\frac{d}{dr}\left(r^4 j \frac{d\bar{\omega}}{dr}\right) + 4r^3 \frac{dj}{dr}\bar{\omega} = 0, \qquad j = e^{-(\Phi+\Lambda)}, \quad (45)$$

where $\bar{\omega}(r) = \Omega - \omega(r)$ is the angular velocity of the internal fluid as measured from the local free-falling frame. In the slowly rotating limit, the moment of inertia of a uniformly rotating star is calculated by

$$I = \frac{J}{\Omega} = \frac{8\pi}{3}\int_0^R r^4 dr j(r)(\rho(r) + p(r))\frac{\bar{\omega}(r)}{\Omega}. \quad (46)$$

In the vacuum region, where matter density and pressure are zero, i.e., $\rho = p = 0$, the lapse functions can be expressed as

$$e^{2\Phi(r)} = e^{-2\Lambda(r)} = f(r) = 1 - \frac{2M}{r}, \quad r \geq R, \quad (47)$$

and the Lense-Thirring angular velocity $\omega(r)$ takes a form

$$\omega(r) = \frac{2J}{r^3}, \qquad r \geq R. \quad (48)$$

It is assumed that the shape of a star spherically symmetric in the slow rotation limit and the deformation due to the stellar rotation is negligibly small. Outside the star is assumed as an electrical vacuum, i.e., $\rho_e = \mathbf{j} = 0$, where ρ_e is the electric charge density and $\mathbf{j}$ is the electric current, while inside the star it has been assumed it consists of the perfect fluid with infinity electric conductivity, i.e., $\sigma \to \infty$. In this section we build up basic stellar model assumptions and in the next section we will concentrate on the Maxwell equations in the external spacetime of the slowly rotating magnetized neutron star.

IV. GENERAL RELATIVISTIC MAXWELL EQUATIONS

In this section, we investigate the general relativistic Maxwell equations in background spacetime (41) of a slowly rotating neutron star. In the slowly rotating limiting case of spacetime, i.e., $g^2_{t\phi} \to 0$, Maxwell equations (24)–(31) for the components of the electric $(E^{\hat{r}}, E^{\theta}, E^{\phi})$ and the magnetic $(B^{\hat{r}}, B^{\theta}, B^{\phi})$ fields in the ZAMO reference frame can be expressed as

$$\frac{e^{-\Lambda}}{r}\partial_r(r^2 B^{\hat{r}}) + \frac{1}{\sin\theta}[\partial_\theta(\sin\theta B^{\hat{\theta}}) + \partial_\phi B^{\hat{\phi}}] = 0, \quad (49a)$$

$$(\partial_t + \omega\partial_\phi)B^{\hat{r}} = \frac{e^{\Phi}}{r\sin\theta}[\partial_\phi E^{\hat{\theta}} - \partial_\theta(\sin\theta E^{\hat{\phi}})], \quad (49b)$$

$$(\partial_t + \omega\partial_\phi)B^{\hat{\theta}} = \frac{e^{-\Lambda}}{r}\partial_r(e^{\Phi} r E^{\hat{\phi}}) - \frac{e^{\Phi}}{r\sin\theta}\partial_\phi E^{\hat{r}}, \quad (49c)$$

$$(\partial_t + \omega\partial_\phi)B^{\hat{\phi}} = \frac{e^{\Phi}}{r}\partial_\theta E^{\hat{r}} - \frac{e^{-\Lambda}}{r}\partial_r(e^{\Phi} r E^{\hat{\theta}}) + e^{-\Lambda}\omega'(r) r\sin\theta B^{\hat{r}}, \quad (49d)$$

and

$$\frac{e^{-\Lambda}}{r}\partial_r(r^2 E^{\hat{r}}) + \frac{1}{\sin\theta}[\partial_\theta(\sin\theta E^{\hat{\theta}}) + \partial_\phi E^{\hat{\phi}}] = 4\pi r J^{\hat{t}}, \quad (50a)$$

$$(\partial_t + \omega\partial_\phi)E^{\hat{r}} = \frac{e^{\Phi}}{r\sin\theta}[\partial_\theta(\sin\theta B^{\hat{\phi}}) - \partial_\phi B^{\hat{\theta}}] - 4\pi e^{\Phi} J^{\hat{r}}, \quad (50b)$$

$$(\partial_t + \omega\partial_\phi)E^{\hat{\theta}} = \frac{e^{\Phi}}{r\sin\theta}\partial_\phi B^{\hat{r}} - \frac{e^{-\Lambda}}{r}\partial_r(e^{\Phi} r B^{\hat{\phi}}) - 4\pi e^{\Phi} J^{\hat{\theta}}, \quad (50c)$$

$$(\partial_t + \omega\partial_\phi)E^{\hat{\phi}} = \frac{e^{-\Lambda}}{r}\partial_r(e^{\Phi} r B^{\hat{\theta}}) - \frac{e^{\Phi}}{r}\partial_\theta B^{\hat{r}} + e^{-\Lambda}\omega'(r) r\sin\theta E^{\hat{r}} - 4\pi\omega r\sin\theta J^{\hat{t}}, \quad (50d)$$

where the components of the source term $J^{\hat{\alpha}}$ take the form

$$J^{\hat{t}} = \rho_e + \sigma \frac{\bar{\omega} r \sin\theta}{e^{\Phi}} E^{\hat{\phi}}, \tag{51a}$$

$$J^{\hat{r}} = \sigma \left(E^{\hat{r}} - \frac{\bar{\omega} r \sin\theta}{e^{\Phi}} B^{\hat{\theta}} \right), \tag{51b}$$

$$J^{\hat{\theta}} = \sigma \left(E^{\hat{\theta}} + \frac{\bar{\omega} r \sin\theta}{e^{\Phi}} B^{\hat{r}} \right), \tag{51c}$$

$$J^{\hat{\phi}} = \sigma E^{\hat{\phi}} + \frac{\bar{\omega} r \sin\theta}{e^{\Phi}} \rho_e. \tag{51d}$$

which have been derived by using Ohm's law. Note that in Ref. [14] very similar expressions have been obtained for the pair of Maxwell equations; however, it is convenient to write Maxwell equations as presented in (49) and (50) in further calculations. From these equations, one can see that due to the rotation of spacetime (due to the Lense-Thirring term) the time derivative "∂_t" in Maxwell equations is modified as "$\partial_t + \omega\partial_\phi$." There are new terms that appear in the last equations of (49) and (50) proportional to $\omega'(r)$ which play an important role to obtain the coupled differential equations for electric and magnetic fields.

A. Exterior electromagnetic fields

Let us focus on the vacuum (exterior) solution of Maxwell equations for electromagnetic fields, which means that the source terms can be safely removed (i.e., $J^{\hat{\alpha}} = 0$, $\alpha = t, r, \theta, \phi$), in Eqs. (50), on the other hand, as we mentioned before that the metric functions take the form $e^{2\Phi} = e^{-2\Lambda} = f(r) = 1\text{–}2M/r$ and $\omega(r) = 2J/r^3$, so that derivative from the Lense-Thirring term takes a form $\omega'(r) = -3\omega(r)/r$. Taking into account the aforementioned facts the pair of Maxwell equations in (49) and (50) can be rewritten as

$$\frac{\sqrt{f}}{r} \partial_r (r^2 B^{\hat{r}}) + \frac{1}{\sin\theta} [\partial_\theta (\sin\theta B^{\hat{\theta}}) + \partial_\phi B^{\hat{\phi}}] = 0, \tag{52a}$$

$$(\partial_t + \omega\partial_\phi) B^{\hat{r}} = \frac{\sqrt{f}}{r\sin\theta} [\partial_\phi E^{\hat{\theta}} - \partial_\theta (\sin\theta E^{\hat{\phi}})], \tag{52b}$$

$$(\partial_t + \omega\partial_\phi) B^{\hat{\theta}} = \frac{\sqrt{f}}{r\sin\theta} [\sin\theta \partial_r (r\sqrt{f} E^{\hat{\phi}}) - \partial_\phi E^{\hat{r}}], \tag{52c}$$

$$(\partial_t + \omega\partial_\phi) B^{\hat{\phi}} = \frac{\sqrt{f}}{r} [\partial_\theta E^{\hat{r}} - \partial_r (r\sqrt{f} E^{\hat{\theta}})] - 3\omega\sqrt{f}\sin\theta B^{\hat{r}}, \tag{52d}$$

and

$$\frac{\sqrt{f}}{r} \partial_r (r^2 E^{\hat{r}}) + \frac{1}{\sin\theta} [\partial_\theta (\sin\theta E^{\hat{\theta}}) + \partial_\phi E^{\hat{\phi}}] = 0, \tag{53a}$$

$$(\partial_t + \omega\partial_\phi) E^{\hat{r}} = -\frac{\sqrt{f}}{r\sin\theta} [\partial_\phi B^{\hat{\theta}} - \partial_\theta (\sin\theta B^{\hat{\phi}})], \tag{53b}$$

$$(\partial_t + \omega\partial_\phi) E^{\hat{\theta}} = -\frac{\sqrt{f}}{r\sin\theta} [\sin\theta \partial_r (r\sqrt{f} B^{\hat{\phi}}) - \partial_\phi B^{\hat{r}}], \tag{53c}$$

$$(\partial_t + \omega\partial_\phi) E^{\hat{\phi}} = -\frac{\sqrt{f}}{r} [\partial_\theta B^{\hat{r}} - \partial_r (r\sqrt{f} B^{\hat{\theta}})] - 3\omega\sqrt{f}\sin\theta E^{\hat{r}}. \tag{53d}$$

In fact, Maxwell equations in vacuum are symmetric under the following transformations $E^{\hat{i}} \to B^{\hat{i}}$ or $B^{\hat{i}} \to E^{\hat{i}}$, in particular, when we have the second order equations for the electromagnetic fields but this symmetry breaks due to the rotation of spacetime. It can easily be seen by comparing Eqs. (52d) and (53d). A more precise way of seeing the rotation effect is to compare the following expressions $(\partial_t + \omega\partial_\phi)^2 B^{\hat{r}}$ and $(\partial_t + \omega\partial_\phi)^2 E^{\hat{r}}$. Using the operator $(\partial_t + \omega\partial_\phi)$ on both side of Eq. (52b) and then using Eqs. (53c), (53d), and (52a) we obtain the following second order equation for the radial component of the magnetic field:

$$(\partial_t + \omega\partial_\phi)^2 B^{\hat{r}} = \frac{f}{r^2} \partial_r [f \partial_r (r^2 B^{\hat{r}})] + \frac{f}{r^2} \Delta_\Omega B^{\hat{r}} + \frac{3\omega f}{r\sin\theta} \partial_\theta (\sin^2\theta E^{\hat{r}}). \tag{54}$$

In a similar way, using the operator $(\partial_t + \omega\partial_\phi)$ on both sides of Eq. (53b) and taking into account Eqs. (52c), (52d), and (53a) one can get the second order differential equation for a radial component of the electric field

$$(\partial_t + \omega\partial_\phi)^2 E^{\hat{r}} = \frac{f}{r^2} \partial_r [f \partial_r (r^2 E^{\hat{r}})] + \frac{f}{r^2} \Delta_\Omega E^{\hat{r}} - \frac{3\omega f}{r\sin\theta} \partial_\theta (\sin^2\theta B^{\hat{r}}), \tag{55}$$

where Δ_Ω is the angular part of the Laplace operator

$$\Delta_\Omega = \frac{1}{\sin\theta} \partial_\theta (\sin\theta \partial_\theta) + \frac{1}{\sin^2\theta} \partial_\phi^2, \tag{56}$$

which satisfies the equation $\Delta_\Omega Y_{\ell m} = -\ell(\ell+1) Y_{\ell m}$, where $Y_{\ell m}(\theta, \phi)$ is the spherical harmonics with $\ell = 0, 1, 2, \ldots$ and $|m| \le \ell$. Our analyses show that the second order equations for the angular components of the electromagnetic fields are not separable, and the radial components of electromagnetic fields appear in the equations for angular components of the electromagnetic fields. Later we will show that the radial equations (54) and (55) are important to find not only the radial components

$(E^{\hat{r}}, B^{\hat{r}})$ but also the angular components $(E^{\hat{\theta}}, E^{\hat{\phi}}, B^{\hat{\theta}}, B^{\hat{\phi}})$ of the electromagnetic fields.

Now we concentrate on the radial equations, and one can easily see that Eqs. (54) and (55) are coupled differential equations for the radial component of the magnetic $B^{\hat{r}}$ and the electric $E^{\hat{r}}$ fields. As we mentioned before, these equations are not symmetric under the following transformation $E^{\hat{r}} \leftrightarrow B^{\hat{r}}$, because of the sign of the last terms in each equation which means that the vacuum symmetry of Maxwell equations is broken due to the frame-dragging effect or due to the Lense-Thirring effect. Of course, finding the solutions of radial equations (54) and (55) is not an easy task; nevertheless, we will try to solve them under the reasonable assumptions and approximations. In Ref. [26] it is shown that the general solution for the electromagnetic fields around NS, without the frame-dragging effect, can be expanded in terms of spherical harmonics $Y_{\ell m}(\theta, \phi)$. In the presence of the Lense-Thirring term, the solutions can be extended as

$$B^{\hat{r}} = \sum_{\ell m} \frac{\ell(\ell+1)}{r^2} R^B_{\ell m}(t,r) Y_{\ell m}(\theta,\phi), \tag{57a}$$

$$B^{\hat{\theta}} = \sum_{\ell m} \frac{\sqrt{f}}{r} \left[\partial_r R^B_{\ell m}(t,r)\partial_\theta Y_{\ell m}(\theta,\phi) - \frac{1}{f\sin\theta}(\partial_t + \omega\partial_\phi) R^E_{\ell m}(t,r)\partial_\phi Y_{\ell m}(\theta,\phi)\right], \tag{57b}$$

$$B^{\hat{\phi}} = \sum_{\ell m} \frac{\sqrt{f}}{r} \left[\frac{1}{\sin\theta}\partial_r R^B_{\ell m}(t,r)\partial_\phi Y_{\ell m}(\theta,\phi) + \frac{1}{f}(\partial_t + \omega\partial_\phi) R^E_{\ell m}(t,r)\partial_\theta Y_{\ell m}(\theta,\phi)\right], \tag{57c}$$

and

$$E^{\hat{r}} = \sum_{\ell m} \frac{\ell(\ell+1)}{r^2} R^E_{\ell m}(t,r) Y_{\ell m}(\theta,\phi), \tag{58a}$$

$$E^{\hat{\theta}} = \sum_{\ell m} \frac{\sqrt{f}}{r} \left[\partial_r R^E_{\ell m}(t,r)\partial_\theta Y_{\ell m}(\theta,\phi) + \frac{1}{f\sin\theta}(\partial_t + \omega\partial_\phi) R^B_{\ell m}(t,r)\partial_\phi Y_{\ell m}(\theta,\phi)\right], \tag{58b}$$

$$E^{\hat{\phi}} = \sum_{\ell m} \frac{\sqrt{f}}{r} \left[\frac{1}{\sin\theta}\partial_r R^E_{\ell m}(t,r)\partial_\phi Y_{\ell m}(\theta,\phi) - \frac{1}{f}(\partial_t + \omega\partial_\phi) R^B_{\ell m}(t,r)\partial_\theta Y_{\ell m}(\theta,\phi)\right], \tag{58c}$$

where the profile functions $R^B_{\ell m}(t,r)$ and $R^E_{\ell m}(t,r)$ are, respectively, responsible for the magnetic and the electric fields, and can be found by solving Eqs. (52) and (53) simultaneously. Note that the solutions (57) and (58) fully satisfy the pair of Maxwell equations (52) and (53), and six components of the electromagnetic field can be expressed in terms of two profile functions $R^B_{\ell m}(t,r)$ and $R^E_{\ell m}(t,r)$, respectively. Now we focus on the radial components of the electromagnetic fields; substituting solutions (57a) and (58a) into Eqs. (54) and (55) we obtain the following set of coupled differential equations:

$$\sum_{\ell m} \ell(\ell+1)[\partial_r(f\partial_r) - \frac{\ell(\ell+1)}{r^2} - \frac{1}{f}(\partial_t + \omega\partial_\phi)^2] R^B_{\ell m} Y_{\ell m} = -\frac{3\omega}{r} \sum_{\ell' m'} \frac{\ell'(\ell'+1)}{\sin\theta} \partial_\theta(\sin^2\theta Y_{\ell' m'}) R^E_{\ell' m'}, \tag{59}$$

$$\sum_{\ell m} \ell(\ell+1)[\partial_r(f\partial_r) - \frac{\ell(\ell+1)}{r^2} - \frac{1}{f}(\partial_t + \omega\partial_\phi)^2] R^E_{\ell m} Y_{\ell m} = \frac{3\omega}{r} \sum_{\ell' m'} \frac{\ell'(\ell'+1)}{\sin\theta} \partial_\theta(\sin^2\theta Y_{\ell' m'}) R^B_{\ell' m'}. \tag{60}$$

Keep in mind that in Eqs. (59) and (60) we have used two different summations for the electric and the magnetic fields, and that is why to distinguish summations, we used the different indices (ℓ, m) and (ℓ', m') for each field. Now, we start to simplify the right-hand side of Eqs. (59) and (60). To do this we can use the following useful formulas for spherical harmonics $Y_{\ell m}(\theta, \phi)$:

$$\frac{1}{\sin\theta}\partial_\theta(\sin^2\theta Y_{\ell m}) = (m+2)\cos\theta Y_{\ell m} + \sqrt{\frac{\Gamma(\ell-m+1)\Gamma(\ell+m+2)}{\Gamma(\ell-m)\Gamma(\ell+m+1)}} \sin\theta Y_{\ell, m+1} e^{-i\phi}, \tag{61}$$

where $\Gamma(n)$ is the Gamma function defined as $\Gamma(n) = (n-1)!$ for any integer number n. Each term of Eq. (61) can be expressed as (see e.g., [52])

$$\cos\theta Y_{\ell m} = \sqrt{\frac{(\ell-m+1)(\ell+m+1)}{(2\ell+1)(2\ell+3)}} Y_{\ell+1,m} + \sqrt{\frac{(\ell-m)(\ell+m)}{(2\ell+1)(2\ell-1)}} Y_{\ell-1,m}, \tag{62a}$$

$$\sin\theta Y_{\ell,m+1} e^{-i\phi} = \sqrt{\frac{(\ell-m)(\ell-m+1)}{(2\ell+1)(2\ell+3)}} Y_{\ell+1,m} - \sqrt{\frac{(\ell+m+1)(\ell+m)}{(2\ell+1)(2\ell-1)}} Y_{\ell-1,m}. \tag{62b}$$

Taking into account Eqs. (61) and (62), we now multiply the complex conjugate of spherical harmonics, $Y^*_{\ell m}(\theta,\phi)$, from both sides of Eqs. (59) and (60), and integrate along the solid angle, i.e., $d\Omega = \sin\theta d\theta d\phi$. Hereafter using the orthogonality condition of spherical harmonics, $\int Y^*_{\ell m} Y_{\ell' m'} d\Omega = \delta_{\ell\ell'}\delta_{mm'}$, we can obtain the following coupled equations:

$$\left[\partial_r(f\partial_r) - \frac{\ell(\ell+1)}{r^2} - \frac{1}{f}(\partial_t - im\bar{\omega})^2\right] R^B_{\ell m} = -\frac{3\omega}{r}[(\ell-1)C_{\ell,m}R^E_{\ell-1,m} - (\ell+2)C_{\ell+1,m}R^E_{\ell+1,m}], \tag{63}$$

$$\left[\partial_r(f\partial_r) - \frac{\ell(\ell+1)}{r^2} - \frac{1}{f}(\partial_t - im\bar{\omega})^2\right] R^E_{\ell m} = \frac{3\omega}{r}[(\ell-1)C_{\ell,m}R^B_{\ell-1,m} - (\ell+2)C_{\ell+1,m}R^B_{\ell+1,m}], \tag{64}$$

where the coefficients $C_{\ell,m}$ are defined as

$$C_{\ell,m} = \sqrt{\frac{\ell^2-m^2}{4\ell^2-1}}, \quad C_{\ell,0} = \frac{\ell}{\sqrt{4\ell^2-1}}, \quad C_{\ell,\ell} = 0. \tag{65}$$

Here one has to emphasize that very similar expressions for the profile functions for the electromagnetic fields has been derived in Refs. [16,17] by using a different approach, and one can also find similar equations in particular for the profile function of the electric field in [11,53].

In section, we have derived in very detailed equations for the profile functions $R^B_{\ell m}(r,t)$ and $R^E_{\ell m}(r,t)$ for the electromagnetic fields. Now we multiply imaginary "i" in both sides of Eq. (63), add it to Eq. (64), and introduce a new function $R_{\ell m} = R^E_{\ell m} + iR^B_{\ell m}$ which satisfies the following equation:

$$\left[\partial_r(f\partial_r) - \frac{\ell(\ell+1)}{r^2} - \frac{1}{f}(\partial_t - i\bar{\omega})^2\right] R_{\ell m} = -\frac{3i\omega}{r}[(\ell-1)C_{\ell,m}R_{\ell-1,m} - (\ell+2)C_{\ell+1,m}R_{\ell+1,m}], \tag{66}$$

which shows that all six components of the electromagnetic fields might be expressed in terms of single profile function $R_{\ell m}(r,t)$. Unfortunately, finding the analytical solution of Eq. (66) is not an easy task. In the next section, we will show an exact analytical solution for the electromagnetic field of the neutron star at the slowly rotating limit.

B. Vector potential of the electromagnetic fields

It is also convenient to find the components of the vector four-potential, A_α, of the electromagnetic fields. To do this we introduce the definition of the electromagnetic field tensor, i.e., $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$. On the other hand, using expressions (6) for the electromagnetic field tensor and general solutions for the components of the electric and the magnetic fields (57) and (58), one can obtain

$$F_{\theta\phi} = \ell(\ell+1)R^B_{\ell m}(t,r)Y_{\ell m}(\theta,\phi)\sin\theta, \tag{67a}$$

$$F_{\phi r} = \partial_r R^B_{\ell m}(t,r)\partial_\theta Y_{\ell m}(\theta,\phi)\sin\theta - \frac{1}{f}(\partial_t + \omega\partial_\phi)R^E_{\ell m}(t,r)\partial_\phi Y_{\ell m}(\theta,\phi), \tag{67b}$$

$$F_{r\theta} = \partial_r R^B_{\ell m}(t,r)\frac{1}{\sin\theta}\partial_\phi Y_{\ell m}(\theta,\phi) + \frac{1}{f}(\partial_t + \omega\partial_\phi)R^E_{\ell m}(t,r)\partial_\theta Y_{\ell m}(\theta,\phi), \tag{67c}$$

which can easily be found for the spatial components of the four-vector potential of electromagnetic fields in the form

$$A_r(t,\mathbf{r}) = -\frac{1}{f}(\partial_t + \omega\partial_\phi)R^E_{\ell m}(t,r)Y_{\ell m}(\theta,\phi), \tag{68a}$$

$$A_\theta(t,\mathbf{r}) = R^B_{\ell m}(t,r)\frac{1}{\sin\theta}\partial_\phi Y_{\ell m}(\theta,\phi), \tag{68b}$$

$$A_\phi(t,\mathbf{r}) = -R^B_{\ell m}(t,r)\sin\theta\partial_\theta Y_{\ell m}(\theta,\phi). \tag{68c}$$

Note that the expressions in (68) are important in order to find the time component, A_t, vector potential. We again use Eqs. (57), (58), (6) and take into account Eqs. (68) where one can find the time component of the vector potential:

$$A_t(t,\mathbf{r}) = \omega R^B_{\ell m}(t,r)\sin\theta\partial_\theta Y_{\ell m}(\theta,\phi) - f\partial_r R^E_{\ell m}(t,r)Y_{\ell m}(\theta,\phi). \tag{69}$$

V. STATIONARY MULTIPOLE SOLUTION IN SLOWLY ROTATING LIMIT

In this section, we study the multipole stationary solution for the electromagnetic fields in the slowly rotating limit of NS [11,14,16,17]. It means that the time derivative from the fields to be zero $\partial_t \mathbf{E} = 0$ and $\partial_t \mathbf{B} = 0$, which are equivalent to $\partial_t R^B_{\ell m} = 0$ and $\partial_t R^E_{\ell m} = 0$. We will safely neglect the highest order for angular velocities and keep only linear terms in Maxwell equations (52) and (53). In fact, the magnetic field of NS does not depend on its rotation, but it is caused due to the perfect fluid interior region of NS. However, the induced electric field is caused due to the magnetic field and rotation of NS and/or it can be found as, i.e., $\mathbf{E} \sim \mathbf{\Omega} \times \mathbf{B}$ (or $\mathbf{E} \sim \boldsymbol{\omega} \times \mathbf{B}$), which is equivalent to the relation $R^E_{\ell m} \sim \omega R^B_{\ell m}$, which means that the terms proportional to $\sim \boldsymbol{\omega} \mathbf{E}$ or $\sim \omega R^E_{\ell m}$ are negligibly small. Then we rewrite Eqs. (63) and (64) in the form

$$r^2 \frac{d}{dr}\left(f \frac{d}{dr} R^B_{\ell m}\right) - \ell(\ell+1) R^B_{\ell m} = 0, \tag{70}$$

$$r^2 \frac{d}{dr}\left(f \frac{d}{dr} R^E_{\ell m}\right) - \ell(\ell+1) R^E_{\ell m} = 3\omega r((\ell-1) C_{\ell,m} R^B_{\ell-1,m} - (\ell+2) C_{\ell+1,m} R^B_{\ell+1,m}). \tag{71}$$

From these equations we can see that once we find the solution of Eq. (70) for $R^B_{\ell m}(r)$, then one can find another function $R^E_{\ell m}(r)$. Equation (70) is quite well known and the solution of this equation can be expressed in different forms. For example, in Refs. [11,53] authors studied the influence of GR in the induced electric field due to the rotation of the magnetic field, and they obtained very similar equations for the profile functions; the solution of Eq. (70) can be expressed as

$$R_{\ell m}(r) = a_{\ell m} \left(\frac{2M}{r}\right)^{\ell} {}_2F_1\left(\ell, \ell+2, 2(\ell+1), \frac{2M}{r}\right), \tag{72}$$

while in another approach the solution of Eq. (70) can be expressed in terms of the Jacobi polynomial (see details in Ref. [9]). Actually the author studied the magnetic generated by a current loop around black hole Schwarzschild space and obtained the same equation as (70):

$$R_{\ell m}(r) = a_{\ell m} \left(\frac{r}{2M}\right)^2 (2M)^{\ell} P^{(2,0)}_{\ell-1}\left(1 - \frac{r}{M}\right). \tag{73}$$

However, in Refs. [14,26] it has been shown that the solution of Eq. (70) written in terms of the Legendre function of the second kind,

$$R^B_{\ell m}(r) = -a_{\ell m} \frac{r^2}{M^{\ell}} \frac{d}{dr}\left[r f(r) \frac{d}{dr} Q_{\ell}\left(1 - \frac{r}{M}\right)\right], \tag{74}$$

where $Q_{\ell}(x)$ is the Legendre polynomial of the second kind, $a_{\ell m}$ is the constant of integration which can be found from the Newtonian limit for the expression of the magnetic field. Note that expressions (72), (73), and (74) for the solution of the radial component of the magnetic field gave the same result. However, in the present paper we wish to present our own solutions for Eqs. (70) and (71). To do this we start finding the exact analytical solution of Eq. (70), and before we go further, we introduce the new function

$$F_{\ell m}(r) = \sqrt{f} \frac{dR^B_{\ell m}(r)}{dr}. \tag{75}$$

Substituting the function (75) into (70) and after performing simple algebraic manipulations, one can have the well-known associated Legendre equation

$$\frac{d}{dx}\left((1-x^2)\frac{dF_{lm}}{dx}\right) + \left(\ell(\ell+1) - \frac{1}{1-x^2}\right) F_{\ell m} = 0, \tag{76}$$

where $x = r/M - 1$. The solution of Eq. (76) is found as

$$F_{\ell m}(x) = c_1 P^1_{\ell}(x) + c_2 Q^1_{\ell}(x), \tag{77}$$

where c_1 and c_2 are constants of integration and $P^m_{\ell}(x)$ and $Q^m_{\ell}(x)$ are associated Legendre functions of the first and second kinds, respectively. From the asymptotic behavior of magnetic field one can obtain that $c_1 = 0$ and $c_2 = b_{\ell m}$. In order to find radial function $R_{\ell m}(r)$ we use again Eq. (70) and obtain

$$R^B_{\ell m}(r) = \frac{r^2}{\ell(\ell+1)} \frac{d}{dr}(\sqrt{f} Q^1_{\ell}(x)) b_{\ell m}. \tag{78}$$

The components of the dipole magnetic field for the misaligned magnetized NS are [14]

$$B^{\hat{r}}(\mathbf{r}) = -\frac{3B_0R^3}{8M^3}\left[\ln f(r) + \frac{2M}{r}\left(1 + \frac{M}{r}\right)\right](\cos\chi\cos\theta + \sin\chi\sin\theta\cos(\phi - \Omega t)), \tag{79}$$

$$B^{\hat{\theta}}(\mathbf{r}) = \frac{3B_0R^3}{8rM^2}\sqrt{f(r)}\left[\frac{r}{M}\ln f(r) + \frac{1}{f(r)} + 1\right](\cos\chi\sin\theta - \sin\chi\cos\theta\cos(\phi - \Omega t)), \tag{80}$$

TABLE I. The analytical expressions for the magnetic field profile function and its Newtonian limit are presented for versus values of the multipole number.

Multipole number	$\ell(\ell+1)R^B_{lm}(r)/r^2$	Newtonian limit
$\ell=1$	$-\frac{3BR^3}{8M^3}[\ln f(r)+\frac{2M}{r}(1+\frac{M}{r})]$	$B(\frac{R}{r})^3$
$\ell=2$	$\frac{5B_QR^4}{4M^4}[(3-\frac{2r}{M})\ln f(r)+\frac{2M^2}{3r^2}+\frac{2M}{r}-4]$	$B_Q(\frac{R}{r})^4$
$\ell=3$	$-\frac{35R^5B_H}{128M^7r^2}[(\frac{45r^2}{M^2}-\frac{120r}{M}+72)\ln f(r)+\frac{4M^2}{r^2}+\frac{24M}{r}+\frac{90r}{M}-150]$	$B_H(\frac{R}{r})^5$
$\ell=4$	$\frac{21R^6B_O}{160M^6}[15(40-\frac{120r}{M}+\frac{105r^2}{M^2}-\frac{28r^3}{M^3})\ln f(r)+2(\frac{6M^2}{r^2}-\frac{420r^2}{M^2}+\frac{60M}{r}+\frac{1155r}{M}-785)]$	$B_O(\frac{R}{r})^5$

$$B^{\hat{\phi}}(\mathbf{r}) = \frac{3B_)R^3}{8rM^2}\sqrt{f(r)}\left[\frac{r}{M}\ln f(r) + \frac{1}{f(r)} + 1\right] \times \sin\chi\sin\theta\cos(\phi-\Omega t), \tag{81}$$

where B_0 is a magnetic field strength at the polar cap of NS and χ is the inclination angle between the magnetic field and rotation axes. However, one can also present the exact solutions of the magnetic field for a higher multipole number. Here we will not show the components of the multipole magnetic field; in Table I it listed the exact expression for the radial profile function of the magnetic field. One has to emphasize that very similar expressions as shown in Table I can be found in Refs. [16,17]. The interesting fact is that even a dipole electric field can be generated as a quadrupole magnetic field in the framework of GR. In order to see the significance of the general relativity in a magnetic field we study the radial dependence magnetic profile functions for different values of the pole numbers. Figure 1 shows the radial dependence of the magnetic profile functions for different pole numbers in Newtonian and general relativistic approaches. From Fig. 1 one can see that due to general relativistic corrections the surface dipole magnetic field ($\ell=1$) increases up to 40% of its Newtonian value. However, for large multipole numbers general relativistic treatments in the surface values of the multipole magnetic field are getting larger.

Equation (71) is a second-order inhomogeneous differential equation for $R^E_{\ell m}$ which has two solutions, the first one is the solution for the homogeneous equation, while the second one is a particular solution which depends on the function on the right-hand side of Eq. (71). The solution for the homogeneous equation is the same as in (78) with the different constant of integration $e_{\ell m}$ instead of $b_{\ell m}$, and the particular solution can be found from the right-hand side of Eq. (71). Hereafter introducing new function $G_{\ell m}(r) = \sqrt{f}(dR^E_{\ell m}/dr)$ and using the solution for the profile of the magnetic field (78), the inhomogeneous equation (71) can be rewritten as

$$\frac{d}{dx}\left((1-x^2)\frac{dG_{lm}}{dx}\right) + \left(\ell(\ell+1) - \frac{1}{1-x^2}\right)G_{\ell m} = \frac{6a}{M}\sqrt{f}\left[b_{\ell+1,m}\frac{C_{\ell+1,m}}{\ell+1}\frac{d^2}{dx^2}(\sqrt{f}Q^1_{\ell+1}(x)) - b_{\ell-1,m}\frac{C_{\ell,m}}{\ell}\frac{d^2}{dx^2}(\sqrt{f}Q^1_{\ell-1}(x))\right]. \tag{82}$$

In order to find the general solution of inhomogeneous equation (82), we use well-known Lagrange's formula (see, for example, [54,55]),

$$G_{\ell m}(x) = c_3P^1_\ell(x) + c_4Q^1_\ell(x) + G^p_{\ell m}(x), \tag{83}$$

where c_3 and c_4 are constants of integration, $G^p_{\ell m}(x)$ is the particular solution of the inhomogeneous equation which can be expressed as

$$G^p_{\ell m}(x) = \frac{6a}{M}Q^1_\ell(x)\int dx\frac{P^1_\ell(x)}{W}\sqrt{f}\left[b_{\ell+1,m}\frac{C_{\ell+1,m}}{\ell+1}\frac{d^2}{dx^2}(\sqrt{f}Q^1_{\ell+1}(x)) - b_{\ell-1,m}\frac{C_{\ell,m}}{\ell}\frac{d^2}{dx^2}(\sqrt{f}Q^1_{\ell-1}(x))\right] - \frac{6a}{M}P^1_\ell(x)\int dx\frac{Q^1_\ell(x)}{W}\sqrt{f}\left[b_{\ell+1,m}\frac{C_{\ell+1,m}}{\ell+1}\frac{d^2}{dx^2}(\sqrt{f}Q^1_{\ell+1}(x)) - b_{\ell-1,m}\frac{C_{\ell,m}}{\ell}\frac{d^2}{dx^2}(\sqrt{f}Q^1_{\ell-1}(x))\right], \tag{84}$$

and the Wronskian W in Eq. (84) is defined as $W = P^1_\ell(x)(Q^1_\ell(x))' - Q^1_\ell(x)(P^1_\ell(x))' = 1-x^2$; here the prime denotes the derivative with respect to x. Note that the induced electric field of NS should vanish at infinity, which means constant c_3 in solution (82) should be zero, i.e., $c_3 = 0$,

$$R^E_{\ell m}(r) = \frac{r^2}{\ell(\ell+1)}\left[\frac{d}{dr}(\sqrt{f}Q^1_\ell(x))e_{\ell m} + \frac{d}{dr}(\sqrt{f}G^p_{\ell m}(x))\right]. \tag{85}$$

So far explicit analytical expressions for the time and azimuthal components of the vector potential of stationary

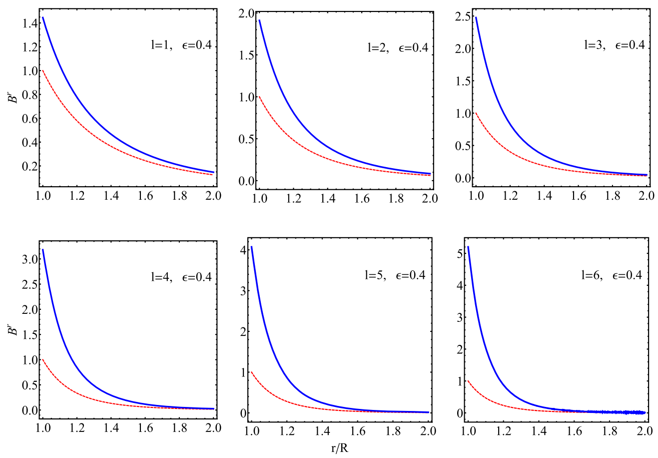

FIG. 1. Radial dependence of a multipole radial magnetic field is presented in GR with blue solid lines and in Newtonian with dashed red lines.

exterior electromagnetic fields surrounding the rotating star endowed with a dipole magnetic field have been obtained in Ref. [56]; however, those solutions diverge in Newtonian approximation. In the previous section, we have obtained the expressions for the vector potential of the electromagnetic fields for given radial profile functions of both electric and magnetic fields. In order to obtain a proper solution for the vector potential, we use expressions (68) and (69) in slowly rotating limiting cases. In the case of a slow rotation limit of the spacetime metric of the magnetized NS, the components of the vector potential take the form

$$A_r(t,\mathbf{r}) = 0, \tag{86a}$$

$$A_\theta(t,\mathbf{r}) = R^B_{\ell m}(t,r)\frac{1}{\sin\theta}\partial_\phi Y_{\ell m}(\theta,\phi), \tag{86b}$$

$$A_\phi(t,\mathbf{r}) = -R^B_{\ell m}(t,r)\sin\theta\partial_\theta Y_{\ell m}(\theta,\phi), \tag{86c}$$

$$A_t(t,\mathbf{r}) = -f\partial_r R^E_{\ell m}(t,r)Y_{\ell m}(\theta,\phi) + \omega R^B_{\ell m}(t,r)\sin\theta\partial_\theta Y_{\ell m}(\theta,\phi). \tag{86d}$$

One of the advantages of the vector potential of the electromagnetic fields is to consider the particle motion of a charged particle; on the other hand, it is also important to produce the field line of the electromagnetic fields. Here we are interested in only magnetic field lines which can be produced from the condition $A_i = 0$, where A_i is the spatial components of the vector potential. Since the vector potential is time dependent, more correctly, orientation magnetic poles always change. Nevertheless, one can draw the magnetic field lines at fixed time and fixed azimuthal angle, or $\phi - \Omega t = \text{const}$. Figure 2 shows the magnetic field lines of the dipole ($\ell = 1$) and quadrupole ($\ell = 2$) magnetic fields for typical NS with stellar compactness $\epsilon = 0.4$ in the framework of Newtonian and GR theories.

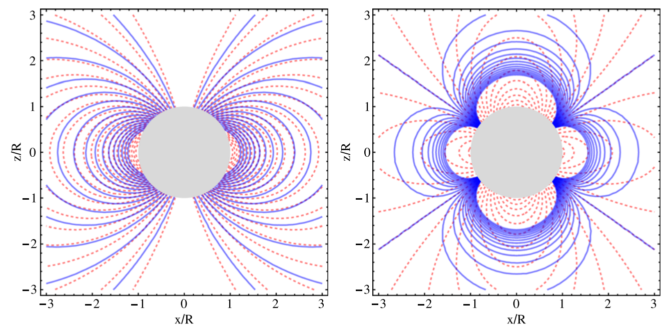

FIG. 2. Left panel: Dipole magnetic field line for typical NS. Right panel: Quadrupole magnetic field line for NS. Solid (blue) lines are the represented general relativistic magnetic field, while dashed (red) lines represent the magnetic field line in the Newtonian case. In both cases stellar compactness to be taken is $\epsilon = 0.4$.

From Fig. 2 one can see that magnetic field lines are more dense in the polar cap of NS in the GR approach in comparison with that in the Newtonian case.

VI. ELECTROMAGNETIC FIELDS IN WAVE ZONE

A. Electromagnetic wave in the spacetime of nonrotating magnetized NS

In this subsection, we study the electromagnetic fields in the wave zone in the spacetime of nonrotating (the Lense-Thirring term is negligibly small in comparison with stellar rotation, i.e., $\omega \ll \Omega$) magnetized NS. So then the absence of the Lense-Thirring term equations (54) and (55) can be expressed as

$$\partial_t^2 B^{\hat r} = \frac{f}{r^2}\partial_r[f\partial_r(r^2 B^{\hat r})] + \frac{f}{r^2}\Delta_\Omega B^{\hat r}, \tag{87a}$$

$$\partial_t^2 E^{\hat r} = \frac{f}{r^2}\partial_r[f\partial_r(r^2 E^{\hat r})] + \frac{f}{r^2}\Delta_\Omega E^{\hat r}. \tag{87b}$$

From Eqs. (87) one can see that wave equations for the electromagnetic fields are decoupled and solutions of these equations can be sought as

$$B^{\hat r}(t,\mathbf{r}) = \frac{U(\mathbf{r})}{r}e^{-iwt}, \qquad E^{\hat r}(t,\mathbf{r}) = \frac{V(\mathbf{r})}{r}e^{-iwt}, \tag{88}$$

where w is the angular frequency of the electromagnetic wave and $U(\mathbf{r})$ and $V(\mathbf{r})$ are unknown functions of spatial coordinates, i.e., (r,θ,ϕ) which satisfy the following wave equations:

$$\frac{f}{r}\partial_r[f\partial_r(rU)] + \frac{f}{r^2}\Delta_\Omega U + w^2 U = 0, \tag{89}$$

$$\frac{f}{r}\partial_r[f\partial_r(rV)] + \frac{f}{r^2}\Delta_\Omega V + w^2 V = 0. \tag{90}$$

Obviously it is difficult to solve Eqs. (89) and (90), analytically, in particular when the lapse function exists; however, in flat spacetime, i.e., $f(r) = 1$, the solutions of wave equations are trivial, and the radial parts of both the electric and the magnetic fields can be expressed in terms of the spherical Bessel (or spherical Hankel) function, i.e., $\{E^{\hat r}, B^{\hat r}\} \sim j_\ell(wr)$ [or $\sim h_\ell(wr)$]. However, in some physical situations, one can solve Eqs. (89) and (90) by using perturbation to obtain approximate solutions for the profile functions (U, V). Before we start the calculations, let us introduce dimensionless radial coordinate $\eta = r/R$ normalized by stellar radius R, and then the lapse function can be expressed as $f(r) = 1 - \epsilon/\eta$, where $\epsilon = 2M/R$ is stellar compactness and for typical NS it is about $\epsilon \le 0.4$. Since ϵ is a small quantity, it can be taken as an expansion parameter and the profile functions $U(\mathbf{r}, \epsilon)$ and $V(\mathbf{r}, \epsilon)$ can be expanded as

$$U(\boldsymbol{\eta}, \epsilon) = U^{(0)}(\boldsymbol{\eta}) + \epsilon U^{(1)}(\boldsymbol{\eta}) + \frac{1}{2}\epsilon^2 U^{(2)}(\boldsymbol{\eta}) + \cdots, \quad (91a)$$

$$V(\boldsymbol{\eta}, \epsilon) = V^{(0)}(\boldsymbol{\eta}) + \epsilon V^{(1)}(\boldsymbol{\eta}) + \frac{1}{2}\epsilon^2 V^{(2)}(\boldsymbol{\eta}) + \cdots. \quad (91b)$$

Substituting solutions (91) into the wave equation (89), one can obtain the following set of equations for $U^{(n)}(\boldsymbol{\eta})$ functions:

$$(\Delta + w_*^2)U^{(0)} = 0, \quad (92a)$$

$$(\Delta + w_*^2)U^{(1)} = \frac{2w_*^2}{\eta}U^{(0)} - \frac{1}{\eta^3}[\partial_\eta(\eta U^{(0)}) + \Delta_\Omega U^{(0)}], \quad (92b)$$

$$(\Delta + w_*^2)U^{(2)} = \frac{2w_*^2}{\eta}U^{(1)} - \frac{1}{\eta^2}\partial_\eta\left[\frac{1}{\eta}\partial_\eta(\eta U^{(0)})\right] - \frac{1}{\eta^3}[\partial_\eta(\eta U^{(1)}) + \Delta_\Omega U^{(1)}], \quad (92c)$$

$$(\Delta + w_*^2)U^{(3)} = \frac{2w_*^2}{\eta}U^{(2)} - \frac{1}{\eta^2}\partial_\eta\left[\frac{1}{\eta}\partial_\eta(\eta U^{(1)})\right] - \frac{1}{\eta^3}[\partial_\eta(\eta U^{(2)}) + \Delta_\Omega U^{(2)}], \quad (92d)$$

$$\cdots, \quad (92e)$$

$$(\Delta + w_*^2)U^{(n)} = \frac{2w_*^2}{\eta}U^{(n-1)} - \frac{1}{\eta^2}\partial_\eta\left[\frac{1}{\eta}\partial_\eta(\eta U^{(n-2)})\right] - \frac{1}{\eta^3}[\partial_\eta(\eta U^{(n-1)}) + \Delta_\Omega U^{(n-1)}], \quad (92f)$$

where $w_* = wR$ and Δ is the Laplace operator,

$$\Delta = \frac{1}{\eta^2}\frac{\partial}{\partial\eta}\left(\eta^2\frac{\partial}{\partial\eta}\right) + \frac{1}{\eta^2}\Delta_\Omega. \quad (93)$$

Here note that $U^{(0)}(\boldsymbol{\eta})$ is a solution for the wavelike equation in flat space. In a similar way one can obtain equations for $V^{(n)}(\boldsymbol{\eta})$ functions as given in (92). We are now in a position to solve wavelike equations (92), and it is easy to see that once we find the function in the zeroth approximation $U^{(0)}(\boldsymbol{\eta})$ (in the case of flat spacetime), then one can find other functions in any high order of n, i.e., $U^{(n)}(\boldsymbol{\eta})$. Before focusing on the solution in flat space let us concentrate on the solutions in the highest order. It is easy to see that Eqs. (92), except the first one, reminds us that the d'Alembert equation and the right-hand side of the equations play the role of the source, which means that each previous two solutions play the role of the source for the function in the next order approximation. The solution for the function in any order can then be expressed in terms of Green's function $G(\boldsymbol{\eta}, \boldsymbol{\eta}')$ and for the differential operator, $\Delta - w_*^2$, one can find the following Green's function:

$$G(\boldsymbol{\eta}, \boldsymbol{\eta}') = -\frac{e^{-iw_*|\boldsymbol{\eta}-\boldsymbol{\eta}'|}}{|\boldsymbol{\eta}-\boldsymbol{\eta}'|}, \quad (94)$$

so then the solution for $U^{(n)}(\boldsymbol{\eta})$ can found as in the following integral form:

$$U^{(n)}(\boldsymbol{\eta}) = -\int d\boldsymbol{\eta}' \frac{e^{-iw_*|\boldsymbol{\eta}-\boldsymbol{\eta}'|}}{|\boldsymbol{\eta}-\boldsymbol{\eta}'|}\left\{\frac{2w_*^2}{\eta'}U^{(n-1)}(\boldsymbol{\eta}') - \frac{1}{\eta'^3}[\partial_{\eta'}(\eta' U^{(n-1)}(\boldsymbol{\eta}')) + \Delta'_\Omega U^{(n-1)}(\boldsymbol{\eta}')] - \frac{1}{\eta'^2}\partial_{\eta'}\left[\frac{1}{\eta'}\partial_{\eta'}(\eta' U^{(n-2)}(\boldsymbol{\eta}'))\right]\right\}, \quad (95)$$

while the solution for the function in the first order correction, i.e., $U^{(1)}(\boldsymbol{\eta})$, is given by

$$U^{(1)}(\boldsymbol{\eta}) = -\int d\boldsymbol{\eta}' \frac{e^{-iw_*|\boldsymbol{\eta}-\boldsymbol{\eta}'|}}{|\boldsymbol{\eta}-\boldsymbol{\eta}'|}\left\{\frac{2w_*^2}{\eta'}U^{(0)}(\boldsymbol{\eta}') - \frac{1}{\eta'^3}[\partial_{\eta'}(\eta' U^{(0)}(\boldsymbol{\eta}')) + \Delta'_\Omega U^{(0)}(\boldsymbol{\eta}')]\right\}, \quad (96)$$

where $d\boldsymbol{\eta}' = \eta'^2 d\eta' \sin\theta' d\theta' d\phi'$ and Δ'_Ω is the angular part of the Laplace operator in primed angular coordinates.

B. Electromagnetic wave in the spacetime of the rotating magnetized NS

Now we focus on the electromagnetic wave equation in the presence of the frame of dragging in where $\omega \neq 0$. It is well known that the solution for this equation in the flat space is in the form of spherical Hankel function $h_\ell(r)$. In the presence of the Lense-Thirring angular velocity the solution for the wave equation can be written as

$$B^{\hat{r}} = \sum_{\ell,m} \frac{U_{\ell m}}{r} Y_{\ell m} e^{-iwt}, \tag{97a}$$

$$B^{\hat{\theta}} = \sum_{\ell,m} \frac{\sqrt{f}}{r\ell(\ell+1)} \left[\partial_r(rU_{\ell m})\partial_\theta Y_{\ell m} - \frac{i(w-m\omega)r}{f\sin\theta} V_{\ell m}\partial_\phi Y_{\ell m}\right] e^{-iwt}, \tag{97b}$$

$$B^{\hat{\phi}} = \sum_{\ell,m} \frac{\sqrt{f}}{r\ell(\ell+1)} \left[\partial_r(rU_{\ell m})\frac{1}{\sin\theta}\partial_\phi Y_{\ell m} + \frac{i(w-m\omega)r}{f} V_{\ell m}\partial_\theta Y_{\ell m}\right] e^{-iwt}, \tag{97c}$$

and

$$E^{\hat{r}} = \sum_{\ell,m} \frac{V_{\ell m}}{r} Y_{\ell m} e^{-iwt}, \tag{98a}$$

$$E^{\hat{\theta}} = \sum_{\ell,m} \frac{\sqrt{f}}{r\ell(\ell+1)} \left[\partial_r(rV_{\ell m})\partial_\theta Y_{\ell m} + \frac{i(w-m\omega)r}{f\sin\theta} U_{\ell m}\partial_\phi Y_{\ell m}\right] e^{-iwt}, \tag{98b}$$

$$E^{\hat{\phi}} = \sum_{\ell,m} \frac{\sqrt{f}}{r\ell(\ell+1)} \left[\partial_r(rV_{\ell m})\frac{1}{\sin\theta}\partial_\phi Y_{\ell m} - \frac{i(w-m\omega)r}{f} U_{\ell m}\partial_\theta Y_{\ell m}\right] e^{-iwt}, \tag{98c}$$

where $U_{\ell m}(r)$ and $V_{\ell m}(r)$ are the radial functions of the magnetic and the electric fields, respectively. Substituting solutions (97) and (98) into Maxwell equations (52) and (53), we obtain the following equations for profile functions $U_{\ell m}$ and $V_{\ell m}$:

$$\frac{f}{r}\partial_r(f\partial_r(rU_{\ell m})) + \left[(w-m\omega)^2 - f\frac{\ell(\ell+1)}{r^2}\right]U_{\ell m} = -\frac{3f\omega}{r}[(\ell-1)C_{\ell,m}V_{\ell-1,m} - (\ell+2)C_{\ell+1,m}V_{\ell+1,m}], \tag{99}$$

$$\frac{f}{r}\partial_r(f\partial_r(rV_{\ell m})) + \left[(w-m\omega)^2 - f\frac{\ell(\ell+1)}{r^2}\right]V_{\ell m} = \frac{3f\omega}{r}[(\ell-1)C_{\ell,m}U_{\ell-1,m} - (\ell+2)C_{\ell+1,m}U_{\ell+1,m}]. \tag{100}$$

Obviously, it is difficult to solve Eqs. (99) and (100), analytically, when lapse function $f(r)$ exists. However, in flat spacetime, i.e., $f(r) = 1$, the solutions of Eqs. (99) and (100) are trivial, and both functions $U_{\ell m}$ and $V_{\ell m}$ can be expressed in terms of the spherical Hankel (or spherical Besel) function, i.e., $(U_{\ell m}, V_{\ell m}) \sim h_\ell(wr)$ [or $\sim j_\ell(wr)$]. However, in some physical situation we can solve Eqs. (99) and (100) by using perturbation in order to obtain approximate solutions for the profile functions $(U_{\ell m}, V_{\ell m})$. Before we start the calculations, let us introduce dimensionless radial coordinate $\eta = r/R$ and spin parameter $a_* = a/R$ normalized by stellar radius R, and then the lapse function can be expressed as $f(\eta) = 1 - \epsilon/\eta$, where $\epsilon = 2M/R$ is stellar compactness and for typical NS it is about $\epsilon \lesssim 0.4$. Since ϵ is a small quantity, it can be taken as an expansion parameter and the profile functions $U_{\ell m}(r,\epsilon)$ and $V_\ell(r,\epsilon)$ can be expanded as

$$U_{\ell m}(\eta,\epsilon) = U^{(0)}_{\ell m}(\eta) + \epsilon U^{(1)}_{\ell m}(\eta) + \frac{1}{2}\epsilon^2 U^{(2)}_{\ell m}(\eta) + \cdots, \tag{101}$$

$$V_{\ell m}(\eta,\epsilon) = V^{(0)}_{\ell m}(\eta) + \epsilon V^{(1)}_{\ell m}(\eta) + \frac{1}{2}\epsilon^2 V^{(2)}_{\ell m}(\eta) + \cdots. \tag{102}$$

Inserting the expressions (101) and (102) into Eqs. (99) and (100), we obtain the following set of recurrence relations:

$$\left(\frac{d^2}{\eta^2} + \frac{2}{\eta}\frac{d}{d\eta} - \frac{\ell(\ell+1)}{\eta^2} + w_*^2\right)U^{(0)}_{\ell m} = 0, \tag{103a}$$

$$\left(\frac{d^2}{\eta^2} + \frac{2}{\eta}\frac{d}{d\eta} - \frac{\ell(\ell+1)}{\eta^2} + w_*^2\right)U^{(1)}_{\ell m} = \frac{1}{\eta^3}\left[\ell^2 + \ell - 1 + 4ma_*w_* - 2w_*^2\eta^2 - \eta\frac{d}{d\eta}\right]U^{(0)}_{\ell m} - \frac{6a_*}{\eta^4}[(\ell-1)C_{\ell,m}V^{(0)}_{\ell-1,m} - (\ell+2)C_{\ell+1,m}V^{(0)}_{\ell+1,m}], \tag{103b}$$

$$\left(\frac{d^2}{\eta^2}+\frac{2}{\eta}\frac{d}{d\eta}-\frac{\ell(\ell+1)}{\eta^2}+w_*^2\right)U^{(2)}_{\ell m}=\frac{2}{\eta^3}\left[\ell^2+\ell-1+4ma_*w_*-2w_*^2\eta^2-\eta\frac{d}{d\eta}\right]U^{(1)}_{\ell m}+\frac{2}{\eta^4}\left[\ell^2+\ell-1+8ma_*w_*-3w_*^2\eta^2-\eta\frac{d}{d\eta}\right]U^{(0)}_{\ell m}-\frac{6a_*}{\eta^4}[(\ell-1)C_{\ell,m}V^{(1)}_{\ell-1,m}-(\ell+2)C_{\ell+1,m}V^{(1)}_{\ell+1,m}]-\frac{12a_*}{\eta^5}[(\ell-1)C_{\ell,m}V^{(0)}_{\ell-1,m}-(\ell+2)C_{\ell+1,m}V^{(0)}_{\ell+1,m}], \tag{103c}$$

and

$$\left(\frac{d^2}{\eta^2}+\frac{2}{\eta}\frac{d}{d\eta}-\frac{\ell(\ell+1)}{\eta^2}+w_*^2\right)V^{(0)}_{\ell m}=0, \tag{104a}$$

$$\left(\frac{d^2}{\eta^2}+\frac{2}{\eta}\frac{d}{d\eta}-\frac{\ell(\ell+1)}{\eta^2}+w_*^2\right)V^{(1)}_{\ell m}=\frac{1}{\eta^3}\left[\ell^2+\ell-1+4ma_*w_*-2w_*^2\eta^2-\eta\frac{d}{d\eta}\right]V^{(0)}_{\ell m}+\frac{6a_*}{\eta^4}[(\ell-1)C_{\ell,m}U^{(0)}_{\ell-1,m}-(\ell+2)C_{\ell+1,m}U^{(0)}_{\ell+1,m}], \tag{104b}$$

$$\left(\frac{d^2}{\eta^2}+\frac{2}{\eta}\frac{d}{d\eta}-\frac{\ell(\ell+1)}{\eta^2}+w_*^2\right)V^{(2)}_{\ell m}=\frac{2}{\eta^3}\left[\ell^2+\ell-1+4ma_*w_*-2w_*^2\eta^2-\eta\frac{d}{d\eta}\right]V^{(1)}_{\ell m}+\frac{2}{\eta^4}\left[\ell^2+\ell-1+8ma_*w_*-3w_*^2\eta^2-\eta\frac{d}{d\eta}\right]V^{(0)}_{\ell m}+\frac{6a_*}{\eta^4}[(\ell-1)C_{\ell,m}U^{(1)}_{\ell-1,m}-(\ell+2)C_{\ell+1,m}U^{(1)}_{\ell+1,m}]+\frac{12a_*}{\eta^5}[(\ell-1)C_{\ell,m}U^{(0)}_{\ell-1,m}-(\ell+2)C_{\ell+1,m}U^{(0)}_{\ell+1,m}], \tag{104c}$$

where $w_*=wR$ is the dimensionless frequency. We are now in a position to solve wavelike equations (103) and (104) at least up the linear order of stellar compactness. It is easy to see that once we find the functions in the zeroth approximation, i.e., $U^{(0)}_{\ell m}$ and $V^{(0)}_{\ell m}(\eta)$, then one can find other functions in any highest order approximation $U^{(n)}(\eta)$. Here note that the functions $U^{(0)}_{\ell m}(\eta)$ and $V^{(0)}_{\ell m}(\eta)$ represent solutions of wavelike equations in flat space that can be found in Ref. [26], and it can be expressed in terms of the spherical Hankel functions $h^{(1)}_\ell$ and $h^{(2)}_\ell$,

$$U^{(0)}_{\ell m}(\eta)=\alpha^{(0)}_{\ell m}h^{(1)}_\ell(w_*\eta)+\beta^{(0)}_{\ell m}h^{(2)}_\ell(w_*\eta), \tag{105}$$

where the first term of solution (105) is responsible for the pure outgoing wave which we are interested in, while the second term is responsible for the pure incoming wave, which can be safely removed by setting $\beta^{(0)}_{\ell m}=0$, so that the solution will be $U^{(0)}_{\ell m}(\eta)=\sqrt{\ell(\ell+1)}h^{(1)}_\ell(\tilde{\omega}\eta)u_{\ell m}$. The solution of Eq. (103) in the first approximation can be found $U^{(1)}_{\ell m}=\alpha^{(1)}_{\ell m}h^{(1)}_\ell+\beta^{(1)}_{\ell m}h^{(2)}_\ell+\mathcal{U}^{(1)}_{\ell m}$, where $\mathcal{U}^{(1)}_{\ell m}$ is the particular solution of Eq. (103). In order to solve this equation we use the Lagrange method, and the general solution can be expressed as

$$U^{(1)}_{\ell m}(\eta)=\alpha^{(1)}_{\ell m}h^{(1)}_\ell(w_*\eta)+\beta^{(1)}_{\ell m}h^{(2)}_\ell(w_*\eta)+h^{(1)}_\ell(w_*\eta)\int d\eta\frac{h^{(2)}_\ell(w_*\eta)}{W}F(\eta)-h^{(2)}_\ell(w_*\eta)\int d\eta\frac{h^{(1)}_\ell(w_*\eta)}{W}F(\eta), \tag{106}$$

where W is the Wronskian of the solutions, i.e., $W=-2i/(\tilde{\omega}\eta)^2$, and the function $F(\eta)$ is defined as

$$F=\frac{1}{\eta^3}\left[\ell^2+\ell-1+4ma_*w_*-2w_*^2\eta^2-\eta\frac{d}{d\eta}\right]U^{(0)}_{\ell m}-\frac{6a_*}{\eta^4}[(\ell-1)C_{\ell,m}V^{(0)}_{\ell-1,m}-(\ell+2)C_{\ell+1,m}V^{(0)}_{\ell+1,m}]. \tag{107}$$

Notice that a very similar equation can be obtained for $V^{(1)}_{\ell m}(\eta)$ in the linear approximation.

C. Electromagnetic radiation

In this section, we investigate the Lense-Thirring effect in the electromagnetic dipole radiation from a rotating magnetized neutron star. Here one has to emphasize that such a phenomenon is at the basis of the observational evidence of radio pulsars identified with the rotating magnetized neutron stars. The classical expression for energy loss through the electromagnetic dipole radiation from the rotating magnetized neutron star reads [57]

$$L_{\rm Newt.} = \frac{B_0^2\Omega^4 R^6}{6c}\sin^2\chi, \tag{108}$$

where χ is the inclination angle of the magnetic moment relative to the rotational axe of the NS.

However, in Ref. [26] the expression for electromagnetic dipole radiation is modified to include general relativistic effects, namely, in magnetic fields at the stellar surface, and take into account that redshift arises due to general relativity. The explicit form of the stellar luminosity can be written as

$$L_{\rm GR} = \frac{\tilde{B}^2\Omega_R^4 R^6}{6c}\sin^2\chi, \qquad \Omega_R = \frac{\Omega}{\sqrt{f_R}}, \tag{109}$$

$$\frac{\tilde{B}}{B_0} = -\frac{3R^3}{4M^3}\left[\ln f_R + \frac{2M}{R}\left(1+\frac{M}{R}\right)\right], \tag{110}$$

where subindex $_R$ denotes the value of the quantities at the stellar surface. In order to investigate the significance of the general relativistic effect one can take the ratio of the expressions for the electromagnetic dipole radiation from a rotating magnetized neutron star, so that

$$\frac{L_{\rm GR}}{L_{\rm Newt.}} = \frac{9R^6}{16M^6 f_R^2}\left[\ln f_R + \frac{2M}{R}\left(1+\frac{M}{R}\right)\right]^2. \tag{111}$$

In the present paper we wish to investigate how the Lense-Thirring effect affects in pure electromagnetic dipole radiation of the rotating magnetized star. In order to estimate the Lense-Thirring effect in the luminosity carried away by the electromagnetic dipole radiation $L_{\rm EM}$ we first have to take the integral from the radial component of the Poynting vector which can be written as $S_r \simeq [\mathbf{E}\times\mathbf{B}^*]_r = E^{\hat\theta}B^{\widehat{\phi *}} - E^{\hat\phi}B^{\widehat{\theta *}}$, and then using the results in the previous subsection we obtain

$$\begin{aligned} L &= \frac{1}{4\pi}\sum_{\ell,m}\frac{(w-m\omega)}{\ell(\ell+1)}(U_{\ell m}DU_{\ell m} + V_{\ell m}DV_{\ell m}) \\ &\simeq \frac{1}{4\pi}\sum_{\ell,m}\frac{(w-m\omega)}{\ell(\ell+1)}Dh_\ell(wr)h_\ell(wr)(|u_{\ell m}|^2 + |v_{\ell m}|^2), \end{aligned} \tag{112}$$

where an operator, D, is defined as $Df = (1/r)\partial_r(rf)$. Using the techniques presented in Ref. [26] and after making lengthy calculations, finally, we obtain the expression for the electromagnetic dipole radiation from a rotating magnetized neutron star in terms of Lense-Thirring angular velocity,

$$L_{\rm LT} = \frac{\tilde{B}^2\Omega_R^4 R^6}{6c}\kappa\sin^2\chi, \tag{113}$$

where κ is defined as

$$\kappa \simeq 1 + \frac{\omega_R^2}{\Omega_R^2 - \omega_R^2} + \frac{4\omega_R^4}{3(\Omega_R^2-\omega_R^2)^2}. \tag{114}$$

Since the stellar angular velocity is larger than Lense-Thirring angular velocity $\Omega_R > \omega_R$, then simple analysis shows that κ is always larger than 1 ($\kappa \geq 1$), which means the luminosity of the magnetized star increases due to the Lense-Thirring effect. Let us express the Lense-Thirring angular velocity in terms of stellar angular velocity

$$\omega(r) = \frac{2J}{r^3} = \frac{2I\Omega}{r^3}. \tag{115}$$

Here we will introduce the new parameter β which characterizes the ratio of the general relativistic and Newtonian value of the moment of inertia, i.e., $\beta = I/I_0$, where $I_0 = MR^2$ is the Newtonian value of the moment of inertia, so then the Lense-Thirring angular velocity in (115) at the stellar surface takes a form, $\omega(R) = \epsilon\beta\Omega$. Finally, the expression for the electromagnetic dipole radiation from a rotating magnetized neutron star takes a form,

$$L_{\rm LT} = \frac{\tilde{B}^2\Omega_R^4 R^6}{6c}\left(1 + \frac{\epsilon^2\beta^2}{1-\epsilon^2\beta^2} + \frac{4\epsilon^4\beta^4}{3(1-\epsilon^2\beta^2)^2}\right)\sin^2\chi. \tag{116}$$

Figure 3 draws dependence on the rate of energy loss from the compactness of the magnetized neutron star in both cases with and without the Lense-Thirring effect. The plot shows that the Lense-Thirring term forces the luminosity to become larger than that in general relativistic cases. Figure 4 illustrates the field line of the magnetic dipole around the typical (i.e., $\epsilon = 0.4$) misaligned neutron star for the different values of the inclination angles. One can easily see that with increases of the inclination angle, magnetic field structures around the misaligned star become more complex; even separate closed magnetic field lines can occur due to the rotation of the star. Another important thing from Fig. 4 is that one can see that due to the general relativistic effect the magnetic field line becomes denser than that in the Newtonian case, in particular, the polar cap region of the star.

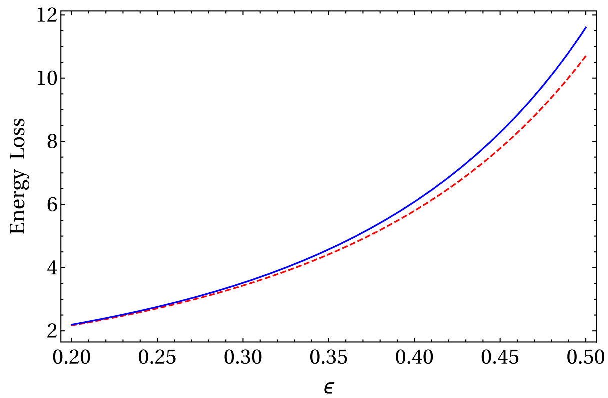

FIG. 3. Dependence of the energy losses $L_{\mathrm{GR}}/L_{\mathrm{Newt}}$. from the stellar compactness at $\beta = 0.4$. Solid (blue) line represents energy loss for the existence of the Lense-Thirring term, while dashed (red) line is responsible for the energy-loss absence of the Lense-Thirring angular velocity.

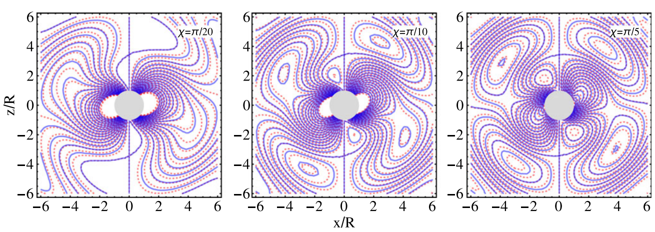

FIG. 4. Magnetic field line of misaligned neutron star for different values of inclination angles at $\epsilon = 0.4$. The rotation of the star is to be taken as $\Omega R = 0.1$

VII. NUMERICAL SOLUTIONS FOR INTERIOR ELECTROMAGNETIC FIELDS

In Sec. IV, we showed in a detailed derivation of the second order differential equations for the electromagnetic fields exterior spacetime of rotating magnetized NS. In this section we will do some calculations for the electromagnetic fields in the interior structure of the star as we did in Sec. IV. Using Eqs. (49), (50), and (51) we obtain the following second order differential equations for radial components of the electromagnetic fields:

$$\tilde{\Box} B^{\hat{r}} - \frac{e^{\Phi-\Lambda}\omega'}{\sin\theta}\partial_\theta(\sin^2\theta E^{\hat{r}}) = \frac{4\pi\Omega e^{\Phi}}{\sin\theta}\partial_\theta(\sin^2\theta\rho_e) + \frac{4\pi\sigma\bar{\omega}\omega r}{\sin\theta}\partial_\theta(\sin^3\theta E^{\hat{\phi}}) \quad (117)$$

$$\tilde{\Box} E^{\hat{r}} + \frac{e^{\Phi-\Lambda}\omega'}{\sin\theta}\partial_\theta(\sin^2\theta B^{\hat{r}}) = -\frac{4\pi e^{\Phi-\Lambda}}{r^2}\partial_r(e^{\Phi}r^2\rho_e) + 4\pi\sigma\sin\theta e^{\Phi-\Lambda}[\bar{\omega}(2-r\phi') + r\bar{\omega}']E^{\hat{\phi}}, \quad (118)$$

where the operator $\tilde{\Box}$ is defined as

$$\tilde{\Box} = (\partial_t + \omega\partial_\phi)^2 + 4\pi\sigma e^{\Phi}(\partial_t + \Omega\partial_\phi) - \frac{e^{\Phi-\Lambda}}{r^2}\partial_r(e^{\Phi-\Lambda}\partial_r r^2) - \frac{e^{2\Phi}}{r^2}\Delta_\Omega. \quad (119)$$

In the absence of the rotation of the star, Eqs. (117) and (118) take the form

$$(\partial_t^2 + 4\pi\sigma e^{\Phi}\partial_t)B^{\hat{r}} = \tilde{\Delta} B^{\hat{r}}, \quad (120)$$

$$(\partial_t^2 + 4\pi\sigma e^{\Phi}\partial_t)E^{\hat{r}} = \tilde{\Delta} E^{\hat{r}} - 4\pi\frac{e^{\Phi-\Lambda}}{r^2}\partial_r(e^{\Phi}r^2\rho_e), \quad (121)$$

where

$$\tilde{\Delta} = \frac{1}{r^2}e^{\Phi-\Lambda}\partial_r(e^{\Phi-\Lambda}\partial_r r^2) - \frac{e^{2\Phi}}{r^2}\Delta_\Omega. \quad (122)$$

A. Toy model

In this subsection we consider a simple model of NS to investigate the interior electromagnetic fields. For simplicity, assume that the density of NS uniform $\rho = \rho_0 = \mathrm{const}$, then the internal solution can be found as [51]

$$e^{\Phi(r)} = \frac{3}{2}\sqrt{1-\frac{2M}{R}} - \frac{1}{2}\sqrt{1-\frac{2Mr^2}{R^3}}, \quad (123)$$

$$e^{-\Lambda(r)} = \sqrt{1-\frac{2Mr^2}{R^3}}, \quad r \le R, \quad (124)$$

and the internal pressure is given by the following equation of state:

$$p(r) = -\rho_0\frac{\sqrt{1-\frac{2M}{R}} - \sqrt{1-\frac{2Mr^2}{R^3}}}{3\sqrt{1-\frac{2M}{R}} - \sqrt{1-\frac{2Mr^2}{R^3}}}, \quad r \le R, \quad (125)$$

which is zero at the surface of star $p(R) = 0$, while at the center of the star it is a constant $p(0) = -\rho_0(\sqrt{1-\epsilon}-1)/(3\sqrt{1-\epsilon}-1)$. One has to emphasize that the pressure arises due to the general relativistic effects in the Tolman-Oppenheimer-Volkoff (TOV) equations, which is why in the Newtonian case, i.e., $\epsilon \to 0$, its correction is negligible. In Fig. 5 (left panel) the radial

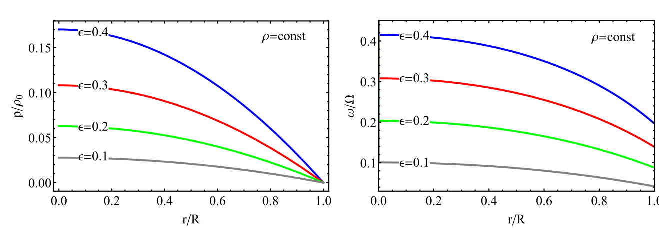

FIG. 5. Left: Radial dependence of pressure $p(r)$ of the medium with uniform density, i.e., $\rho = \mathrm{const}$, for different values of compactness of the star. Right: Radial dependence of normalized Lense-Thirring angular velocity for different values of compactness of the star.

dependence of the pressure inside the star for different values of stellar compactness is illustrated. From Fig. 5 one can see that the central value of the pressure is getting smaller for the smaller values of stellar compactness. As we mentioned before, it vanishes for the zero value of compactness.

Before we calculate the moment of inertia of the star, we first need to solve the differential equation for the angular velocity of frame dragging in (45), which can be rewritten as

$$\frac{d}{dr}(\eta^4 j(\eta)\bar{\omega}'(\eta)) + 4\eta^3 j'(\eta)\bar{\omega}(\eta) = 0, \tag{126}$$

$$j(\eta) = \frac{2\sqrt{1-\epsilon\eta^2}}{3\sqrt{1-\epsilon} - \sqrt{1-\epsilon\eta^2}}. \tag{127}$$

Of course, we cannot analytically solve this equation; however, it can be numerically solved with the following boundary conditions (see, for example, [58]):

$$\bar{\omega}(0) = 0, \bar{\omega}(1) + \frac{1}{3}\bar{\omega}'(1) = \Omega. \tag{128}$$

Hereafter carefully making a numerical calculation we can obtain radial dependence of the Lense-Thiring angular velocity, as shown in the right panel of Fig. 5. Indeed, the angular velocity of frame dragging arises from the general relativistic effect, and it is shown that for the value $\epsilon = 0.4$, one gets $\omega/\Omega \simeq 0.4$, while this fractional angular velocity decreases with decreasing the stellar compactness and it will be zero in the Newtonian limit (i.e., $\epsilon = 0$).

We now calculate the moment of inertia of the neutron star in the case of uniform density. To do this we rewrite Eq. (46) in the form

$$I(\epsilon) = I_0 \int_0^1 \eta^4 d\eta \frac{20\sqrt{1-\epsilon}}{(3\sqrt{1-\epsilon} - \sqrt{1-\epsilon\eta^2})^2} \frac{\bar{\omega}(\eta)}{\Omega}, \tag{129}$$

where $I_0 = (2/5)MR^2$ is the Newtonian value of the moment inertia which can be obtained by setting $\epsilon = 0$ and $\bar{\omega}/\Omega = 1$; here we have used the following standard relation for the neutron star mass $M = \int \rho d\mathbf{r} = (4\pi/3)R^3\rho_0$ for uniform density matter. As we can see from Eq. (129) the ratio of the moment of inertia depends on the stellar compactness only. Table II shows the dependence of the fractional moment of inertia from the stellar compactness. One can easily see that the general relativistic effect is not large in the case of the neutron star with uniform density and in this case the moment of inertia of the relativistic star is larger than that in the Newtonian by about 3% as shown in Table II.

TABLE II. Dependence of the fractional moment of inertia from the stellar compactness.

Stellar compactness, ϵ	0.1	0.2	0.3	0.4
Ratio of moment of inertia, I/I_0	1.00731	1.01501	1.02319	1.03201
Difference of moment of inertia, $(I - I_0)/I_0$	0.731%	1.501%	2.319%	3.201%

Now we are a position to determine the interior magnetic and electric fields inside the neutron star with uniform density. For simplicity we assume that the angular velocity of frame dragging is relatively smaller than the stellar angular velocity $\omega \leq \Omega$. On the other hand, as we mentioned before the conductivity in the stellar medium is very high $\sigma \to \infty$, which allows one to rewrite Eq. (117) in the form

$$\left[\partial_r(e^{\Phi-\Lambda}\partial_r) - e^{\Phi+\Lambda}\frac{\ell(\ell+1)}{r^2}\right]R^B_{\ell m}(r) = 0, \tag{130}$$

where $R^B_{\ell m}(r)$ is a radial part of the radial component of the magnetic field as shown in Eq. (57a). In this paper, we are interested in finding the interior dipole ($\ell = 1$) magnetic field of the nonrotating and high conducting ideal star with uniform density. Obviously we cannot find an analytical solution of Eq. (130) even for the internal background spacetime of the star with uniform density. Nevertheless, we can make numerical calculations in order to find the interior magnetic field. Since Eq. (130) is the second-order differential equation, one has to require two boundary conditions: (i) continuity of the normal magnetic field across the surface of the star, i.e., $B^{\hat{r}}_{\rm Int}(R) = B^{\hat{r}}_{\rm Ext}(R)$, where the radial part of the normal magnetic field ($\ell = 1$) is given in Table I; (ii) the critical value of the magnetic field strength at a given position inside the star $B_c = B^{\hat{r}}(r_c)$, where the critical magnetic field indicates the maximum value of the field inside of the star overall. Finally, after making numerical calculation, we found that the interior magnetic field increases as shown in Fig. 6.

We now concentrate on the interior electric field of the nonrotating ideal star. Since the effect of the frame dragging is small, then the induced electric field generated by the magnetic field can also be negligible; again under the assumption of a high conducting star, the equation for the electric field (118) can be written as

$$\left[\frac{1}{r^2}\partial_r(e^{\Phi-\Lambda}\partial_r r^2) - \frac{e^{\Phi+\Lambda}}{r^2}\Delta_\Omega\right]E^{\hat{r}} = \frac{4\pi}{r^2}\partial_r(e^{\Phi}r^2\rho_e). \tag{131}$$

From Eq. (131) one can see that the interior electric field depends on the charge distribution inside the star. Since mass density inside the star is uniform, then we also can assume that the charge density can also be uniform $\rho_e(\mathbf{r}) = \rho_{\mathbf{e0}}$. Following this fact, one can conclude that the electric field will be only radially dependent (i.e., $\Delta_\Omega E^{\hat{r}} = 0$) and Eq. (131) can be expressed as

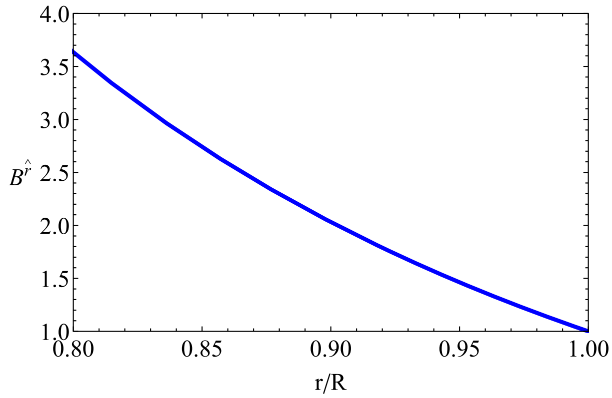

FIG. 6. Radial dependence of the interior dipole ($\ell = 1$) magnetic field of nonrotating ideal star with uniform density.

$$\frac{1}{r^2}\partial_r(r^2 E^{\hat{r}}) = 4\pi\rho_{e0}e^{\Lambda}. \tag{132}$$

Note that Eq. (132) can be directly obtained from the first equation in (50) in the case of the monopole electric field and on the right-hand side of Eq. (132), and e^{Λ} stands to represent a general relativistic correction to the equation for the monopole electric field in Newtonian theory. Hereafter taking integration from Eq. (132) one can get

$$E^{\hat{r}} = \frac{\pi\rho_{e0}R^3}{Mr}\left[\left(\frac{2Mr^2}{R^3}\right)^{-1/2}\sin^{-1}\left(\frac{2Mr^2}{R^3}\right)^{1/2} - \left(1 - \frac{2Mr^2}{R^3}\right)^{1/2}\right]. \tag{133}$$

The Newtonian limit of the expression for the monopole electric field is rather simple: $E^{\hat{r}} = 4\pi\rho_{e0}r/3$. In order to see the general relativistic effect in the interior monopole electric field of nonrotating and high conducting ideal star one can expand expression (133) in the power of the Newtonian potential, i.e., M/R (or stellar compactness),

$$E^{\hat{r}} \simeq \frac{4\pi\rho_{e0}r}{3}\left(1 + \frac{3Mr^2}{5R^3} + \frac{9M^2r^4}{14R^6} + \mathcal{O}(M^3/R^3)\right). \tag{134}$$

From expression (134) one can immediately see that the interior electric field of the ideal star with uniform density increases due to the general relativistic effect. Figure 7 draws radial dependence of the interior monopole electric field of the nonrotating and highly conducting ideal star for different values of the stellar compactness. From Fig. 7 one can see that GR effects are negligible at a small distance interior region of the star. However, near the stellar surface we can see that the absolute value of the radial electric field increases up to 15% due to GR effects.

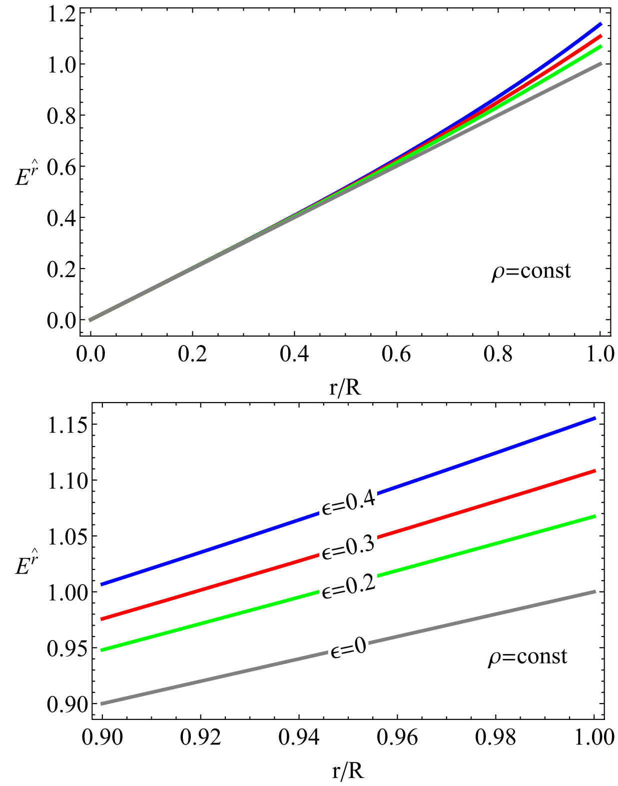

FIG. 7. Top panel: Radial dependence of interior monopole electric field of the nonrotating and highly conducting ideal star with uniform density for different values of stellar compactness whole region. Bottom panel: Same figure near the stellar surface.

VIII. CONCLUSION

In the present research paper, we have investigated general relativistic effects in the electromagnetic fields around rotating magnetized NS. We considered general relativistic Maxwell equations in the background spacetime of the slowly rotating magnetized NS. We have first shown the explicit derivation of the second-order differential equations for the profile functions of the electromagnetic fields in the background spacetime of rotating magnetized NS. Finally, we made the general relativistic correction on the components of the multipole electromagnetic fields. Obtained results are summarized as follows:

(i) It has been shown that the Lense-Thirring term plays a very important role to obtain coupled equations for the radial components of electromagnetic fields. Due to the Lense-Thirring effect, symmetry of the second-order wavelike equations for the electromagnetic field is broken unlike in the nonrotating spacetime.

(ii) It has been shown that even in the background of a slowly rotating spacetime the components of the electromagnetic fields can be expanded in terms of the spherical harmonics as usually done in nonrotating spacetime, and in the presence of the frame-dragging effect (Lense-Thirring effect) all six components of the electromagnetic fields can be expressed in terms of two radial profile functions which satisfy the coupled wavelike equations.

(iii) It is shown that in the presence of the frame dragging, it is impossible to get an analytical solution of Maxwell equations for the electromagnetic fields; however, in the slowly rotating limit of the spacetime of NS one can obtain analytical expressions for the components of multipole electromagnetic fields. We also showed that general relativistic effects are larger for the high order of multipole solutions. Another interesting fact is shown that even the dipole electric field can be measured in slow rotation approximation of the spacetime of NS.

(iv) Finally, as an astrophysical sequence, we have computed the magnetodipole radiation's luminosity of magnetized NS. The semianalytical expression for the luminosity of the magnetized NS has been derived at small stellar compactness and slow rotation approximation spacetime.

(v) We derive a general expression for the radial components of the electromagnetic fields which also allows computing interior electromagnetic fields of the neutron star. We give a description to investigate the time evaluation of the magnetic field of the slowly rotating magnetized neutron star. As a toy model, we compute the interior dipole magnetic field as well as the monopole electric field interior region of NS described by internal Schwarzschild spacetime.

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APPENDIX: GENERAL RELATIVISTIC MAXWELL EQUATIONS

The explicit form of the components of the tensor of the electromagnetic field in terms of the magnetic $B^{\hat{r}}$ and electric $E^{\hat{r}}$ fields in the ZAMO frame can be expressed as [59]

$$F_{\alpha\beta} = \begin{pmatrix} 0 & -e^{\Phi+\Lambda}E^{\hat{r}} - \omega e^{\Lambda} r\sin\theta B^{\hat{\theta}} & -e^{\Phi} rE^{\hat{\theta}} + \omega r^2 \sin\theta B^{\hat{r}} & -e^{\Phi} r\sin\theta E^{\hat{\phi}} \\ e^{\Phi+\Lambda}E^{\hat{r}} + \omega e^{\Lambda} r\sin\theta B^{\hat{\theta}} & 0 & e^{\Lambda} rB^{\hat{\phi}} & -e^{\Lambda} r\sin\theta B^{\hat{\theta}} \\ e^{\Phi} rE^{\hat{\theta}} - \omega r^2\sin\theta B^{\hat{r}} & -e^{\Lambda} rB^{\hat{\phi}} & 0 & r^2\sin\theta B^{\hat{r}} \\ e^{\Phi} r\sin\theta E^{\hat{\phi}} & e^{\Lambda} r\sin\theta B^{\hat{\theta}} & -r^2\sin\theta B^{\hat{r}} & 0 \end{pmatrix} \tag{A1}$$

and

$$^{*}F_{\alpha\beta} = \begin{pmatrix} 0 & -e^{\Phi+\Lambda}B^{\hat{r}} - \omega e^{\Lambda} r\sin\theta E^{\hat{\theta}} & -e^{\Phi} rB^{\hat{\theta}} + \omega r^2 \sin\theta E^{\hat{r}} & -e^{\Phi} r\sin\theta B^{\hat{\phi}} \\ e^{\Phi+\Lambda}B^{\hat{r}} + \omega e^{\Lambda} r\sin\theta E^{\hat{\theta}} & 0 & e^{\Lambda} rE^{\hat{\phi}} & -e^{\Lambda} r\sin\theta E^{\hat{\theta}} \\ e^{\Phi} rB^{\hat{\theta}} - \omega r^2\sin\theta E^{\hat{r}} & -e^{\Lambda} rE^{\hat{\phi}} & 0 & r^2\sin\theta E^{\hat{r}} \\ e^{\Phi} r\sin\theta B^{\hat{\phi}} & e^{\Lambda} r\sin\theta E^{\hat{\theta}} & -r^2\sin\theta E^{\hat{r}} & 0 \end{pmatrix}. \tag{A2}$$

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