Scalar-multitensor approach to teleparallel modified theories of gravity

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The analysis of teleparallel $f(T, \nabla_{\mu_1} T, \ldots, \nabla_{\mu_n} \cdots \nabla_{\mu_1} T)$ gravity in the Jordan and Einstein frames is presented. The equivalence between $f(T, \nabla_{\mu_1} T, \ldots, \nabla_{\mu_n} \cdots \nabla_{\mu_1} T)$ gravity and a scalar-multitensor theory is proved in both frames for systems with a regular Hessian matrix. For each order of derivative an auxiliary tensor of the same order is introduced. As a consequence, the order of the differential equation for the tetrad field is reduced to an equation of order two, but the price to be paid is the analysis of a system of coupled equations for the auxiliary fields.

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I. INTRODUCTION

Despite the success of general relativity (GR) in predicting the existence of gravitational waves, light deflection, etc., some problems still remain unanswered by this theory. For instance, the nonrenormalizability [1,2] and the existence of singularities prevent the construction of a self-consistent theory of quantum gravity. Two other problems are related to galaxy rotation curves and the accelerated expansion of the universe, both of them not accommodated by GR in the presence of ordinary matter [3–6]. A possible solution to these last two problems is the introduction of exotic contents of matter-energy, known as *dark energy* and *dark matter* [7–11]. Accordingly, these two should correspond to 95% of the total amount of matter and energy of the Universe.

As an alternative to the "dark side" (and also as an attempt to circumvent the problems of the quantization of gravity), there are some proposals which suggest modifications to the geometric content of the field equations. Several of these modifications also end up presenting some problems like the existence of ghosts, etc., but they are still of interest either as toy models or as effective theories. In the Riemann manifold, these modifications include the so called "$f(R)$ theories", as well as other modifications in which the Lagrangian has a functional dependence with the scalar curvature, the Riemann, Ricci or Weyl tensors, and eventually with their derivatives [12–51].

Other proposals include modifying the underlying manifold in which the theory is built. In particular, if the Riemann manifold is replaced, then other modifications can be studied. Here, the interest is devoted to theories constructed on the Weitzeböck manifold [52,53], which is a manifold that presents torsion instead of curvature. These theories are known as teleparallel gravities and among them one was proposed to be equivalent to GR, the so called "teleparallel equivalent of general relativity" (TEGR) [53–59]. The Lagrangian of this theory is linear in the scalar torsion T, just like the Einstein-Hilbert Lagrangian is linear in the scalar curvature R. Since TEGR is in many ways equivalent to GR, many of the problems found on the latter still remain on the former. This way, extensions of TEGR (and their applications) have been considered in modifications that include "$f(T)$ theories" and other models where the Lagrangian depends on derivatives of the torsion scalar [60–73].

Extensions of the teleparallel gravity usually modify the number of degrees of freedom (d.o.f.) of the physical system when comparing with the number obtained in TEGR. Determining the d.o.f. is not a trivial task as has been discussed in Refs. [65,68,69,71]. An interesting approach to deal with this problem is to consider the system in the Jordan and/or Einstein frames. In both representations, auxiliary variables are introduced and in some cases, the analysis can be carried out in a simpler way. For instance, for $f(T)$ theories (in which the field equations are second order differential equations in the geometric representation), a scalar field is introduced and the analysis of the field equations or the analysis of the Hamiltonian structure can clarify the number of d.o.f. of the problem [65,68].

Theories that include derivatives of the scalar torsion usually lead to differential equations of order higher than two. In cases like these, the analysis of the system in the Jordan and Einstein frames can be even more interesting once the order of the differential equations can be reduced in these representations. As it will be seen, besides the usual scalar field that usually one obtains in $f(T)$ theories, other auxiliary tensor fields will be introduced. In both frames, the result is a scalar-multitensor theory. The results

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obtained here are somehow similar to those obtained in Riemann manifolds for $f(R, \nabla_{\mu_1} R, \ldots, \nabla_{\mu_n} \cdots \nabla_{\mu_1} R)$ theories [47,48], in which a complicated higher-order differential equation is replaced by a system of coupled equations of lower order of derivatives.

As far as the author is concerned, the analysis of $f(T, \nabla_{\mu_1} T, \ldots, \nabla_{\mu_n} \cdots \nabla_{\mu_1} T)$ theories in the Jordan and Einstein frames has not been addressed in the literature. This is what is going to be explored here. The aim of this work is to present the Jordan and Einstein representations for these theories. In Sec. II, $f(T, \nabla_\mu T, e^a_\mu)$ theories will analyzed in both Jordan and Einstein frames. In Sec. III, the analysis of Sec. II is extended to include higher order derivatives. In Sec. IV, an example is presented. Section V will be dedicated to the final remarks.

II. FOURTH-ORDER TELEPARALLEL THEORIES OF GRAVITY

The fundamental variables that will be considered here are the tetrad fields e^a_μ. Latin indexes $a, b, c, \ldots, g = (0), (1), (2), (3)$ label the internal/tangent space coordinates; greek indexes refer to the Weitzenböck spacetime coordinates running from 0 to 3, while latin indexes $i, j, k, \ldots = 1, 2, 3$ label space coordinates.

The tetrads and the spacetime metric tensor $g_{\mu\nu}$ are related by

$$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu, \qquad \eta_{ab} = e^\mu_a e^\nu_b g_{\mu\nu}, \tag{1}$$

where η_{ab} is the Minkowski metric tensor and the tetrad e^μ_a and e^a_μ satisfy the duality relation:

$$e^a_\mu e^\nu_a = \delta^\nu_\mu, \qquad e^a_\mu e^\mu_b = \delta^a_b. \tag{2}$$

The covariant derivative is defined in terms of the Weitzenböck connection

$$\Gamma^\alpha_{\mu\nu} = e^\alpha_a \partial_\mu e^a_\nu, \tag{3}$$

which is a connection that presents torsion,

$$T^\alpha{}_{\mu\nu} \equiv \Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu} = e^\alpha_a [\partial_\mu e^a_\nu - \partial_\nu e^a_\mu], \tag{4}$$

but no curvature.

A proper index contraction of the torsion tensor defines the vector

$$T_\mu \equiv T^\nu{}_{\nu\mu}, \tag{5}$$

while a quadratic combination defines the *scalar torsion*:

$$T \equiv \frac{1}{4} T^{\beta\rho\mu} T_{\beta\rho\mu} + \frac{1}{2} T^{\beta\rho\mu} T_{\rho\beta\mu} - T^\beta T_\beta \tag{6}$$

This scalar can also be expressed in terms of the *superpotential* [63,64] or *dual torsion* [66]:

$$\Sigma^{\nu\alpha\mu} \equiv \frac{1}{4}(T^{\nu\alpha\mu} + T^{\alpha\nu\mu} - T^{\mu\nu\alpha}) + \frac{1}{2}(g^{\mu\nu} T^\alpha - g^{\alpha\nu} T^\mu) \tag{7}$$

so that $T = \Sigma^{\beta\rho\mu} T_{\beta\rho\mu}$.

In order to study a gravitation theory equivalent to general relativity in a Weitzenböck manifold, one has to consider the action

$$S = \int d^4 x e T + S_{\mathrm{matter}}, \tag{8}$$

where $e = \det e^a_\mu = \sqrt{-g}$ and S_{matter} stands for the action of a matter field. This leads to the so called TEGR.

The simplest modification that can be made in teleparallel gravity is the replacement of T in the Lagrangian by a function of T. This is known in the literature as "$f(T)$ gravity". In this section, the interest is devoted to a more general class of teleparallel gravity, in which the gravitational Lagrangian depends on a general function of T and its first derivative:

$$S_g = \int d^4 x e f(T, \nabla_\mu T, e^a_\mu). \tag{9}$$

The explicit dependence of f with e^a_μ is necessary in order to consider the contribution of the tetrad which is not present in T and $\nabla_\mu T$. Actually, in order to build a scalar quantity with $\nabla_\mu T$, a contraction of indexes needs to take place. The tetrad can be used for this, and this is the reason why it is considered separately [74]. It is important to stress that covariant derivative considered here is build with the Weitzenböck connection and not the Levi-Civita one. This class of modified gravities will be analyzed in the geometric, Jordan and Einstein frames.

A. Geometric frame

In this subsection, the field equations will be analyzed considering only the tetrad fields as the independent variables. Since the Lagrangian depends on the derivative of the torsion, fourth-order equations differential equations for the tetrad are expected.

The variation of the action integral leads to

$$\delta S_g = \int d^4 x \left[e e^\alpha_a \delta e^a_\alpha f + \frac{\partial f}{\partial T} \delta T + \frac{\partial f}{\partial (\nabla_\mu T)} \delta(\nabla_\mu T) + \frac{\partial f}{\partial e^d_\alpha} \delta e^d_\alpha \right]. \tag{10}$$

The third term on the right-hand side (rhs) can be rewritten as

$$e\frac{\partial f}{\partial(\nabla_\mu T)}\delta(\nabla_\mu T) = \partial_\mu B^\mu_{(1)} - e\nabla^T_\mu \frac{\partial f}{\partial(\nabla_\mu T)}\delta T, \quad (11)$$

where the modified covariant derivative operator ∇^T_μ is defined as $\nabla^T_\mu \equiv \nabla_\mu - T_\mu$. After a straightforward calculation, the field equations are obtained, leading to:

$$\frac{4}{e}\partial_\sigma[e\bar{\Phi}_{(1)}\Sigma_d{}^{\sigma\alpha}] + 4\bar{\Phi}_{(1)}\Sigma^{\sigma\rho\alpha}T_{\sigma\rho d} - e^\alpha_d f - \frac{\partial f}{\partial e^d_\alpha} = 0, \quad (12)$$

where

$$\bar{\Phi}_{(1)} \equiv \left(\frac{\partial f}{\partial T} - \nabla^T_\mu \frac{\partial f}{\partial(\nabla_\mu T)}\right). \quad (13)$$

These equations can be up to fourth-order in the derivatives of the tetrad field according to the functional dependence of f with the derivative of the scalar torsion. For instance, this will be the case if this dependence is quadratic or of higher order. It is interesting to note that the structure of the field equations is pretty similar to the ones obtained in $f(T)$ theories [75]—one should replace $f'(T) \to (\frac{\partial f}{\partial T} - \nabla^T_\mu \frac{\partial f}{\partial(\nabla_\mu T)})$ and subtract the term $\frac{\partial f}{\partial e^d_\alpha}$.

Dealing with nonlinear fourth-order differential equations may be a hard task. As an alternative, the action can be studied in other frames where auxiliary fields are introduced. This will be explored in the next subsections.

B. Jordan frame

The analysis of the action (9) in the Jordan frame consists in introducing auxiliary variables in order to reduce the differential order of the field equations. The price to be paid is the presence of extra field equations that compose a system of coupled equations. In the present case, a scalar and a vector fields are introduced.

The starting point is to write an action S'_g that will be equivalent to S_g. This is done with the help of a scalar and a vector quantities which must be considered as independent fields; T and $\nabla_\mu T$ are taken as parameters:

$$S'_g = \int d^4xe\left[f(\xi,\xi_\mu,e^a_\mu) - \frac{\partial f}{\partial \xi}(\xi - T) - \frac{\partial f}{\partial \xi_\mu}(\xi_\mu - \nabla_\mu T)\right]. \quad (14)$$

The variations of this action (with respect to ξ and ξ_μ) under the assumption of the minimal action principle lead to the set of equations:

$$\begin{pmatrix} \frac{\partial^2 f}{\partial \xi^2} & \frac{\partial^2 f}{\partial \xi \partial \xi_\mu} \\ \frac{\partial^2 f}{\partial \xi_\nu \partial \xi} & \frac{\partial^2 f}{\partial \xi_\nu \partial \xi_\mu} \end{pmatrix} \begin{pmatrix} (\xi - T) \\ (\xi_\mu - \nabla_\mu T) \end{pmatrix} = 0. \quad (15)$$

The equivalence between S'_g and S_g is attained when

$$\det\begin{pmatrix} \frac{\partial^2 f}{\partial \xi^2} & \frac{\partial^2 f}{\partial \xi \partial \xi_\mu} \\ \frac{\partial^2 f}{\partial \xi_\nu \partial \xi} & \frac{\partial^2 f}{\partial \xi_\nu \partial \xi_\mu} \end{pmatrix} \neq 0, \quad (16)$$

i.e. when the Hessian matrix in regular. This implies

$$\begin{cases} \xi - T = 0, \\ \xi_\mu - \nabla_\mu T = 0. \end{cases} \quad (17)$$

Clearly, $\frac{\partial f}{\partial \xi}$ and $\frac{\partial f}{\partial \xi_\mu}$ play the role of Lagrangian multipliers, so they can be considered as independent fields

$$\begin{cases} \phi \equiv \frac{\partial f}{\partial \xi}, \\ \phi^\nu \equiv \frac{\partial f}{\partial \xi_\nu}, \end{cases} \quad (18)$$

These new quantities have an inversible relation with ξ and ξ_ν due to Eq. (16). In terms of these new fields ϕ and ϕ^μ, S'_g is given by

$$S'_g = \int d^4xe[\phi T + \phi^\mu \nabla_\mu T - \bar{U}(\phi,\phi_\mu,e^a_\mu)], \quad (19)$$

where

$$\bar{U}(\phi,\phi_\mu,e^a_\mu) \equiv \phi\xi + \phi^\mu\xi_\mu - f(\xi,\xi_\mu,e^a_\mu). \quad (20)$$

In this expression, one has to consider $\xi = \xi(\phi,\phi_\mu)$ and $\xi_\mu = \xi_\mu(\phi,\phi_\mu)$.

In Eq. (19), the vector field couples to the derivative of the scalar torsion. This coupling is such that the action still depends on the second derivative of the tetrad. However, this is a linear dependence so that a fourth-order equation will not be obtained for the tetrad field.

Notwithstanding, this derivative coupling can be eliminated with the introduction of a new scalar field Φ. For this, an integral by parts has to be performed in the term $e\phi^\mu\nabla_\mu T$ [similarly to what was done in Eq. (11)]. This new scalar field is defined as a linear combination of ϕ, the covariant derivative of ϕ^μ and the contraction of the vector field with the torsion (in its vector form):

$$\Phi \equiv \phi - \nabla^T_\mu \phi^\mu. \quad (21)$$

The action becomes

$$S'_g = \int d^4xe(\Phi T - U + \partial_\mu B^\mu), \quad (22)$$

where the potential U is given by

$$U = U(\Phi, \phi^\mu, \nabla^T_\mu \phi^\mu, e^a_\mu) \equiv (\Phi + \nabla^T_\mu \phi^\mu)\xi + \phi^\mu \xi_\mu - f(\xi, \xi_\mu, e^a_\mu), \tag{23}$$

and B^μ stands for a surface term.

The action given in Eq. (22) resembles a teleparallel scalar-tensor (Brans-Dicke-like) theory. However, one must recall that a vector field is also present in the potential. In principle, there are no kinetic terms for Φ and ϕ^μ, so one could infer that the field equations for these fields are not dynamical and will only set constraints. This may be misleading and the manipulation of the field equations can show otherwise. This way, the analysis of the dynamics and the counting of the degrees of freedom has to be done carefully for each case.

Finally, the field equations will be obtained. The fundamental fields are e^a_α, Φ, and ϕ^μ and the respective field equations are

$$\begin{cases} \frac{4}{e}\partial_\sigma(\Phi e \Sigma_a{}^{\sigma\alpha}) + 4\Phi\Sigma^{\sigma\rho\alpha}T_{\sigma\rho a} - e^\alpha_a(\Phi T - U) + \frac{\delta U}{\delta e^a_\alpha} = 0, \\ T - \frac{\delta U}{\delta \Phi} = 0, \\ \frac{\delta U}{\delta \phi^\mu} = 0. \end{cases} \tag{24}$$

The first of these equations presents several similarities with Eq. (12). However, it is clearly a second order differential equation for the tetrad field. The order of the second equation has to be determined for each case, but it seems reasonable to suppose it can depend only on the first derivative of the vector field. For the last equation, it can be up to second order since the potential U has a functional dependence with the covariant divergence of ϕ^μ.

This system can also be analyzed in the Einstein frame. This will be done in the next subsection.

C. Einstein frame

Now the analysis of the system in the Einstein frame is presented. Although using the term "Einstein frame" may be a "forced terminology" (as discussed in Ref. [65]), it will be applied in the same context that it was used in this reference. This means that the starting point is the action obtained in the Jordan frame—Eq. (22)—except by a surface term which is neglected. Next, a conformal transformation is applied to the tetrad field:

$$\tilde{e}^a_\mu(x) = \Omega(x) e^a_\mu(x), \qquad \tilde{e}^\mu_a(x) = \Omega^{-1}(x) e^\mu_a(x). \tag{25}$$

The conformal transformation is propagated to the Weitzenböck connection, torsion, scalar torsion, covariant derivative and so on. Some useful relations are

$$\tilde{\Gamma}^\alpha_{\nu\mu} = \Gamma^\alpha_{\nu\mu} + \delta^\alpha_\mu \partial_\nu \ln\Omega, \tag{26}$$

$$\tilde{T}^\alpha_{\nu\mu} = T^\alpha_{\nu\mu} + \delta^\alpha_\mu \partial_\nu \ln\Omega - \delta^\alpha_\nu \partial_\mu \ln\Omega, \tag{27}$$

$$\tilde{T}_\mu = T_\mu - 3\partial_\mu \ln\Omega, \tag{28}$$

$$\tilde{T} = \Omega^{-2}T + 4\Omega^{-2}T^\nu \partial_\nu \ln\Omega - 6\Omega^{-2} g^{\mu\nu}\partial_\nu \ln\Omega \partial_\mu \ln\Omega, \tag{29}$$

$$\tilde{\nabla}^T_\mu \phi^\mu = \nabla^T_\mu \phi^\mu + 4\partial_\mu \ln\Omega \phi^\mu. \tag{30}$$

The conformal factor is chosen such that

$$\Omega^2 = \Phi. \tag{31}$$

As a consequence, the coupling between the scalar torsion and the scalar field is absorbed by the transformed quantities:

$$S'_g = \int d^4x \tilde{e}\left(\tilde{T} - 2\tilde{T}^\nu \partial_\nu \ln\Phi - \frac{3}{2}\tilde{g}^{\mu\nu}\partial_\nu \ln\Phi \partial_\mu \ln\Phi - \tilde{V}\right), \tag{32}$$

where

$$\tilde{V} \equiv \Phi^{-2}U(\Phi, \phi^\mu, \tilde{\nabla}^T_\mu \phi^\mu, \tilde{e}^a_\mu) - 4\Phi^{-2}\partial_\mu \ln\Omega \phi^\mu \xi. \tag{33}$$

If a new scalar field is defined,

$$\tilde{\phi} \equiv \sqrt{3}\ln\Phi \Rightarrow \Phi = e^{\frac{\tilde{\phi}}{\sqrt{3}}}, \tag{34}$$

then a canonical kinetic term appears for this scalar field. Up to a surface term, the transformed Lagrangian is expressed as a linear combination of the scalar torsion (i.e., the same of TEGR) with a canonical kinetic term for $\tilde{\phi}$. There is also a nonminimal coupling term between the scalar field, the torsion and its derivative. The potential $\tilde{V}$ includes the contribution of the vector field:

$$S'_g = \int d^4x \tilde{e}\left(\tilde{T} - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\nu\tilde{\phi}\partial_\mu\tilde{\phi} + \frac{2}{\sqrt{3}}\tilde{\phi}\tilde{\nabla}^T_\nu \tilde{T}^\nu - \tilde{V}\right), \tag{35}$$

where $\tilde{\nabla}^T_\nu \equiv (\tilde{\nabla}_\nu - \tilde{T}_\nu)$.

Now the variation of the action for the fields $\tilde{e}^a_\mu, \tilde{\phi}$, and ϕ^μ leads to the following field equations:

$$-4\tilde{e}\tilde{\Sigma}^{\sigma\rho\alpha}\tilde{T}_{\sigma\rho a} - 4\partial_\sigma(\tilde{e}\tilde{\Sigma}_a{}^{\sigma\alpha}) + \tilde{e}\tilde{e}^\alpha_a\left(\tilde{T} - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\nu\tilde{\phi}\partial_\mu\tilde{\phi} - \tilde{V} - \frac{2}{\sqrt{3}}\tilde{\Box}\tilde{\phi}\right) + \tilde{e}\frac{2}{\sqrt{3}}\left[\tilde{\nabla}_\sigma\tilde{\phi}\tilde{T}^\alpha + \tilde{\nabla}_\sigma\tilde{\nabla}^\alpha\tilde{\phi} + \frac{\sqrt{3}}{2}\tilde{\nabla}^\alpha\tilde{\phi}\partial_\sigma\tilde{\phi}\right]\tilde{e}^\sigma_a - \tilde{e}\frac{\delta\tilde{V}}{\delta\tilde{e}^a_\alpha} = 0, \tag{36}$$

$$\begin{cases} \tilde{\Box}\tilde{\phi} - \tilde{T}^{\nu}\partial_{\nu}\tilde{\phi} + \frac{2}{\sqrt{3}}\tilde{\nabla}^{T}_{\nu}\tilde{T}^{\nu} - \frac{\delta\tilde{V}}{\delta\tilde{\phi}} = 0, \\ \frac{\delta\tilde{V}}{\delta\phi^{\mu}} = 0. \end{cases} \tag{37}$$

Alternatively, Eq. (36) can be presented in a more compact form:

$$\frac{4}{\tilde{e}}\partial_{\sigma}(\tilde{e}\tilde{\Sigma}_{a}{}^{\sigma\alpha}) + 4\tilde{\Sigma}^{\sigma\rho\alpha}\tilde{T}_{\sigma\rho a} - \tilde{e}^{\alpha}_{a}\tilde{T} = \kappa\tilde{T}^{\alpha}_{(\mathrm{eff})a}, \tag{38}$$

where

$$\begin{aligned}\tilde{T}^{\alpha}_{(\mathrm{eff})a} \equiv& \frac{1}{\kappa}\tilde{e}^{\alpha}_{a}\left(-\frac{1}{2}\tilde{g}^{\mu\nu}\partial_{\nu}\tilde{\phi}\partial_{\mu}\tilde{\phi} - \tilde{V} - \frac{2}{\sqrt{3}}\tilde{\Box}\tilde{\phi}\right) \\ &+ \frac{1}{\kappa}\frac{2}{\sqrt{3}}\left[\tilde{\nabla}_{\sigma}\tilde{\phi}\tilde{T}^{\alpha} + \tilde{\nabla}_{\sigma}\tilde{\nabla}^{\alpha}\tilde{\phi} + \frac{\sqrt{3}}{2}\tilde{\nabla}^{\alpha}\tilde{\phi}\partial_{\sigma}\tilde{\phi}\right]\tilde{e}^{\sigma}_{a} \\ &- \frac{1}{\kappa}\frac{\delta\tilde{V}}{\delta\tilde{e}^{a}_{\alpha}}.\end{aligned} \tag{39}$$

Equation (38) is exactly the same one obtained in TEGR where $\tilde{T}^{\alpha}_{(\mathrm{eff})a}$ is interpreted as an effective energy-momentum tensor for the auxiliary fields $\tilde{\phi}, \phi^{\mu}$. The first equation of Eq. (37) is a dynamical equation for $\tilde{\phi}$ while the second one may be dynamical (or not) for ϕ^{μ}, depending on the potential $\tilde{V}$. It can be a differential equation up to second-order, once $\tilde{V}$ depends on the first derivative of the vector field.

III. HIGHER-ORDER TELEPARALLEL THEORIES OF GRAVITY

The analysis performed in the previous section can be extended to a more general class of modified gravities in which the function f depends on derivatives of the scalar torsion up to order n. Essentially, this extension consists in taking an action for the gravitational field given by:

$$S_{g} = \int d^{4}xef(T, \nabla_{\mu_1}T, \nabla_{\mu_1}\nabla_{\mu_2}T, \ldots, \nabla_{\mu_1}\ldots\nabla_{\mu_n}T, e^{g}_{\alpha}). \tag{40}$$

As before, the explicit dependence of f on the tetrad ensures that a scalar quantity can be built with the tensors of order 1 to n that are now present in this Lagrangian. The analysis in the geometric, Jordan, and Einstein frames is presented below.

A. Geometric frame

Once again the analysis is carried out by considering the tetrad fields as the independent variables. The Lagrangian now depends on the derivatives up to order $(n+1)$ of the tetrad, meaning that the field equations will be differential equations of order up to $2(n+1)$.

The variation of the action (40) gives:

$$\delta S_{g} = \int d^{4}x\delta ef + \int d^{4}xe\left[\frac{\partial f}{\partial T}\delta T + \frac{\partial f}{\partial(\nabla_{\mu_1}T)}\delta(\nabla_{\mu_1}T) + \ldots + \frac{\partial f}{\partial(\nabla_{\mu_1}\ldots\nabla_{\mu_n}T)}\delta(\nabla_{\mu_1}\ldots\nabla_{\mu_n}T) + \frac{\partial f}{\partial e^{d}_{\alpha}}\delta e^{d}_{\alpha}\right]. \tag{41}$$

The successive use of integration by parts must be carried out in order to let the action in a more suitable form. In general, each term containing the variation of a derivative of T will result in a boundary term combined with a term proportional to the variation δT and other terms proportional to the variation of the tetrad field. To illustrate this, the expression obtained for the term involving the second derivative of T is presented:

$$e\frac{\partial f}{\partial(\nabla_{\mu_1}\nabla_{\mu_2}T)}\delta(\nabla_{\mu_1}\nabla_{\mu_2}T) = \partial_{\mu}B^{\mu}_{(2)} + e\nabla^{T}_{\mu_2}\nabla^{T}_{\mu_1}\frac{\partial f}{\partial(\nabla_{\mu_1}\nabla_{\mu_2}T)}\delta T + e\nabla^{T}_{\mu_1}\left[\frac{\partial f}{\partial(\nabla_{\mu_1}\nabla_{\alpha}T)}e^{\rho}_{g}\nabla_{\rho}T\right]\delta e^{g}_{\alpha}. \tag{42}$$

This result can be extended so that:

$$\begin{aligned}& e\frac{\partial f}{\partial(\nabla_{\mu_1}\ldots\nabla_{\mu_n}T)}\delta(\nabla_{\mu_1}\ldots\nabla_{\mu_n}T) \\ &= \partial_{\mu}B^{\mu}_{(n)} + (-1)^{n}e\nabla^{T}_{\mu_n}\ldots\nabla^{T}_{\mu_1}\frac{\partial f}{\partial(\nabla_{\mu_1}\ldots\nabla_{\mu_n}T)}\delta T \\ &\quad - (-1)^{1}e\nabla^{T}_{\mu_1}\left[(\delta^{\alpha}_{\nu_2}\ldots\delta^{\mu_n}_{\nu_n}e^{\mu_2}_{d} + \cdots + \delta^{\mu_2}_{\nu_2}\ldots\delta^{\mu_{n-1}}_{\nu_{n-1}}\delta^{\alpha}_{\nu_n}e^{\mu_n}_{d})\frac{\partial f}{\partial(\nabla_{\mu_1}\nabla_{\nu_2}\ldots\nabla_{\nu_n}T)}\nabla_{\mu_2}\ldots\nabla_{\mu_n}T\right]\delta e^{d}_{\alpha}\end{aligned}$$

$$
\begin{aligned}
&-(-1)^2 e\nabla^T_{\mu_2}\nabla^T_{\mu_1}\left[(\delta^\alpha_{\nu_3}\ldots\delta^{\mu_n}_{\nu_n}e^{\mu_3}_d+\cdots+\delta^{\mu_3}_{\nu_3}\ldots\delta^{\mu_{n-1}}_{\nu_{n-1}}\delta^\alpha_{\nu_n}e^{\mu_n}_d)\frac{\partial f}{\partial(\nabla_{\mu_1}\nabla_{\mu_2}\nabla_{\nu_3}\ldots\nabla_{\nu_n}T)}\nabla_{\mu_3}\ldots\nabla_{\mu_n}T\right]\delta e^d_\alpha\\
&-\ldots\\
&-(-1)^{n-1}e\nabla^T_{\mu_{n-1}}\ldots\nabla^T_{\mu_1}\left[\delta^\alpha_{\nu_n}e^{\mu_n}_d\frac{\partial f}{\partial(\nabla_{\mu_1}\ldots\nabla_{\mu_{n-1}}\nabla_{\nu_n}T)}\nabla_{\mu_n}T\right]\delta e^d_\alpha. \qquad (43)
\end{aligned}
$$

After a long calculation and defining the quantity

$$
\bar{\Phi}_{(n)}\equiv\left(\frac{\partial f}{\partial T}-\nabla^T_{\mu_1}\frac{\partial f}{\partial(\nabla_{\mu_1}T)}+\nabla^T_{\mu_2}\nabla^T_{\mu_1}\frac{\partial f}{\partial(\nabla_{\mu_1}\nabla_{\mu_2}T)}+\cdots+(-1)^n\nabla^T_{\mu_n}\ldots\nabla^T_{\mu_1}\frac{\partial f}{\partial(\nabla_{\mu_1}\ldots\nabla_{\mu_n}T)}\right), \qquad (44)
$$

the field equations are obtained:

$$
\begin{aligned}
0=&\frac{4}{e}\partial_\sigma(\bar{\Phi}_{(n)}e\Sigma_d{}^{\sigma\alpha})+4\bar{\Phi}_{(n)}\Sigma^{\sigma\rho\alpha}T_{\sigma\rho d}-e^\alpha_d f-\frac{\partial f}{\partial e^d_\alpha}-\nabla^T_{\mu_1}\left[\frac{\partial f}{\partial(\nabla_{\mu_1}\nabla_\alpha T)}e^{\mu_2}_d\nabla_{\mu_2}T\right]+\ldots\\
&+(-1)^1\nabla^T_{\mu_1}\left[(\delta^\alpha_{\nu_2}\ldots\delta^{\mu_n}_{\nu_n}e^{\mu_2}_d+\cdots+\delta^{\mu_2}_{\nu_2}\ldots\delta^{\mu_{n-1}}_{\nu_{n-1}}\delta^\alpha_{\nu_n}e^{\mu_n}_d)\frac{\partial f}{\partial(\nabla_{\mu_1}\nabla_{\nu_2}\ldots\nabla_{\nu_n}T)}\nabla_{\mu_2}\ldots\nabla_{\mu_n}T\right]\\
&+(-1)^2\nabla^T_{\mu_2}\nabla^T_{\mu_1}\left[(\delta^\alpha_{\nu_3}\ldots\delta^{\mu_n}_{\nu_n}e^{\mu_3}_d+\cdots+\delta^{\mu_3}_{\nu_3}\ldots\delta^{\mu_{n-1}}_{\nu_{n-1}}\delta^\alpha_{\nu_n}e^{\mu_n}_d)\frac{\partial f}{\partial(\nabla_{\mu_1}\nabla_{\mu_2}\nabla_{\nu_3}\ldots\nabla_{\nu_n}T)}\nabla_{\mu_3}\ldots\nabla_{\mu_n}T\right]+\cdots\\
&+(-1)^{n-1}\nabla^T_{\mu_{n-1}}\ldots\nabla^T_{\mu_1}\left[\delta^\alpha_{\nu_n}e^{\mu_n}_d\frac{\partial f}{\partial(\nabla_{\mu_1}\ldots\nabla_{\mu_{n-1}}\nabla_{\nu_n}T)}\nabla_{\mu_n}T\right]. \qquad (45)
\end{aligned}
$$

In this case, the order of derivative of the field equations can be up to $2(n+1)$. Of course, this is determined by the form of the function f. In particular, in order to have a $2(n+1)$ differential equation, f needs to present (at least) a quadratic dependence with the higher derivative of the scalar torsion. By comparing these field equations with Eq. (12), one sees that the structure of the first line of Eq. (45) is essentially the same of Eq. (12), with $\bar{\Phi}_{(1)}$ replaced by $\bar{\Phi}_{(n)}$. The remaining lines of Eq. (45) have no analog in Eq. (12) since they all are related to higher order terms. It is worth noting that the number of terms in these remaining lines increases considerably with n.

As before, in order to reduce the order of the differential equations, one can introduce auxiliary fields. This is done next, where the system is analyzed in the Jordan and Einstein frame representations.

B. Jordan frame

The lesson that one can take from the analysis of the Jordan frame of $f(T)$ theories and the previous analysis of $f(T,\nabla_\mu T)$ theories is: If f depends only on the scalar T, then an auxiliary tensor of order zero (i.e., a scalar field) should be considered; if f depends on the scalar field and its first derivative, then a tensor of order zero (a scalar field) and a tensor of order 1 (a vector field) have to be introduced. If this line of reasoning is extended to $f(T,\nabla_{\mu_1}T,\nabla_{\mu_1}\nabla_{\mu_2}T,\ldots,\nabla_{\mu_1}\ldots\nabla_{\mu_n}T,e^g_\alpha)$, then for each order of derivative, a tensor of the same order should be proposed. In fact, in order to build an action equivalent to Eq. (40) with auxiliary fields, one must introduce $n+1$ tensors $\xi,\xi_{\mu_1},\xi_{\mu_1\mu_2},\ldots,\xi_{\mu_1\ldots\mu_n}$ and $n+1$ Lagrange multipliers $\frac{\partial f}{\partial\xi},\ldots,\frac{\partial f}{\partial\xi_{\mu_1\ldots\mu_n}}$ so that

$$
\begin{aligned}
S'_g=\int d^4xe\Bigg[&f(\xi,\xi_{\mu_1},\xi_{\mu_1\mu_2},\ldots,\xi_{\mu_1\ldots\mu_n})-\frac{\partial f}{\partial\xi}(\xi-T)-\frac{\partial f}{\partial\xi_{\mu_1}}(\xi_{\mu_1}-\nabla_{\mu_1}T)\\
&-\frac{\partial f}{\partial\xi_{\mu_1\mu_2}}(\xi_{\mu_1\mu_2}-\nabla_{\mu_1}\nabla_{\mu_2}T)-\ldots-\frac{\partial f}{\partial\xi_{\mu_1\ldots\mu_n}}(\xi_{\mu_1\ldots\mu_n}-\nabla_{\mu_1}\ldots\nabla_{\mu_n}T)\Bigg]. \qquad (46)
\end{aligned}
$$

The null variations of S_g' considering the independence of $\xi, \xi_{\nu_1}, \ldots, \xi_{\nu_1\ldots\nu_n}$ lead to

$$H\begin{pmatrix} (\xi - T) \\ (\xi_{\mu_1} - \nabla_{\mu_1} T) \\ \vdots \\ (\xi_{\mu_1\ldots\mu_n} - \nabla_{\mu_1}\ldots\nabla_{\mu_n} T) \end{pmatrix} = 0, \tag{47}$$

where

$$H \equiv \begin{pmatrix} \frac{\partial^2 f}{\partial\xi^2} & \frac{\partial^2 f}{\partial\xi\partial\xi_{\mu_1}} & \cdots & \frac{\partial^2 f}{\partial\xi\partial\xi_{\mu_1\ldots\mu_n}} \\ \frac{\partial^2 f}{\partial\xi_{\nu_1}\partial\xi} & \frac{\partial^2 f}{\partial\xi_{\nu_1}\partial\xi_{\mu_1}} & \cdots & \frac{\partial^2 f}{\partial\xi_{\nu_1}\partial\xi_{\mu_1\ldots\mu_n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial\xi_{\nu_1\ldots\nu_n}\partial\xi} & \frac{\partial^2 f}{\partial\xi_{\nu_1\ldots\nu_n}\partial\xi_{\mu_1}} & \cdots & \frac{\partial^2 f}{\partial\xi_{\nu_1\ldots\nu_n}\partial\xi_{\mu_1\ldots\mu_n}} \end{pmatrix}.$$

Whenever H is regular (i.e. $\det H \neq 0$), the equivalence of S_g' and S_g is proved and the Lagrange multipliers can be considered as independent fields with an inversible relation with the quantities $\xi, \xi_{\mu_1}, \xi_{\mu_1\mu_2}, \ldots, \xi_{\mu_1\ldots\mu_n}$:

$$\begin{cases} \phi \equiv \frac{\partial f}{\partial \xi}, \\ \phi^{\nu_1} \equiv \frac{\partial f}{\partial \xi_{\nu_1}}, \\ \vdots \\ \phi^{\nu_1\ldots\nu_n} \equiv \frac{\partial f}{\partial \xi_{\nu_1\ldots\nu_n}}, \end{cases} \Rightarrow \begin{cases} \xi = \xi(\phi, \phi^{\mu_1}, \ldots, \phi^{\mu_1\cdots\mu_n}), \\ \xi_{\nu_1} = \xi_{\nu_1}(\phi, \phi^{\mu_1}, \ldots, \phi^{\mu_1\cdots\mu_n}), \\ \vdots \\ \xi_{\nu_1\ldots\nu_n} = \xi_{\nu_1\ldots\nu_n}(\phi, \phi^{\mu_1}, \ldots, \phi^{\mu_1\cdots\mu_n}). \end{cases} \tag{48}$$

Equation (46) then is rewritten as

$$\begin{aligned} S_g' = \int d^4xe[&f(\xi, \xi_{\mu_1}, \xi_{\mu_1\mu_2}, \ldots, \xi_{\mu_1\ldots\mu_n}) \\ &- \phi\xi - \phi^{\mu_1}\xi_{\mu_1} - \phi^{\mu_1\mu_2}\xi_{\mu_1\mu_2} - \phi^{\mu_1\cdots\mu_n}\xi_{\mu_1\ldots\mu_n} \\ &+ \phi T + \phi^{\mu_1}\nabla_{\mu_1}T + \phi^{\mu_1\mu_2}\nabla_{\mu_1}\nabla_{\mu_2}T + \cdots \\ &+ \phi^{\mu_1\cdots\mu_n}\nabla_{\mu_1}\ldots\nabla_{\mu_n}T]. \end{aligned} \tag{49}$$

Recurrent integration by parts of the terms involving derivatives of the scalar torsion shows that this action can be expressed in a very compact form

$$S' = \int d^4xe(\Phi T - U + \partial_\mu B^\mu), \tag{50}$$

where $\partial_\mu B^\mu$ stands for a surface term. The quantities Φ and U are defined as

$$\begin{aligned} \Phi \equiv \phi - \nabla^T_{\mu_1}\phi^{\mu_1} + \nabla^T_{\mu_2}\nabla^T_{\mu_1}\phi^{\mu_1\mu_2} + \cdots \\ + (-1)^n\nabla^T_{\mu_n}\ldots\nabla^T_{\mu_1}\phi^{\mu_1\cdots\mu_n}, \end{aligned} \tag{51}$$

and

$$\begin{aligned} U &\equiv U(\Phi, \phi^{\mu_1}, \ldots, \phi^{\mu_1\cdots\mu_n}, \nabla^T_{\mu_n}\phi^{\mu_1}, \nabla^T_{\mu_2}\nabla^T_{\mu_1}\phi^{\mu_1\mu_2}, \ldots, \nabla^T_{\mu_n}\ldots\nabla^T_{\mu_1}\phi^{\mu_1\cdots\mu_n}) \\ &\equiv [\Phi + \nabla^T_{\mu_1}\phi^{\mu_1} - \nabla^T_{\mu_2}\nabla^T_{\mu_1}\phi^{\mu_1\mu_2} + \cdots - (-1)^n\nabla^T_{\mu_n}\ldots\nabla^T_{\mu_1}\phi^{\mu_1\cdots\mu_n}]\xi \\ &\quad + \phi^{\mu_1}\xi_{\mu_1} + \phi^{\mu_1\mu_2}\xi_{\mu_1\mu_2} + \cdots + \phi^{\mu_1\cdots\mu_n}\xi_{\mu_1\ldots\mu_n} - (\xi, \xi_{\mu_1}, \xi_{\mu_1\mu_2}, \ldots, \xi_{\mu_1\ldots\mu_n}). \end{aligned} \tag{52}$$

In this expression, $\xi, \xi_{\mu_1}, \xi_{\mu_1\mu_2}, \ldots, \xi_{\mu_1\ldots\mu_n}$ have to be taken as functions of $\phi, \phi^{\mu_1}, \ldots, \phi^{\mu_1\cdots\mu_n}$.

The action presented in Eq. (46) has the same structure of the one presented in Eq. (22)—a nonminimal coupling term between the scalar field and the scalar torsion, no explicit kinetic terms for the auxiliary fields and a potential that includes the dependence with the auxiliary tensors. It is interesting to note that the potential U has a dependence with the recurrent divergence of each auxiliary tensor field, i.e., $\nabla^T_{\mu_n}\phi^{\mu_1}$, $\nabla^T_{\mu_2}\nabla^T_{\mu_1}\phi^{\mu_1\mu_2}$, …, $\nabla^T_{\mu_n}\ldots\nabla^T_{\mu_1}\phi^{\mu_1\cdots\mu_n}$.

The field equations obtained from variations of Eq. (46) with respect to $e^a_\alpha, \Phi, \phi^{\mu_1}, \ldots, \phi^{\mu_1\cdots\mu_n}$ are

$$\begin{cases} 4\partial_\sigma(\Phi e\Sigma_a{}^{\sigma\alpha}) + 4e\Phi\Sigma^{\sigma\rho\alpha}T_{\sigma\rho a} - ee^\alpha_a(\Phi T - U) + e\frac{\delta U}{\delta e^a_\alpha} = 0, \\ T - \frac{\delta U}{\delta\Phi} = 0, \\ \frac{\delta U}{\delta\phi^{\mu_1}} = 0, \\ \vdots \\ \frac{\delta U}{\delta\phi^{\mu_1\cdots\mu_n}} = 0. \end{cases} \tag{53}$$

The first of these equations is a second order differential equation for the tetrad field, while the order of the remaining equations has to be determined in each case. In principle, the maximum order of derivative for the auxiliary fields is n due to presence of the term $\nabla^T_{\mu_n}\ldots\nabla^T_{\mu_1}\phi^{\mu_1\cdots\mu_n}\xi$ in the potential U. It is important to stress that derivatives of the tetrad field of order higher than two may eventually appear in the other equations since the covariant derivative is recurrently applied.

C. Einstein frame

The analysis of the system in the Einstein frame follows precisely the same steps that were taken in the previous Section. All the equations from Eq. (25) until Eq. (31) are valid in the present case. After the application of the conformal transformation, the action becomes

$$S'_g = \int d^4x\tilde{e}\Big(\tilde{T} - 2\tilde{T}^\nu\partial_\nu \ln\Phi - \frac{3}{2}\tilde{g}^{\mu\nu}\partial_\nu \ln\Phi\partial_\mu \ln\Phi - \tilde{V} + \Phi^{-2}\tilde{\mathcal{L}}_M\Big), \tag{54}$$

where the potential $\tilde{V}$ is extended as:

$$\begin{aligned}\tilde{V} &= \Phi^{-2}U(\Phi,\phi^{\mu_1},\ldots,\phi^{\mu_1\cdots\mu_n},\tilde{\nabla}^T_{\mu_n}\phi^{\mu_1},\tilde{\nabla}^T_{\mu_2}\tilde{\nabla}^T_{\mu_1}\phi^{\mu_1\mu_2},\ldots,\tilde{\nabla}^T_{\mu_n}\ldots\tilde{\nabla}^T_{\mu_1}\phi^{\mu_1\cdots\mu_n})\\ &\quad - \Phi^{-2}\Big[2\partial_\mu \ln\Phi\phi^\mu - \frac{5}{2}\nabla^T_{\mu_2}(\partial_{\mu_1}\ln\Phi\phi^{\mu_1\mu_2}) - 2\partial_{\mu_2}\ln\Phi\nabla^T_{\mu_1}\phi^{\mu_1\mu_2} - 5\partial_{\mu_2}\ln\Phi\partial_{\mu_1}\ln\Phi\phi^{\mu_1\mu_2}\\ &\quad + h_{(3)}(\phi^{\mu_1\mu_2\mu_3},\ldots,\partial_{\mu_1}\ln\Phi,\partial_{\mu_1}\partial_{\mu_2}\ln\Phi,\partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3}\ln\Phi)\\ &\quad + \cdots + h_{(n)}(\phi^{\mu_1\cdots\mu_n},\ldots,\partial_\mu\ln\Phi,\ldots,\partial_{\mu_1}\ldots\partial_{\mu_n}\ln\Phi)\Big]\xi.\end{aligned} \tag{55}$$

Above, $h_{(3)}(\phi^{\mu_1\mu_2\mu_3},\ldots,\partial_{\mu_1}\ln\Phi,\partial_{\mu_1}\partial_{\mu_2}\ln\Phi,\partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3}\ln\Phi)$, $\ldots, h_{(n)}(\phi^{\mu_1\cdots\mu_n},\ldots,\partial_\mu\ln\Phi,,\ldots\partial_{\mu_1}\ldots\partial_{\mu_n}\ln\Phi)$ stand for functions containing contractions of the tensors (and their derivatives) with the derivatives of the conformal factor (or equivalently derivatives of Φ).

As before, a new scalar field is defined as in Eq. (34) and the action becomes (disregarding surface terms):

$$S'_g = \int d^4x\tilde{e}\Big(\tilde{T} + \frac{2}{\sqrt{3}}\tilde{\phi}\tilde{\nabla}^T_\nu\tilde{T}^\nu - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\nu\tilde{\phi}\partial_\mu\tilde{\phi} - \tilde{V}\Big). \tag{56}$$

This is formally the same action obtained in Eq. (35). This way, the field equations have the same structure of the equations Eq. (36) and Eq. (37):

$$\frac{4}{\tilde{e}}\partial_\sigma(\tilde{e}\tilde{\Sigma}_a{}^{\sigma\alpha}) + 4\tilde{\Sigma}^{\sigma\rho\alpha}\tilde{T}_{\sigma\rho a} - \tilde{e}^\alpha_a\tilde{T} = \kappa\tilde{T}^\alpha_{(\mathrm{eff})a}, \tag{57}$$

$$\begin{cases}\frac{\delta\tilde{V}}{\delta\phi^{\mu_1}} = 0,\\ \vdots\\ \frac{\delta\tilde{V}}{\delta\phi^{\mu_1\cdots\mu_n}} = 0.\end{cases} \tag{58}$$

The effective energy-momentum tensor is defined just like in Eq. (39). Again, a TEGR-like equation is obtained and the order of derivative of each field has to be determined for the particular cases of interest. In the next section, an example will be presented.

IV. EXAMPLE

In order to illustrate how to proceed in each of the frames presented previously, an example will be considered. Let the Lagrangian be composed by a combination of a linear and a quadratic terms in the scalar torsion and a quadratic contraction of the first derivative of T:

$$f(T,\nabla_\mu T,e^a_\mu) = \alpha T + \frac{\beta}{2}T^2 + \frac{\gamma}{2}g^{\mu\nu}\nabla_\mu T\nabla_\nu T. \tag{59}$$

As mentioned before, in order to build a scalar quantity with $\nabla_\mu T$, a contraction using explicitly the tetrad field (via $g^{\mu\nu}$) had to be considered. This is the reason why the term $\frac{\partial f}{\partial e^d_\alpha}$ appears in Eq. (12) [and also in Eq. (45)]. This is the term that makes the structure of the field equation obtained here slightly different from the one obtained in $f(T)$ theories. Now the analysis for each representation frame is presented.

A. Geometric frame

In the geometric frame, the partial derivatives of f with respect to $T, \nabla_\mu T$, and e^d_α are

$$\begin{cases}\frac{\partial f}{\partial T} = \alpha + \beta T,\\ \frac{\partial f}{\partial(\nabla_\mu T)} = \gamma g^{\mu\nu}\partial_\nu T,\\ \frac{\partial f}{\partial e^d_\alpha} = -\gamma g^{\alpha\nu}e^\mu_d\nabla_\mu T\nabla_\nu T,\end{cases} \tag{60}$$

so that $\bar{\Phi}_{(1)}$ given in Eq. (13) is

$$\bar{\Phi}_{(1)} = \alpha + \beta T - \gamma g^{\mu\nu} \nabla_\mu^T \nabla_\nu T. \tag{61}$$

Finally, the field equation is

$$\begin{aligned} 0 = {} & e_d^\alpha \left[\alpha T + \frac{\beta}{2} T^2 + \frac{\gamma}{2} g^{\mu\nu} \partial_\mu T \partial_\nu T \right] - \gamma g^{\alpha\nu} e_d^\mu \partial_\mu T \partial_\nu T \\ & - 4[\alpha + \beta T - \gamma g^{\mu\nu} \nabla_\mu^T \partial_\nu T] \Sigma^{\sigma\rho\alpha} T_{\sigma\rho d} \\ & - \frac{4}{e} \partial_\sigma (e[\alpha + \beta T - \gamma g^{\mu\nu} \nabla_\mu^T \partial_\nu T] \Sigma_d{}^{\sigma\alpha}). \end{aligned} \tag{62}$$

This is a differential equation of order 4 in the tetrad fields—see the term $4\gamma g^{\mu\nu} \Sigma_d{}^{\sigma\alpha} (\partial_\sigma \partial_\mu \partial_\nu T)$ of the last line of the equation above. A quick analysis of the Cauchy problem [76] shows that the fourth-order time derivative of tetrad fields are present only for the components e_i^a. Actually, there are no time derivatives of e_0^a in T, so these variables cannot be considered as dynamical from the point of view of the Cauchy problem. If one takes into account the general covariance of the theory, it seems reasonable to suppose that one can get rid of 4 of the remaining d.o.f. The global symmetry by Lorentz transformations may be useful to reduce even more the number of d.o.f. In principle, we can suppose that up to 4 of the remaining degrees of freedom could be potentially eliminated. Were this case, the system would be left with 4 d.o.f. However, it is not clear if this can be achieved. Actually, determining the number of d.o.f. that can be eliminated is not trivial and deserves a careful analysis. At this point, what can be said is that the number of d.o.f. lies between 4 and 8.

B. Jordan frame

The analysis in the Jordan frame begins with the introduction of the auxiliary scalar and vector quantities ξ and ξ_μ. This way,

$$f(\xi, \xi_\mu, e_\mu^a) = \alpha \xi + \frac{\beta}{2} \xi^2 + \frac{\gamma}{2} g^{\mu\nu} \xi_\mu \xi_\nu. \tag{63}$$

The Hessian matrix in this case is given by

$$H \equiv \begin{pmatrix} \frac{\partial^2 f}{\partial \xi^2} & \frac{\partial^2 f}{\partial \xi \partial \xi_\mu} \\ \frac{\partial^2 f}{\partial \xi_\nu \partial \xi} & \frac{\partial^2 f}{\partial \xi_\nu \partial \xi_\mu} \end{pmatrix} = \begin{pmatrix} \beta & 0 \\ 0 & \gamma g^{\mu\nu} \end{pmatrix}. \tag{64}$$

As long as β and γ are non-null, $\det H \neq 0$. This way, the scalar and vector fields ϕ and ϕ^μ are well defined and establish an inversible relation with ξ and ξ_μ:

$$\begin{cases} \phi \equiv \frac{\partial f}{\partial \xi} = \alpha + \beta \xi, \\ \phi^\nu \equiv \frac{\partial f}{\partial \xi_\nu} = \gamma g^{\mu\nu} \xi_\mu, \end{cases} \Rightarrow \begin{cases} \xi = \frac{\phi - \alpha}{\beta}, \\ \xi_\mu = \frac{1}{\gamma} g_{\mu\nu} \phi^\nu. \end{cases} \tag{65}$$

The scalar field Φ is defined by Eq. (21) so that the potential U [Eq. (23)] is given by

$$U(\Phi, \phi^\mu, \nabla_\mu^T \phi^\mu, e_\mu^a) = \frac{1}{2\beta} [\Phi + \nabla_\mu^T \phi^{\mu_1} - \alpha]^2 + \frac{1}{2\gamma} g_{\mu\nu} \phi^\nu \phi^\mu. \tag{66}$$

With this potential, the action can be written as

$$S' = \int d^4 x e \left[\Phi T - \frac{1}{2\beta} (\Phi + \nabla_{\mu_1}^T \phi^{\mu_1} - \alpha)^2 - \frac{1}{2\gamma} g_{\mu\nu} \phi^\nu \phi^\mu \right]. \tag{67}$$

Finally, the variation of S' leads to the field equations, given by:

$$\begin{aligned} 0 = {} & -4\Phi \Sigma^{\sigma\rho\alpha} T_{\sigma\rho a} - \frac{4}{e} \partial_\sigma (\Phi e \Sigma_a{}^{\sigma\alpha}) - \frac{1}{\gamma} \phi_a \phi^\alpha + e_a^\alpha \Phi T \\ & - e_a^\alpha \frac{1}{2\beta} [\Phi + \nabla_\mu^T \phi^\mu - \alpha]^2 - e_a^\alpha \frac{1}{2\gamma} g_{\mu\nu} \phi^\nu \phi^\mu \\ & + \frac{1}{\beta} e_a^\alpha [\phi^\rho \partial_\rho (\Phi + \nabla_\mu^T \phi^\mu) + (\Phi + \nabla_\mu^T \phi^\mu - \alpha) \nabla_\rho^T \phi^\rho]. \end{aligned} \tag{68}$$

$$0 = T - \frac{1}{\beta} (\Phi + \nabla_\mu^T \phi^\mu - \alpha), \tag{69}$$

$$0 = \frac{1}{\beta} \partial_\rho (\Phi + \nabla_\mu^T \phi^\mu) - \frac{1}{\gamma} \phi_\rho. \tag{70}$$

As one can directly verify, this is a set of second order differential equations. To be more precise, Eqs. (68) and (70) are second order equations (for e_μ^a and ϕ^μ) while Eq. (69) is of first order. From the point of view of the Cauchy problem, the latter should be understood as a constraint on the initial data. Equation (70) sets four equations for ϕ^μ but only when $\rho = 0$ a dynamical equation is obtained. Besides, only the component ϕ^0 can be considered a dynamical variable since this is the only component of ϕ^μ that is twice derived in time. When $\rho = 1$, 2, 3, a set of constraints on the initial data are obtained for ϕ^i. Finally, Eq. (68) is a second order differential equation for both the tetrad and the vector fields (while it is of first order for Φ). In this equation, the second-order time derivatives of the tetrad field are only present for the variables e_i^a. This way, e_0^a cannot be considered as dynamical variables. In addition, if one considers general covariance of the theory, then 4 degrees of freedom can be eliminated. The global Lorentz invariance could be used to eliminate additional d.o.f.—between 0 and 4 (as discussed previously, the number of d.o.f. associated to this symmetry that can be eliminated is not easily determined). A naive counting of the degrees of freedom would consider between 4 and 8 coming from the tetrad fields and one from ϕ^μ (i.e., ϕ^0). However, this is misleading. If, for instance, the trace of Eq. (68) is considered (by a contraction with e_α^a), then a combination between $\partial_0 \partial_0 \phi^0$ and $\partial_0 \partial_0 e_i^a$ emerges and one of the degrees of freedom can be further eliminated.

This way, as it happens in the geometric approach, this system presents between 4 and 8 degrees of freedom.

C. Einstein frame

As presented previously, the passage from the Jordan to the Einstein frame is performed by a conformal transformation of the tetrad field. A new scalar field $\tilde{\phi}$ is also introduced in such a way that the fundamental fields are $\tilde{\phi}$, the vector field ϕ^μ and the transformed tetrad field, $\tilde{e}^a_\mu$. The action integral, Eq. (35), is completely determined when the potential $\tilde{V}$ is set. For the present case, one finds

$$\tilde{V} = e^{-\frac{2\tilde{\phi}}{\sqrt{3}}}\frac{1}{2\beta}\left[e^{\frac{\tilde{\phi}}{\sqrt{3}}} + (\tilde{\nabla}_\mu - \tilde{T}_\mu)\phi^\mu - \frac{2}{\sqrt{3}}\phi^\mu\partial_\mu\tilde{\phi} - \alpha\right]^2 + e^{-\frac{3\tilde{\phi}}{\sqrt{3}}}\frac{1}{2\gamma}\tilde{g}_{\mu\nu}\phi^\nu\phi^\mu, \tag{71}$$

so that

$$S' = \int d^4x\tilde{e}\left\{\tilde{T} + \frac{2}{\sqrt{3}}\tilde{\phi}\tilde{\nabla}^T_\nu\tilde{T}^\nu - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\nu\tilde{\phi}\partial_\mu\tilde{\phi} - e^{-\sqrt{3}\tilde{\phi}}\frac{1}{2\gamma}\tilde{g}_{\mu\nu}\phi^\nu\phi^\mu - e^{-\frac{2\tilde{\phi}}{\sqrt{3}}}\frac{1}{2\beta}\left[e^{\frac{\tilde{\phi}}{\sqrt{3}}} + \tilde{\nabla}^T_\mu\phi^\mu - \frac{2}{\sqrt{3}}\phi^\mu\partial_\mu\tilde{\phi} - \alpha\right]^2\right\}. \tag{72}$$

Now the field equations are obtained respectively for $\tilde{\phi}$, ϕ^μ, and $\tilde{e}^a_\mu$:

$$\begin{aligned} 0 = {}& \frac{1}{\sqrt{3}\beta}e^{-\frac{2\tilde{\phi}}{\sqrt{3}}}\left[e^{\frac{\tilde{\phi}}{\sqrt{3}}} + \tilde{\nabla}^T_\mu\phi^\mu - \frac{2}{\sqrt{3}}\phi^\mu\partial_\mu\tilde{\phi} - \alpha\right]^2 + \tilde{\Box}\tilde{\phi} - \tilde{T}^\mu\partial_\mu\tilde{\phi} + \frac{2}{\sqrt{3}}\tilde{\nabla}_\nu\tilde{T}^\nu - \frac{2}{\sqrt{3}}\tilde{T}_\nu\tilde{T}^\nu + \frac{\sqrt{3}}{2\gamma}e^{-\sqrt{3}\tilde{\phi}}\tilde{g}_{\mu\nu}\phi^\nu\phi^\mu \\ & - \frac{1}{\sqrt{3}\beta}e^{-\frac{\tilde{\phi}}{\sqrt{3}}}\left(e^{\frac{\tilde{\phi}}{\sqrt{3}}} + \tilde{\nabla}^T_\mu\phi^\mu - \frac{2}{\sqrt{3}}\phi^\mu\partial_\mu\tilde{\phi} - \alpha\right) - \frac{2}{\sqrt{3}\beta}\tilde{\nabla}^T_\nu\left[e^{-\frac{2\tilde{\phi}}{\sqrt{3}}}\left(e^{\frac{\tilde{\phi}}{\sqrt{3}}} + \tilde{\nabla}^T_\mu\phi^\mu - \frac{2}{\sqrt{3}}\phi^\mu\partial_\mu\tilde{\phi} - \alpha\right)\phi^\nu\right], \end{aligned} \tag{73}$$

$$0 = -\frac{1}{\gamma}e^{-\sqrt{3}\tilde{\phi}}\phi_\nu + \frac{1}{\beta}e^{-\frac{2\tilde{\phi}}{\sqrt{3}}}\partial_\nu\left(e^{\frac{\tilde{\phi}}{\sqrt{3}}} + \tilde{\nabla}^T_\mu\phi^\mu - \frac{2}{\sqrt{3}}\phi^\mu\partial_\mu\tilde{\phi} - \alpha\right), \tag{74}$$

$$4\frac{1}{\tilde{e}}\partial_\sigma(\tilde{e}\tilde{\Sigma}_a{}^{\sigma\alpha}) + 4\tilde{\Sigma}^{\sigma\rho\alpha}\tilde{T}_{\sigma\rho a} - \tilde{e}^\alpha_a\tilde{T} = \chi T^\alpha_{(\mathrm{eff})a}. \tag{75}$$

The effective energy-momentum tensor is:

$$\begin{aligned} \chi T^\alpha_{(\mathrm{eff})a} \equiv {}& \tilde{e}^\alpha_a\left\{\left[-\frac{1}{2}\tilde{g}^{\mu\nu}\partial_\nu\tilde{\phi}\partial_\mu\tilde{\phi} - \frac{2}{\sqrt{3}}\tilde{\Box}\tilde{\phi} - \frac{1}{2\gamma}e^{-\sqrt{3}\tilde{\phi}}\tilde{g}_{\mu\nu}\phi^\nu\phi^\mu - \frac{1}{2\beta}e^{-\frac{2\tilde{\phi}}{\sqrt{3}}}\left(e^{\frac{\tilde{\phi}}{\sqrt{3}}} + \tilde{\nabla}^T_\mu\phi^\mu - \frac{2}{\sqrt{3}}\phi^\mu\partial_\mu\tilde{\phi} - \alpha\right)^2\right.\right. \\ & \left. + \frac{1}{\beta}\phi^\nu\tilde{\nabla}_\nu\left[e^{-\frac{2\tilde{\phi}}{\sqrt{3}}}\left(e^{\frac{\tilde{\phi}}{\sqrt{3}}} + \tilde{\nabla}^T_\mu\phi^\mu - \frac{2}{\sqrt{3}}\phi^\mu\partial_\mu\tilde{\phi} - \alpha\right)\right]\right] + \frac{1}{\beta}e^{-\frac{2\tilde{\phi}}{\sqrt{3}}}\left[e^{\frac{\tilde{\phi}}{\sqrt{3}}} + \tilde{\nabla}^T_\mu\phi^\mu - \frac{2}{\sqrt{3}}\phi^\mu\partial_\mu\tilde{\phi} - \alpha\right]\tilde{\nabla}^T_\nu\phi^\nu\Bigg\} \\ & + \tilde{g}^{\alpha\nu}\tilde{e}^\mu_a\partial_\nu\tilde{\phi}\partial_\mu\tilde{\phi} - \frac{1}{\gamma}e^{-\sqrt{3}\tilde{\phi}}\eta_{ab}\tilde{e}^b_\mu\phi^\alpha\phi^\mu + \frac{2}{\sqrt{3}}[\tilde{\nabla}_\sigma\tilde{\nabla}^\alpha\partial_\mu\tilde{\phi} + \tilde{\nabla}_\sigma\tilde{\phi}\tilde{T}^\alpha]\tilde{e}^\sigma_a. \end{aligned} \tag{76}$$

Now all equations are second order for the scalar, vector and tetrad fields. The equations for the tetrad fields have the same structure of those obtained in TEGR. Concerning the Cauchy problem, the second time derivatives of $\tilde{e}^a_0$ and ϕ^i are absent in these equations, so in principle only the fields $\tilde{\phi}$, ϕ^0 and $\tilde{e}^a_i$ seem to be dynamical. However, similarly to what happens in the analysis of the Jordan frame, if the trace of the equation for the tetrad field is taken into account, then a constraint between the second time derivatives of these fields emerges. As a consequence one degree of freedom is removed. Besides, for $\nu = 0$ in Eq. (74), a combination of the second time derivatives of $\tilde{\phi}$ and ϕ^0 emerges, showing that these quantities are not dynamically independent. Again, considering the Lorentz global invariance and the general covariance of the theory, the counting of the degrees of freedom results a number between 4 and 8, as it was observed in the previous cases.

V. FINAL REMARKS

In this paper, the Jordan and Einstein representations for systems in which the Lagrangian is a function of the scalar torsion and its derivatives have been presented. The analysis started with $f(T, \nabla_\mu T, e^a_\mu)$ systems. It was shown that in the Jordan frame two auxiliary fields had to introduced—a scalar and a vector fields. The theory resembles a scalar-tensor theory with no kinetic term for the scalar field; also the potential U has a dependence with the divergence of the vector field. Using the results from the Jordan frame as the starting point, a conformal transformation was applied to the tetrad fields leading the system to its Einstein representation. In this case, a kinetic term for a new scalar field (that was introduced to replace the scalar field of the Jordan frame) is present. The field equations for the tetrads have the same form of the equations obtained in TEGR where an effective energy-momentum tensor for the auxiliary fields was defined.

These results were extended to $f(T, \nabla_{\mu_1} T, \ldots, \nabla_{\mu_n} \cdots \nabla_{\mu_1} T)$ theories. In order to change the representation from the geometric to the Jordan frame, $n+1$ tensors (of order zero to n) had to be introduced. The theory also resembles a scalar-tensor theory similar to $f(T, \nabla_\mu T, e^a_\mu)$, however the potential U now has the contribution of the divergences of the auxiliary tensor fields. The representation in the Einstein frame was obtained by a conformal transformation of the tetrad field. As before, a kinetic term for the scalar field was present in the action and the equations for the tetrad field was basically the same of the TEGR with an effective energy-momentum tensor for the auxiliary fields.

Finally, an example was presented in order to illustrate how to proceed in order to obtain each of the representations considered here. A quick analysis of the Cauchy problem in each frame has shown that the number of d.o.f. in each case seems to be consistent and limited to the same number. A rigorous approach to this problem should consider the constraint analysis, for instance in the Hamiltonian or in Hamilton-Jacobi approaches. A question that was not considered here concerns the equivalence between the Jordan and Einstein frames. A final answer for this problem is not yet known for simpler system like $f(R)$ or $f(T)$ theories. For the present case, the proof of equivalence (or not) seems to be more intricate. This is a problem for future investigations.

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[74] For instance, if f depends on a quadratic combination of $\nabla_\mu T$, i.e., $f \sim (\nabla_\mu T)^2$, the contraction of indexes are carried out by a metric tensor $f \sim g^{\mu\nu}\nabla_\nu T \nabla_\mu T$ where $g^{\mu\nu}$ is determined in terms of the tetrad. Its contribution is not present neither in T nor in $\nabla_\mu T$.

[75] For instance, see Eq. (4.2) of Ref. [63] or Eq. (29) of Ref. [65].

[76] A rigorous analysis of the Cauchy problem can be performed in the same lines presented in Ref. [67].