Maximum force and cosmic censorship

Valerio Faraoni[*]

Department of Physics and Astronomy, Bishop's University,
2600 College Street, Sherbrooke, Québec J1M 1Z7, Canada

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Although the idea that there is a maximum force in nature seems untenable, we explore whether this concept can make sense in the restricted context of black holes. We discuss uniformly accelerated and cosmological black holes and we find that, although a maximum force acting on these black holes can in principle be introduced, this concept is rather tautological.

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I. INTRODUCTION

The idea that there exists a maximum value,

$$F_{\rm max} = \frac{c^4}{4G}, \tag{1.1}$$

to any physically attainable force (or tension) was advanced by Schiller [1–3] and Gibbons [4] and further explored by several authors [2,3,5–10]. Extensions to include a cosmological constant [6], angular momentum [8], and a maximum force in Brans-Dicke gravity [10] have also been discussed.

The key idea leading to the sharp factor $1/4$ in Eq. (1.1) comes from the fact that there is a maximum deficit angle in the geometry of a cosmic string [4,6]. A static cosmic string aligned with the z-axis is described by a locally flat spacetime with line element

$$ds^2 = -dt^2 + dr^2 + dz^2 + r^2 d\varphi^2 \tag{1.2}$$

with $0 \leq \varphi < 2\pi(1 - \frac{4G\mu}{c^2})$ and with a conical singularity along the z-axis and a deficit angle

$$\delta = \frac{8\pi G\mu}{c^2}, \tag{1.3}$$

where μ is the string tension. The range of the φ coordinate in cylindrical coordinates (t, r, φ, z) is $0 \leq \varphi \leq 2\pi(1 - \delta)$. To prevent δ from spanning the entire three-dimensional space it must be $\delta < 2\pi$, which generates the upper bound (1.1) on the string tension μ. It is not clear, however, how this bound on a string tension has risen to the role of a universal bound on any possible force acting on a particle.

Sometimes the factor $1/4$ is replaced by $1/2$, or by factors of order unity, in weaker formulations of the upper bound [9], while other works involving black holes reproduce the $1/4$ factor [7].

Another argument involves the Planck scale. Using only dimensional analysis and the fundamental constants G, c, and $\hbar$, one can construct the "Planck force" $F_{\rm Pl} = c^4/G$ and the "Planck power" $P_{\rm Pl} = c^5/G$. Contrary to the Planck length, energy, temperature, and energy density, these two new Planck scale quantities do not contain $\hbar$ and are purely classical. The construction of Planck scale quantities, however, may lead to very vague concepts: for example, ordinary objects like a football have masses exceeding the Planck mass and, therefore, a force resulting from the product of a mass times an acceleration could easily exceed the Planck force even if there was an upper limit to the acceleration of a particle (such a limit was claimed by Caianiello on the basis of a generalized uncertainty principle [11]).

The idea of a maximum force has been extended to hypothesizing a maximum power for any system in general relativity (GR), the Dyson luminosity [12]

$$P_{\rm max} = cF_{\rm max} = \frac{c^5}{4G}, \tag{1.4}$$

which would be the maximum possible luminosity of an isolated system [1,3] (for example, its luminosity in gravitational waves [13,14]). An upper limit on entropy production rates is discussed in Ref. [15].

Schiller's maximum force proposal in [1] (see also [2,3]) even included the idea that the existence of a maximum force implies general relativity as a consequence, in the same way that a maximum possible velocity (c) leads to special relativity [2]. The derivation parallels Jacobson's derivation of the Einstein equations as an equation of state that began the area of research now known as the thermodynamics of spacetime [16]. This derivation is associated with the fact that the Einstein equations

[*]vfaraoni@ubishops.ca

$$G_{ab} = \frac{8\pi G}{c^4} T_{ab} \tag{1.5}$$

contain the constant G/c^4 and can be written as $T_{ab} = \frac{F_{\max}}{2\pi} G_{ab}$ [17].

Recently, the existence of a maximum force has been criticized by Jowsey and Visser [17] who provide counterexamples violating the Schiller-Gibbons upper bound using fluid spheres, or the TOV equation, in GR. Counterexamples to the maximum luminosity conjecture are given by Cardoso, Ikeda, Moore and Yoo [18].

Indeed, Barrow and Gibbons themselves mention possible counterexamples related with sudden future singularities in cosmology, which cause unbounded pressure forces. They mention the fact that the maximum force bound is restricted to situations in which a suitable energy condition is imposed to prevent sudden future singularities [6]. However, this example is rather vague and it is not clear what the cosmological "force" applies to. Section 1.5 of [6] begins with the usual Newtonian analogy for an expanding universe, consisting of a Newtonian ball of material with radius proportional to the scale factor $a(t)$ of a (spatially flat) expanding universe. Since the "force" is proportional to $\ddot{a}$, assuming a power-law expansion $a(t) \simeq t^n$, the "force" F grows like

$$F = F_{\rm Pl} \left(\frac{t}{t_{\rm Pl}} \right)^{n-2}. \tag{1.6}$$

If $n > 2$ the Planck force is exceeded. In our opinion, however, this Newtonian argument does not suffice to draw firm conclusions. In fact, the only way to make the analogy with a Newtonian ball work in GR is if the analogous relativistic universe is filled with dust [19]. Then it is $n = 2/3$ and the "force" decreases after the Planck time. This entire argument, however, is too vague and needs to be put on a firmer basis (in particular specifying what is the force acting on what).

Varying constants and entropic force theories, as well as black hole thermodynamics in generalized entropy models, also escape the force bound [20], while attempts have been made to relate it to the holographic principle [21].

On the one hand, in view of the counterexamples provided, it is hard to argue with Jowsey and Visser that the maximum force acting on particles, in GR or in other theories of gravity, is doomed. On the other hand, Gibbon's argument in the restricted context of cosmic strings is sound and perhaps versions of the maximum force conjecture restricted to well defined contexts may be true. Here we explore this direction: although there is no universal maximum force on particles and other objects, there may exist an upper bound to the force (or to a similar quantity with the dimensions of a force) acting on more fundamental objects: *black holes*. We report two instances that corroborate this idea. They include (i) uniformly accelerated black holes and (ii) black holes embedded in Friedmann-Lemaître-Robertson-Walker (FLRW) universes and described by the McVittie metric [22]. For these two categories of black holes, we present first a purely classical argument and then a second argument based on horizon thermodynamics (therefore, ultimately based on quantum field theory in curved space).

We follow the notation of Ref. [23].

II. UNIFORMLY ACCELERATED BLACK HOLE

The static C-metric discovered by Levi-Civita in 1917 [24], a member of the Weyl class of cylindrically symmetric solutions of the Einstein equations [25], describes a pair of uniformly accelerated black holes and has been generalized to include electrically and magnetically charged and/or spinning black holes, possibly with a positive or negative cosmological constant. The C-metric has been the subject of a long literature [26]. It can be studied in several coordinate systems, each one of which is more suitable for certain purposes than others [27].

The C-metric can be written in coordinates (t, p, q, φ) and in units in which $c = G = 1$ as the 2-parameter family of solutions [28]

$$ds^2 = \frac{1}{a^2(p+q)^2} \left[-F(q)dt^2 + \frac{dq^2}{F(q)} + \frac{dp^2}{G(p)} + G(p)d\varphi^2 \right], \tag{2.1}$$

where

$$F(q) = -2amq^3 + q^2 - 1, \tag{2.2}$$

$$G(p) = -2amp^3 - p^2 + 1. \tag{2.3}$$

The parameters m and a are related with the black hole mass and the uniform acceleration. The coordinates have the range $-\infty < t < +\infty$, $0 \le \varphi \le 2\pi$, while p and q have different ranges chosen to satisfy $G(p) > 0$. The cubic polynomial $G(x) = -F(-x)$ has three real roots if $am < 1/\sqrt{27}$ [28]. If $q_{\rm R,S}$ and $p_{0,\pi}$ denote the roots of $F(q) = 0$ and $G(p) = 0$, it is $q_{\rm R} \le q \le q_{\rm S}$ and $p_0 \le p \le p_\pi$.

In the limit $m \to 0$, the line element (2.1) reduces to the Minkowski one in Rindler accelerated coordinates, which allows the identification of the parameter a with the uniform acceleration [28]. The limit $a \to 0$ reproduces the Schwarzschild metric, but a transformation to different coordinates that reduce to the usual Schwarzschild ones when $a = 0$ is necessary to see that [27]. The metric is of algebraic type D, has a spacetime singularity inside the black hole horizon, and is static and cylindrically symmetric since it admits a timelike Killing vector ξ_t and a rotational Killing vector ξ_φ, plus a boost vector [28].

The timelike and boost Killing vectors generate null Killing horizons, located by the vanishing of their norms. The first is a black hole horizon distorted by the acceleration, while the second is an acceleration (Rindler) horizon distorted by the presence of the black hole. These Killing horizons are best studied in the accelerated coordinates associated with observers comoving with the uniformly accelerated black hole [28]. If $ma < 1/(3\sqrt{3})$, the black hole and the Rindler horizon are separated, the black hole horizon is elongated along the direction of its motion, and the Rindler horizon is an infinite open surface around the black hole horizon (most of the literature on accelerated black holes restricts to this situation [26]). A second black hole, joined to the first by a cosmic string, is located in the region behind the Rindler horizon [26]. If $ma = 1/(3\sqrt{3})$, the two horizons touch each other while, if $ma > 1/(3\sqrt{3})$ the Rindler horizon penetrates the black hole horizon and the black hole effectively disappears from this region of spacetime.

The geometry has a conical singularity along the axis corresponding to the direction of motion and to a cosmic string, aligned with this axis, that pulls the black hole. There is a deficit angle δ that manifests itself when the length of a circumference in a plane orthogonal to this axis is computed for $0 \leq \varphi < 2\pi$. The string tension is $\mu = \frac{c^4}{G}\frac{\delta}{8\pi}$; imposing Gibbons' argument [4] that this deficit angle be less than 2π yields $\mu \leq \frac{c^4}{4G}$. This bound is not the same as the bound on am; however, m cannot be interpreted literally as the black hole mass since the spacetime is not asymptotically flat and the mass parameter is redefined in a complicated way [29] when the coordinates are transformed. In any case, the quantity ma no doubt has the dimensions of a force acting on the black hole horizon, although the physical interpretation of the C-metric compels one to regard the string tension as the force acting on the black hole.

From the point of view of accelerated observers comoving with the black hole, the black hole is destroyed (i.e., it disappears from these observers' world) if the parameter a becomes too large. The condition $ma \leq \frac{c^4}{3\sqrt{3}G}$ can also be obtained naively by remembering that the distance of a uniformly accelerated observer to the Rindler horizon is $d \leq \frac{c^2}{2a}$, which implies that the size of a uniformly accelerated object must be not larger[1] than $c^2/(2a)$. When the "object" is a Schwarzschild black hole of radius $R_S = 2Gm/c^2$, the bound $R_S \leq c^2/(2a)$ reproduces $ma \leq \frac{c^4}{4G}$. This is, however, rather hand-waving while the derivation using the Farhoosh-Zimmerman solution is exact.

A thermodynamical argument can be given. The black hole and acceleration horizons are distorted,[2] however one can still approximate their temperatures with the quantities pertaining to an isolated black hole and a Rindler horizon in Minkowski space. The result is not rigorous but is very suggestive.

The Unruh temperature of the thermal bath perceived by a uniformly accelerated observer is

$$T_{\mathrm{U}} = \frac{\hbar a}{2\pi K_B c}, \tag{2.4}$$

while the Hawking temperature of a Schwarzschild black hole of mass m is

$$T_{\mathrm{H}} = \frac{\hbar c^3}{8\pi G K_B m}. \tag{2.5}$$

The requirement $T_{\mathrm{U}} \leq T_{\mathrm{H}}$ translates to

$$ma \leq \frac{c^4}{4G}, \tag{2.6}$$

which is exactly the maximum force proposed by Gibbons and Schiller for a particle of mass m. However, in their proposals, the particle is not a black hole and the acceleration a is not uniform.

A physical interpretation of this argument can be given as follows. The typical quanta of Hawking radiation from the black hole have wavelength $\lambda \sim R_s$. When $T_{\mathrm{U}} > T_{\mathrm{H}}$, this wavelength is larger than the acceleration horizon and one can no longer talk about thermal emission. This is consistent, of course, with the fact that when the black hole horizon becomes larger than the acceleration horizon, it does not make sense to talk about a black hole. This echoes a situation discussed by Barrow [37] in the independent context of varying speed of light cosmologies, in which the Compton wavelength of a particle crosses outside the particle horizon of the universe.

The thermodynamical argument is conceptually different from, and independent of, Gibbons' argument based on the deficit angle caused by the cosmic string pulling the black hole. It is interesting that it provides the same bound, although the coefficient of c^4/G cannot be taken literally because the Unruh and Hawking temperatures employed are approximations. In the next section we examine a physical situation in which black hole horizons appear in the complete absence of cosmic strings, but similar considerations ensue.

[1]Incidentally, by imposing that the Compton wavelength $\lambda = h/(mc)$ of a particle of mass m lies outside its Schwarzschild radius, one obtains the bound $a \leq a_c(m)/(8\pi)$ on the acceleration a, where $a_c(m) \equiv 2mc^3/h$ is Caianiello's maximal acceleration [11]. (Loose arguments like this are very unlikely to produce exactly the same numerical factors.) This argument, however, brings in quantum physics.

[2]See Refs. [30–36] for the exact thermodynamics of accelerating black holes in anti–de Sitter space.

III. BLACK HOLES EMBEDDED IN FLRW UNIVERSES

The McVittie spacetime [22] generalizes the Schwarzschild–de Sitter (or Kottler) solution of the Einstein equations and is interpreted as describing a central object embedded in a generic FLRW space [38–50]. The geometry in the region between black hole and cosmological apparent horizons is time-dependent.

The McVittie line element in isotropic coordinates is

$$ds^2 = -\frac{[1-\frac{m_0}{2\bar{r}a(t)}]^2}{[1+\frac{m_0}{2\bar{r}a(t)}]^2}dt^2 + a^2(t) \times \left[1+\frac{m_0}{2\bar{r}a(t)}\right]^4 (d\bar{r}^2 + \bar{r}^2 d\Omega_{(2)}^2). \quad (3.1)$$

Apart from the special case of a de Sitter "background" (in which McVittie reduces to Schwarzschild–de Sitter), there is a spacetime singularity at $\bar{r} = m/2$ (which reduces to the Schwarzschild horizon if $a \equiv 1$) [38,40–42,51], which is spacelike [40–42]. The pressure of the fluid source,

$$P = -\frac{1}{8\pi}\left[3H^2 + \frac{2\dot{H}(1+\frac{m}{2\bar{r}})}{1-\frac{m}{2\bar{r}}}\right], \quad (3.2)$$

diverges at $\bar{r} = m/2$ together with the Ricci scalar $\mathcal{R} = 8\pi(\rho - 3P)$ [38,40–42,51–53].

The areal radius is

$$R \equiv a(t)\bar{r}\left(1+\frac{m}{2\bar{r}}\right)^2; \quad (3.3)$$

restricting to a spatially flat FLRW universe for simplicity, the apparent horizons of the McVittie metric are the roots of the equation $\nabla^c R \nabla_c R = 0$, or

$$H^2(t)R^3 - R + 2m_0 = 0, \quad (3.4)$$

which is familiar from the Schwarzschild–de Sitter case if $H = \text{const}$. Since here $H = H(t)$, the apparent horizon radii are time-dependent. There are two real and positive roots $R_1(t)$ and $R_2(t)$ if $m_0 H(t) < 1/(3\sqrt{3})$. To fix the ideas, consider a dust-dominated FLRW universe with $H(t) = 2/(3t)$. Then there is a critical time $t_* = 2\sqrt{3}m_0$ at which $m_0 H(t) = 1/(3\sqrt{3})$. We can distinguish three situations occurring during the history of the McVittie universe:

(i) At early times $t < t_*$ we have $m_0 > \frac{1}{3\sqrt{3}H(t)}$, $R_1(t)$ and $R_2(t)$ are complex and there are no apparent horizons.

(ii) At $t = t_*$ we have $m_0 = \frac{1}{3\sqrt{3}H(t)}$ and two coincident apparent horizons $R_1 = R_2 = \frac{1}{\sqrt{3}H(t)}$ appear simultaneously.

(iii) As $t > t_*$ it is $m_0 < \frac{1}{3\sqrt{3}H(t)}$ and there are two distinct apparent horizons $R_{1,2}(t)$: a black hole and a cosmological horizon.

The physical interpretation is that at late times the black hole fits inside the cosmological horizon and can properly be called a black hole, while this is impossible as $t \leq t_*$, when the "force" caused by the attraction of cosmic matter enlarges the black hole horizon.

Instead of considering a "force" acting on the black hole apparent horizon and stretching it, we can consider a quantity with the dimensions of a force, i.e., the product $Gm/c^2 \cdot H/c$ that measures the ratio of the sizes of the black hole apparent horizon ($\sim Gm/c^2$) and the cosmological apparent horizon ($\sim c/H$). The black hole apparent horizon is smaller than the cosmological apparent horizon, or touches it, when

$$mH \leq \frac{c^4}{3\sqrt{3}G}. \quad (3.5)$$

If $mH > c^4/(3\sqrt{3}G)$, the black hole horizon is outside the would-be cosmological horizon and the comoving observers of the underlying FLRW cosmology do not see a black hole at all.

One can consider again the corresponding thermodynamics. Approximating the temperature of the black hole with that of a Schwarzschild black hole of the same mass m, $T_{\mathrm{H}} = \frac{\hbar c^3}{8\pi G K_B m}$, and the temperature of the FLRW apparent horizon with the Gibbons-Hawking temperature (originally obtained for de Sitter space with $H = \text{const}$. [54]), $T_{\mathrm{GH}} = \frac{\hbar H}{2\pi K_B c}$, the requirement that $T_{\mathrm{GH}} \leq T_{\mathrm{H}}$ is equivalent to

$$mH \leq \frac{c^4}{4G} \quad (3.6)$$

which is, again, the upper bound proposed by Gibbons and Schiller. The coincidence of the numerical coefficients of c^4/G is, again, not significant because the black hole and cosmological horizons are modified with respect to the GR solutions containing only one or the other, and so will their temperatures. The following physical interpretation can be given: when the wavelength of the thermal Hawking radiation emitted by the black hole becomes larger than the cosmological apparent horizon, corresponding to $T_{\mathrm{GH}} > T_{\mathrm{H}}$, the concept of Hawking radiation loses meaning, consistent with the fact that the black hole itself is no longer seen by observers comoving with the FLRW cosmic fluid. Particles beyond the Hubble horizon cannot fall into the black hole singularity.

IV. CONCLUSIONS AND OUTLOOKS

In both cases the Schwarzschild null event horizon is altered by the black hole environment: in the first case, it is distorted by the pull of the cosmic string causing the black hole to accelerate while, in the second case, the horizon remains spherical but it becomes a time-dependent apparent horizon due to the gravitational pull of the cosmic fluid. It is enlarged with respect to the Schwarzschild radius of a black hole with the same mass (the central singularity is also stretched to a finite radius singularity [50]).

Technically, in our two examples, the maximum force principle is respected by black holes in the sense that when it is violated the spacetime regions under consideration are no longer black holes according to the observers located in them. However, both examples are legitimate solutions of the Einstein equations also in the spacetime regions beyond the acceleration horizon (in the first case) or the cosmological apparent horizon (in the second case) and, as such, the maximum force principle is violated. Alternatively, there is no *a priori* reason why the Unruh temperature of an accelerated black hole cannot be larger than its Hawking temperature. According to Wien's law of displacement, $\lambda T = b$ for a black body, where b is constant and λ is the wavelength corresponding to the maximum of the black body spectral energy density (taken here as a typical wavelength). Then, the situation $T_{\mathrm{U}} > T_{\mathrm{H}}$ corresponds to the typical wavelength $\lambda_{\mathrm{U}} = b/T_{\mathrm{U}}$ of Unruh quanta being larger than the typical wavelength λ_{H} of Hawking quanta. However it seems that, as long as observers can indisputably establish the existence of a black hole horizon, a restricted maximum force principle limited to these horizons could be valid. This conclusion is not surprising: although no universal force limits exists, the very fact that observers are required to see a black hole imposes an upper bound on the force (rather loosely defined) acting on this black hole horizon. In both cases considered, the solution of the Einstein equations is still mathematically admissible, but it contains a naked singularity. This statement is rather tautological, as we have decided to consider a situation in which the black hole horizon exists, but is located beyond the Rindler horizon or the cosmological apparent horizon. Imposing that the black hole horizon is not removed and that the singularity inside it remains invisible to what are deemed to be physically relevant observers located outside the black hole horizon amounts to limiting the force acting on the black hole horizon. The lesson seems to be that the existence a maximal force acting *on black hole horizons* is (a bit tautologically) tied to cosmic censorship. The reference to forces acting on event, apparent, or Killing horizons, however, establishes a very special and restricted context within which to talk about maximal forces and in no way implies the existence of universal upper bound on the forces acting on (classical or quantum) particles or bodies, in agreement with the conclusions of Ref. [17].

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