

Lyra scalar-tensor theory: A scalar-tensor theory of gravity on Lyra manifold

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We present a scalar-tensor theory of gravity on a torsion-free and metric compatible Lyra manifold. This is obtained by generalizing the concept of physical reference frame by considering a scale function defined over the manifold. The choice of a specific frame induces a local base, naturally nonholonomic, whose structure constants give rise to extra terms in the expression of the connection coefficients and in the expression for the covariant derivative. In the Lyra manifold, transformations between reference frames involving both coordinates and scale change the transformation law of tensor fields, when compared to those of the Riemann manifold. From a direct generalization of the Einstein-Hilbert minimal action coupled with a matter term, it was possible to build a Lyra invariant action, which gives rise to the associated Lyra scalar-tensor theory of gravity (LyST), with field equations for $g_{\mu\nu}$ and ϕ . These equations have a well-defined Newtonian limit, from which it can be seen that both metric and scale play a role in the description of gravitational interaction. We present a spherically symmetric solution for the LyST gravity field equations. It depends on two parameters m and r_L , whose physical meaning is carefully investigated. We highlight the properties of LyST spherically symmetric line element and compare it to Schwarzschild solution.

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I. INTRODUCTION

In the formulation of general relativity, Einstein presented a geometric structure resulting from a set of adopted hypotheses, among which are the adoption of Riemannian manifold for modeling space-time, the shape of space-time infinitesimal interval, the integrability of vector lengths and the absence of torsion [1]. This fact opens avenues for formulating alternative theories of gravity through changing these considerations [2–5]. In this scenario, Hermann Weyl's unified theory, published in 1918, gains prominence, since it raises the possibility of using the freedom to adopt these geometric hypotheses to formulate a structure that accommodates both gravitation and electromagnetism [6]. In Weyl geometry, the change in the length of vectors under parallel transport is nonzero and depends on a new vector quantity A_μ that plays the role of electromagnetic potential. An important finding is that the presented structure exhibits a new type of symmetry, called gauge symmetry, in addition to invariance by diffeomorphisms.

In spite of its simplicity, mathematical beauty and great potential for unifying fundamental theories based on simple geometric concepts, Weyl's theory presents problems of a physical nature. In the first place, as pointed out by Einstein, the hypothesis of nonintegrability of length makes the frequency of spectral lines emitted by atoms depend on their past history and, as such, would not remain constant [7]. Moreover, the Lagrangian density invariance under diffeomorphisms and gauge transformations gives rise to fourth order field equations, which is not desirable in a physical theory [8]. Notwithstanding, Weyl's work is widely recognized, since he was the pioneer in the approach to gauge theories, a concept on which much of the work in modern physics is based. One way to reduce the problems inherent in Weyl's theory is to impose that the Weyl displacement vector is irrotational, that is, $A_\mu = \partial_\mu \sigma$ [9]. Theories with this characteristic, called WIST (Weyl integrable spacetime) have attracted the attention of researchers in recent years [10]. In this approach, the unification between gravity and electromagnetism is set aside, and WIST is seen as a scalar-tensor theory of gravity.

Another way of maintaining the vector length integrability was proposed by Lyra in 1951 through the adoption of a gauge function ϕ as an intrinsic part of the manifold's

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geometric structure [11]. This approach naturally alters the definition of physical frames, since these, in Lyra's geometry, depend on both coordinates and the gauge function. Lyra proposes that the affine connection describing parallel transport be defined by the sum of two sectors; one of them depending on the metric, and the second one depending exclusively on a quantity ϕ , such that the curvature tensor, the torsion tensor and their respective contracted forms will be functions, not only of the metric, but also of ϕ .

Although he described the geometric structure, Lyra did not formulate a field theory where geometric objects play the role of gravitational field. The first proposal to address the matter was made by D. K. Sen, in 1957, in the formulation of a static cosmological model where the scale function appears as responsible for the redshift of the galactic spectral lines [12]. For that, Sen proposed to obtain the field equations through the variational principle:

$$\delta \left(\int d^4x \phi^4 \sqrt{-g} \mathcal{R} \right) = 0, \quad (1)$$

where both the integration element $d^4x \phi^4$ and the scalar curvature \mathcal{R} are invariant under Lyra's reference frame transformation. The equations presented as a result were:

$$\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} + \frac{3}{2} A_\mu A_\nu - \frac{3}{4} g_{\mu\nu} A_\lambda A^\lambda = \kappa T_{\mu\nu} \quad (2)$$

where A_μ is a vector field that, according to the author, is a direct consequence of the gauge function ϕ parallel transport, although he does not make it clear how A_μ is related to ϕ . In this equation, $\mathcal{R}_{\mu\nu}$ and \mathcal{R} are the Ricci tensor and the scalar curvature calculated with Christoffel symbols, as in general relativity. In order to derive (2), Sen used the gauge fixing condition $\phi = 1$, keeping the components of A_μ free and varying the action with respect to the metric components. Subsequently, Sen showed that geodesics on Lyra manifold are not, in general, autoparallel curves [13]. However, as a specific case, it is possible to guarantee the autoparallelism of the geodesics by imposing the condition $A_\mu = \phi^{-1} \partial_\mu \ln \phi^2$. After the measuring of the cosmic background radiation and confirmation of the big bang model [14], Eq. (2) was widely used for modeling the cosmological dynamics [15–20].

Later, in 1971, Sen and Dunn took a step further in an attempt to formalize a gravity theory in Lyra manifold [21]. To this end, the authors, recognized the simultaneous influence of $g_{\mu\nu}$ and A_μ on gravitational phenomena and performed the variational procedure without fixing the gauge. Instead, they varied the action with respect to the metric components and the components of A_μ . The field equation from metric variations recovers (2) in the limit $\phi \rightarrow 1$. The second set of equations, given by

$$3A_\mu + \frac{3}{2} \phi^{-1} \partial_\mu \ln \phi^2 = 0, \quad (3)$$

directly relates A_μ to ϕ . This equation is problematic. First, it is explicitly incompatible with the condition of geodesic autoparallelism, and, as a consequence, metric geodesics and affine geodesics do not coincide in the gravity theory formulated by Sen and Dunn. Secondly, the gauge-fixing condition used in the derivation of (2), namely $\phi = 1$, leads to the vanishing of A_μ . In other words, by imposing the gauge condition in the Sen and Dunn analogue of Einstein's equations, one recovers (2). However, the application of the same condition on (3) causes $A_\mu = 0$. So $\phi = 1$ does not act as a gauge-fixing condition, but most properly as a general relativity limit condition for Sen and Dunn gravity. In fact, disregarding the gauge-fixing condition in this theory makes it possible to work with coupled differential equations for the metric coefficients and the ϕ function. In this case, Sen and Dunn set aside Eq. (2) and combined the two sets of equations in a single expression:

$$\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} - \frac{3}{2\phi^2} \phi_{,\mu} \phi_{,\nu} + \frac{3g_{\mu\nu}}{4\phi^2} \phi_{,\lambda} \phi^{,\lambda} = \kappa \phi^2 T_{\mu\nu}. \quad (4)$$

This is an interesting expression, since it can be interpreted as a specific case of Brans-Dicke's theory [22]. The authors proposed a spherically symmetric solution for (4) written in terms of power series [21]. The following year, Halford found a spherically symmetric analytical solution in isotropic coordinates [23].

In 1975, Jeavons, McIntosh and Sen pointed out the heuristic importance of Eq. (4) but showed that it cannot be obtained from the principle of least action [24], unlike what is proposed in [21]. This equation was obtained by neglecting the contribution of the terms $\int d^4x \phi^2 \delta(\sqrt{-g} \mathcal{R})$ and $\int d^4x \phi^3 \delta((\sqrt{-g} A_\mu g^{\mu\nu})_{,\nu})$. The field equation that respects the aforementioned principle, arises from the variation of the action with respect to the components of the metric tensor, while the condition of self-parallelism of geodesics $A_\mu = \phi^{-1} \partial_\mu \ln \phi^2$ arises naturally from the variation of the action with respect to A_μ .

Through the knowledge on theories of gravity in the Lyra manifold, one can notice that the presence of a scale in the vector transformation law naturally induces extra terms to Christoffel symbols. However, the introduction of a vector field A_μ as an entity uncorrelated to ϕ describing these terms does not seem a strictly necessary procedure, since the field equations themselves establish a direct relation between A_μ and ϕ , both in the work of Sen and Dunn of 1971 and in the theory of Jeavons, McIntosh, and Sen of 1975. Thus, this work proposes a new interpretation on the description of the Lyra manifold, assuming ϕ , and the metric tensor as the fundamental fields of a scalar-tensor theory of gravity, here called LyST, a shorthand notation for Lyra scalar-tensor theory of gravity. In order to do so, in Sec. II we build a manifold where the scale function ϕ and the coordinates define a Lyra reference. The choice of a specific reference frame induces a local basis, naturally

nonholonomic, where the structure constants of the Lie algebra depend exclusively on ϕ . This geometric structure leads naturally to Lyra's vector transformation law and induces extra terms in the related affine connection. Once this framework is established, the Lyra manifold is adopted as a space-time model, in Sec. III, where we impose suitable geometric conditions and calculate curvature properties. In the Sec. IV, a variational principle is proposed through which field equations can be obtained for the metric coefficients and the scale function. Such equations have an appropriate Newtonian limit, as shown in Sec. IV A. A symmetrically symmetrical solution for LyST gravity is presented in Sec. V. The vacuum solution is built up in Sec. V A. It is shown in Sec. V B that it reduces to the ordinary Schwarzschild line element in the appropriate limit. The long-distance regime of LyST spherically symmetric solution accommodates the tantalizing possibility of reproducing a cosmological constantlike behavior, either in the ways of a de Sitter solution or similar to a anti-de Sitter space-time. We identify two different classes of LyST spherically symmetric metrics in Sec. V C. Section V D is dedicate to study the singularities of this metric and to discuss its causal structure. Section VI contains our final comments and future perspectives.

II. LYRA GEOMETRY

A Lyra differential manifold \mathcal{M} is a second-countable Hausdorff topological space of dimension n . A reference frame in this geometry is represented by the triad (U_i, χ_i, Φ_i) , where $U_i \subset \mathcal{M}$ is a open subset of \mathcal{M} , $\{\chi_i: U_i \rightarrow \mathbb{R}^n | n \in \mathbb{Z}_+\}$ is a map parametrizing the coordinates and $\Phi: U_i \rightarrow \mathbb{R}^*$ a scale map. Thus, considering $P \in \mathcal{M}$, then the coordinates of P are defined as the map $x := \chi \circ P: \mathcal{M} \rightarrow \mathbb{R}^n$ and the scale function as $\phi := \Phi \circ \chi^{-1}: \mathbb{R}^n \rightarrow \mathbb{R}$. Therefore, $\Phi \circ P = \Phi \circ \chi^{-1} \circ \chi \circ P$ can be written as $\phi(x_P) := \phi \circ x_P$.

The reference frames $\{(U_i, \chi_i, \Phi_i)\}$ on \mathcal{M} must respect [25]:

- (i) $\cup_{i \in I} U_i = \mathcal{M}$, where i is an index that covers a total account I of frames;
- (ii) the maps $\chi_i \circ \chi_j^{-1}$ and $\chi_j \circ \chi_i^{-1}$ must be C^∞ functions in their domains whenever $U_i \cap U_j \neq \emptyset$;
- (iii) the maps $\Phi_i \circ \chi_i$ must be C^∞ functions in U_i for each $i \in I$; and
- (iv) the family of reference frames is maximal with respect to (i), (ii), and (iii).

Consider a curve $\alpha(t)$ over \mathcal{M} through the differential map $\alpha: \mathbb{R} \rightarrow \mathcal{M}$, where $\alpha(0) := P \in \mathcal{M}$, and a C^∞ function $f: \mathcal{M} \rightarrow \mathbb{R}$. It is possible to define a *tangent vector* $\mathbf{v}_{(\alpha)}$ in P , as a map $\mathbf{v}_{(\alpha)}: C^\infty \rightarrow \mathbb{R}$ expressed by:

$$\mathbf{v}_{(\alpha)}f = \left. \frac{d(f \circ \alpha)}{dt} \right|_P. \quad (5)$$

By considering an specific reference frame (U, χ, Φ) , Eq. (5) can be represented by $\mathbf{v}_{(\alpha)}f = x^{\mu'}(0)\partial_{\mu'}f$.

In Lyra's geometry, the coordinates x^μ and scale function ϕ related to the choice of a specific reference frame can be used to define a natural basis as:

$$\mathbf{e}_\mu = \frac{1}{\phi(x)} \partial_\mu. \quad (6)$$

Given the basis, any tangent vector $\mathbf{v}_{(\alpha)}$ can be represented as $\mathbf{v}_{(\alpha)} = v_{(\alpha)}^\mu \mathbf{e}_\mu$, where $v_{(\alpha)}^\mu$, given by:

$$v_{(\alpha)}^\mu := \phi(x)x^{\mu'}(0) \quad (7)$$

are the components of $\mathbf{v}_{(\alpha)}$. The set $\{\mathbf{v}_1, \dots, \mathbf{v}_K\}$ of all tangent vectors in P forms a tangent vector space $\mathbf{T}_P\mathcal{M} = \cup_{k \in K} \mathbf{v}_k$.

A direct consequence of the expression (6) is the noncommutativity of the basis elements induced by the choice of a reference frame. Indeed, the Lie algebra can be characterized by:

$$[\mathbf{e}_\mu, \mathbf{e}_\nu] = \phi^{-2}(\delta_\mu^\alpha \partial_\nu \phi - \delta_\nu^\alpha \partial_\mu \phi) \mathbf{e}_\alpha. \quad (8)$$

The fact that the structure constants are non-null leads to important consequences to the geometric properties of the manifold. It is convenient to define the notation $\gamma^\alpha_{\mu\nu} = \phi^{-2}(\delta_\mu^\alpha \partial_\nu \phi - \delta_\nu^\alpha \partial_\mu \phi)$.

Let (U, χ, Φ) e $(\bar{U}, \bar{\chi}, \bar{\Phi})$ two reference frames, such that $U \cap \bar{U} \neq \emptyset$ and $P \in U \cap \bar{U}$. The relation between $\mathbf{e}_\mu \in \mathbf{T}_P\mathcal{M}$, according to (6), is

$$\bar{\mathbf{e}}_\mu = \frac{\phi(x)}{\bar{\phi}(\bar{x})} \frac{\partial x^\nu}{\partial \bar{x}^\mu} \mathbf{e}_\nu. \quad (9)$$

The components of the tangent vector $\mathbf{v}_{(\alpha)} \in \mathbf{T}_P\mathcal{M}$ in reference frame U relate to its components in reference \bar{U} via [11]:

$$\bar{v}_{(\alpha)}^\mu = \frac{\bar{\phi}(\bar{x})}{\phi(x)} \frac{\partial \bar{x}^\mu}{\partial x^\nu} v_{(\alpha)}^\nu. \quad (10)$$

A covector ω over the point $P \in \mathcal{M}$ is defined as a linear map $\omega: \mathbf{T}_P\mathcal{M} \rightarrow \mathbb{R}$, which takes tangent vectors as arguments. The set of all covectors in P form a vector space, called a cotangent space $\mathbf{T}_P^*\mathcal{M}$. A covector ω can be represented in a specific basis, denoted by $\{\theta^\nu\}$, as $\omega = \omega_\nu \theta^\nu$. Applying this in a given vector $\mathbf{v} \in \mathbf{T}_P\mathcal{M}$, the result is given by:

$$\omega_P \circ \mathbf{v}_P = (\omega_\nu \theta^\nu) \circ (v^\mu \mathbf{e}_\mu) = v^\mu \omega_\mu \quad (11)$$

where $\omega_\mu := \omega \circ \mathbf{e}_\mu$ are the components of the covector ω and it was assumed the orthonormality condition $\theta^\nu \circ \mathbf{e}_\mu = \delta_\mu^\nu$ to the basis.

¹The set C^∞ consists of all differentiable functions $f: \mathcal{M} \rightarrow U$ in P .

Let a curve $\alpha(t)$ in \mathcal{M} given by the differentiable map $\alpha: \mathbb{R} \rightarrow \mathcal{M}$, with $\alpha(0) := P \in \mathcal{M}$ and a smooth function $f: \mathcal{M} \rightarrow \mathbb{R}$. Thus, one can check that the map $f \circ \alpha: \mathbb{R} \rightarrow \mathbb{R}$, where $f \circ \alpha = f \circ \chi^{-1} \circ (\chi \circ \alpha) = \tilde{f}(x(t))$ is an ordinary function of t that describes the behavior of f along the curve. Taking the derivative of this function in P , cf. (5), it turns out that the value of f at $t = 0$ depends only on the tangent vector $\mathbf{v}_{(\alpha)}$ to the α curve. That is, the operation $d(f \circ \alpha)/dt|_P = \mathbf{v}_{(\alpha)}f$ defines a vector $\mathbf{d}f: \mathbf{T}_P\mathcal{M} \rightarrow \mathbb{R}$. By choosing a reference frame, $\mathbf{d}f$ is represented by:

$$\mathbf{d}f \circ \mathbf{v} = v^\lambda \phi^{-1} \partial_\lambda \tilde{f}, \quad (12)$$

whence, according to the Eq. (11), $\phi^{-1} \partial_\mu f$ are the components of $\mathbf{d}f$.

As mentioned above, a basis $\boldsymbol{\theta}^\mu$ to cotangent space $\mathbf{T}_P^*\mathcal{M}$ is obtained by requiring the orthonormality condition. This is done by defining n smooth functions $x^\mu: \mathcal{M} \rightarrow \mathbb{R}$, parametrized in a specific reference frame which receive a point in \mathcal{M} as argument and return the μ th coordinate. Let us take \tilde{f} as x^μ and \mathbf{v} as \mathbf{e}_ν in Eq. (12). It follows that $\mathbf{d}x^\mu \circ \mathbf{e}_\nu = \delta_\mu^\lambda \phi^{-1} \partial_\lambda x^\nu = \delta_\mu^\nu \phi^{-1}$. Consequently, an adequate basis for $\mathbf{T}_P^*\mathcal{M}$ will be

$$\boldsymbol{\theta}^\mu = \phi \mathbf{d}x^\mu. \quad (13)$$

So, a general covector $\boldsymbol{\omega}$ can be written as $\boldsymbol{\omega} = \omega_\mu \phi \mathbf{d}x^\mu$ in a local frame. Under a change of frames, its components are transformed as:

$$\bar{\omega}_\mu = \frac{\phi(x)}{\phi(\bar{x})} \frac{\partial x^\nu}{\partial \bar{x}^\mu} \omega_\nu. \quad (14)$$

The tensors \mathbf{W} of type (p, q) are defined as applications receiving p tangent covectors and q tangent vectors, mapping them to a real number:

$$\mathbf{W}: \underbrace{\mathbf{T}_P^*\mathcal{M} \times \cdots \times \mathbf{T}_P^*\mathcal{M}}_{p \text{ factors}} \times \underbrace{\mathbf{T}_P\mathcal{M} \times \cdots \times \mathbf{T}_P\mathcal{M}}_{q \text{ factors}} \rightarrow \mathbb{R}.$$

A tensor like this can be expanded on a basis

$$\{\boldsymbol{\theta}^{\mu_1} \otimes \cdots \otimes \boldsymbol{\theta}^{\mu_p} \otimes \mathbf{e}_{\nu_1} \otimes \cdots \otimes \mathbf{e}_{\nu_q}\}$$

and its components transform as:

$$\begin{aligned} \bar{W}_{\beta_1 \cdots \beta_p}^{\alpha_1 \cdots \alpha_q} &= \left(\frac{\phi(\bar{x})}{\phi(x)} \right)^{q-p} \frac{\partial \bar{x}^{\alpha_1}}{\partial x^{\mu_1}} \cdots \frac{\partial \bar{x}^{\alpha_q}}{\partial x^{\mu_q}} \\ &\times \frac{\partial x^{\nu_1}}{\partial \bar{x}^{\beta_1}} \cdots \frac{\partial x^{\nu_p}}{\partial \bar{x}^{\beta_p}} W_{\nu_1 \cdots \nu_p}^{\mu_1 \cdots \mu_q} \end{aligned} \quad (15)$$

under changes of Lyra frames.

A. The metric

The differentiable structure over \mathcal{M} presented so far is not capable of defining scalar products between tangent vectors, a fact that makes it impossible to determine lengths and angles. Since it is desired to use Lyra's geometry as a space-time model, it is necessary to consider an additional structure on \mathcal{M} so that the inner product of vectors $\{\mathbf{u}, \mathbf{v}\} \in \mathbf{T}_P\mathcal{M}$ is defined. This can be accomplished by equipping the Lyra manifold with a $(0,2)$ metric tensor $\mathbf{g}: \mathbf{T}_P\mathcal{M} \times \mathbf{T}_P\mathcal{M} \rightarrow \mathbb{R}$, characterized as a bilinear, symmetric, and nondegenerate map. By introducing a reference frame, its components $g_{\mu\nu}$ will be obtained through the relation $g_{\mu\nu} = \mathbf{g}(\mathbf{e}_\mu, \mathbf{e}_\nu)$, and the dot product between the two above vectors will be $\mathbf{g}(\mathbf{v}, \mathbf{u}) = g_{\mu\nu} u^\mu v^\nu$. The components $g_{\mu\nu}$ form a nonsingular matrix, whose inverse components $g^{\mu\nu}$ transform like the components of a $(2,0)$ -tensor.

The definition of an inner product naturally leads to a canonical association between $\mathbf{T}_P^*\mathcal{M}$ and $\mathbf{T}_P\mathcal{M}$. Take a vector $\mathbf{u} \in \mathbf{T}_P\mathcal{M}$. A covector $\mathbf{T}_P\mathcal{M} \rightarrow \mathbb{R}$ is defined through $\mathbf{g}(\mathbf{u}, \cdot)$. Choose a frame, such that $\mathbf{u} = u^\mu \mathbf{e}_\mu$. Then, the covector components u_ν , can be obtained by the referred application to the base vectors of $\mathbf{T}_P\mathcal{M}$: $u_\nu = \mathbf{g}(\mathbf{u}, \mathbf{e}_\nu) = g_{\mu\nu} u^\mu$.

As in Riemannian geometry, the matrix of components $g_{\mu\nu}$ has p positive eigenvalues, q negative eigenvalues and r null eigenvalues, such that $p + q + r = n$, where $n = \dim \mathcal{M}$. The proportion of each of these quantities determines the signature (p, q, r) for the metric that does not depend on the choice of frames and is constant throughout \mathcal{M} , provided that the metric is nondegenerate and smooth at all points of the manifold. In the case of $(3, 0, 0)$ and $(0, 3, 0)$ the signature is said to be Riemannian whereas $(1, 3, 0)$ or $(3, 1, 0)$ are said to be Lorentzian signatures [26].

The introduction of a metric allows the definition of essential physical elements in the construction of a gravity theory. The first one is the concept of length. Let be a curve $\lambda: \mathbb{R} \rightarrow \mathcal{M}$, with tangent vectors $\mathbf{v} = \phi(x) x'^\mu \mathbf{e}_\mu$. The curve segment s connecting a and b is defined as:

$$s = \int_a^b dt \mathbf{g}(\mathbf{v}, \mathbf{v})^{1/2} = \int_a^b dt \sqrt{\phi^2 g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}}, \quad (16)$$

where $[a, b] \subset \mathbb{R}$. A metric geodesic is defined as the curve through points a and b whose distance s is stationary under fixed infinitesimal variations at the extremes. By performing the variations in (16) imposing $\delta s = 0$, we obtain the geodesic equation in Lyra manifold:

$$\begin{aligned} \frac{d^2 x^\mu}{dt^2} + \left\{ \begin{array}{c} \mu \\ \alpha\beta \end{array} \right\} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} \\ + \frac{1}{\phi} (\partial_\alpha \phi \delta_\beta^\mu + \partial_\beta \phi \delta_\alpha^\mu - \partial_\nu \phi g^{\mu\nu} g_{\alpha\beta}) \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} = 0 \end{aligned} \quad (17)$$

where $\left\{ \begin{array}{c} \mu \\ \alpha\beta \end{array} \right\}$ are the Christoffel symbols.

In Lyra geometry, the presence of the scale influences the volume invariant element. Starting from the transformation law for the metric tensor components—which is a $(0, 2)$ tensor—it can be shown that the determinant of the metric transforms under a change of frames as:

$$\tilde{g} = \frac{\phi^{2n}}{\bar{\phi}^{2n}} \det \left(\frac{\partial x^\alpha}{\partial \bar{x}^\mu} \right)^2 g,$$

where $g := \det g_{\mu\nu}$ and n is the dimension of the space. Thus, the volume integration element will be $d^n x \phi^n \sqrt{|g|}$ and the volume of a given region $R \subset \mathcal{M}$ will be

$$\text{vol}(R) = \int_{\mathcal{X}(R)} d^n x \phi^n \sqrt{|g|}.$$

B. Linear connection

A linear connection over \mathcal{M} is defined as a bilinear map $\nabla: \mathbf{T}_p \mathcal{M} \times \mathbf{T}_p \mathcal{M} \rightarrow \mathbf{T}_p \mathcal{M}$ relating two tangent vectors \mathbf{u} and \mathbf{v} to a $(1, 1)$ —tensor $\nabla_{\mathbf{u}} \mathbf{v}$ [27]. Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{T}_p \mathcal{M}$ and a smooth function $f \in C^\infty$. The connection respects the following properties:

- (i) $\nabla_{\mathbf{u}}(\mathbf{v} + \mathbf{w}) = \nabla_{\mathbf{u}} \mathbf{v} + \nabla_{\mathbf{u}} \mathbf{w}$;
- (ii) $\nabla_{\mathbf{u}+\mathbf{v}}(\mathbf{w}) = \nabla_{\mathbf{u}} \mathbf{w} + \nabla_{\mathbf{v}} \mathbf{w}$;
- (iii) $\nabla_{f\mathbf{u}}(\mathbf{v}) = f \nabla_{\mathbf{u}} \mathbf{v}$; and
- (iv) $\nabla_{\mathbf{u}}(f\mathbf{v}) = (\mathbf{u}f)\mathbf{v} + f \nabla_{\mathbf{u}} \mathbf{v}$.

Let $\sigma: \mathbb{R} \rightarrow \mathcal{M}$ and $\lambda: \mathbb{R} \rightarrow \mathcal{M}$ be two smooth curves over \mathcal{M} which intersect at the point $P = \sigma(0) = \lambda(0) \in \mathcal{M}$.

Through Eq. (5), one can define the tangent vectors $\mathbf{v}_{(\sigma)}$ to σ and $\mathbf{u}_{(\lambda)}$ to λ , where $f \in C^\infty: \mathcal{M} \rightarrow \mathbb{R}$. Taking a specific frame (U, \mathcal{X}, ϕ) , such that $P \in U$, the vectors can be represented as $\mathbf{v}_{(\sigma)} = v_{(\sigma)}^\mu \mathbf{e}_\mu$ and $\mathbf{u}_{(\lambda)} = u_{(\lambda)}^\mu \mathbf{e}_\mu$, where their respective components are given by $v_{(\sigma)}^\mu = \phi(x)(dx^\mu/dt)(0)$ and $u_{(\lambda)}^\mu = \phi(x)(dx^\mu/dt)(0)$. The operation $\nabla_{\mathbf{u}_{(\lambda)}} \mathbf{v}_{(\sigma)}$ represents the change in the vector field $\mathbf{v}_{(\sigma)}$ in the direction of $\mathbf{u}_{(\lambda)}$. Therefore, the related connection naturally brings with it the concept of parallel vector fields along a curve. If $\nabla_{\mathbf{u}_{(\lambda)}} \mathbf{v}_{(\sigma)} = 0$, it is said that $\mathbf{v}_{(\sigma)}$ is paralleled in the direction of λ .

In this frame, the tensor $\nabla_{\mathbf{u}_{(\lambda)}} \mathbf{v}_{(\sigma)}$ will be

$$\nabla_{\mathbf{u}_{(\lambda)}} \mathbf{v}_{(\sigma)} = u_{(\lambda)}^\nu (\nabla_{\mathbf{e}_\nu} v_{(\sigma)}^\mu) \mathbf{e}_\mu + u_{(\lambda)}^\nu v_{(\sigma)}^\mu \Gamma_{\mu\nu}^\alpha \mathbf{e}_\alpha,$$

where the connection coefficients were defined as:

$$\Gamma_{\mu\nu}^\alpha \mathbf{e}_\alpha := \nabla_{\mathbf{e}_\nu} \mathbf{e}_\mu. \quad (18)$$

Using expression (6), one finds:

$$\nabla_{\mathbf{u}_{(\lambda)}} \mathbf{v}_{(\sigma)} = u_{(\lambda)}^\nu (\phi^{-1} \partial_\nu v_{(\sigma)}^\alpha + \Gamma_{\mu\nu}^\alpha v_{(\sigma)}^\mu) \mathbf{e}_\alpha. \quad (19)$$

Therefore, the change in the components of $\mathbf{v}_{(\sigma)}$ under parallel transport comes as a result of the differential equation:

$$\phi^{-1} u_{(\lambda)}^\nu \partial_\nu v_{(\sigma)}^\alpha + \Gamma_{\mu\nu}^\alpha v_{(\sigma)}^\mu u_{(\lambda)}^\nu = 0. \quad (20)$$

I. Covariant derivative

Consider the curves $\lambda_{(\nu)}$ generating the base vectors \mathbf{e}_ν in a given frame. The covariant derivative of a vector \mathbf{v} in the direction of \mathbf{e}_ν will be written as:

$$\nabla_{\mathbf{e}_\nu} \mathbf{v} = (\phi^{-1} \partial_\nu v^\alpha + \Gamma_{\mu\nu}^\alpha v^\mu) \mathbf{e}_\alpha. \quad (21)$$

Its components

$$\nabla_\nu v^\alpha = \frac{1}{\phi} \partial_\nu v^\alpha + \Gamma_{\mu\nu}^\alpha v^\mu, \quad (22)$$

differ from the Riemannian case, by the presence of the factor ϕ^{-1} in the ordinary derivative.

The covariant derivative of a covector $\boldsymbol{\omega}$ will be written as $\nabla_{\mathbf{e}_\mu} \boldsymbol{\omega} = \nabla_\mu \omega_\alpha \boldsymbol{\theta}^\alpha$. By using the fact that the covariant derivative of a scalar f is $\nabla_{\mathbf{u}} f = \mathbf{u}f$ and that the application of a $\boldsymbol{\omega}$ vector to a tangent vector returns a scalar, we find that:

$$\nabla_\mu \omega_\nu = \frac{1}{\phi} \partial_\mu \omega_\nu - \Gamma_{\nu\mu}^\lambda \omega_\lambda. \quad (23)$$

The concept of covariant derivative can be extended to a general (p, q) —tensor \mathbf{T} . In this case, its covariant derivative will be referred to by $\nabla_{\mathbf{e}_\alpha} \mathbf{T}$ and can be expanded as:

$$\nabla_{\mathbf{e}_\alpha} \mathbf{T} = \nabla_\alpha T_{\mu_1 \dots \mu_p}^{\nu_1 \dots \nu_q} (\mathbf{e}_\alpha \otimes \boldsymbol{\theta}^{\mu_1} \otimes \dots \otimes \boldsymbol{\theta}^{\mu_p} \otimes \mathbf{e}_{\nu_1} \otimes \dots \otimes \mathbf{e}_{\nu_q})$$

where the components $\nabla_\alpha T_{\mu_1 \dots \mu_p}^{\nu_1 \dots \nu_q}$ are given by:

$$\begin{aligned} \nabla_\alpha T_{\mu_1 \dots \mu_p}^{\nu_1 \dots \nu_q} &= \frac{1}{\phi} \partial_\alpha T_{\mu_1 \dots \mu_p}^{\nu_1 \dots \nu_q} + \Gamma^{\nu_1}_{\lambda\alpha} T_{\mu_1 \dots \mu_p}^{\lambda \dots \nu_q} \\ &+ \dots + \Gamma^{\nu_q}_{\lambda\alpha} T_{\mu_1 \dots \mu_p}^{\nu_1 \dots \lambda} - \Gamma_{\mu_1 \alpha}^\lambda T_{\lambda \dots \mu_p}^{\nu_1 \dots \nu_q} \\ &- \dots - \Gamma_{\mu_p \alpha}^\lambda T_{\mu_1 \dots \lambda}^{\nu_1 \dots \nu_q}. \end{aligned} \quad (24)$$

2. Autoparallel Curves

Let $\sigma: \mathbb{R} \rightarrow \mathcal{M}$ be a smooth curve over \mathcal{M} , with tangent vector $\mathbf{v}_{(\sigma)} = v_{(\sigma)}^\mu \mathbf{e}_\mu$. This will be called an autoparallel curve if $\mathbf{v}_{(\sigma)}$ is parallel transported in relation to its own generating curve σ ; that is, $\nabla_{\mathbf{v}_{(\sigma)}} \mathbf{v}_{(\sigma)} = 0$. With the Eq. (20) and the expression for the components of the tangent vector in terms of the coordinates $v_{(\sigma)}^\mu = \phi(x)(dx^\mu/dt)$, we find the differential equation for a autoparallel curve is

$$\frac{d^2 x^\nu}{dt^2} + (\phi \Gamma_{\rho\mu}^\nu + \nabla_\mu \phi \delta_\rho^\nu) \frac{dx^\mu}{dt} \frac{dx^\rho}{dt} = 0, \quad (25)$$

where $\nabla_\mu \phi := \phi^{-1} \partial_\mu \phi$.

3. Curvature and torsion

The curvature tensor is defined by the linear map $\mathbf{R}: \mathbf{T}_p\mathcal{M} \times \mathbf{T}_p\mathcal{M} \times \mathbf{T}_p\mathcal{M} \rightarrow \mathbf{T}_p\mathcal{M}$ through [25]:

$$\mathbf{R}(\mathbf{u}, \mathbf{v})\mathbf{w} = \nabla_{\mathbf{u}}\nabla_{\mathbf{v}}\mathbf{w} - \nabla_{\mathbf{v}}\nabla_{\mathbf{u}}\mathbf{w} - \nabla_{[\mathbf{v}, \mathbf{u}]}\mathbf{w}. \quad (26)$$

It is clear from this equation the antisymmetric property $\mathbf{R}(\mathbf{u}, \mathbf{v})\mathbf{w} = -\mathbf{R}(\mathbf{v}, \mathbf{u})\mathbf{w}$. The curvature components can be obtained through the expression:

$$R^{\lambda}{}_{\mu\nu\alpha} = \frac{1}{\phi^2}\partial_{\mu}(\phi\Gamma^{\lambda}{}_{\alpha\nu}) - \frac{1}{\phi^2}\partial_{\nu}(\phi\Gamma^{\lambda}{}_{\alpha\mu}) + \Gamma^{\rho}{}_{\alpha\nu}\Gamma^{\lambda}{}_{\rho\mu} - \Gamma^{\rho}{}_{\alpha\mu}\Gamma^{\lambda}{}_{\rho\nu} \quad (27)$$

The torsion tensor is defined as a map:

$$\boldsymbol{\tau}: \mathbf{T}_p\mathcal{M} \times \mathbf{T}_p\mathcal{M} \rightarrow \mathbf{T}_p\mathcal{M}$$

expressed as [25]:

$$\boldsymbol{\tau}(\mathbf{u}, \mathbf{v}) = \nabla_{\mathbf{u}}\mathbf{v} - \nabla_{\mathbf{v}}\mathbf{u} - [\mathbf{u}, \mathbf{v}], \quad (28)$$

from where it can be directly checked that $\boldsymbol{\tau}(\mathbf{u}, \mathbf{v}) = -\boldsymbol{\tau}(\mathbf{v}, \mathbf{u})$. The torsion tensor components in a local frame $\tau^{\alpha}{}_{\mu\nu}$ are obtained applying the expression (28) to the vector basis:

$$\tau^{\alpha}{}_{\mu\nu}\mathbf{e}_{\alpha} = \boldsymbol{\tau}(\mathbf{e}_{\mu}, \mathbf{e}_{\nu}) = (\Gamma^{\alpha}{}_{\nu\mu} - \Gamma^{\alpha}{}_{\mu\nu} - \gamma^{\alpha}{}_{\mu\nu})\mathbf{e}_{\alpha},$$

where $\gamma^{\alpha}{}_{\mu\nu}$ are the structure constants of the Lie algebra of local basis vectors, given in (8). Furthermore, the components of the torsion tensor will be:

$$\tau^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}{}_{\nu\mu} - \Gamma^{\alpha}{}_{\mu\nu} + \phi^{-1}(\nabla_{\mu}\phi\delta_{\nu}^{\alpha} - \nabla_{\nu}\phi\delta_{\mu}^{\alpha}). \quad (29)$$

Starting from Eqs. (28) and (26), considering \mathfrak{S} as a cyclic sum symbol, it can be shown that curvature and torsion respect [25]:

(i) the first Bianchi identity:

$$\mathfrak{S}[\mathbf{R}(\mathbf{u}, \mathbf{v})\mathbf{w}] = \mathfrak{S}[\boldsymbol{\tau}(\mathbf{u}, \mathbf{v}), \mathbf{w}] + (\nabla_{\mathbf{u}}\boldsymbol{\tau})(\mathbf{v}, \mathbf{w}); \quad (30)$$

(ii) the second Bianchi identity:

$$\mathfrak{S}[(\nabla_{\mathbf{u}}\mathbf{R})(\mathbf{v}, \mathbf{w})] + \mathbf{R}(\boldsymbol{\tau}(\mathbf{u}, \mathbf{v}), \mathbf{w}) = 0. \quad (31)$$

III. ASSUMPTIONS ON LYRA SPACETIME

The geometric structure of the Lyra manifold $(\mathcal{M}, \mathbf{g}, \phi, \nabla)$ can be used as a model for relativistic spacetime. A four-dimensional manifold is adopted, equipped with a Lorentzian metric of signature $(+, -, -, -)$. A line element in this spacetime model will be

$$ds^2 = g_{\mu\nu}\phi dx^{\mu}\phi dx^{\nu},$$

where ds^2 is invariant by general coordinate and scale transformations.

Although the geometric structure of Lyra manifold is very rich in terms of geometric conceptualization, it depends on many degrees of freedom to describe the properties of space-time. Looking for an analogue in the case of pseudo-Riemannian manifolds, it can be concluded that, although the theories of Einstein-Cartan and Weyl, for example, present more daring proposals for geometrization of physics, it is the theory of general relativity that best provides a description, according to scientific method, of the gravitational effects on the scales of the solar system. The spacetime of general relativity is obtained from the hypotheses of metric compatibility and torsion-free manifold, which considerably simplifies the geometric structure and field equations.

Likewise, it is desirable to impose a few hypotheses to restrict the degrees of freedom of Lyra manifold. We would like that Lyra gravitodynamics steams from Lyra manifold in the same way general relativity is the theory of gravity coming from the geometry in a pseudo-Riemannian manifold. That is, we require Lyra spacetime to be a metric-compatible

$$\nabla_{\mathbf{u}}\mathbf{g}(\mathbf{v}, \mathbf{w}) = 0 \quad (32)$$

and torsion-free spacetime:

$$\nabla_{\mathbf{u}}\mathbf{v} - \nabla_{\mathbf{v}}\mathbf{u} = [\mathbf{u}, \mathbf{v}]. \quad (33)$$

As a consequence, the connection will be dependent only on the metric and the scale function. One verify this claim by taking into account the possible permutations of (32) and building a tensor $\mathfrak{B}[\nabla_{\mathbf{u}}\mathbf{g}(\mathbf{v}, \mathbf{w})]$ defined by:

$$\mathfrak{B}[\nabla_{\mathbf{u}}\mathbf{g}(\mathbf{v}, \mathbf{w})] := \nabla_{\mathbf{u}}\mathbf{g}(\mathbf{v}, \mathbf{w}) + \nabla_{\mathbf{v}}\mathbf{g}(\mathbf{u}, \mathbf{w}) - \nabla_{\mathbf{w}}\mathbf{g}(\mathbf{u}, \mathbf{v}),$$

which is symmetric with respect to \mathbf{u} and \mathbf{v} . Since the connection respects Leibniz's rule, the covariant directional derivative applied to the metric can be rewritten as [25]:

$$\nabla_{\mathbf{u}}\mathbf{g}(\mathbf{v}, \mathbf{w}) = \mathbf{u}[\mathbf{g}(\mathbf{v}, \mathbf{w})] - \mathbf{g}(\nabla_{\mathbf{u}}\mathbf{v}, \mathbf{w}) - \mathbf{g}(\mathbf{v}, \nabla_{\mathbf{u}}\mathbf{w}).$$

This relation together with the metric-compatibility and torsion-free conditions, defined on Eqs. (32) and (33), cast $\mathfrak{B}[\nabla_{\mathbf{u}}\mathbf{g}(\mathbf{v}, \mathbf{w})]$ in the form:

$$\begin{aligned} 2\mathbf{g}(\nabla_{\mathbf{v}}\mathbf{u}, \mathbf{w}) &= \mathbf{u}[\mathbf{g}(\mathbf{v}, \mathbf{w})] + \mathbf{v}[\mathbf{g}(\mathbf{w}, \mathbf{u})] \\ &\quad - \mathbf{w}[\mathbf{g}(\mathbf{u}, \mathbf{v})] + \mathbf{g}([\mathbf{v}, \mathbf{u}], \mathbf{w}) \\ &\quad + \mathbf{g}([\mathbf{w}, \mathbf{u}], \mathbf{v}) + \mathbf{g}([\mathbf{w}, \mathbf{v}], \mathbf{u}). \end{aligned}$$

The components of this equation can be obtained by taking a frame where $\mathbf{u} = \mathbf{e}_{\mu}$, $\mathbf{v} = \mathbf{e}_{\nu}$ and $\mathbf{w} = \mathbf{e}_{\lambda}$, along with Eqs. (8) and (18):

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{\phi} \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} + \phi^{-1} (\nabla_\mu \phi \delta_\nu^\alpha - g_{\mu\nu} \nabla^\alpha \phi). \quad (34)$$

Another important property of the connection coefficients is the following. By replacing them (34) in the Eq. (25), the autoparallel curve equation is reduced to:

$$\begin{aligned} \frac{d^2 x^\alpha}{dt^2} + \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} \\ + \frac{1}{\phi} (\partial_\mu \phi \delta_\nu^\alpha + \partial_\nu \phi \delta_\mu^\alpha - g^{\alpha\lambda} g_{\mu\nu} \partial_\lambda \phi) \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} = 0 \end{aligned} \quad (35)$$

which is exactly the Eq. (17) of the geodesic curve. Consequently, in a metric-compatible and torsion-free Lyra manifold, metric and affine geodesics are equivalent.

Just like it happens for the connection coefficients, the curvature tensor components (27) in a specific frame depend on both the metric and the scale function. Plugging Eq. (34) into Eq. (27), one finds:

$$\begin{aligned} R^\alpha_{\mu\nu\beta} = \frac{1}{\phi^2} \mathcal{R}^\alpha_{\mu\nu\beta} - \frac{2}{\phi} \delta^\alpha_{[\nu} \nabla_{\beta]} \nabla_\mu \phi + \frac{2}{\phi} g_{\mu[\nu} \nabla_{\beta]} \nabla^\alpha \phi \\ + \frac{2}{\phi^2} \delta^\alpha_{[\nu} g_{\beta]\mu} \nabla_\lambda \phi \nabla^\lambda \phi, \end{aligned} \quad (36)$$

where $\nabla_\mu \phi := \phi^{-1} \partial_\mu \phi$ and $\mathcal{R}^\alpha_{\mu\nu\beta}$ is the curvature tensor evaluated with the Christoffel symbols of general relativity [28]. Lyra's analogue for the Ricci tensor is obtained by contracting the indexes α e β of (36):

$$\begin{aligned} R_{\mu\nu} = \frac{1}{\phi^2} \mathcal{R}_{\mu\nu} + \frac{2}{\phi} \nabla_\nu \nabla_\mu \phi + \frac{1}{\phi} g_{\mu\nu} \nabla_\alpha \nabla^\alpha \phi \\ - \frac{3}{\phi^2} g_{\mu\nu} \nabla_\lambda \phi \nabla^\lambda \phi. \end{aligned} \quad (37)$$

The curvature scalar for Lyra manifold is the trace of $R_{\mu\nu}$:

$$R = \frac{1}{\phi^2} \mathcal{R} + \frac{6}{\phi} \nabla_\lambda \nabla^\lambda \phi - \frac{12}{\phi^2} \nabla_\lambda \phi \nabla^\lambda \phi. \quad (38)$$

Lyra's curvature tensor has the following antisymmetry properties:

$$R_{\mu\nu\alpha\beta} = R_{[\mu\nu]\alpha\beta} \quad \text{and} \quad R_{\mu\nu\alpha\beta} = R_{\mu\nu[\alpha\beta]}. \quad (39)$$

Moreover, it is symmetrical under the exchange between the first two indexes and the last two indexes:

$$R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu}. \quad (40)$$

Once the torsion-free condition is assumed, the Bianchi identities (30) and (31) projected in a specific frame are expressed as:

$$R^\alpha_{\mu\nu\beta} + R^\alpha_{\nu\beta\mu} + R^\alpha_{\beta\mu\nu} = 0 \quad (41)$$

and

$$\nabla_\lambda R^\alpha_{\mu\nu\beta} + \nabla_\nu R^\alpha_{\mu\beta\lambda} + \nabla_\beta R^\alpha_{\mu\lambda\nu} = 0. \quad (42)$$

From the contracted form for of Eq. (42), one finds that

$$\nabla^\mu G_{\mu\nu} = 0 \quad (43)$$

where the Einstein's tensor analogue for Lyra manifold $G_{\mu\nu}$ was defined as:

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \quad (44)$$

and, through Eqs. (37) and (38), it can be rewritten as:

$$\begin{aligned} G_{\mu\nu} = \frac{1}{\phi^2} \mathcal{G}_{\mu\nu} + \frac{2}{\phi} \nabla_\nu \nabla_\mu \phi - \frac{2}{\phi} g_{\mu\nu} \nabla_\lambda \nabla^\lambda \phi \\ + \frac{3}{\phi^2} g_{\mu\nu} \nabla_\lambda \phi \nabla^\lambda \phi \end{aligned} \quad (45)$$

where $\mathcal{G}_{\mu\nu} := \mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R}$ is the Einstein tensor in general relativity.

IV. LYRA SCALAR-TENSOR THEORY OF GRAVITY

The framework presented so far allows the construction of a scalar-tensor theory of gravity where the basic fields are the reference frame scalar function ϕ and the manifold metric $g_{\mu\nu}$. The action from which the field equations emerge via the variational principle will be taken as direct generalization of Einstein-Hilbert action. We take as ingredients the Lyra's invariant integration element for a 4-dimensional space-time of Lorentzian metric $d^4 x \phi^4 \sqrt{-g}$, the scalar curvature (38) plus a Lagrangian density $\mathcal{L}_m = \mathcal{L}_m(\psi_i, \nabla \psi_i, \phi, \nabla \phi)$, admittedly a Lyra scalar, which describes the contribution of matter terms. Here, it is assumed that \mathcal{L}_m also depends on the scale function, in addition to the ψ_i matter fields and their first derivatives. Thus, the action will be

$$\begin{aligned} S = \int d^4 x \phi^4 \sqrt{-g} \left[\mathcal{L}_m(\psi_i, \nabla \psi_i, \phi, \nabla \phi) \right. \\ \left. - \frac{1}{16\pi G} \left(\frac{1}{\phi^2} \mathcal{R} + \frac{6}{\phi} \nabla_\lambda \nabla^\lambda \phi - \frac{12}{\phi^2} \nabla_\lambda \phi \nabla^\lambda \phi \right) \right]. \end{aligned} \quad (46)$$

The variation of (46) with respect to the components of the metric tensor and the scale function leads respectively to the equations:

$$\begin{aligned} \mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} + 2\phi\nabla_{(\mu}\nabla_{\nu)}\phi - 2\phi g_{\mu\nu}\nabla_\lambda\nabla^\lambda\phi \\ + 3g_{\mu\nu}\nabla_\lambda\phi\nabla^\lambda\phi = -8\pi G\phi^2 T_{\mu\nu} \end{aligned} \quad (47)$$

and

$$6\phi\nabla_\mu\nabla^\mu\phi - 12\nabla_\alpha\phi\nabla^\alpha\phi + \mathcal{R} = -8\pi G\phi^2 M \quad (48)$$

where the stress-energy tensor $T_{\mu\nu}$ was defined as:

$$T_{\mu\nu} = -2\frac{\delta\mathcal{L}_m}{\delta g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_m \quad (49)$$

and the Lyra scalar M is

$$M = -4\mathcal{L}_m - \phi\left(\frac{\partial\mathcal{L}_m}{\partial\phi} - \nabla_\alpha\frac{\partial\mathcal{L}_m}{\partial(\nabla_\alpha\phi)}\right). \quad (50)$$

In addition, variations with respect to the matter fields ψ_i give rise to the covariant form of the Euler-Lagrange equations:

$$\frac{\partial\mathcal{L}_m}{\partial\psi_i} - \nabla_\mu\frac{\partial\mathcal{L}_m}{\partial(\nabla_\mu\psi_i)} = 0.$$

Through (45), the Eq. (47), which is the Lyra analogue of Einstein's equations, can be simply rewritten as $G_{\mu\nu} = 8\pi GT_{\mu\nu}$. Therefore, LyST gravity can be interpreted as the direct generalization of the theory of general relativity to Lyra space-time, where the quantities $G_{\mu\nu}$ and $T_{\mu\nu}$ are tensors, not only by general transformations of coordinates, but also by scale transformations.

A. Newtonian limit

The field equations for the LyST theory of gravity must be consistent with the gravitational phenomenology accessible to experiments and observations. As is known, the gravitational effects on celestial mechanics are, with the exception of Mercury's perihelion precession and light deflection, perfectly described by Newtonian gravity at the level of the solar system within observational uncertainties. A successful gravitational theory must, therefore, recover Newton's equations on a given scale of validity, coined here as *Newtonian limit*.

Under the phenomenological aspect, three basic requirements define the Newtonian limit to a gravity theory [29]. The first is to keep the focus on nonrelativistic movements, in such a way that the spatial components of the 4-velocity of the test particles can be neglected, within the required approximation order. The second condition requires the gravitational field to be static, which means that the time derivatives of metric that appear in the Christoffel symbol, as well as the time derivative of ϕ , can be neglected. Thus, the equation for geodesics (35) is reduced to:

$$\frac{d^2x^\mu}{dt^2} = \frac{1}{2}g^{\alpha\mu}\partial_\alpha g_{00} + g^{\mu\nu}\frac{1}{\phi}\partial_\nu\phi g_{00}. \quad (51)$$

Finally, the third requirement establishes that the gravitational field is weak and, as a consequence, must be interpreted as a perturbation to Minkowski's space-time. In LyST, this limit is obtained based on the simultaneous assumption of two conditions:

$$g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu} \quad \text{and} \quad \phi \approx 1 + \delta\phi(x), \quad (52)$$

where $h_{\mu\nu} \ll \eta_{\mu\nu}$ and $\delta\phi(x) \ll 1$. The contravariant components of $h^{\alpha\beta}$ are obtained by imposing the reciprocal relation between covariant and contravariant metric coefficients. Considering Eqs. (52) in (51), keeping only the first order terms and focusing on the spatial components, one finds $\ddot{\mathbf{x}} = -\nabla U$, where U is the Newtonian potential that depends on both the metric and the $\delta\phi$ field:

$$U \equiv \frac{1}{2}h_{00} + \delta\phi. \quad (53)$$

Once this important finding has been obtained, attention should be turned to the field equations. By evaluating the trace of Eq. (47), one can cast it into the form:

$$\begin{aligned} \mathcal{R}_{\mu\nu} + 2\phi\nabla_\mu\nabla_\nu\phi + g_{\mu\nu}\phi\nabla_\lambda\nabla^\lambda\phi - 3g_{\mu\nu}\nabla^\lambda\phi\nabla_\lambda\phi \\ = 8\pi G\phi^2\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right). \end{aligned} \quad (54)$$

In the case of low velocities and weak fields, the gravitational field is dominated by the time components of (54). The stress-energy momentum tensor, in this case, is dominated by the energy density $T \approx T_{00} = \rho$, such that the sector 00 in (54) is reduced to:

$$\mathcal{R}_{00} - \nabla^2(\delta\phi) = -4\pi G\rho. \quad (55)$$

The expression for \mathcal{R}_{00} can be obtained using the equation for the Ricci tensor in terms of Christoffel's symbols and applying the conditions that define the Newtonian limit:

$$\mathcal{R}_{00} \approx -\frac{1}{2}\nabla^2 h_{00}. \quad (56)$$

Replacing (56) in (55), one obtains:

$$\nabla^2\left(\frac{1}{2}h_{00} + \delta\phi\right) = 4\pi G\rho, \quad (57)$$

which shows that the Newtonian potential obtained in (53) respects the Poisson equation.

V. VACUUM SPHERICALLY SYMMETRIC SOLUTIONS

A first case of interest is the one in which space-time is stationary and spherically symmetric. In general relativity, the solutions of Einstein's equations around a stationary black hole can be written with a metric of the form:

$$g_{\mu\nu} = \text{diag}(\alpha(r), -\alpha^{-1}(r), -r^2, -r^2 \sin^2 \theta). \quad (58)$$

Even in the presence of the cosmological constant in the field equations of general relativity, this shape for the line element remains unchanged. Since the observations points to a very significant correspondence between gravitational phenomena and GR predictions, it must surely be assumed that a gravitational theory in the Lyra manifold must recover Einstein's equations on scales of the observed data. Following this point of view, a working approach can be formulated for the Lyra manifold, assuming a metric such as (58), called *Schwarzschild-type* solutions. Furthermore, it is considered $\phi = \phi(r)$, in order to maintain the scale function the invariant of SO(3). This is not a more general approach, but this consideration greatly simplifies the work and, as will be shown later, gives rise to solutions that tend to the Schwarzschild space within specific limits.

The field equations in this framework are reduced to:

$$\frac{\alpha' \phi'}{\alpha \phi} - \frac{1}{\alpha r^2} + \frac{1}{r^2} + \frac{\alpha'}{\alpha r} + \frac{4\phi'}{r\phi} + \frac{2\phi''}{\phi} - \frac{\phi'^2}{\phi^2} = 0 \quad (59a)$$

$$-\frac{\alpha' \phi'}{\alpha \phi} + \frac{1}{\alpha r^2} - \frac{1}{r^2} - \frac{\alpha'}{\alpha r} - \frac{4\phi'}{r\phi} - \frac{3\phi'^2}{\phi^2} = 0 \quad (59b)$$

$$-\frac{\alpha''}{2\alpha} - 2\frac{\alpha' \phi'}{\alpha \phi} - \frac{\alpha'}{\alpha r} - \frac{2\phi'}{r\phi} - \frac{2\phi''}{\phi} + \frac{\phi'^2}{\phi^2} = 0. \quad (59c)$$

Adding the Eq. (59a) to (59b), one finds:

$$\frac{\phi''}{\phi} - 2\frac{\phi'^2}{\phi^2} = 0. \quad (60)$$

Solving this differential equation for ϕ , it results:

$$\phi(r) = \frac{r_0}{r_L - r}, \quad (61)$$

where $r_L \in \mathbb{R}$ and $r_0 \in \mathbb{R}$ are integration constants. Parameter r_L is a new constant appearing in the context of LyST: it will be later identified as Lyra radius and will be key to characterize the causal structure of Lyra-based vacuum spherically symmetric solutions. With the solution (61), it is possible to use one of the Eqs. (59) to find α . Replacing (61) in (59b):

$$r_L \alpha' + \alpha \frac{r_L (1 + 2r/r_L)}{r (1 - r/r_L)} = -\left(1 - \frac{r_L}{r}\right).$$

This is a first order linear differential equation, which solution depends on a new integration constant r_s :

$$\alpha(r) = \frac{(1 - r/r_L)^2 (1 - r_s/r)}{(1 - r_s/r_L)}. \quad (62a)$$

(Parameter r_s will be later related to the familiar Schwarzschild radius.)

The next step is to work with the Eq. (61) for ϕ . As argued before, the theory must recover general relativity when the scale function is constant. By writing (61) as $\phi = 1/(r_L/r_0 - r/r_0)$ we see that r_L and r_0 must go to ∞ while keeping the ratio r_L/r_0 constant in order to satisfy this condition. Since the r_0 appears only in the equation for ϕ , its possible to factorize the ratio r_0/r_L in the line element equation; this leads to $ds^2 = \frac{r_0^2}{r_L^2} \tilde{\phi}^2 g_{\mu\nu} dx^\mu dx^\nu$, where both $\tilde{\phi} := 1/(1 - r/r_L)$ and $g_{\mu\nu}$ do not depend on r_0 . Because of this, it is possible to work with Lyra frames where $r_0 = r_L$, in which case the scale function becomes

$$\phi = \frac{1}{1 - r/r_L}. \quad (62b)$$

In this approach, it is evident that Schwarzschild solution of GR arises by taking the limit $r_L \rightarrow \infty$ in Eqs. (62). Up to this point, there are no additional constraints on r_s and r_L . The only information about this constants is that they are the roots of $\alpha(r)$ in \mathbb{R} . For more insight into the physical meaning of these parameters, let us study the weak field limit of LyST gravity.

A. Corrections to Newtonian gravity

Under the requirement of spherical symmetry, the Eq. (51) reduces to $\dot{r} = -\partial_r U$. The spherically symmetric solution for LyST gravity is entirely determined by the functions $\alpha(r)$ and $\phi(r)$, cf. Eqs. (62). By expanding them up to the second order in r , one finds $\phi = 1 + \delta\phi$ and $\alpha = 1 + h_{00}$, where:

$$\delta\phi \approx \frac{r}{r_L} + \frac{r^2}{r_L^2} \quad (63)$$

and

$$h_{00} \approx 3\left(\frac{r_s}{r_L} + \frac{r_s^2}{r_L^2}\right) - \left(1 + \frac{r_s}{r_L}\right) \frac{r_s}{r} + -2\frac{r}{r_L} \left(1 + \frac{3r_s}{2r_L}\right) + \frac{r^2}{r_L^2}. \quad (64)$$

With these expressions, the radial sector of the equation of motion will be

$$\ddot{r} = -\frac{1}{2} \frac{r_s}{(1 - \frac{r_s}{r_L})} \frac{1}{r^2} + \frac{3}{2} \frac{r_s}{r_L^2} - \frac{3r}{r_L^2}. \quad (65)$$

The coefficient of the term scaling as r^{-2} in Eq. (65) may be defined as a new geometrical parameter m :

$$m := \frac{1}{2} \frac{r_s}{(1 - \frac{r_s}{r_L})}. \quad (66)$$

This quantity can be interpreted as the mass in the Newtonian description of gravity. The mass parameter is related not only to Schwarzschild radius r_s (as is the case of general relativity), but also to Lyra radius r_L . According to this interpretation, the first order effect of r_L is simply to change the way m relates to r_s . Second order effects include a constant repulsive acceleration $3r_s/2r_L^2$ and an anti-de Sitter type attractive term $-3r/r_L^2$.

B. Properties of the spherically symmetric solution

As explained in the previous section, the existence of a Newtonian limit allows the recognition of a parameter m that can be interpreted as the mass. Following this line of reasoning, this parameter must be taken as a real and positive quantity based on phenomenological principles. Thus, one can rewrite the solution for α in Eq. (62) by replacing the constant r_s in terms of m and r_L . The set of solutions will be

$$\phi(r) = \frac{1}{1 - \frac{r}{r_L}} \quad \text{and} \quad \alpha(r) = \left(1 - \frac{r}{r_L}\right)^2 \left(1 - \frac{2m}{r} + \frac{2m}{r_L}\right), \quad (67)$$

which, according to Eq. (10), lead to the line element:

$$\begin{aligned} ds^2 = & \left(1 - \frac{2m}{r} + \frac{2m}{r_L}\right) dt^2 \\ & - \left(1 - \frac{r}{r_L}\right)^{-4} \left(1 - \frac{2m}{r} + \frac{2m}{r_L}\right)^{-1} dr^2 \\ & - \left(1 - \frac{r}{r_L}\right)^{-2} r^2 d\Omega^2 \end{aligned} \quad (68)$$

1. Schwarzschild limit

At this point, it should be checked whether Eqs (67) are physically plausible. It is known that General Relativity must be the emerging theory from LyST when scale function ϕ is constant. The limit $r_L \rightarrow \pm\infty$ in (67), leads directly to:

$$\alpha(r) \simeq 1 - \frac{2m}{r} \quad (69)$$

which is the conventional result from GR. This shows the existence of a well-defined Schwarzschild limit stemming

from LyST spherically symmetric solution. This very fact brings about the possibility of interpreting the parameter $r_s = (1/2m + 1/r_L)^{-1}$ as Lyra's generalization of the Schwarzschild radius.

2. Long-distance regime

The form of the metric coefficients in the infinitely remote regions from the source can be obtained by taking the limit $r \rightarrow \infty$ in Eq. (67). This process depends almost exclusively on a quadratic term in r , namely $\alpha(r) \simeq (1 + 2m/r_L)r^2/r_L^2$. If Lyra's space-time were completely characterized by the metric, it could be understood as an emerging "cosmological constant":

$$\Lambda \simeq -\frac{3}{r_L^2} \left(1 + \frac{2m}{r_L}\right), \quad (70)$$

induced by the presence of the Lyra scale. Moreover, the observed small absolute value of Λ would be explained by a large value r_L , something that is reassuringly consistent with the Schwarzschild limit discussed above. It is clear from Eq. (70) that the sign of r_L determines whether the constant Λ will be positive or negative. In the case where $1 + 2m/r_L < 0$ the solution for the metric would be asymptotically de Sitter; on the other hand, if $1 + 2m/r_L > 0$, it would be anti-de Sitter type solution.

This is all too compelling. However, one should recall that in LyST gravitational theory, space-time is specified not only by the metric tensor, as in the case of general relativity, but also by the scale function. Considering the line element (68), it turns out that, in fact, Lyra's spherically symmetric space-time does not recover the characteristics of either de Sitter or anti-de Sitter space-times in the limits of large distances from the source.

C. Classes of solutions

It is considered that $r \in \mathbb{R}_+$, the line element (68) is singular at the points $r = r_s = (1/2m + 1/r_L)^{-1}$ and $r = r_L$. These values can be both negative or positive. Accordingly, Lyra's spherically symmetric space-time (68) may have different properties in different regions of the parameters' range of values. These properties will be explored later on.

The Newtonian limit offers the interpretation of parameter m as the mass of the source; therefore, it must be a real and positive quantity. Consequently, it should be $r_s > 0$ for $r_L > 0$ according to $r_s = (1/2m + 1/r_L)^{-1}$.

In the range $r_L < -2m$, the Schwarzschild radius remains positive and the region $0 < r < \infty$ contains one singular point.

The interval $-2m < r_L < 0$ does not induce any singularities in $\{r \in \mathbb{R} | r > 0\}$. However, the coefficients of dt^2 and dr^2 in the line element change their signs in this range.

This type of mathematical solution has no physical interest and will not be considered in the rest of the paper.

Assuming that neither r_s nor r_L diverge,² LyST spherically symmetric solutions can be classified as:

- (i) Class 1: $r_s > 0$ and $r_L > 0$, space-time with two singularities;
- (ii) Class 2: $r_s > 0$ and $r_L < -2m$, space-time with one singularity.

It is necessary to characterize the divergent points in the line element to learn if they are essential singularities of space-time, or if they are removable divergences arising from particular choices of Lyra frames. This will be done in the upcoming section.

D. Singularities and causal structure

LyST Class 1 solution exhibits two singularities, which seems to indicate a line element with multiple horizons. Thus, this section aims to study these concepts to determine whether there are physical singularities and, if so, what their nature is.

A first case of interest is the determination of light-cone distortions, a task that is accomplished through the study of null geodesics. A local light cone is defined by the vanishing of the line element in Eq. (10); this corresponds to the geodesics of nonmassive particles. The relation between coordinates t and r for purely radial motion is given by:

$$\frac{dt}{dr} = \pm \left(1 - \frac{r}{r_L}\right)^{-2} \left(1 - \frac{2m}{r} + \frac{2m}{r_L}\right)^{-1}. \quad (71)$$

The solution to Eq. (71) is

$$t_{\pm}(r) = c_1 \pm \left[\frac{r}{\left(1 - \frac{r}{r_L}\right)} + 2m \ln \left(\frac{r}{1 - \frac{r}{r_L}} - 2m \right) \right]. \quad (72)$$

The curves with signal + (−) are called the *outgoing* (*ingoing*) *null geodesics*. Note that the limit $r_L \rightarrow \infty$ naturally leads to the result of general relativity [30]. LyST Class 1 solution does not tend to Minkowski space-time for regions very distant from the source; in fact, for $r \gg r_L$ and $r \gg r_s$, it approaches an anti-de Sitter space-time. This feature directly influences the radial propagation of light beams, since $dr/dt \neq 1$ away from the source.

LyST Class 1 space-time diagrams are built from Eq. (72) through the overlap of ingoing and outgoing null geodesics. This procedure unveils local light cones in strategic regions. Figure 1 displays some null-geodesic curves from Eq. (72) and local light cones in multiple space-time events for the case where $r_s < r_L$ in LyST Class 1

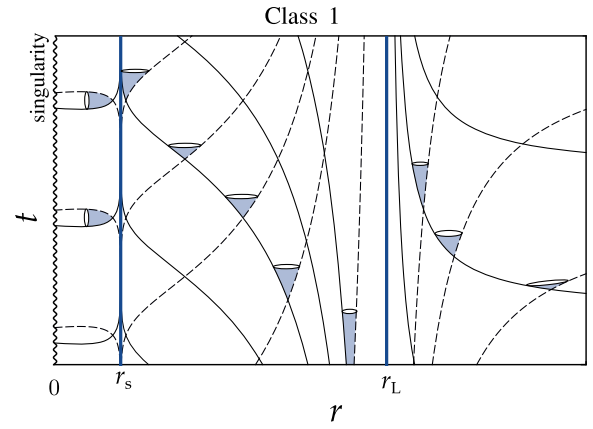


FIG. 1. Null geodesics for the spherically symmetric solution of Class 1 in LyST gravity, in Schwarzschild coordinates.

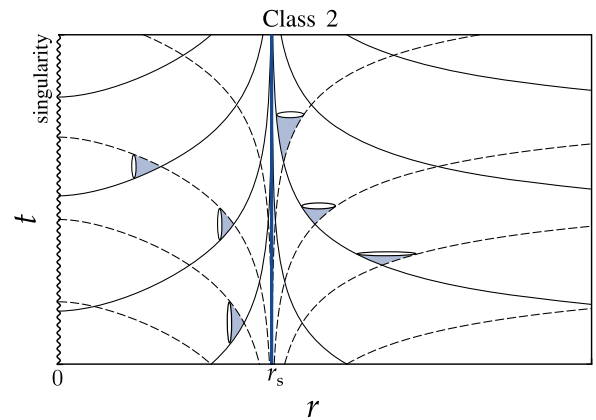


FIG. 2. Space-time diagram for null geodesics in the spherically symmetric solution of Class 2 in LyST gravity with Schwarzschild coordinates.

solutions. We can see a significant distortion of the light cones in the immediate vicinity of both r_s and r_L . This is a direct consequence of the fact that dt/dr diverges at r_s and r_L , the roots of $\alpha(r)$. The space-time diagram for LyST Class 2 solutions is shown in Fig. 2. It seems that this class give rise to a space-time diagram qualitatively indistinguishable from that derived from Schwarzschild solution of general relativity.

Through the diagrams in Figs. 1 and 2, it can be checked that the light cones rotate at 90 degrees counterclockwise to the left of r_s ; in the region $r < r_s$ coordinates r and t switch roles.

The distortion of the light cones in the vicinity of $r = r_s$ and $r = r_L$ suggests that it takes an infinite coordinated time t for a test particle to reach the surfaces corresponding to these radii. However, this does not automatically mean that the horizons at r_s and r_L are essential singularities of LyST spherically symmetric space-time. They might as well be removable coordinate singularities (recall that Kruskal-Szekeres coordinates [26] eliminate the divergence

²For if $r_L \rightarrow \infty$, the Schwarzschild solution of general relativity is recovered, which is also a solution of LyST field equations, for the constant scale.

in GR's Schwarzschild line element at $r = r_s$). In order to check this, it is necessary to look for divergences in the gravitational field invariants at the radial values r_s and r_L .

The simplest invariant that can be thought of would be the curvature scalar R . However, in the case of vacuum solutions, the trace of Einstein's equations shows that $R = 0$ in the entire space; therefore, R is useless to assess the nature of the horizons at r_s and r_L . Alternatively, Kretschmann scalar $K = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ can be used for this purpose [31]. Utilizing Eqs. (15), it follows that:

$$K = \frac{12r_s^2}{r^6} \frac{(1 - \frac{r}{r_L})^6}{(1 - \frac{r_s}{r_L})^2}. \quad (73)$$

Notice that this Kretschmann scalar tends dutifully to the value $K = 48m^2/r^6$ of the ordinary GR's Schwarzschild solution in the limit $r_L \rightarrow \pm\infty$. The Kretschmann scalar (73) remains finite at $r = r_s$, which leads to the conclusion that the horizon at r_s indicated by Eq. (71) is not an essential singularity. Consequently, it can be removed through an appropriate choice of Lyra reference frames. On the other hand, Kretschmann invariant for LyST Schwarzschild-type metric approaches zero as $r \rightarrow r_L$. Let us investigate the meaning of this fact by considering what happens to a particle in radially free-falling motion.

The equations for the geodesic motion (71) in a spherically symmetric Lyra space-time are given by:

$$\phi^2 \dot{\alpha} = k; \quad (74a)$$

$$\begin{aligned} \ddot{r} + \alpha^2 \dot{r}^2 \left(\frac{\phi'}{\phi} + \frac{\alpha'}{2\alpha} \right) + \dot{r}^2 \left(\frac{\phi'}{\phi} - \frac{\alpha'}{2\alpha} \right) \\ - \alpha r^2 \dot{\phi}^2 \left(\frac{\phi'}{\phi} + \frac{1}{r} \right) = 0; \end{aligned} \quad (74b)$$

$$\phi^2 r^2 \dot{\phi} = h; \quad (74c)$$

where $\dot{(\)} := \frac{d}{dt}$ e $(\)' := \frac{d}{dr}$, k and h are respectively the specific energy and specific angular momentum of the test particle, respectively; and, the functions α and ϕ are given in (67). The components u^μ of the four-velocity respect the condition $u_\mu u^\mu = 1$ in natural units. By combining this constraint and Eqs. (74a)–(74b), we obtain a differential equation for the evolution of the radial coordinate:

$$\frac{k^2}{\phi^2 \alpha} - \frac{\phi^2}{\alpha} \dot{r}^2 - \frac{h^2}{\phi^2 r^2} = 1. \quad (74d)$$

In a pure radial motion, the angular momentum vanishes, and the equation of motion reduces to:

$$\dot{r}(\tau) = \pm \left(1 - \frac{r}{r_L} \right)^2 \sqrt{\frac{2m}{r} - \frac{2m}{r_0}}. \quad (75)$$

Here r_0 is the value of the radial coordinate where the test particle is at rest.

E. Class 1 solutions

The allowed types of motion are those that keep the argument of the square root in Eq. (75) positive. Thus, the region accessible to time-like trajectories are those where $r \leq r_0$. In order to study the neighboring regions of r_L for $r_s < r_0 \leq r_L$, we take $r_0 \rightarrow r_L$ and consider a free-fall trajectory of a particle that leaves r_L from rest. In this case, Eq. (75) can reads $\dot{r}(\tau) = -\sqrt{2mr^2(1/r - 1/r_L)^{5/2}}$ and the proper time for the particle to come out of rest at r_L to reach any point of radial coordinate $r_{\text{end}} < r_L$ is

$$\begin{aligned} \Delta\tau &= - \int_{r_L}^{r_{\text{end}}} dr (2m)^{-1/2} r^{-2} \left(\frac{1}{r} - \frac{1}{r_L} \right)^{-5/2} \\ &= \sqrt{\frac{2}{9m}} \left(\frac{1}{r} - \frac{1}{r_L} \right)^{-3/2} \Big|_{r_{\text{end}}}^{r_L} \rightarrow \infty. \end{aligned}$$

In the instance where $r_0 > r_L$, the physically allowed region is $r_L \leq r \leq r_0$. The proper time elapsed during a free fall from rest at r_0 will be see Eq. (75):

$$\Delta\tau = \int_{r_L}^{r_0} \frac{dr}{(1 - \frac{r}{r_L})^2 \sqrt{\frac{2m}{r} - \frac{2m}{r_0}}} \rightarrow \infty;$$

that is, a particle starting from rest off the radius $r_0 > r_L$ takes an infinite time to reach r_L . Along with the fact that a particle at rest in r_L cannot move to values of $r < r_L$, these findings lead to the conclusion that the regions $r < r_L$ and $r > r_L$ are not in causal contact.

F. Class 2 solutions

The solutions in this class are constrained to $r_s > 0$ and $r_L < -2m$. Thus, parameter r_L is negative and cannot be interpreted as a radial distance since it lays outside the validity domain of the radial coordinate. The singularity at $r_s = (1/2m + 1/r_L)^{-1}$ can be removed with an appropriate choice of frame as indicated by the Kretschmann scalar analysis, and the causal structure of this class of solution is equivalent to the already known case of GR's Schwarzschild solution. The influence of r_L with respect to Schwarzschild's event horizon is to induce changes in the expression involving r_s and m .

VI. FINAL COMMENTS

LyST gravity is a scalar-tensor theory whose fundamental fields are the metric $g_{\mu\nu}$ and the scale function ϕ . Unlike established scalar-tensor theories, LyST is built on a non-Riemannian Lyra manifold. In fact, the main change on the geometric framework is regarding the definition of the basis of the tangent and cotangent spaces. It makes the reference

frames dependent on the choice of both the coordinates and the scale function. In Lyra manifolds, the transformation law of tensors is distinct from that on the Riemannian manifold.

LyST theory of gravity was built under the geometric hypotheses of metric-compatibility and torsion-free connection. For this reason, LyST gravity can be interpreted as the equivalent of general relativity in a Lyra manifold. Its field equations were obtained via a Lyra invariant action (i.e., through an action invariant both under scale and coordinates transformations) given by the simple generalization of the Einstein-Hilbert action. By writing the contributions of ϕ in the action (46) explicitly, we note that LyST is quite different from conventional theories like Brans-Dicke and its generalizations [32]. In the first place, this is true due to the presence of the higher order derivative term. Furthermore, there is also the factor ϕ^4 that appears to guarantee the symmetry properties of the integration element, which induces terms dependent on ϕ in the field equations during the process of integration by parts.

It was found that the linearized and static approximations of LyST theory tends to a well-defined Newtonian limit for nonrelativistic motion. In fact, we have obtained $U = (1/2)h_{00} + \delta\phi$, where $\delta\phi$ is the perturbation on the Lyra scale. Trajectories and field equations were characterized by the single SO(3) scalar function U , which is recognized as the Newtonian gravitational potential, cf. $\ddot{\mathbf{x}} = -\nabla U$ and Eq. (57). A remarkable feature of this potential is its dependence on both the linearized metric h_{00} and the scale function perturbation $\delta\phi$, which is a natural consequence of the fact that the curvature depends on these two quantities.

LyST gravity presents a spherically symmetric solution dependent on two parameters m and r_L . Parameter m is the geometrical mass and r_L is a distance related quantity dubbed Lyra radius. LyST spherical metric tends naturally to Schwarzschild solution of general relativity if $r_L \rightarrow \pm\infty$, as it can be seen from Eq. (67). The equations of motion show a well-defined Newtonian limit: the non-relativistic gravitational regime is equivalent to Newtonian gravity up to orders 0 and 1 of r_L^{-1} . First order effects change the customary relation between mass and Schwarzschild radius to $r_s = (1/2m + 1/r_L)^{-1}$, while higher order effects steaming from terms scaling as r_L^{-2} induce de-Sitter-type correction terms to the line element.

LyST spherically symmetric solution splits into two mains cases of physical interests. In Class 1, r_L is a

positive parameter greater than r_s , and LyST space-time exhibits two singular points. The singularity at $r = r_s$ is non-essential just like it happens in General Relativity and can be removed through an appropriate choice of frame. On the other hand, there is nonremovable horizon at r_L , where the space-time is divided into two regions without mutual causal contact. LyST spherical metric of Class 2 features $r_L < -2m$ so that r_L lays outside the domain of radial coordinates. In this case, LyST space-time presents a single Schwarzschild-type singularity and the only influence of the Lyra scale ϕ is to change the expression for the event horizon radius r_s .

The fact that LyST gravity exhibits spherically symmetric solutions with a well-defined limit to Schwarzschild solution of general relativity is a very encouraging finding. Since a significant part of the gravitational phenomena can be described via general relativity, the uncertainties in the measurements in solar-system-size scales could impose lower limits to the absolute value of LyST parameter r_L . Future perspectives toward this goal are to build the equations for PPN approximation in the context of LyST.

On the one hand, local measurements of gravitational physics constrain LyST free parameters. On the other hand, large-scale gravitational observations bring an opportunity to the new theoretical framework opened up by LyST. Indeed, GR faces difficulties in cosmology while, for instance, describing the current cosmic acceleration and the σ_8 -tension [33]. These open problems could be addressed via LyST analogous to FLRW metric plus an SO(3)-invariant scale function ϕ .

In addition, studies on geodesic motion in spherically symmetric LyST space-time and on gravitational waves in LyST gravity are being carried out. The relaxation of imposing a torsion-free connection and the metric compatibility condition is also being investigated in order to determine the Lyra equivalents to Einstein-Cartan [34–36] and teleparallel [37] theories of gravity.

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