Optical equations for null strings

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An optical equation for null strings is derived. The equation is similar to Sachs's optical equations for null geodesic congruences. The string optical equation is given in terms of a single complex scalar function Z, which is a combination of spin coefficients at the string trajectory. Real and imaginary parts of Z determine the expansion and rotation of strings. Trajectories of strings can be represented by diagrams in a complex Z plane. Such diagrams allow one to draw some universal features of null strings in different backgrounds. For example, in asymptotically flat space-times Z vanishes as future null infinity is approached, that is, gradually shapes of strings are "freezing out." Outgoing gravitational radiation and flows of matter leave ripples on the strings. These effects are encoded in subleading terms of Z. String diagrams are demonstrated for rotating and expanding strings in a flat space-time and in cosmological models.

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I. INTRODUCTION

One-dimensional objects, strings, which move with the speed of light have been studied since the 1970s. Points of null strings move along trajectories of light rays, orthogonally to strings. Null strings were introduced by Schild [1] as microscopic objects in the theory of strong interactions. Later on such strings attracted much attention as a tensionless limit of the string theory, since they may capture different features of fundamental strings at Planckian energies [2,3]. Modern scenarios concerning how fundamental tensionless strings may emerge in a quantum gravity theory have been discussed in [4,5].

If fundamental tensionless strings were produced in the early Universe, then they might be stretched to cosmological scales and became cosmic strings. At the present moment, such mechanisms are known for fundamental strings with a finite tension, called tensile strings [6,7].

Cosmic tensile strings [8,9] have a finite rest mass per unit length. Tensionless strings have zero rest mass but a finite energy per unit length. So one can also call these two types of cosmic strings, respectively, massive and massless strings [10,11]. These names are more appropriate in studying physical effects caused by the gravitational field of the stings on the surrounding matter.

Massless cosmic strings in a flat space-time can be obtained from massive cosmic strings as a limiting case, when the velocity of the string reaches the speed of light, mass tends to zero, while energy remains finite [12]. As a result of this limit, a holonomy along a closed contour around a massive string is transformed into a nontrivial holonomy [13], which belongs to the parabolic subgroup associated to null rotations. Effects caused by massless strings look like mutual transformations of trajectories of massive bodies or light rays when the string moves in between two trajectories.

A method describing the physical effects around massless cosmic strings has been developed in [10,11] for strings in flat and de Sitter space-times. Here backreaction effects can be described analytically due to maximal isometries. For some extension of these results see [14].

Massless cosmic strings generate perturbations of the velocities of bodies resulting in overdensities of matter. The strings also shift energies of photons and may yield additional anisotropy of cosmic microwave background. These effects are direct analogs of, respectively, wake effects [15] and the Kaiser-Stebbins effect [16,17] known for tensile cosmic strings. Thus, if cosmic strings which move with velocity of light do exist in nature, then their physical effects can be discovered in future cosmological and astrophysical observations.

In the rest of this paper we consider general properties of trajectories of null strings introduced in [1]. We use name "null strings" instead of "massless strings" to keep connection to earlier publications. As we will see, the massless strings in space-times with parabolic isometries studied in [10,11,14] are a subclass of the null strings.

Solutions to equations of motion for null strings in various gravitational backgrounds have been presented in many publications, see e.g., [18–24]. The solutions are coordinate and parametrization dependent. To extract useful physical information one needs some invariant characteristics of the string trajectories (world sheets).

The aim of the present work is to identify such characteristics and establish universal features of string trajectories.

We treat them as 1-parameter null geodesic congruences and derive equations analogous to Sachs's optical equations. The string optical equations are given in terms of a single complex function Z, which determines expansion and rotation of the string.

The paper is organized as follows. We start in Sec. II with description of trajectories of stringlike objects (not necessarily null strings) which move with the speed of light. The trajectories are specified by a set of spin coefficients introduced with respect to a tetrad l, n, p, q, where n, l are null, l and p are tangent to the trajectory, and l is the velocity of the string. Rotations of the tetrad and reparametrizations make a 2-parameter group of l-preserving null rotations of n, p, q, accompanied with rescalings of l and n. In general, the spin coefficients are not invariant with respect to this group. For null strings, however, two spin-coefficients along the trajectories, $\theta_s = (p \cdot \nabla_p l)$ and $\kappa_2 = (q \cdot \nabla_p l)$, are invariant up to boost rescalings. In Sec. III we demonstrate that the complex quantity $Z = \theta_s + i\kappa_2$ satisfies an equation analogous to Sachs's optical equations for null geodesic congruences (NGC). We call equation for Z the string optical equation. In fact, Sachs's equations taken at the trajectory follow from string equations and vice versa. Z is a linear combination of complex divergence ρ and complex shear σ of NGC where string trajectory belongs to. Scalars θ_s and κ_2 determine expansion and rotation of the string congruence. Trajectory of the string in a space of parameters θ_s, κ_2, the Z plane, looks as a sequence of diagrams, see Sec. IV. Examples of null strings in flat and cosmological backgrounds and their diagrams are presented in Sec. V. In Sec. VI we discuss null strings in asymptotically flat space-times. We consider strings on outgoing null hypersurfaces, and, in particular, strings in the Bondi-Sachs formalism. We show that the leading asymptotics of Z capture the amplitude of the outgoing gravitational radiation. The gravitational memory effect in Z caused by an ingoing flux of energy is calculated for string trajectories in the weak field approximation. Short discussion of our results can be found in Sec. VII.

II. STRING TRAJECTORIES

A. Spin coefficients and symmetries

The "trajectory" of a string in a space-time $\mathcal{M}$ with coordinates x^μ is defined as $x^\mu = x^\mu(\lambda, \tau)$. Two real parameters, λ and τ, numerate points on the string. It is implied that the string trajectory has no caustics, or, in the case of caustics, the definitions below are not applied at their locations. One can introduce the tangent vectors $l^\mu \equiv x^\mu{}_{,\lambda}$ and $\eta^\mu \equiv x^\mu{}_{,\tau}$. For a string which moves with the speed of light we require that

$$(l \cdot l) = 0, \tag{2.1}$$

$$(l \cdot \eta) = 0, \tag{2.2}$$

where the notation $(.. \cdot ..)$ stands for the scalar product in the tangent space of $\mathcal{M}$. We also assume that l is future directed, η is spacelike, and $(\eta \cdot \eta) > 0$. Thus, λ is lightlike and τ is a spatial coordinate on the world sheet. The velocity of the string is directed along l.

Parameters (τ, λ) can be denoted as χ^a, $a = 1, 2$. The matrix $h_{ab} = x^\mu{}_{,a} g_{\mu\nu} x^\nu{}_{,b}$ is degenerate, $\det h = 0$. One can demand that $\det h = 0$ at first. Then (2.2) follows from (2.1), or (2.1) follows from (2.2).

The strings which are considered in this work obey (2.1), (2.2) and the condition that each point of the string at fixed τ moves along a null geodesic,

$$\nabla_l l \sim l. \tag{2.3}$$

To put it another way, the string is a 1-parameter family of rays. Such a definition was first given by Schild [1], and the corresponding strings are called "null strings."

Trajectories of null strings can be considered as a certain type of NGC. NGC play an important role in general relativity. They have been extensively studied since the mid-fifties of the last century. The material of this section follows in part the classical monograph [25], where η is called the connecting vector between two rays. Given (2.2) a pair of neighboring rays are called abreast. Equation (2.2) ensures that two points of the string with neighboring worldlines always lie in a 2-plane orthogonal to the worldlines.

To study congruences which obey (2.1), (2.2) we introduce a tetrad l, n, p, q at the each point of the string trajectory. Here p is a unit vector directed along η, $p = \eta/N$, $N^2 = (\eta \cdot \eta)$. Vector n is null, orthogonal to p, and normalized as $(n \cdot l) = -2$. Vector q is spacelike, unit, and orthogonal to l, n, p. Note that the tetrad cannot be introduced at fixed point sets of η, where $\eta = 0$, and $N = 0$. These points are caustics, where the interval between two neighboring rays vanishes. We discuss these cases later, see Sec. V A. From now on we assume that $N \neq 0$.

The tetrad is defined up to null rotations (Lorentz transformations of the parabolic type) of n, q:

$$n = n' + 2\omega q' + \omega^2 l', \qquad q = q' + \omega l', \tag{2.4}$$

where ω may vary along the trajectory. The transformations leave invariant the tangent vectors, $l = l'$ and $p = p'$.

There is an arbitrariness in the choice of l and p. Conditions (2.1), (2.2) allow reparametrizations

$$\lambda' = g(\lambda, \tau), \qquad \tau' = \phi(\tau), \tag{2.5}$$

which change tangent vectors to l', η',

$$l = g_{,\lambda} l', \qquad \eta = \phi_{,\tau} \eta' + g_{,\tau} l'. \tag{2.6}$$

Then (2.6) requires change of p and n of the tetrad

$$n=\frac{1}{g_{,\lambda}}(n'+2\bar{\omega}p'+\bar{\omega}^2 l'),\quad p=p'+\bar{\omega}l',\quad \bar{\omega}=\frac{g_{,\tau}}{N},\tag{2.7}$$

similar to (2.4). Reparametrizations (2.5) generate null rotations of p and n and rescalings of l and n.

Therefore, the tetrad l, n, p, q is defined up to a 2-parameter family of l-preserving null rotations of n, p, q, and rescalings of l and n. These are class I and class III transformations, respectively, according to [26]. We call (2.4)–(2.7) "S transformations" for brevity. Our aim is to construct from l, n, p, q quantities which are S-transform invariant.

We define $X_\lambda \equiv l$, $X_\tau \equiv \eta$, and denote reparametrizations (2.5) as $(\chi')^a = (\chi')^a(\chi)$. One can introduce a set of vectors $\nabla_a X_b$ ($\nabla_a = X^\mu_a \nabla_\mu$) tangent to the string world sheet. Since $\nabla_l \eta = \nabla_\eta l$,

$$\nabla_a X_b = \nabla_b X_a.\tag{2.8}$$

$\nabla_a X_b$ yield three independent vectors which can be decomposed as

$$\nabla_a X_b = f^c_{ab} X_c + \kappa^n_{ab} n + \kappa^q_{ab} q.\tag{2.9}$$

Coefficients f^c_{ab}, κ^n_{ab}, κ^q_{ab} are symmetric with respect to the permutation of a and b, and are related to spin coefficients for the chosen tetrad.

With the help of (2.1) and (2.2) one finds

$$\kappa^n_{\tau\tau} = \frac{1}{2}N_{,\sigma}N,\qquad \kappa^n_{\tau\lambda} = \kappa^n_{\lambda\lambda} = 0,\tag{2.10}$$

$$f^\tau_{\tau\tau} = \partial_\tau(\ln N),\qquad f^\tau_{\lambda\tau} = \partial_\lambda(\ln N),\qquad f^\tau_{\lambda\lambda} = 0.\tag{2.11}$$

The rest six coefficients, f^λ_{ab}, κ^q_{ab}, are not all independent and get mixed under S transforms. Null rotations (2.4) yield relations

$$\kappa^q_{ab} = (q\cdot\nabla_a X_b) = (\kappa')^q_{ab} - 2\omega(\kappa')^n_{ab},\tag{2.12}$$

$$\begin{aligned} f^\lambda_{ab} &= -\frac{1}{2}(n\cdot\nabla_a X_b)\\ &= (f')^\lambda_{ab} - \omega(\kappa')^q_{ab} + \omega^2(\kappa')^n_{ab}. \end{aligned}\tag{2.13}$$

Reparametrizations (2.5) imply that

$$\kappa^q_{ab} = (\chi')^{a'}{}_{,a}(\chi')^{b'}{}_{,b}(\kappa')^q_{a'b'},\tag{2.14}$$

$$\begin{aligned} f^\lambda_{ab} = {}&\frac{1}{g_{,\lambda}}(\chi')^{a'}{}_{,a}(\chi')^{b'}{}_{,b}\left((f')^\lambda_{a'b'} - \frac{g_{,\tau}}{\phi_{,\tau}}(f')^\tau_{a'b'} + \frac{g^2_{,\tau}}{N^2}(\kappa')^n_{a'b'}\right)\\ &+\frac{1}{g_{,\lambda}}\left(\lambda'_{,ab} - \frac{g_{,\tau}}{\phi_{,\tau}}\tau'_{,ab}\right), \end{aligned}\tag{2.15}$$

where we used (2.7) and (2.9).

B. Null strings and boost-weighted scalars

A boost-weighted scalar (a b scalar) along the string trajectory is defined as a scalar Q which changes under S transformations (2.4)–(2.7) as $Q = (g_{,\lambda})^b Q'$. Parameter b is called the boost weight of Q. Boost-weighted and spin-weighted quantities are discussed in [25] in the context of spin-coefficient formalism.

The b scalars play a key role in our analysis since they allow one to construct physical quantities measured by specific observers. In a frame of reference related to observers with velocities u_o ($u_o^2 = -1$) one can introduce a b scalar $(u_o\cdot l)$ which is nonvanishing, since u_o is timelike, and has a boost weight $b=1$. If Q is b scalar, then $Q_o = (u_o\cdot l)^{-b}Q$ is S-transform invariant scalar, $b=0$. In the given frame, Q_o can be interpreted as a physical observable.

We construct b scalars from κ^q_{ab}. Coefficients in (2.10), (2.11) depend only on N. Parameter N determines length of a small segment of the string, $dL = N d\tau$.S transformations are $N = \phi_{,\tau}N'$, $dL = dL'$. One can introduce the expansion parameter

$$\theta_s \equiv \partial_\lambda(\ln N) = (p\cdot\nabla_p l),\tag{2.16}$$

which is $b=1$ scalar. θ_s measures how fast dL changes along the worldline of a point on the string, $\partial_\lambda(dL) = \theta_s(dL)$. The next set of scalars is related to κ^q_{ab}. It follows from (2.12), (2.14), and (2.6) that

$$\kappa_1 \equiv \kappa^q_{\lambda\lambda}\tag{2.17}$$

is $b=2$ scalar. According to (2.9), (2.10), (2.11)

$$\nabla_l l = f^\lambda_{\lambda\lambda} l + \kappa_1 q.\tag{2.18}$$

Hence, the condition $\kappa_1 = 0$ implies that the string is null.

Coefficient $\kappa^q_{\lambda\tau}$ does not change under null rotations (2.12) but transforms as

$$\kappa^q_{\lambda\tau} = g_{,\lambda}\phi_{,\tau}(\kappa')^q_{\lambda'\tau'} + g_{,\lambda}g_{,\tau}(\kappa')^q_{\lambda'\lambda'}\tag{2.19}$$

under reparametrizations (2.6). If the string is null the last term in the rhs of (2.19) is zero and

$$\kappa_2 \equiv (q\cdot\nabla_p l) = N^{-1}\kappa^q_{\lambda\tau},\tag{2.20}$$

is weight $b=1$ scalar. Scalars κ_2 and θ_s play an important role in the subsequent analysis. We show that κ_2 measures the rotation of η in a plane orthogonal to the velocity of the string under a parallel transport of η along a worldline, see Sec. IV C.

One can continue in this way to come to other b scalars under certain restrictions. For example,

$$\kappa_3 \equiv (q \cdot \nabla_p p) = N^{-2} \kappa^q_{\tau\tau} \tag{2.21}$$

is $b = 0$ scalar, if $\theta_s = \kappa_1 = \kappa_2 = 0$. When the string is not null ($\kappa_1 \neq 0$),

$$\kappa_4 \equiv N^{-2} \det \kappa^q_{ab} \tag{2.22}$$

is $b = 2$ scalar, if $\theta_s = 0$.

Coefficients f^λ_{ab} yield no scalars due to the last term in the rhs of (2.15).

Note that θ_s and κ_1 are the b scalars which do not require any conditions. Other κ_i requires the vanishing of spin coefficients. Condition $\kappa_1 = 0$ can be imposed in any space-time: null strings, like rays, can always be constructed. Hence, κ_2 can be introduced for null strings.

In certain space-times, null strings include a subclass of strings with $\theta_s = \kappa_2 = 0$. Then κ_3 can be considered as a physical parameter. This subclass includes other subclasses with $\theta_s = \kappa_2 = \kappa_3$.

Strings in space-times with global parabolic isometries [10,11,14] are of special interest since they allow for the explicit description of backreaction effects. They were called massless strings. The world sheets of such strings are null 2-surfaces which are fixed points sets of null rotations. The metric of a space-time which allows a global parabolic isometry is [14]

$$ds^2 = -2e^f (dudv - dy^2) + h(dz + sdu)^2, \tag{2.23}$$

where f, h, and s are functions of u, z, and $\theta = uv - y^2$. The isometries of (2.23) are null rotations of coordinates

$$u' = u, \qquad v' = v + 2\omega y + \omega^2 u,$$
$$y' = y + \omega u, \qquad z' = z, \tag{2.24}$$

which leave θ invariant. The trajectory of a massless string is given by simple equations: $v = \lambda$, $z = \tau$, $u = y = 0$. One can check that it fulfills (2.1), (2.2), (2.3). Hence the string is null.

Now let ζ be the Killing field which generates (2.24) and the corresponding null rotation of the tetrad (2.4). Since $\zeta = 0$ on the world sheet one gets the set of conditions:

$$0 = \delta_\zeta(\nabla_a X_b \cdot n) = (\nabla_a X_b \cdot \delta_\zeta n)$$
$$= 2\omega(\nabla_a X_b \cdot q) = 2\omega \kappa^q_{ab}, \tag{2.25}$$

$$0 = \delta_\zeta(\nabla_a X_b \cdot q) = (\nabla_a X_b \cdot \delta_\zeta q)$$
$$= \omega(\nabla_a X_b \cdot l) = -2\omega \kappa^n_{ab}. \tag{2.26}$$

By taking into account (2.10) one concludes that strings studied in [10,11,14] are null strings for which $\theta_s = \kappa_1 = \kappa_2 = \kappa_3 = 0$.

III. OPTICAL EQUATIONS

A. String scalar and its equation

Consider now the relation between spin coefficients at the string trajectory and the curvature of the background space-time $\mathcal{M}$. It follows from (2.8) that

$$[\nabla_a, \nabla_b] X^\mu_c = -R^\mu{}_{\alpha\rho\nu} X^\alpha_c X^\rho_a X^\nu_b, \tag{3.1}$$

where $R^\mu{}_{\alpha\rho\nu}$ is the Riemann tensor of $\mathcal{M}$ at the string trajectory (we use the definition $R^\mu{}_{\alpha\rho\nu} V^\alpha \equiv [\nabla_\nu, \nabla_\rho] V^\mu$). The left-hand side of (3.1) can be calculated with the help of (2.9). After some algebra, replacing η with p gets us a set of relations for components of the Riemann tensor at the string trajectory

$$R_{plpl} = (\partial_l - \beta_1)\theta_s + \theta_s^2 + \kappa_1\kappa_3 - \kappa_2^2, \tag{3.2}$$

$$R_{qlpl} = (\partial_l - \beta_1)\kappa_2 + 2\theta_s\kappa_2 - (\partial_p - \beta_2)\kappa_1, \tag{3.3}$$

$$R_{qplp} = (\partial_p + \beta_2)\kappa_2 - (\partial_l + \theta_s)\kappa_3 - \beta_3\kappa_1 + \frac{1}{2}\theta_s\alpha_1, \tag{3.4}$$

$$R_{nllp} = (\partial_l + 2\theta_s)\beta_2 - \kappa_2\alpha_1 + \kappa_1\alpha_2, \tag{3.5}$$

$$R_{nplp} = 2((\partial_l + \theta_s)\beta_3 - \partial_p\beta_2 + \beta_3\beta_1 - \beta_2^2)$$
$$+ \kappa_2\alpha_2 - \kappa_3\alpha_1. \tag{3.6}$$

We put $\beta_1 = -\frac{1}{2}(n \cdot \nabla_l l) = f^\lambda_{\lambda\lambda}$, $\beta_2 = -\frac{1}{2}(n \cdot \nabla_p l)$, $\beta_3 = -\frac{1}{2}(n \cdot \nabla_p p)$, $\alpha_1 = (\nabla_l q \cdot n)$, $\alpha_2 = (\nabla_p q \cdot n)$, and use notation $R_{abcd} = R_{\mu\alpha\rho\nu} e^\mu_a e^\alpha_b e^\rho_c e^\nu_d$ for $e_a = l,\ p,\ n,\ q$.

Equations (3.2), (3.3) are of the most interest since R_{plpl}, R_{qlpl} are $b = 2$ scalars. One can use these equations to get an analog of Raychaudhuri-type equations and draw a physical information. Other curvatures in (3.4)–(3.3) are not boost weighted scalars, in general. Under S transformations R_{qplp}, R_{nllp} get mixed with R_{plpl}, R_{qlpl}. Thus, R_{qplp}, R_{nllp} are $b = 2$ and $b = 1$ scalars, respectively, only if $R_{plpl} = R_{qlpl} = 0$. Analogously R_{nplp} is $b = 1$ scalar, if $R_{plpl} = R_{qlpl} = R_{nllp} = 0$.

From now on we assume that the string is null, $\kappa_1 = 0$.

It is convenient to introduce a pair of null complex vectors $\hat{m}$, $\hat{\bar{m}}$, $(\hat{m} \cdot \hat{\bar{m}}) = 1$,

$$\hat{m} = \frac{1}{\sqrt{2}}(p + iq), \qquad \hat{\bar{m}} = \frac{1}{\sqrt{2}}(p - iq). \tag{3.7}$$

By following the standard procedure [25,26] one defines invariants C_{abcd} and R_{ab} constructed from the Weyl tensor and Ricci tensor, respectively,

$$\hat{\Psi}_0 = -C_{\hat{m}l\hat{m}l}, \qquad \Phi_{00} = -\frac{1}{2}R_{ll}.$$

Since $C_{\hat{m}l\hat{\bar{m}}l} = 0$, see [26], it follows that

$$R_{plpl} = C_{plpl} + \frac{1}{2}R_{ll} = -\mathrm{Re}\,\hat{\Psi}_0 - \hat{\Phi}_{00}, \tag{3.8}$$

$$R_{qlpl} = C_{qlpl} = -\mathrm{Im}\,\hat{\Psi}_0. \tag{3.9}$$

We also introduce complex $b = 1$ scalar

$$Z \equiv \theta_s + i\kappa_2 = ((p + iq) \cdot \nabla_p l). \tag{3.10}$$

With these definitions Eqs. (3.2), (3.3) in the case of null strings take the following simple form:

$$D_l Z + Z^2 = -\hat{\Psi}_0 - \Phi_{00}, \tag{3.11}$$

where $D_l \equiv \partial_l - \beta_1$ is a covariant derivative with respect to boost transformations.

We call Z the string scalar. This parameter plays an important role for the rest of the article. By taking square of the left- and right-hand sides of (2.9) for $a = \tau$, $b = \lambda$, one gets a useful identity

$$(\nabla_\eta l \cdot \nabla_\eta l) = N^2(\theta_s^2 + \kappa_2^2), \tag{3.12}$$

where we took into account (2.11). Then it follows from (3.10) that

$$|Z| = |\nabla_p l|. \tag{3.13}$$

We call (3.11) an optical equation for strings, by analogy with optical equations of Sachs for NGC. The importance of (3.11) is that it can be used to draw some universal features of string trajectories, for example, in asymptotically flat or asymptotically de Sitter space-times, where curvatures $\hat{\Psi}_0$, Φ_{00} decay fast enough at null infinities.

B. Relation to Sachs's equations

To establish the relation between (3.11) and the Sachs equations we introduce another complex dyad at the string trajectory, m and $\bar{m}$,

$$(m \cdot \bar{m}) = 1, \qquad (m \cdot l) = (m \cdot n) = 0. \tag{3.14}$$

The set $(n, l, m, \bar{m})$ is a doubly null tetrad. Note that m and $\bar{m}$ are arbitrary vectors while dyad $\hat{m}$, $\hat{\bar{m}}$ is connected with η and q, and implies the condition $\nabla_l \eta = \nabla_\eta l$. The fact that $(n, l, m, \bar{m})$ are not restricted by any conditions allows one to use the Newman-Penrose formalism [25,26] and require that m is parallel transported along the worldlines, $\nabla_l m = 0$. We put $\hat{m} = N\zeta^{-1}m$, where ζ is a complex parameter, $|\zeta| = N$. The connecting vector in the new basis is

$$\eta = \frac{1}{\sqrt{2}}(\zeta\bar{m} + \bar{\zeta}m). \tag{3.15}$$

When a point moves along a worldline the phase of ζ determines orientation of η in the plane $(m, \bar{m})$. The condition $\nabla_l \eta = \nabla_\eta l$ requires that

$$\partial_l \zeta = -\rho\zeta - \sigma\bar{\zeta}, \tag{3.16}$$

$$\rho \equiv -(m \cdot \nabla_{\bar{m}} l), \qquad \sigma \equiv -(m \cdot \nabla_m l). \tag{3.17}$$

Here we took into account that $\nabla_l m = 0$. Spin coefficients ρ, σ are known as optical scalars which are defined for NGC with velocity vector l. One can check that ρ, σ are $b = 1$ boost-weighted scalars on trajectories of null strings. It then follows from definition (3.10) that

$$\zeta Z = -\zeta\rho - \bar{\zeta}\sigma. \tag{3.18}$$

So (3.16) implies

$$\partial_l \zeta = Z\zeta. \tag{3.19}$$

If a string optical equation (3.11) is satisfied, then one can take the derivative D_l of the left and right parts of (3.18), and use (3.16), (3.19), to get

$$-\zeta\Phi_{00} - \bar{\zeta}\Psi_0 = \zeta(\rho^2 + |\sigma|^2 - D_l\rho) + \bar{\zeta}(\sigma(\rho + \bar{\rho}) - D_l\sigma), \tag{3.20}$$

$$\Psi_0 = -C_{mlml} = \bar{\zeta}^{-1}\zeta\hat{\Psi}_0. \tag{3.21}$$

Equation (3.20) requires the following relations known as optical equations of Sachs:

$$D_l\rho = \rho^2 + |\sigma|^2 + \Phi_{00}, \tag{3.22}$$

$$D_l\sigma = \sigma(\rho + \bar{\rho}) + \Psi_0. \tag{3.23}$$

Sachs equations (3.22), (3.23) are derived for general NGC and are not related to string trajectories. Given a string trajectory, Sachs equations at the trajectory may follow from (3.11). Opposite is true as well: if a string trajectory is a 1-parameter family of rays in a NGC, then the string equation (3.11) follows from Sachs equations for the given NGC.

IV. REPRESENTATION OF STRING TRAJECTORIES

A. Diagram description of string trajectories

The physical meaning of θ_s, the real part of the string scalar Z, is related to local expansion (contraction) of string segment moving along a worldline, see Sec. II B. To find the interpretation of κ_2, the imaginary part of Z, we use

(3.19). When a point of the string moves along the worldline, the components $(\zeta, \bar{\zeta})$ of the connecting vector with respect to dyad $(m, \bar{m})$ change,

$$\zeta(\lambda + \delta\lambda) \simeq (1 + Z\delta\lambda)\zeta(\lambda). \tag{4.1}$$

If $\zeta = |\zeta|e^{i\alpha}$, then Eq. (4.1) implies change of the phase

$$\alpha(\lambda + \delta\lambda) \simeq \alpha(\lambda) + \mathrm{Im}\, Z\delta\lambda. \tag{4.2}$$

Therefore, ImZ determines rotation of η in the plane $(m, \bar{m})$. The rotation angle of η under the shift $\delta\lambda$ is $\kappa_2\delta\lambda$. The sign of the rotation is connected with the sign of κ_2. It changes if q is replaced with $-q$. This reflection arbitrariness can be eliminated by additional arguments. For example, one can introduce a null vector $\tilde{l}_\mu = \epsilon_{\mu\nu\lambda\rho} p^\nu q^\lambda l^\rho$. Since $\tilde{l} = al$ one can fix the sign of q by requiring, for example, that $a > 0$.

In the next sections we study constant λ slices of the string world sheet. These slices are curves that determine shape of the string. If λ is fixed, then $Z = Z(\lambda, \tau)$ is a curve in a space of parameters θ_s, κ_2, or in a complex Z plane. String scalar Z can be used to describe the evolution of the shape of the string in a given slicing.

We call such curves diagrams of the string trajectory. The diagrams can be written locally as $\theta_s = \theta_s(\kappa_2)$ or $\kappa_2 = \kappa_2(\theta_s)$, so they do not depend on spatial parametrization of the trajectory by τ. String diagrams yield "portraits" of the string trajectories.

To construct the string diagrams one needs to remember that λ is not uniquely defined and allows transformations (2.6). One can use physical arguments and relate λ to a frame of reference where the string trajectory is considered. If u_o are 4-velocities of observers which make a certain frame of reference, then it is natural to require that

$$(u_o \cdot \eta) = 0. \tag{4.3}$$

Condition (4.3) fixes λ up to rescalings $\lambda' = g(\lambda)$ which leave constant λ slices invariant and, hence, do not change string diagrams. Alternatively, one can require that constant λ slices coincide with constant time slices. [This condition is reduced to (4.3) when u_o is orthogonal to constant time slices.] The rescalings can be restricted by further arguments. For instance, in some cases one can require that $(u_o \cdot l) = -1$. Examples of string trajectories with such conditions are presented in Secs. IV B and V B.

B. Exact solutions for Z

To give an idea of the string diagrams consider strings in space-times where $\hat{\Psi}_0 = \Phi_{00} = 0$. One can choose affine parametrization, $D_l = \partial_l$, to get a general solution to (3.11)

$$Z(\lambda, \tau) = \frac{1}{\lambda + z(\tau)}, \tag{4.4}$$

where z is a complex function. Scalars θ_s and κ_2 are real and imaginary parts of the rhs of (4.4), respectively. Solutions like (4.4) hold for strings in conformally flat space-times with $\Phi_{00} = 0$. Equation (4.4) holds for strings in de Sitter and anti–de Sitter geometries. Singularities of Z at $\lambda = -z(\tau)$ may appear in different models. They correspond to caustics at fixed points of the vector field η, see below.

As an example, consider the case when $z = ce^{i\tau}$, where c is a positive constant. One finds for $|\lambda| \neq c$:

$$\theta_s(\lambda, \tau) = \frac{\lambda + c\cos\tau}{(\lambda + c\cos\tau)^2 + c^2\sin^2\tau},$$
$$\kappa_2(\lambda, \tau) = -\frac{c\sin\tau}{(\lambda + c\cos\tau)^2 + c^2\sin^2\tau}, \tag{4.5}$$

which is equivalent to

$$(\theta_s - d)^2 + \kappa_2^2 = R^2, \tag{4.6}$$

$$d = d(\lambda) = \frac{\lambda}{\lambda^2 - c^2}, \qquad R = R(\lambda) = \frac{c}{|\lambda^2 - c^2|}. \tag{4.7}$$

One can represent Z as

$$Z(\lambda, \tau) = d(\lambda) + R(\lambda)e^{i\xi}, \tag{4.8}$$

$$e^{i\xi} = \pm\frac{c + \lambda e^{i\tau}}{\lambda + ce^{i\tau}}, \tag{4.9}$$

where signs $+$ and $-$ correspond to the cases $\lambda < c$ and $\lambda > c$, respectively. Equation (4.9) is a linear fractional transformation from $z = e^{i\tau}$ to $w = e^{i\xi}$. Therefore, the diagram of the string at fixed λ is a circle. The diagrams for $|\lambda| \neq c$ are shrinking or expanding circles shown on Fig. 1.

The case $\lambda = c$ is special,

$$Z(c, \tau) = \frac{e^{-i\tau/2}}{2c\cos\frac{\tau}{2}}.$$

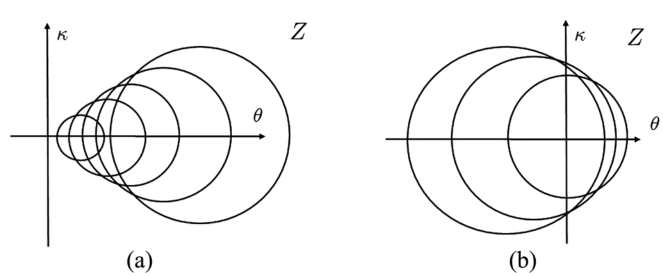

FIG. 1. String diagrams for string scalar Z given by (4.4) with $z = ce^{i\tau}$. For $\lambda > r$, (a), circles shrink as λ grows. For $0 < \lambda < r$, (b), circles expand as λ grows.

Let us emphasize that the diagrams depend on the choice of λ. Examples given above are for λ being an affine parameter. The residual freedom of changing λ to $\lambda' = \lambda + f(\tau)$ can be eliminated by additional conditions like (4.3). Examples are given in Sec. V A.

C. Closed strings

To see how parameters θ_s and κ_2 determine the transformation of the shape of the string consider, as an illustration, a closed string, $Z(\lambda, \tau + 2\pi) = Z(\lambda, \tau)$. Suppose that λ slices are fixed by (4.3).

One can use (2.4) to choose vector q orthogonal to u_o. Then the pair (p, q) yield a 2-plane in the frame of reference of the observers. If one considers the Fourier transform

$$Z(\lambda, \tau) = \sum_n e^{in\tau} c_n(\lambda), \tag{4.10}$$

then the coefficients c_n determine different transformations of the shape of the string, the string modes. Assume that only the single mode with $n \neq 0$ is present in (4.10) and $c_n > 0$. The corresponding expansion and rotation scalars are

$$\theta_s(\lambda, \tau) = c_n(\lambda) \cos n\tau, \quad \kappa_2(\lambda, \tau) = c_n(\lambda) \sin n\tau. \tag{4.11}$$

At fixed λ, parameters $\tau = 2k\pi/n$, $k = 0, 1, 2\ldots$ are the points of maximal expansion, $\theta_s > 0$, while maximal contraction occurs at $\tau = (2k+1)\pi/n$, $\theta_s < 0$. There is no rotation of η at these points, $\kappa_2 = 0$. Between $\tau = 2k\pi/n$ and $\tau = (2k+1)\pi/n$ rotation of η is, say, counterclockwise, $\kappa_2 > 0$, from $\tau = 2k\pi/n$ to $\tau = (2k+1)\pi/n$. Between $\tau = (2k-1)\pi/n$ and $\tau = 2k\pi/n$ rotation is clockwise, $\kappa_2 < 0$, from $\tau = 2k\pi/n$ to $\tau = (2k-1)\pi/n$. That is, points of the string rotate toward a nearby point of maximal contraction.

String modes have a simple form if the string is a circle. String modes $n = 1, 2, 3, 4$ are shown on Fig. 2.

V. EXAMPLES OF STRING TRAJECTORIES

A. Strings in Minkowsky space-time

A general solution to Eqs. (2.1)–(2.3) for a null string in a flat space-time, for the choice of λ as an affine parameter, is

$$X^\mu(\lambda, \tau) = \lambda b^\mu(\tau) + a^\mu(\tau), \tag{5.1}$$

where b^μ is an arbitrary null vector, $b^2 = 0$. Restrictions on a^μ are $(b \cdot \dot{a}) = 0$, see (2.2), and $\dot{a}^2 > 0$, $\dot{a} \equiv a_{,\tau}$. One finds

$$N^2 = \lambda^2 \dot{b}^2 + 2\lambda(\dot{b} \cdot \dot{a}) + \dot{a}^2, \tag{5.2}$$

$$\theta_s = N^{-2}(\lambda \dot{b}^2 + (\dot{b} \cdot \dot{a})). \tag{5.3}$$

To calculate the rotation scalar we use (3.12)

$$\kappa_2^2 = N^{-2} \dot{b}^2 - \theta_s^2 = N^{-4}(\dot{a}^2 \dot{b}^2 - (\dot{b} \cdot \dot{a})^2). \tag{5.4}$$

In the flat space-time we can fix the parametrization of the string-world sheet for an inertial frame of reference by conditions $(l \cdot u_o) = -1$, $(\eta \cdot u_o) = 0$. In Minkowsky coordinates, where velocity of observers is $u_o^\mu = \delta_0^\mu$, these conditions are ensured if $t(\lambda, \tau) = \lambda$, that is, $\dot{a}$ and $\dot{b}$ have only spatial components. Let $|\dot{b}| \neq 0$, and

$$\cos\varphi \equiv \frac{(\dot{b} \cdot \dot{a})}{|\dot{a}||\dot{b}|}, \qquad r \equiv \frac{|\dot{a}|}{|\dot{b}|}. \tag{5.5}$$

Then under the appropriate choice of the sign of κ_2 the string scalar is

$$Z(\lambda, \tau) = \frac{1}{\lambda + r(\tau) e^{i\varphi(\tau)}}, \tag{5.6}$$

in accord with (4.4). The rotation scalar vanishes when $\dot{a} = 0$, or $\varphi = 0$.

Let us consider several examples.

1. Take Eq. (5.1) in Minkowsky coordinates as $t = x = \lambda$, $y = y(\tau)$, $z = z(\tau)$. The corresponding string moves along the x axis, the string lies in a 2-plane orthogonal to the direction of motion. One can check that $Z \equiv 0$. The string diagrams are a single dot, since the strings do not change their shape.

2. An interesting string trajectory is

$$t = \lambda, \qquad x = \lambda \cos\tau + \frac{c}{4}\cos 2\tau,$$
$$y = \lambda \sin\tau + \frac{c}{4}(\sin 2\tau + 2\tau), \qquad z = c\cos\tau, \tag{5.7}$$

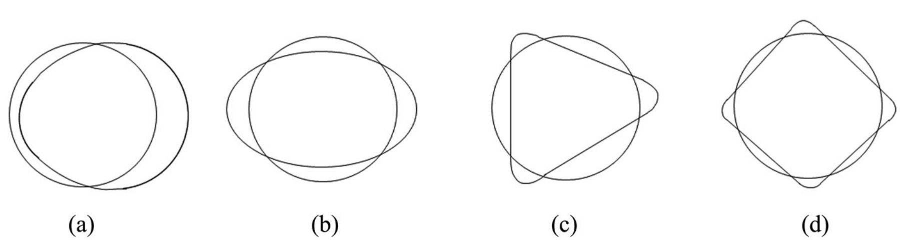

FIG. 2. Shows schematically first four deformation modes of a circular string in the (p, q) plane when λ is slightly increased. Deformations a,b,c,d correspond to $n = 1, 2, 3, 4$.

where $c > 0$ is a constant. At fixed λ the string is twisted along y axis. It is instructive to calculate the corresponding connecting vector field $\eta^\mu = \dot{x}^\mu$:

$$\eta^t = 0, \qquad \eta^x = -\sin\tau(\lambda + c\cos\tau),$$
$$\eta^y = \cos\tau(\lambda + c\cos\tau), \qquad \eta^z = -c\sin\tau. \tag{5.8}$$

This vector field vanishes, $\eta = 0$, at two fixed points of the congruence: $\lambda = r$, $\tau = \pi$ and $\lambda = -r$, $\tau = 0$. One can check that

$$b^2 = (b \cdot \dot{a}) = 0, \qquad \dot{b}^2 = 1,$$
$$r = |\dot{a}| = c, \qquad (\dot{b} \cdot \dot{a}) = c\cos\tau = c\cos\varphi. \tag{5.9}$$

Therefore, θ_s and κ_2 in the Z plane satisfy (4.6), (4.7), for $\lambda \neq r$. Z has simple poles at fixed points of η.

The diagrams of the string are shrinking or expanding circles shown on Fig. 1.

3. Consider a closed string described by equations

$$t = \lambda, \qquad x = \lambda\cos\tau,$$
$$y = \lambda\sin\tau, \qquad z = c\sin\tau, \tag{5.10}$$

where $c > 0$ is a constant. For such a string

$$b^2 = (b \cdot \dot{a}) = 0, \qquad \dot{b}^2 = 1,$$
$$r = |\dot{a}| = c|\cos\tau|, \qquad (\dot{b} \cdot \dot{a}) = 0. \tag{5.11}$$

$$Z(\lambda, \tau) = \frac{1}{\lambda + ic\cos\tau}. \tag{5.12}$$

It follows from (5.12) that θ_s and κ_2 are related with

$$(\theta_s - R)^2 + \kappa_2^2 = R^2, \qquad R = \frac{1}{2\lambda}. \tag{5.13}$$

$$Z(\lambda, \tau) = R(\lambda) + R(\lambda)e^{i\xi}, \tag{5.14}$$

$$e^{i\xi} = \frac{\lambda - ic\cos\tau}{\lambda + ic\cos\tau}. \tag{5.15}$$

Map (5.15) from $z = e^{i\tau}$ to $w = e^{i\xi}$ is not a linear fractional transformation. At fixed λ, the string portrait is only an arc of the circle (5.13). Ends of the arc are at points $\xi = \pm\xi_0$, where $\partial_\tau \xi = 0$. This corresponds to $\tau = 0, \pi$, and

$$\sin\xi_0 = \frac{2\lambda c}{\lambda^2 + c^2}.$$

Each point on the arc corresponds to two values of τ. The diagrams of the string are shown on Fig. 3 for positive λ.

In the above examples string diagrams have a common feature: the diagrams shrink to point $Z = 0$ as λ grows. At large λ, or as null future infinity is approached, all deformations of string's shape gradually decay. One can say that the null strings are "freezing out."

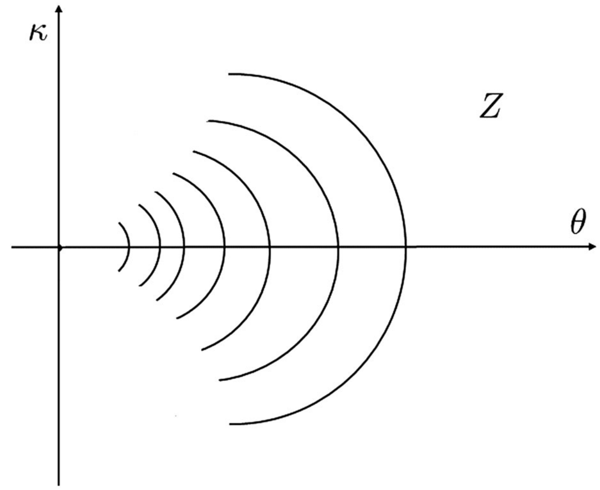

FIG. 3. Diagrams of a closed string in Minkowsky space-time. The diagrams are arcs of circles that shrink as λ grows.

B. Strings in cosmological models

Studying null cosmic strings in an expanding universe is of particular interest, since such objects may result in observable physical effects [11]. We consider conformally flat cosmologies with

$$ds^2 = -dt^2 + a^2(t)(dx^i)^2, \tag{5.16}$$

where the scale factor $a(t)$ is determined by a concrete model. Coordinate t is the cosmological time. The string equations are defined as

$$t = \lambda, \qquad x^i(\lambda, \tau) = f(\lambda)b^i(\tau) + a^i(\tau), \tag{5.17}$$

$$\partial_\lambda f = a^{-1}(\lambda), \tag{5.18}$$

where $(b^i)^2 = 1$, $b^i \dot{a}^i = 0$. The tangent vectors are $l = \partial_t + f' b^i \partial_i$, $\eta = (f\dot{b}^i + \dot{a}^i)\partial_i$. With the help of (5.18) one checks that

$$l^2 = 0, \qquad \nabla_l l = \chi l, \qquad (\eta \cdot l) = 0, \tag{5.19}$$

where $\chi = \partial_\lambda a/a$. Thus, (5.17) describes a null string. In the frame of freely moving observers with 4-velocities $u_o^\mu = \delta_t^\mu$ the following conditions hold:

$$(u_o \cdot l) = -1, \qquad (u_o \cdot \eta) = 0. \tag{5.20}$$

Equation (5.20) can be used to fix the parametrization. Relation between λ and the affine parameter $\bar{\lambda}$ is $\partial_\lambda \bar{\lambda} = C(\tau)a(\lambda)$, where $C(\tau)$ is an arbitrary real function.

A straightforward calculation yields

$$Z(\lambda,\tau)=\frac{1}{a(\lambda)}\frac{1}{f(\lambda)+z(\tau)}+\partial_\lambda \ln a, \qquad (5.21)$$

where $z(\tau)=r(\tau)e^{i\varphi(\tau)}$ is defined by (5.5). For a universe filled with a matter with the equation of state $p=w\rho$, $a\sim\lambda^p$, $f\sim\lambda^{1-p}$, where $p=2/(3(1+w))$. One concludes that, for a dust or radiation dominated universe, $Z(\lambda,\tau)=O(\lambda^{-1})$ at large λ. This means that strings do not change their shape at large cosmological time t.

For a flat de Sitter universe, $a(\lambda)=e^{H\lambda}$,

$$Z(\lambda,\tau)=\frac{H^2 z(\tau)}{Hz(\tau)-e^{-H\lambda}}. \qquad (5.22)$$

At large λ, $Z(\lambda,\tau)\simeq H$. Cosmic strings at late times do not rotate but expand exponentially, similarly to other scales.

Scalar Z vanishes if a^i is a constant and $z=0$ in (5.22). Trajectories of these strings lie on the cosmological horizon $|x^i(t)-a^i|=H^{-1}e^{-Ht}$. A particular type of such strings, massless strings on the equator of the horizon sphere, admit an exact analysis of backreaction effects, and has been studied in detail in [11].

String diagrams for (5.17) can be constructed analogously to the case of string trajectories in Minkowsky space-time.

VI. STRINGS IN ASYMPTOTICALLY FLAT SPACE-TIMES

A. Strings on null hypersurfaces

String trajectories may lie on null hypersurfaces. Then l is directed along a normal vector to the surface. This case can be studied by using the Bondi-Sachs formalism. One can always choose a tetrad basis in the entire space-time so that it coincides with n, l, $\hat{m}$, $\hat{\bar{m}}$ at the string trajectory, see Sec. III, and introduce corresponding spin-coefficients. As we saw, Eq. (3.18) relates the string scalar Z to the spin coefficients ρ and σ, which are divergence and shear of a null geodesic congruence. In the chosen basis, $\zeta=1$ in (3.18).

This fact is important since asymptotic form of the shear at future null infinity $\mathcal{I}^+$ together with asymptotics of the complex tetrad components of the Weyl tensor Ψ_k are used to extract interior physical properties of the space-time, see [27], for a review. We return to this issue in Sec. VI B.

In asymptotically flat space-times curvature scalars behave as $\Psi_0=O(\lambda^{-5})$, $\Phi_{00}=O(\lambda^{-6})$ at $\mathcal{I}^+$. Therefore the rhs of (3.11) can be ignored and (4.4) can be used to get for expansion and rotation scalars the following asymptotics at large $|\lambda|$:

$$Z\simeq\frac{1}{\lambda}-\frac{z(\tau)}{\lambda^2}+\cdots,\qquad \theta_s\simeq\frac{1}{\lambda}-\frac{z_1(\tau)}{\lambda^2}+\cdots,$$
$$\kappa_2\simeq-\frac{z_2(\tau)}{\lambda^2}+\cdots, \qquad (6.1)$$

where $z_1=\mathrm{Re}z$, $z_2=\mathrm{Im}z$. We see from (6.1) the already familiar phenomenon: shapes of strings in asymptotically flat space-times cease to vary at large λ, as $\mathcal{I}^+$ is approached. Possible ripples on the strings are encoded in subleading terms in Z, and are determined in (6.1) by a complex parameter $z(\tau)$.

Asymptotic behavior of a NGC on an outgoing null hypersurface is [27]

$$\rho=\bar{\rho}\simeq-\frac{1}{\lambda}-\frac{\sigma^0\bar{\sigma}^0}{\lambda^3}+\cdots,\qquad \sigma\simeq\frac{\sigma^0}{\lambda^2}+\cdots, \qquad (6.2)$$

where σ^0 is called the asymptotic complex shear of the NGC. If the string belongs to the given NGC and the tetrads are chosen so that they coincide with n, l, $\hat{m}$, $\hat{\bar{m}}$ at the string trajectory, then one can use (3.18) to find the simple relation

$$z=\sigma^0. \qquad (6.3)$$

That is, the subleading term in the string scalar Z is determined by the asymptotic shear σ^0 calculated in the special basis.

Note that the string trajectory itself may be forming a null 2 surface. This surface is defined by conditions $f=0$, $t=0$, and $l_\mu=af_{,\mu}$, $q_\mu=bt_{,\mu}$, where a and b are functions set on the surface. Then $(q\cdot\nabla_q l)=-\partial_l\ln b=0$, and one gets an additional relation

$$\theta\equiv\nabla_l l=(p\cdot\nabla_p l)+(q\cdot\nabla_q l)-\frac{1}{2}(n\cdot\nabla_l l)=\theta_s. \qquad (6.4)$$

For a general NGC with velocity l parameter θ measures expansion or contraction of the area of 2D spacelike sections of the congruence. Thus, if the NGC is null surface forming, the string trajectory belongs to NGC and it is 2-surface forming, then the string expansion θ_s coincides with the expansion of NGC.

B. Strings in the Bondi-Sachs formalism and gravitational waves

The asymptotic shear σ^0 is known to measure the amplitude of gravity waves far from the source. We now find out an explicit relation between asymptotic form of Z, see (6.1), and outgoing gravitational radiation. We consider strings in asymptotically flat space-times, far from a source of gravity waves. It is convenient to use the Bondi-Sachs coordinates $x^\mu=(u,r,x^A)$, $A=1$, 2, based on a family of outgoing null hypersurafces, see [28] for a review. The corresponding metric is

$$ds^2 = -Udu^2 - 2e^{2\beta}dudr + g_{AB}(dx^A - V^A du)(dx^B - V^B du). \tag{6.5}$$

The null hypersurfaces in question are $u = c$, where c is a constant. Coordinate r which varies along null rays is chosen to be a real coordinate. The future null infinity is at $r \to +\infty$.

Metric (6.5) is flat when $U = 1$, $\beta = V^A = 0$, $g_{AB} = r^2\gamma_{AB}$, with γ_{AB} being a metric on a unit 2-sphere. In an asymptotically flat space-time at large r

$$U = 1 - \frac{2M}{r} + O(r^{-2}), \tag{6.6}$$

$$g_{AB} = r^2\left(\gamma_{AB} + \frac{C_{AB}}{r} + O(r^{-2})\right), \tag{6.7}$$

Other parameters, β and V^A, are $O(r^{-2})$, see [28]. The parameter $M = M(u, x^A)$ is the mass aspect. $C_{AB} = C_{AB}(u, x^A)$ is a traceless tensor in a tangent space to S^2. The term C_{AB}/r in (6.7) is a perturbation of the metric caused by the outgoing gravitational radiation.

Vector $l = \partial_r = -e^{-2\beta}\nabla u$ is a tangent vector to null geodesics on the null hypersurfaces $u = c$. We choose the "gauge" $\beta = 0$. Then $\nabla_l l = 0$, and r can be identified with an affine parameter. The string equations in coordinates u, r, x^A are

$$u = c, \qquad r = \lambda, \qquad x^A = x^A(\tau). \tag{6.8}$$

The connecting vector is $\eta = \eta^A\partial_A$, where $\eta^A = \dot{x}^A$. One can check with the help of (6.5) that (2.1), (2.2) are satisfied and

$$\theta_s = \frac{1}{2}p^A p^B \partial_r g_{AB}, \qquad p^A = \frac{\eta^A}{(\eta^A\eta^B g_{AB})^{1/2}}. \tag{6.9}$$

The simplest choice for the vector q in the tetrad at the string trajectory is $q = q^A\partial_A$,

$$q^A = g^{AC}\epsilon_{CB}p^B, \qquad \epsilon_{AB} = -\epsilon_{BA},$$
$$\epsilon_{12} = (\det g_{AB})^{1/2}. \tag{6.10}$$

A, B are risen and lowered with the help of g_{AB} and its inverse matrix. Equation (6.10) is not a unique choice, but it is enough to find κ_2. A straightforward calculation with the help of (2.20), (6.5) yields

$$\kappa_2 = \frac{1}{2}q^A p^B \partial_r g_{AB}, \tag{6.11}$$

$$Z(\lambda, \tau) = \frac{1}{2}(p^A + iq^A)p^B\partial_r g_{AB}. \tag{6.12}$$

We are interested in asymptotic properties of Z at large $r = \lambda$, which easily follow from (6.6),

$$\partial_r g_{AB} = \frac{2g_{AB}}{r} - C_{AB} + O(r^{-1}),$$
$$Z(\lambda, \tau) = \frac{1}{\lambda} - \frac{z}{\lambda^2} + O(\lambda^{-3}), \tag{6.13}$$

$$z = \frac{1}{2}(\bar{p}^A + i\bar{q}^A)\bar{p}^B C_{AB}. \tag{6.14}$$

Here we took into account that $p^A = \bar{p}^A/r + O(r^{-2})$, $q^A = \bar{q}^A/r + O(r^{-2})$. In fact, $\bar{p}^A$, $\bar{q}^A$ are mutually orthogonal vectors tangent to S^2

$$\bar{p}^A = \frac{\eta^A}{(\eta^A\eta^B\gamma_{AB})^{1/2}}, \qquad \bar{q}_A = \bar{\epsilon}_{AB}\bar{p}^B,$$
$$\bar{\epsilon}_{12} = (\det\gamma_{AB})^{1/2}. \tag{6.15}$$

Since the metric can be written as $\gamma_{AB} = \bar{p}_A\bar{p}_B + \bar{q}_A\bar{q}_B$, and C_{AB} is traceless, $C_{AB}\gamma^{AB} = 0$, one can decompose

$$C_{AB} = (\bar{p}_A\bar{p}_B - \bar{q}_A\bar{q}_B)C_+ + (\bar{p}_A\bar{q}_B + \bar{q}_A\bar{p}_B)C_\times, \tag{6.16}$$

and rewrite (6.14) in a simple form

$$z = -\frac{1}{2}(C_+ + iC_\times). \tag{6.17}$$

Coefficients C_+ and $C_\times$ correspond to "+" and "x" polarizations of a gravity wave in the given basis. Expression (6.17) is in accord with the fact that "+" and "x" polarizations are related to real and imaginary parts of the shear.

C. Interaction of the string with ingoing flux

Consider an outgoing null string which moves toward $\mathcal{I}^+$ and an ingoing flux of matter which crosses the string trajectory. Our aim is to find out how the flux affects parameter $z(\tau)$ in asymptotic (6.1). We assume that the flux is weak and its interaction with all points of the string happens in a short interval $(\lambda_\star, \lambda_\star + \Delta\lambda)$.

Substitution $Z = Y^{-1}$ brings (3.11) to the form

$$\partial_\lambda Y = 1 + \mathcal{R}Y^2, \qquad \mathcal{R} = \hat{\Psi}_0 + \Phi_{00}, \tag{6.18}$$

in the affine parametrization. We assume that $\mathcal{R}$ is entirely due to the flux. Solution to (6.18) can be found perturbatively,

$$Y = Y_0 + Y_1 + Y_2 + \cdots, \tag{6.19}$$

$$\partial_\lambda Y_0 = 1, \qquad \partial_\lambda Y_1 = \mathcal{R}Y_0^2, \tag{6.20}$$

where $Y_k = O(\mathcal{R}^k)$. Solutions to (6.19), (6.20) are

$$Y_0(\lambda,\tau) = \lambda + z(\tau),$$
$$Y_1(\lambda,\tau) = \int_{\lambda_\star}^{\lambda} \mathcal{R}(\tau,\lambda')(\lambda' + z(\tau))^2 d\lambda'. \qquad (6.21)$$

Before the interaction with the flux, $\lambda < \lambda_\star$

$$Z(\lambda,\tau) \simeq \frac{1}{\lambda} - \frac{z(\tau)}{\lambda^2}. \qquad (6.22)$$

After the interaction, at $\lambda > \lambda_\star + \Delta\lambda$, (6.19), (6.21) imply that

$$Z(\lambda,\tau) \simeq \frac{1}{Y_0} - \frac{Y_1}{Y_0^2} \simeq \frac{1}{\lambda} - \frac{z(\tau) + \Delta z}{\lambda^2}, \qquad (6.23)$$

$$\Delta z(\tau) \simeq \mathcal{R}(\tau,\lambda_\star)\lambda_\star^2 \Delta\lambda. \qquad (6.24)$$

The shift Δz is the gravitational memory left after the interaction with the flux.

In the Einstein gravity, if $\Psi_0 = 0$, then the memory effect is due to $\mathcal{R} = -4\pi G T_{ll}$, where T_{ll} is the null component of the stress-energy tensor of ingoing matter. In this case Δz is real, and the flux does not cause an additional rotation of the string.

VII. SUMMARY

The aim of this paper is to develop a coordinate, parametrization, and basis independent description of null strings. We identified a complex spin coefficient Z, which is a boost-weighted scalar. Z obeys a string optical equation and determines the transformation of the string along its trajectory.

The string diagrams introduced in the Z plane may be a useful tool to study the string trajectories in simple terms. The diagrams are portraits of the strings. Some examples of the diagrams have been presented for strings in flat spacetimes and in conformally flat cosmological models. Constructing diagrams for other physically interesting situations, such as null strings interacting with black holes or null strings in realistic cosmological models, is left for further research.

Null cosmic strings, similarly to tensile strings, may result in a number of observable effects in their environment. The strings create overdensities of matter, additional anisotropy of CMB, etc. All these effects are related to a backreaction of the geometry caused by strings.

In the present paper we were interested in the opposite effects: how geometry affects null strings, and, in particular, asymptotic properties of their trajectories. It follows from the string optical equation that all transformations of the shape of strings in asymptotically flat space-times gradually decay. Strings are "freezing out" as the future null infinity $\mathcal{I}^+$ is approached. The same property holds for null strings in cosmological models of dust or matter dominated flat universes. In asymptotically de Sitter space-times strings stop rotating but stretch accordingly with the cosmological expansion.

Interactions of null strings with the background curvature and flows of matter cause ripples on the strings near $\mathcal{I}^+$. These gravitational memory effects are encoded in the subleading terms of Z. If null cosmic strings are fundamental tensionless strings produced in the early Universe and stretched to cosmological scales the ripples may carry an important information about the Planckian physics. This would be an intriguing feature of null cosmic strings, analogous to features of relic gravitational waves, and would be an interesting future research topic.

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