Cosmological perturbations in Gauss-Bonnet quasi-dilaton massive gravity

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We present the cosmological analysis of the Gauss-Bonnet quasi-dilaton massive gravity theory. This offers a gravitational theory with a nonzero graviton mass. We calculate the complete set of background equations of motion. Also, we obtain the self-accelerating background solutions and we present the constraints on parameters to indicate the correct sign of parameters. In addition, we analyse tensor perturbations and calculate the mass of graviton and find the dispersion relation of gravitational waves for two cases. Finally, we investigate the propagation of gravitational perturbation in the Friedman-Lemaître-Robertson-Walker cosmology in the Gauss-Bonnet quasi-dilaton massive gravity.

I. INTRODUCTION

It is clear that the general theory of relativity has great successes in Solar System tests [1–3] and various astronomical observations [4–6]. However, there remain open questions in gravity, cosmology and particle physics, such as the hierarchy problem [7], the cosmological constant problem [8,9], and the origin of the current accelerated expansion of the Universe [10]. Therefore, there are enough motivations for modifying general relativity. For instance, a modification of general relativity can provide a plausible way to explain the late-time acceleration of the Universe without a dark energy component [11,12].

In the context of modern particle physics, general relativity can be considered as a unique theory of a massless Lorentz-invariant spin-2 particle (i.e., the graviton) in four dimensions [13]. Actually, to find alternative theories to general relativity, we need to break one of the underlying assumptions. One possible way is, breaking Lorentz invariance where these theories contain additional degrees of freedom [14]. Another possible way is, maintaining Lorentz invariance and considering gravity as a representation of a higher spin [15]. Here we consider a valuable alternative theory, the massive gravity theory. In this theory, the gravity is propagated by a spin-2 massive graviton.

The mass of graviton in massive gravity theory determines the speed of gravitational wave propagation. Recently, gravitational waves have been detected from merging of two neutron stars [16]. These events give us an opportunity to have electromagnetic waves besides the gravitational waves. One significance of these observations lays in the fact that the speed of these waves can be compared and can give us the constraints on the modified gravity theories.

It is well known that the massive gravity theory was introduced by Fierz and Pauli in 1939 [17]. They found the unique Lorentz-invariant linear theory without ghost. In the following, the massive gravity theory has undergone tremendous changes throughout decades. The striking changes are discoveries of the van Dam-Veltman-Zakharov (vDVZ) discontinuity [18,19], the Vainshtein mechanism [20], and Boulware-Deser ghost [21]. Eventually, the de Rham-Gabadadze-Tolley (dRGT) theory, which is a fully nonlinear massive gravity without Boulware-Deser ghost, was introduced in 2010 by de Rham, Gabadadze, and Tolley [22,23].

It is expected that the extended massive gravity theories can explain the cosmic acceleration without dark energy. As all homogeneous and isotropic cosmological solutions in dRGT theory are unstable [24], there are two alternative approaches. In the first approach, either homogeneity or isotropy of background can be broken [25–27]. In the second approach, we can consider the extra degrees of freedom such as an extra scalar field or an additional spin-2 field [28–31]. The quasi-dilaton massive gravity theory is classified in the second approach. This theory introduces an extra scalar degree of freedom to the dRGT theory [31]. Meanwhile, it should be pointed out that there are efforts to extend the quasi-dilaton massive gravity theory [32–34]. In this paper, we propose a new extended quasi-dilaton massive gravity theory which is achieved by adding a Gauss-Bonnet term.

Actually, Gauss-Bonnet theory was introduced by Lanczos [35], and Lovelock studied more details of this theory [36]. It is interesting to note that the Gauss-Bonnet

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gravity includes curvature-squared terms which have quadratic order of derivatives with respect to the metric [37,38]. Generally, it can be mentioned that this theory is ghost-free and can solve some problems in general relativity [39,40]. In addition, we point out that Gauss-Bonnet theory arises from the low-energy limit of heterotic string theories [41,42]. It is worth noting that there are valuable investigations which have something to do with the inflation in Gauss-Bonnet theory and cosmological perturbations [43–51]. Actually in Refs. [43,44], the slow-roll inflation with a nonminimally coupled Gauss-Bonner term, was investigated analytically and numerically. They analyzed and constrained their models and results by the 7-year Wilkinson Microwave Anisotropy Probe, Planck and BICEP2 data, respectively.

There has been a trend toward cosmological and perturbation analysis of extended quasi-dilaton massive gravity theories. For example, the cosmological perturbations in extended massive gravity were studied in Ref. [34]; the self-accelerated solutions in quasi-dilaton massive gravity were purposed in Ref. [52]; the stability of self-accelerating solutions in that theory in the presence of matter was investigated in Ref. [53]; other investigations can be found in Refs. [32,54–57].

The goal of this paper is introducing a new extension of quasi-dilaton massive gravity theory which is achieved by adding the Gauss-Bonnet term. In this paper, we introduce the cosmological analysis and tensor perturbation in order to calculate the mass of graviton. Actually, we analyse the constraint on the mass of graviton according to this new action. The paper is organized as follows. In Sec. II we introduce the new action which contains the quasi-dilaton massive gravity and Gauss-Bonnet terms. In following stage, we derive the background equations of motion and self-accelerating solutions elaborately. In Sec. III we perform perturbation analysis for determining the mass of graviton in this theory and we discuss the graviton mass bounds in comparison with gravitational-wave data. Finally, in Sec. IV we conclude with a discussion.

II. COSMOLOGICAL BACKGROUND

In this section, we review the quasi-dilaton dRGT massive gravity theory which is extended by the Gauss-Bonnet term, and we discuss the evolution of a cosmological background. The action includes Planck mass M_{Pl}, the Ricci scalar R, the cosmological constant Λ, a dynamical metric $g_{\mu\nu}$ and its determinant $\sqrt{-g}$. The action is given by

$$S = \frac{M_{\text{Pl}}^2}{2} \int d^4x \left\{ \sqrt{-g} \left[R - 2\Lambda + 2m_g^2 U(\mathcal{K}) \right. \right.$$
$$\left. \left. - \frac{\omega}{M_{\text{Pl}}^2} g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma + \xi(\sigma) G(R) \right] \right\}. \qquad (1)$$

In the following, we introduce two main parts—namely $U(\mathcal{K})$ and $G(R)$—of this action separately.

A. Quasi-dilaton massive gravity term

In the first part, we start out with introducing the quasi-dilaton massive gravity theory which includes the massive graviton term and the quasi-dilaton term [31]. Let us now introduce these two parts as a single theory. It is clear that the mass of graviton comes up with the potential U which consists of three parts.

$$U(\mathcal{K}) = U_2 + \alpha_3 U_3 + \alpha_4 U_4, \qquad (2)$$

where α_3 and α_4 are dimensionless free parameters of the theory. U_i ($i = 2, 3, 4$) is given by,

$$U_2 = \frac{1}{2}([\mathcal{K}]^2 - [\mathcal{K}^2]),$$

$$U_3 = \frac{1}{6}([\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3]),$$

$$U_4 = \frac{1}{24}([\mathcal{K}]^4 - 6[\mathcal{K}]^2[\mathcal{K}^2] + 8[\mathcal{K}][\mathcal{K}^3] + 3[\mathcal{K}^2]^2 - 6[\mathcal{K}^4]),$$
$$\qquad (3)$$

where the quantity "$[\cdot]$" is interpreted as the trace of the tensor inside brackets. It is essential to mention that the building block tensor $\mathcal{K}$ is defined as

$$\mathcal{K}^\mu_\nu = \delta^\mu_\nu - e^{\sigma/M_{\text{Pl}}} \sqrt{g^{\mu\alpha} f_{\alpha\nu}}, \qquad (4)$$

where $f_{\alpha\nu}$ is the fiducial metric, which is defined through

$$f_{\alpha\nu} = \partial_\alpha \phi^c \partial_\nu \phi^d \eta_{cd}. \qquad (5)$$

Here $g^{\mu\nu}$ is the physical metric, η_{cd} is the Minkowski metric with $c, d = 0, 1, 2, 3$ and ϕ^c are the Stueckelberg fields which are introduced to restore general covariance. Also, it is important to note that σ is the quasi-dilaton scalar and ω is a dimensionless constant. Moreover, the theory is invariant under a global dilation transformation, $\sigma \to \sigma + \sigma_0$.

According to our cosmological application purpose, we adopt the Friedman-Lemaître-Robertson-Walker (FLRW) Universe. So, the general expression of the corresponding dynamical and fiducial metrics are given as follows,

$$g_{\mu\nu} = \text{diag}[-N^2, a^2, a^2, a^2], \qquad (6)$$

$$f_{\mu\nu} = \text{diag}[-\dot{f}(t)^2, 1, 1, 1]. \qquad (7)$$

Here it is worth pointing out that N is the lapse function of the dynamical metric, and it is similar to a gauge function. Also, it is clear that the scale factor is represented by a, and $\dot{a}$ is the derivative with respect to time. Furthermore, the lapse function relates the coordinate-time dt and the proper-time $d\tau$ via $d\tau = Ndt$ [58,59]. Function $f(t)$ is the Stueckelberg scalar function whereas $\phi^0 = f(t)$ and $\frac{\partial \phi^0}{\partial t} = \dot{f}(t)$ [60].

Therefore, the Lagrangian of the quasi-dilaton massive gravity in FLRW cosmology is

$$
\mathcal{L}_{QD} = M_{Pl}^2 \left[-3\frac{a\dot{a}^2}{N} - \Lambda a^3 N \right] + m_g^2 M_{Pl}^2 \{ N a^3 (X-1)
$$
$$
\times [3(X-2) - (X-4)(X-1)\alpha_3 - (X-1)^2\alpha_4]
$$
$$
+ \dot{f}(t) a^4 X(X-1)[3 - 3(X-1)\alpha_3 + (X-1)^2\alpha_4] \}
$$
$$
+ \frac{\omega a^3}{2N} \dot{\sigma}^2, \tag{8}
$$

where

$$
X \equiv \frac{e^{\sigma/M_{Pl}}}{a}. \tag{9}
$$

B. Gauss-Bonnet term

Here, we introduce the Gauss-Bonnet term which we add to the quasi-dilaton massive gravity theory. This term consists of the Gauss-Bonnet invariant, $G(R) = R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta} - 4R_{\mu\nu}R^{\mu\nu} + R^2$, and a coupling function $\xi(\sigma)$. It should be noted that in $D = 4$ if we consider ξ as a dimensionless coupling constant instead of a coupling function $\xi(\sigma)$, the Gauss-Bonnet term does not contribute to the gravitational dynamics. The reason of this lays in the fact that the Gauss-Bonnet invariant is a total derivative [61]. So, in this paper, we adopt the coupling function $\xi(\sigma)$ similar to Ref. [62]. Using integration by parts, we can convert the second derivative terms into the first order derivatives. The part of the Lagrangian which is related to the Gauss-Bonnet term is

$$
\mathcal{L}_{GB} = \frac{M_{Pl}^2}{2}\sqrt{-g}\xi(\sigma)G(R) \rightarrow -4M_{Pl}^2\frac{\dot{a}^3}{N^3}\xi'(\sigma)\dot{\sigma}, \tag{10}
$$

where the last expression shows its form in the FLRW background. As a result, the total Lagrangian includes two parts,

$$
\mathcal{L} = \mathcal{L}_{QD} + \mathcal{L}_{GB}. \tag{11}
$$

So, the point-like Lagrangian for cosmology is

$$
\mathcal{L} = M_{Pl}^2 \left[-3\frac{a\dot{a}^2}{N} - \Lambda a^3 N \right] + m_g^2 M_{Pl}^2 [N a^3 (X-1)
$$
$$
\times [3(X-2) - (X-4)(X-1)\alpha_3 - (X-1)^2\alpha_4]
$$
$$
+ \dot{f}(t) a^4 X(X-1)[3 - 3(X-1)\alpha_3 + (X-1)^2\alpha_4]]
$$
$$
+ \frac{\omega a^3}{2N} \dot{\sigma}^2 - 4M_{Pl}^2\frac{\dot{a}^3}{N^3}\xi'(\sigma)\dot{\sigma}. \tag{12}
$$

In order to simplify expressions later, we define

$$
H \equiv \frac{\dot{a}}{Na}. \tag{13}
$$

C. Background equations of motion

In order to achieve a constraint equation we should take the unitary gauge into consideration, which means that we choose $f(t) = t$. The significance of the unitary gauge lays in the fact that on the classical level the unphysical fields could be eliminated from the Lagrangian with use of gauge transformations [63]. In this procedure, a constraint equation can be derived by varying with respect to f. So, that equation is given by

$$
m_g^2 M_{Pl}^2 \frac{d}{dt}[a^4 X(X-1)[3 - 3(X-1)\alpha_3 + (X-1)^2\alpha_4]] = 0. \tag{14}
$$

In this stage, the Friedman equation is achieved by varying with respect to the lapse N,

$$
3H^2 - \Lambda - \frac{\omega}{2}\left(H + \frac{\dot{X}}{XN}\right)^2 - m_g^2(X-1)[-3(X-2)
$$
$$
+ (X-4)(X-1)\alpha_3 + (X-1)^2\alpha_4]
$$
$$
+ 12H^3 M_{Pl}\left(H + \frac{\dot{X}}{XN}\right)\xi'(\sigma) = 0. \tag{15}
$$

The equation of motion for σ is

$$
12M_{Pl}H^2\left(H^2 + \frac{\dot{H}}{N}\right)\xi'(\sigma) - \frac{\omega}{M_{Pl}}\left[3H\frac{\dot{\sigma}}{N} + \frac{1}{N}\frac{d}{dt}\left(\frac{\dot{\sigma}}{N}\right)\right]
$$
$$
+ 3m_g^2 X\bigg\{ (2X - 3 + r(2X - 1))
$$
$$
+ (X-1)\bigg[\alpha_3(3 - X + r(1 - 3X))
$$
$$
- \frac{1}{3}\alpha_4(X-1)(3 + r(1 - 4X))\bigg]\bigg\} = 0, \tag{16}
$$

where

$$
r \equiv \frac{a}{N}. \tag{17}
$$

Using the notation in Eq. (9), the following equations can be derived

$$
\frac{\dot{\sigma}}{NM_{Pl}} = H + \frac{\dot{X}}{NX}, \qquad \frac{\ddot{\sigma}}{M_{Pl}} = \frac{d}{dt}\left(NH + \frac{\dot{X}}{X}\right), \tag{18}
$$

and the last equation of motion could be obtained by varying with respect to the scale factor a,

$$3H^2 - \Lambda + \frac{\omega}{2M_{\rm Pl}^2}\left(\frac{\dot{\sigma}}{N}\right)^2 + 4\frac{d}{dt}\left(H^2\xi'\frac{\ddot{\sigma}}{N^2}\right)$$
$$-4\frac{H^2}{N}\xi'\left[\frac{d}{dt}\left(\frac{\dot{\sigma}}{N}\right) - \left(\frac{\ddot{\sigma}}{N}\right) - 2H\dot{\sigma}\right] + 2\frac{\dot{H}}{N}$$
$$+ m_g^2\{1 + rX(2X - 3) + (X - 1)[X - 5$$
$$- \alpha_3(4 - 2X + rX(X - 3))$$
$$- \alpha_4(X - 1)(rX - 1)]\} = 0. \tag{19}$$

In the last part of this subsection, it should be noted that the Stuckelberg field f introduces time reparametrization invariance. So, there is a Bianchi identity which relates the four equations of motion,

$$\frac{\delta S}{\delta\sigma}\dot{\sigma} + \frac{\delta S}{\delta f}\dot{f} - N\frac{d}{dt}\frac{\delta S}{\delta N} + \dot{a}\frac{\delta S}{\delta a} = 0. \tag{20}$$

So, one equation is redundant and can be eliminated.

D. Self-accelerating background solutions

In this step, we want to discuss solutions. It could be started with the Stueckelberg constraint in Eq. (14). After integrating the equation we have

$$X(X - 1)[3 - 3(X - 1)\alpha_3 + (X - 1)^2\alpha_4] \propto a^{-4}. \tag{21}$$

It would be suitable to mention that the constant solutions of X lead to the effective energy density and behave similar to a cosmological constant. If we consider an expanding universe, according to the a^{-4} behavior in Eq. (21), the right-hand side of that equation will decrease. Therefore, after a long enough time, X leads to a constant value, $X_{\rm SA}$, which is a root of the left-hand side of Eq. (21).

One of the solutions for Eq. (21) is $X = 0$ which leads to $\sigma \to -\infty$. Meanwhile, this solution multiplies to the perturbations of the auxiliary scalars which means that we encounter strong coupling in the vector and scalar sectors. Thus, in order to avoid strong coupling, we discard this solution [31]. So, we are left with,

$$(X - 1)[3 - 3(X - 1)\alpha_3 + (X - 1)^2\alpha_4]|_{X=X_{\rm SA}} = 0. \tag{22}$$

An obvious solution is $X = 1$ which leads to a vanishing cosmological constant and because of inconsistency it is unacceptable. So, this solution should be discarded too [31].

As a result, the two remaining solutions of Eq. (21) are

$$X_{\rm SA}^{\pm} = \frac{3\alpha_3 + 2\alpha_4 \pm \sqrt{9\alpha_3^2 - 12\alpha_4}}{2\alpha_4}. \tag{23}$$

The Friedman equation (15) could be written in a different form,

$$\left(3 - \frac{\omega}{2} + 12M_{\rm Pl}\xi'(\sigma)H^2\right)H^2 = \Lambda + \Lambda_{\rm SA}^{\pm}. \tag{24}$$

Considering self-accelerating solutions, in the case of $\xi'(\sigma) = 0$, a condition on the parameter ω is provided by the Friedman equation (24). So, we need to consider $\omega < 6$ to keep the left hand side of Eq. (24) positive. The importance of this issue lays in the fact that when we add ordinary matters to the right hand side, throughout the matter dominated era, we will have the standard cosmology.

It is worth mentioning that the effective cosmological constant from the mass term is

$$\Lambda_{\rm SA}^{\pm} \equiv m_g^2(X_{\rm SA}^{\pm} - 1)[-3X_{\rm SA}^{\pm} + 6 + (X_{\rm SA}^{\pm} - 4)(X_{\rm SA}^{\pm} - 1)\alpha_3$$
$$+ (X_{\rm SA}^{\pm} - 1)^2\alpha_4]. \tag{25}$$

According to Eq. (23), the above equation can be written as

$$\Lambda_{\rm SA}^{\pm} = \frac{3m_g^2}{2\alpha_4^3}[9\alpha_3^4 \pm 3\alpha_3^3\sqrt{9\alpha_3^2 - 12\alpha_4} - 18\alpha_3^2\alpha_4$$
$$\mp 4\alpha_3\alpha_4\sqrt{9\alpha_3^2 - 12\alpha_4} + 6\alpha_4^2]. \tag{26}$$

Therefore, H^2 is obtained via Eq. (24),

$$H^2 = \frac{1}{24M_{\rm Pl}\xi'(\sigma)}\left\{-\left(3 - \frac{\omega}{2}\right)\right.$$
$$\left.\mp\left[\left(3 - \frac{\omega}{2}\right)^2 + 48M_{\rm Pl}(\Lambda + \Lambda_{\rm SA}^{\pm})\xi'(\sigma)\right]^{\frac{1}{2}}\right\}. \tag{27}$$

Therefore, for the self-accelerating solutions, there are two cases.

In the first case, $\xi'(\sigma)$ is a constant so $\xi''(\sigma)$ is equal to zero. Therefore, from Eq. (16) we have,

$$r_{\rm SA1} = 1 + \frac{H^2[\omega - 4M_{\rm Pl}H^2\xi'(\sigma)]}{m_g^2 X_{\rm SA}^{2\pm}(-2 - \alpha_3 + \alpha_3 X_{\rm SA}^{\pm})}. \tag{28}$$

It is important to note that, in this case, we can consider $\xi'(\sigma) = \xi_0$ where ξ_0 is a constant parameter. In the following, $\bar{\Lambda}$ is redefined as $\bar{\Lambda} = 48M_{\rm Pl}(\Lambda + \Lambda_{\rm SA}^{\pm})\xi_0$. Therefore, in order to keep the right-hand side of Eq. (27) positive, the below conditions should be satisfied in "$\pm$" cases respectively:

(i) In the case of "$-$" sign, we should have $\bar{\Lambda} < 0$ and $\omega \geq 6 + 2\sqrt{-\bar{\Lambda}}$; in other words, ξ_0 has to be negative.

(ii) In the case of "$+$" sign, there are two conditions: (a) it can be considered $\bar{\Lambda} < 0$, $\omega \geq 6 + \sqrt{-\bar{\Lambda}}$, and also ξ_0 should be smaller than zero; (b) if we consider $\bar{\Lambda} \geq 0$, the right-hand side of Eq. (27) is positive for any ω.

In the second case, $\xi'(\sigma)$ is an arbitrary function so $\xi''(\sigma)$ is not zero. In the following, we calculated $\dot{H}$ using Eq. (24) and we substitute it into Eq. (28). As a result, we obtain,

$$r_{\mathrm{SA2}} = 1 + \frac{H^2}{m_g^2 X_{\mathrm{SA}}^{2\pm}(-2-\alpha_3+\alpha_3 X_{\mathrm{SA}}^{\pm})}\left[\omega - 4M_{\mathrm{Pl}}\xi'(\sigma)\left(H^2 + \frac{12H^4 M_{\mathrm{Pl}}^2\xi''(\sigma)}{-6+\omega-48H^2 M_{\mathrm{Pl}}\xi'(\sigma)}\right)\right]. \tag{29}$$

Actually, we have used the Stuckelberg equation (21) in order to eliminate α_4. Finally, we should take this into account, and if we consider $\xi(\sigma) = 0$, r_{SA1} and r_{SA2} convert to the equation in Ref. [34] in its unexpanded form.

III. TENSOR PERTURBATION

In this section, we would like to analyse tensor perturbation in order to calculate the mass of graviton for our theory which we introduced in the previous section.

In order to find the action for quadratic perturbation, the physical metric is expanded in small fluctuation, $\delta g_{\mu\nu}$, around a solution $g_{\mu\nu}^{(0)}$,

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + \delta g_{\mu\nu}. \tag{30}$$

In the following analysis, we keep terms to quadratic order in $\delta g_{\mu\nu}$. As we demonstrate all analysis in the unitary gauge, there are not any problems concerning the form of gauge invariant combinations. Moreover, we write the actions expanded in the Fourier domain with plane waves, i.e., $\vec{\nabla}^2 \to -k^2$, $d^3x \to d^3k$. We raise and lower the spatial indices on perturbations by δ^{ij} and δ_{ij}. It should be mentioned that we would like to consider $N = 1$ which means that the derivatives are with respect to time.

We start by considering tensor perturbations around the background,

$$\delta g_{ij} = a^2 h_{ij}^{\mathrm{TT}}, \tag{31}$$

where

$$\partial^i h_{ij} = 0 \quad \text{and} \quad g^{ij}h_{ij} = 0. \tag{32}$$

Therefore, the action quadratic in h_{ij} is

$$S = \frac{M_{\mathrm{Pl}}^2}{8}\int d^3kdta^3\left[(1-4H\xi'(\sigma))\dot{h}^{ij}\dot{h}_{ij}\right.$$
$$\left. - \left(\frac{k^2}{a^2}[1-4\xi''(\sigma)]+M_{\mathrm{GW}}^2\right)h^{ij}h_{ij}\right]. \tag{33}$$

As we have r_{SA1} and r_{SA2}, in order to calculate the dispersion relation of gravitational waves we have two cases. In the first case, we obtain α_3 using Eq. (28) and α_4 using Eq. (23). So, in this case, the dispersion relation of gravitational waves is

$$M_{\mathrm{GW}_1}^2 = 4\dot{H} + 6H^2 - 2\Lambda - 16\xi_0 H(\dot{H}+H^2)$$
$$+ \omega H^2 + \Upsilon_1, \tag{34}$$

where

$$\Upsilon_1 = \frac{1}{(r_{\mathrm{SA1}}-1)(X_{\mathrm{SA}}^{\pm}-1)(X_{\mathrm{SA}}^{\pm})^2}\{\omega H^2[X_{\mathrm{SA}}^{\pm}(X_{\mathrm{SA}}^{\pm}-3)(r_{\mathrm{SA1}}X_{\mathrm{SA}}^{\pm}-2)-2]$$
$$+ m_g^2(r_{\mathrm{SA1}}-1)X_{\mathrm{SA}}^{\pm}[X_{\mathrm{SA}}^{\pm}(6+X_{\mathrm{SA}}^{\pm}[X_{\mathrm{SA}}^{\pm}(1+r_{\mathrm{SA1}})-6])-2] + 4H^4 M_{\mathrm{Pl}}\xi_0[2-X_{\mathrm{SA}}^{\pm}(X_{\mathrm{SA}}^{\pm}-3)(r_{\mathrm{SA1}}X_{\mathrm{SA}}^{\pm}-2)]\}. \tag{35}$$

In the second case, α_3 can be gotten from Eq. (29), and similar to the last case we obtain α_4 from Eq. (23). Therefore, the dispersion relation of gravitational waves is obtained for the second case,

$$M_{\mathrm{GW}_2}^2 = 4\dot{H} + 6H^2 - 2\Lambda + 8\xi''(\sigma)H^2 - 16\xi'(\sigma)H(\dot{H}+H^2) + \omega H^2 + \Upsilon_2, \tag{36}$$

where

$$\Upsilon_2 = \frac{1}{(r_{\mathrm{SA2}}-1)(X_{\mathrm{SA}}^{\pm}-1)(X_{\mathrm{SA}}^{\pm})^2}\frac{1}{\omega-48H^2M_{\mathrm{Pl}}\xi'(\sigma)-6}\{m_g^2(\omega-6)(r_{\mathrm{SA2}}-1)X_{\mathrm{SA}}^{\pm}[X_{\mathrm{SA}}^{\pm}[6+X_{\mathrm{SA}}^{\pm}(X_{\mathrm{SA}}^{\pm}(r_{\mathrm{SA2}}+1)-6)]-2]$$
$$- 4(13\omega-6)H^4 M_{\mathrm{Pl}}\xi'(\sigma)[X_{\mathrm{SA}}^{\pm}(X_{\mathrm{SA}}^{\pm}-3)(r_{\mathrm{SA2}}X_{\mathrm{SA}}^{\pm}-2)-2] + H^2[\omega(\omega-6)(X_{\mathrm{SA}}^{\pm}(X_{\mathrm{SA}}^{\pm}-3)(r_{\mathrm{SA2}}X_{\mathrm{SA}}^{\pm}-2)-2)$$
$$- 48m_g^2 M_{\mathrm{Pl}}\xi'(\sigma)(X_{\mathrm{SA}}^{\pm})^2(r_{\mathrm{SA2}}-1)(X_{\mathrm{SA}}^{\pm}[X_{\mathrm{SA}}^{\pm}(r_{\mathrm{SA2}}+1)-6]-2)]$$
$$- 48H^6 M_{\mathrm{Pl}}^2\xi'(\sigma)(M_{\mathrm{Pl}}\xi''(\sigma)-4\xi'(\sigma))[X_{\mathrm{SA}}^{\pm}(r_{\mathrm{SA2}}X_{\mathrm{SA}}^{\pm}-2)(X_{\mathrm{SA}}^{\pm}-3)-2]\}. \tag{37}$$

As we mentioned before, we eliminate α_3 and α_4 using Eq. (23), and Eqs. (28)–(29). It can be pointed out that if the mass square of gravitational waves is positive, the stability of long-wavelength gravitational waves is guaranteed. On the other hand, if it is negative, it should be tachyonic. Therefore, as the mass of tachyon is of the order of Hubble scale, the instability should take the age of the Universe to develop.

The main results of this section are the modified dispersion relations of gravitational waves, given in Eq. (34) and Eq. (36). They represent the propagation of gravitational perturbations in the FLRW cosmology in the Gauss-Bonnet quasi-dilaton massive gravity. In principle, the propagation can be tested with cosmological events, notably by gravitational wave observations. These modifications will introduce extra contribution to the phase evolution of gravitational waveform [64,65], and to be detected with the accurate matched-filtering techniques in the data analysis.

After the first discovery of gravitational waves in a merging binary black hole (the so-called GW150914) by the LIGO/Virgo Collaboration, tests of graviton mass are revived [66–69]. The latest constraint on the graviton mass from the combination of gravitational wave events from the first and second gravitational wave transient catalogs is $m_g \leq 1.76 \times 10^{-23}$ eV/c^2 at 90% credibility [68]. The corresponding Compton wavelength is still much smaller than the Hubble scale, thus the relevance to modified cosmology is restricted at present. Nevertheless, with future prospects in mind, we shall keep testing this important aspect of gravitation with more and more gravitational events at different wavelengths, notably with future space-based gravitational-wave detectors which are more sensitive to the graviton mass [64].

In particular, modified propagation of gravitational waves in the cosmological setting was investigated by Nishizawa and Arai in a parametrized framework, considering a running Planck mass, a modified speed for gravitation, as well as anisotropic source terms [70–72]. Our results here, representing theory-specific analysis in Gauss-Bonnet quasi-dilaton massive gravity, are complementary to their work, and will provide a target for future detailed analysis.

IV. CONCLUSION

In this work, we have presented a new extension of quasi-dilaton massive gravity theory which is constructed by adding the Gauss-Bonnet term. As the quasi-dilaton massive gravity and its extensions have a rich phenomenology, we have been motivated to investigate some cosmological analysis of Gauss-Bonnet quasi-dilaton massive gravity.

At the first, we have introduced the details of the new action and total Lagrangian. We also presented the full set of equations of motion for a FLRW background. Notice that the investigation of extended massive gravity is important in order to understand the late-time acceleration of the Universe. Therefore, we have discussed the self-accelerating background solutions elaborately. We have provided a way to explain the late-time acceleration of the Universe within the Gauss-Bonnet quasi-dilaton massive gravity.

To study the mass of graviton for the Gauss-Bonnet quasi-dilaton massive gravity theory, we have presented the tensor perturbation calculation and have shown the dispersion relation of gravitational waves for two cases. In other words, we have represented the propagation of gravitational perturbation in the FLRW cosmology in the Gauss-Bonnet quasi-dilaton massive gravity. Such an analysis will be a useful addition to probe alternative gravity theories in the era of gravitational waves. In addition, a detailed direct comparison with observational data (e.g., from type Ia supernovae and the cosmic microwave background) for the late-time acceleration of the Universe will be extremely interesting to check for the valid parameter space of the Gauss-Bonnet quasi-dilaton massive gravity theory in this work. It will also be useful for an insightful comparison with the canonical ΛCDM cosmology model. However, such a statistical study dedicated to data analysis is beyond the scope of the current paper, thus we leave it for future study.

At the end, we think that other possible extensions of quasi-dilaton massive gravity theory can be considered for future investigations.

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