

Scenario for a singularity-free generic cosmological solution

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We develop a scenario for the emergence of a nonsingular generic cosmological solution based on the WKB characterization of one of the two anisotropy degrees of freedom. We investigate the dynamics of the so-called inhomogeneous mixmaster in the “corner” configuration and inferring that one of the two anisotropic variables becomes small enough to explore the uncertainty principle. Then, we apply a standard WKB approximation to the dynamics of the Universe which has macroscopic volume, one macroscopic anisotropy and one microscopic quantum degree of freedom. Our study demonstrates the possibility that the Universe acquires a nonsingular classical behavior, retaining the quantum degree of freedom as a small oscillating ripple on a stationary Universe. The role of the so-called fragmentation process is also taken into account in outlining the generality of such a behavior in independent local space regions.

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I. INTRODUCTION

One of the most important contributions of the Landau school to theoretical cosmology consisted of the dynamical characterization of the generic cosmological solution in the vicinity of the primordial singularity [1–3]. These studies, together with the general theorem derived by Hawking and Penrose [4], allowed us to understand that the presence of a singular point in the past of our actual Universe should be regarded as a general feature of the Einsteinian cosmology, not induced by the high symmetry of the isotropic Universe geometry.

In 1963, Khalatanikov and Lifshitz published a paper [5] which, apart from the Lifshitz investigations on the isotropic Universe stability, contains a relevant analysis about relativistic cosmology in a general framework.

In particular, they derived the so-called generalized Kasner solution, i.e., the inhomogeneous extension of the Kasner solution, which describes the dynamics of the Bianchi I model [3,6,7]. They concluded that the asymptotic behavior of a generic inhomogeneous Universe toward the singularity is Kasner-like. However this conclusion was only partially correct. In fact, if on one hand the generalized Kasner solution could possess the right number of four physically independent space functions, which are required for dealing in vacuum in the general case; on the other hand, in order to survive up to the initial singularity, an inhomogeneous Kasner regime needs the imposition of an additional restriction, therefore losing its general character.

The subsequent studies in [1] (for a detailed review see [3,8]), about the Bianchi VIII and IX models, clarified how the asymptotic regime to the singularity requires an infinite

sequence of Kasner regimes (called Kasner epochs), in which parameters are related by a map having stochastic properties. This picture was translated into an Hamiltonian formulation by Misner in [9]. For a detailed discussion of the link existing between the Belinskii-Khalatnikov-Lifshitz (BKL) map between two Kasner epochs and the Hamiltonian formulation in the so-called Misner-Chitré-like variables, see [10]. Misner called this Hamiltonian formulation of the original oscillatory regime, presented in [1], the “mixmaster universe” (for a covariant characterization of the mixmaster chaos see [11]).

This idea of an infinite sequence of Kasner regimes toward the cosmological singularity was then implemented to the asymptotic dynamics of a generic inhomogeneous model in [2], see also [3] for a detailed reanalysis of this scenario. This work completed the investigation in [5], by precisizing the original statement: the generic inhomogeneous cosmological solution approaches the cosmological singularity via an infinite sequence of Kasner regimes related, point by point in space, via a stochastic map.

However, in [2], the inhomogeneous dynamics was described assuming the existence of a single relevant spatial scale of inhomogeneity and the standard time evolution, associated to the oscillatory regime, was recovered on a smaller spatial scale, roughly identified with the average horizon size. However in [12,13] it was shown that the coupling between the space and time dependence of the metric tensor implies that smaller and smaller inhomogeneous scales are generated approaching the singularity, see [14] for a discussion of the impact that such a phenomenon can have on the primordial Universe turbulence, see also [15].

In [16] it was demonstrated (see [3] for a simplified discussion) that the spatial gradients growth cannot destroy the standard oscillatory regime because they grow slowly (in a logarithmic way) with respect to the terms which induce the instability of a Kasner regime and the transition to a new one.

However, more recent studies, see [17,18], demonstrated, mainly on a numerical ground, the emergence of real spikes in the spatial gradients, which put doubts on the nature of the generic inhomogeneous mixmaster, as the inhomogeneous extension of the oscillatory regime is commonly dubbed.

On how to reconcile the generic mixmaster universe with the highly symmetric isotropic model, at least on a local spatial regime, see [19] where the role of an inflationary regime is modeled via the effect of a massless scalar field plus a cosmological constant.

All these studies seem however to claim that the cosmological singularity is clearly present in the generic inhomogeneous solution, as described in the Einsteinian picture. Canonical quantum gravity in the metric approach seems unable to significantly change this situation, see the original work of Misner [20] or the more recent analysis in [21]. The situation is different if the canonical quantization scheme is reformulated in loop quantum cosmology, see for instance the discussion in [22]. A singularity-free generic cosmological solution has been constructed in [23], where the semiclassical Polymer dynamics (to be thought as the quasiclassical behavior of loop quantum cosmology) is considered for the evolution toward the singularity. For other approaches in extended theories of gravity, able to induce a bounce cosmology, see [24–26].

Here, the possibility for a singularity-free inhomogeneous mixmaster is based on a different scenario, in which the behavior of the Universe during the so-called long era is examined.¹ We investigate the possibility that, when such a configuration is addressed (according to the analysis in [27], see also [28] where the analysis was generalized and partially corrected), one of the two anisotropic degrees of freedom is small enough to approach a quantum behavior since it can explore the uncertainty principle in its own phase space. Then, we apply the WKB scenario proposed in [29] and we demonstrate that the resulting Universe is a classical nonsingular one, plus a small oscillating quantum anisotropy.

Using the language of the standard Hamiltonian formulation we outline how, when the Universe performs a long era in the corner of the potential term, a separation takes place between classical macroscopic components of the inhomogeneous mixmaster and a small quantum subset, made up of one of the two anisotropic degrees of freedom.

¹The “long era” is intended as the Kasner era that the Universe undergoes when it is in the corner configuration.

The assessment of this scenario will rely on two main well-known results:

- (i) In [27,28] it has been shown that, studying the statistical properties of the BKL map, there always exists a significant probability that the parameter u , characterizing a Kasner regime, acquires values large enough for the system dynamics to be deeply trapped in a corner configuration (where one of the two anisotropy degrees of freedom is very small, and when the Universe dynamics involve two close potential walls of the space curvature only).
- (ii) The existence of the fragmentation process discussed in [13]. According to this, rational values taken by the function $u(x^i)^2$ across space cannot be excluded from the evolution of the BKL map, therefore even a few steps of the BKL map ensure the existence of large values of u in the neighborhood of certain space surface.

We observe how, while the result (i) has a statistical character, being associated to the asymptotic iteration of the BKL map, the result in (ii) can also be guaranteed by a finite deterministic implementation of the BKL map across the space.

Our analysis is developed toward the singularity, but we can consider a time reversed picture which is able to connect the standard inhomogeneous mixmaster to a primordial nonsingular generic solution as soon as the small quantum anisotropy degree of freedom is able to become a classical variable, i.e., as soon as the Universe escapes the corner.

II. INHOMOGENEOUS MIXMASTER

In the ADM formalism, the line element of a generic inhomogeneous cosmological model, described by Misner variables α , β_+ and β_- , reads as

$$ds^2 = N^2 dt^2 - h_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \quad (1)$$

with

$$h_{ij} = e^{2\alpha}(e^{2\beta})_{ab}\ell_i^a\ell_j^b, \\ \beta = \text{diag}\{\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+\}. \quad (2)$$

Here, N denotes the lapse function and N^i the shift vector (these, together with the Misner variables, are space-time functions), while the vectors \vec{l}^a ($a = 1, 2, 3$) are linearly independent and they have generic space-dependent components. It has been shown in [2] that the time dependence of the vectors \vec{l}^a is dynamically of higher order

²Actually, the Kasner metric scale factors are, in the synchronous time, power laws, which three exponents are constrained by two conditions and hence expressed by a single parameter u (here being a function of the space coordinates, due to inhomogeneity).

and it is associated with their rotation in space. In the following we assume $8\pi G = 1$.

The action associated to this generic model takes the following Hamiltonian representation [3,16]:

$$S_G = \int dt d^3x \{ p_\alpha \partial_t \alpha + p_+ \partial_t \beta_+ + p_- \partial_t \beta_- - N \mathcal{H} - N^i \mathcal{H}_i \}, \quad (3)$$

p_α , p_+ and p_- are the conjugate momenta to α , β_+ and β_- , respectively. The super-Hamiltonian \mathcal{H} admits the simplified expression:

$$\mathcal{H} = \frac{1}{12} e^{-3\alpha} \{ -p_\alpha^2 + p_+^2 + p_-^2 + e^{4\alpha} V_G(\beta_+, \beta_-) \}, \quad (4)$$

where the potential term V_G is obtained neglecting all the spatial gradients of the Misner variables in the spatial curvature. On a classical level, this approximation is justified *a posteriori* by demonstrating that such gradients increase slowly towards the singularity with respect to the time derivatives of the configurational variables [16]. This scenario leads to the so-called inhomogeneous mixmaster model, i.e., within each smooth spatial scale (roughly the horizon scale), the dynamics is isomorphic to that one of a homogeneous mixmaster [1,20]. However in [12,13,16], it has been shown that, in the inhomogeneous mixmaster, the chaotic time evolution couples to the spatial dependence and increasingly small scales are generated for the space variation of the Misner variables, but without destroying the dynamical scheme of infinite sequence of Kasner regimes.

The classical dynamics of a generic cosmological models is described by the Hamilton equations associated to the Misner variables and by the constraints obtained varying the action S_G respect to N and N^i , namely:

$$\mathcal{H} = \mathcal{H}_i = 0. \quad (5)$$

In the inhomogeneous mixmaster approximation, the supermomentum constraint reduces to the following dominant contribution:

$$p_\alpha \partial_i \alpha + p_+ \partial_i \beta_+ + p_- \partial_i \beta_- = 0. \quad (6)$$

This constraint is consistent with the scalar nature of the Misner variables under reparametrization of the spatial coordinates, which acts on the vectors \vec{l}^a only.

We conclude this dynamical picture assigning the explicit expression for the potential term V_G , namely:

$$V_G = \frac{1}{4} (C_1^2 e^{4\beta_+ + 4\sqrt{3}\beta_-} + C_2^2 e^{4\beta_+ - 4\sqrt{3}\beta_-} + C_3^2 e^{-8\beta_+}) - \frac{1}{2} (C_1 C_2 e^{4\beta_+} - C_1 C_3 e^{-2\beta_+ + 2\sqrt{3}\beta_-} - C_2 C_3 e^{-2\beta_+ - 2\sqrt{3}\beta_-}). \quad (7)$$

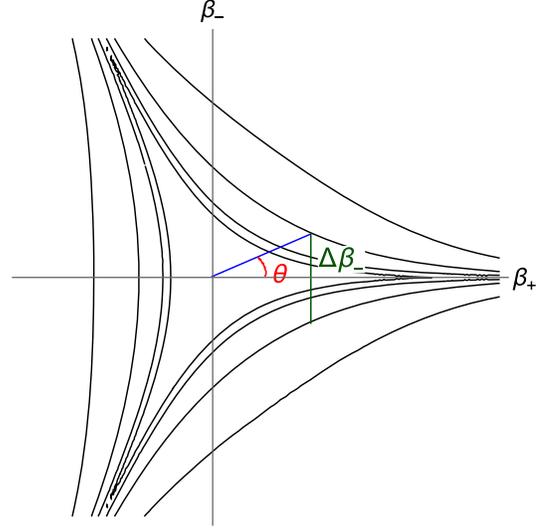


FIG. 1. Equilateral triangle formed by the equipotential lines in each space point. The segment $\Delta\beta_-$ is in green, while the angle θ is represented in red.

Above, the generic functions $C_a(x^i)$ defining the inhomogeneity character of the cosmological model, can be expressed via the vectors \vec{l}^a as $C_a \equiv \vec{l}^a \cdot \text{rot} \vec{l}^a$ (expressions to be intended in Euclidean sense with respect to the coordinates x^i and the vector components l_i^a).

The equipotential lines associated to this potential form, in each space point a curvilinear equilateral triangle (Fig. 1), having three open corners reaching infinity. Here, we will focus our analysis on the system dynamics when the interior of one of these corners is considered.

The three corners are equivalent, this can be shown simply rotating the coordinate plane $\{\beta_+, \beta_-\}$ by $\pi/3$ to map one corner into another. Therefore without loss of generality we consider the corner along the axis $\beta_- = 0$.

From a geometrical point of view, the corner configuration corresponds to deal with two space directions scaling essentially with the same oscillating time law, while the remaining one decays monotonically toward the singularity. It is worth noticing that, on a classical level, an inhomogeneous mixmaster scheme is well established [13,14,16], apart from the emergence of spikes in the spatial gradients [17,18], while, on a quantum level, it stands as an ansatz to be validated *a posteriori* and it is commonly referred as the BKL conjecture.

A. Generalized Kasner solution

If the initial singularity is identified with the instant of time when the spatial volume of the Universe (i.e., the three-metric determinant) vanishes, then we can fix that singularity with the limiting value $\alpha \rightarrow -\infty$.

In such a limit the potential V_G tends to become an infinite well, in which center $\beta_+ \sim \beta_- \sim 0$ (actually for an increasing region as the singularity is approached) the

potential term can be neglected and the generic inhomogeneous Universe is described by the so-called *generalized Kasner solution* [5].

It is immediate to recognize that, when V_G is negligible, the momenta p_α and the two p_\pm are all constant in time and the following relations, obtained making use of the first Hamilton equations, hold

$$\frac{d\beta_\pm}{d\alpha} = \frac{p_\pm}{p_\alpha} \equiv \pi_\pm(x^i), \quad \Rightarrow \beta_\pm = \pi_\pm(x^i)\alpha + \bar{\beta}_\pm(x^i), \quad (8)$$

where $\bar{\beta}_\pm$ denote generic space functions.

Since the functions π_\pm must satisfy, by definition, the relation $\pi_+^2 + \pi_-^2 = 1$, we can set $\pi_+ = \cos\theta$ and $\pi_- = \sin\theta$. The function $\theta(x^i)$ changes at each bounce against the potential walls and it acquires a random behavior. Therefore the system can reach a configuration deeply in the corner $\beta_- = 0$, if $\sin\theta \simeq \theta \simeq \epsilon \ll 1$.

III. QUANTUM SMALL OSCILLATIONS

Let us now investigate more in detail the structure of the generic inhomogeneous model Hamiltonian in the corner configuration.

If we choose a space coordinate system \bar{x}^i , such that $C_1(\bar{x}^i) = C_2(\bar{x}^i) \equiv C(\bar{x}^i)$, then, the super-Hamiltonian constraint reads, inside the corner ($\{\beta_+ \gg 1, \beta_- \ll 1\}$), as

$$-p_\alpha^2 + p_+^2 + \mathcal{H}^- = 0, \quad (9)$$

where \mathcal{H}^- is a small contribution and is defined as

$$\mathcal{H}^- \equiv p_-^2 + 6C^2 e^{A(\alpha+\beta_+)} \beta_-^2. \quad (10)$$

At a fixed α value, the coordinate interval for the variable β_- in the corner is of the order $\Delta\beta_- \sim 2\beta_+\theta = \beta_+\epsilon$. as shown in Fig. 1. Furthermore, according to the generalized Kasner solution (i.e., comparing the kinetic and the potential term in \mathcal{H}^-), we get

$$\frac{\mathcal{H}^-}{p_\alpha^2} \sim \frac{\mathcal{H}^-}{p_+^2} \sim \pi_-^2 \sim \theta^2 = \epsilon^2, \quad (11)$$

where, in the first part we used that, from the super-Hamiltonian constraint applied to the semiclassical Misner variables, $p_\alpha^2 = p_+^2$.

If the BKL map generates a small value of ϵ of order $\sqrt{\hbar}$ (here we disregard the physical dimensions of \hbar to avoid the use of two small parameters, one physical and one dimensionless), then the Hamiltonian constraint (9) can be decoupled, according to the analysis in [29], into a classical part, associated to the variables α and β_+ plus a quantum small subsystem, constituted by the anisotropy degree of freedom β_- , which lives in space-time defined by the classical components. By other words, we are inferring that the variable β_- is enough small to explore the uncertainty

principle with $\Delta\beta_- \leq 2\sqrt{\hbar}\beta_+$ and $\Delta p_- \geq 2\sqrt{\hbar}/\beta_+$. The quantum subsystem shows to possess the ‘‘smallness’’ requirement postulated in [29] and precised in [30]. Under the hypotheses above, the Universe state functional can be written as follows

$$\Psi = \exp\{i\Sigma(\alpha, \beta_+)/\hbar\} \Phi(\alpha, \beta_+, \beta_-), \quad (12)$$

where Σ is associated to the classical system, while Φ describes the quantum subcomponent. According to the scheme developed by [29], the functional derivative of Φ with respect to the space field $\beta_-(x^i)$ are of order of $1/\sqrt{\hbar}$, therefore $\mathcal{H}^- \Phi \propto \mathcal{O}(\hbar)$.

To obtain the dynamical implications of the state function (12), we need to apply the canonical operator version of the constraint (9) and of the super-momentum constraint (6), i.e.,

$$\left[\hbar^2 \frac{\delta^2}{\delta\alpha^2} - \hbar^2 \frac{\delta^2}{\delta\beta_+^2} + \hat{\mathcal{H}}^- \right] \Psi = 0, \quad (13)$$

$$i\hbar \left(\partial_i \alpha \frac{\delta}{\delta\alpha} + \partial_i \beta_+ \frac{\delta}{\delta\beta_+} + \partial_i \beta_- \frac{\delta}{\delta\beta_-} \right) \Psi = 0, \quad (14)$$

where the symbol δ denotes functional derivatives.

At the zero approximation order in \hbar we get the classical Hamilton-Jacobi super-Hamiltonian and supermomentum equations for the variables α and β_+ : i.e., the following system of functional differential equations

$$-\left(\frac{\delta\Sigma}{\delta\alpha} \right)^2 + \left(\frac{\delta\Sigma}{\delta\beta_+} \right)^2 = 0, \quad (15)$$

$$\frac{\delta\Sigma}{\delta\alpha} \partial_i \alpha + \frac{\delta\Sigma}{\delta\beta_+} \partial_i \beta_+ = 0. \quad (16)$$

In other words, the classical component is associated to the reduced action

$$S_{\text{Class}} = \int dt d^3x \left\{ p_\alpha \partial_t \alpha + p_+ \partial_t \beta_+ - \frac{N}{12} e^{-3\alpha} (-p_\alpha^2 + p_+^2) - N^i (p_\alpha \partial_i \alpha + p_+ \partial_i \beta_+) \right\}. \quad (17)$$

By a simple algebra, it is possible to show that the quantum functional Φ obeys the equation

$$i\hbar \partial_t \Phi = \left\{ \int d^3x \frac{N}{12} e^{-3\alpha} \left(\hat{\mathcal{H}}^- \Phi + i\hbar \left(\frac{\delta^2 \Sigma}{\delta\alpha^2} - \frac{\delta^2 \Sigma}{\delta\beta_+^2} \right) \Phi \right) \right\}, \quad (18)$$

where

$$\partial_t \Phi \equiv \int d^3x \left\{ \left(\partial_t \alpha \frac{\delta}{\delta \alpha} + \partial_t \beta_+ \frac{\delta}{\delta \beta_+} \right) \Phi \right\}, \quad (19)$$

$\partial_t \alpha$ and $\partial_t \beta_+$ being calculated from the action (17) and via the identification of the momenta with the corresponding functional derivatives of Σ . To derive (18), we made also use of the semiclassical part of the order \hbar of the supermomentum constraint (14), i.e.,

$$\partial_t \alpha \frac{\delta \Phi}{\delta \alpha} + \partial_t \beta_+ \frac{\delta \Phi}{\delta \beta_+} = 0, \quad (20)$$

which states the invariance of the wave functional Φ with respect to the space coordinates in the classical line element.

The present analysis differs from the approach presented in [29] (see also [31–33]) because we are dealing with a functional formalism, due to the inhomogeneity of the considered model, and we are taking the variables α and β_+ as strictly classical. This last difference results in the last term in parentheses of Eq. (18).

It is immediate to check that Eq. (15) admits the following solution:

$$\Sigma = \int d^3x K(x^i) (\alpha + \beta_+), \quad (21)$$

which, according to the Hamilton-Jacobi method, yields the classical relation

$$\alpha + \beta_+ = \beta_0(x^i), \quad (22)$$

which, substituted in (10), leads to

$$\hat{\mathcal{H}}^- = p_-^2 + 6C^2 e^{4\beta_0} \beta_-^2. \quad (23)$$

In order for the solution (21) to satisfy the supermomentum equation (16), it is enough to require $\beta_0 = \text{const}$.

From the classical action (17), we recognize the following relation between the variable α and the synchronous time T :

$$\alpha = \frac{1}{3} \ln \frac{T}{T_0}, \quad (24)$$

where T_0 is a generic instant.

Choosing, without loss of generality, the vector \vec{l}^3 along the coordinate x^3 , the classical solution above makes the line element (1) of the form

$$ds^2 = dT^2 - \left(\frac{T}{T_0} \right)^2 e^{-4\beta_0} (dx^3)^2 - (dl_2)^2, \quad (25)$$

where $(dl_2)^2$ is a static two-dimensional line element on the plane $\{x^1, x^2\}$. As is well known [1,2], the line element

above is associated to a nonsingular cosmological model, and it becomes static as soon as we pass to new coordinates $T' = (T/T_0) \cosh x^3$ and $x'^3 = (T/T_0) \sinh x^3$. Actually, when u is an integer, the BKL map induces the value “ $u = \infty$,” corresponding to $\beta_- \equiv 0$, de facto a nonsingular Taub model in the homogeneous sector. Here β_- is zero on the nonsingular classical metric, surviving as a negligible quantum fluctuation only.

Using the expression (21) of Σ and introducing the time variable τ defined via the lapse function $N = 12e^{3\alpha}$, the quantum functional equation (18) reduces to the form

$$i\hbar \partial_\tau \Phi = \int d^3x \hat{\mathcal{H}}^- \Phi, \quad (26)$$

with $\hat{\mathcal{H}}^-$ defined in (23).

The dynamical decoupling of the space points, i.e., of each space region sufficiently smooth (so that spatial gradients are negligible), allows to reduce the superspace to the collection of local minisuperspace, each for each point x^i . Thus, we can write:

$$\Phi = \prod_{x^i} \phi_{x^i}(\tau, \beta_-), \quad (27)$$

where the local wave functions ϕ_{x^i} satisfy the equations

$$i\hbar \partial_\tau \phi_{x^i} = \{-\hbar^2 \partial_{\beta_-}^2 + 6C^2(x^i) e^{4\beta_0} \beta_-^2\} \phi_{x^i}. \quad (28)$$

The functional Φ must also satisfy the quantum component of the supermomentum constraint, i.e.:

$$-i\hbar \frac{\delta \Phi}{\delta \beta_-} \partial_i \beta_- = 0. \quad (29)$$

However, when we take the functional Φ in the factorized form (27), we are inferring that it is naturally satisfying Eq. (29), simply because that approximation corresponds to deal locally with the condition $\partial_i \beta_- \simeq 0$. Here, we are implementing the BKL conjecture, based on the idea that the scale of spatial gradients is larger than the quantum correlation length. In this sense, we are reintroducing the concept of “quantum causality”: space regions that evolve independently are not in causal contact.

Equation (27) has the morphology of quantum harmonic oscillators in each space point and it is well known that localized nonspreading states can be always constructed. We expect that the variable β_- can be represented by localized state because when it enters the corner is a classical degree of freedom and its available domain remains of the order $\hbar \beta_+$ in that configuration.

Thus, we can conclude that, if our scheme is reliably applicable to the Universe dynamics deeply entering the corner, the cosmological singularity is removed because we get a classical nonsingular space-time on which very small quantum fluctuations of the variable β_- live. Such an

intriguing picture is well-established when it is referred to in a given spatial point (causal region), but to understand how it works in the Universe as a whole, we need to develop some considerations on the BKL map [1,2] and on the so-called fragmentation process [13] (see also [12]).

We conclude this section by emphasizing that the picture proposed above can be reversed in time and we could start with a nonsingular classical Universe with a small quantum anisotropy and, as the space volume increases (i.e., α increases), this degree of freedom becomes classical, so that the dynamics comes out of the corner configuration and the full configurational domain is restored. By other words, in this scenario, the generic inhomogeneous cosmological solution can emerge from a nonsingular initial configuration and then evolves toward the standard oscillatory regime discussed in [1,2].

IV. INHOMOGENEOUS BKL MAP

If we introduce the parameter $u(x^i)$ [2], then the quantities $\pi_+(x^i)$ and $\pi_-(x^i)$ take the following explicit form:

$$\pi_+ = \frac{u^2 + u - 1/2}{u^2 + u + 1}, \quad \pi_- = \frac{\sqrt{3}}{2} \frac{2u + 1}{1 + u + u^2}. \quad (30)$$

In the present context, we can restrict these expressions to $u \gg 1$ since the considered corner corresponds to very large values of u (note that $\theta \simeq \sqrt{3}/u$). To understand when such large values appear, we have to consider the BKL map [1] which provides the value u' generated from the initial value u via the effect of the potential wall in the standard oscillatory regime, i.e.:

$$\text{for } u > 1u' = u - 1, \quad \text{for } u \leq 1u' = \frac{1}{u - 1}. \quad (31)$$

All the initial rational values of the parameter $u = u^0$ are evolved for a finite number of the BKL map steps, after which the value $u \rightarrow \infty$ (i.e., $\theta = 0$) is recovered. Instead, the irrational values of the initial parameter u^0 evolve indefinitely and the BKL map outlines a strong (exponential) instability with respect to the initial condition: if we modify the value u^0 by a very small amount, the sequence of values generated by the map iteration becomes uncorrelated with respect to the sequence associated to u^0 after a few steps. We stress that the rational values of u were excluded in the original analysis in [1,2], because, being of zero measure on the real axis, they turn out to be a nongeneral initial condition. However, if we assign, over the inhomogeneous space, the initial condition $u = u^0(x^i)$, the rational values cannot clearly be excluded simply for continuity reasons. Thus, each spatial region containing surfaces on which u is rational enters deeply into the corner after a certain number of iterations of the map and our

scenario can be implemented close enough to one of such regions.

Actually, when the parameter u is thought as a physical parameter, we have to assign its values with a given uncertainty, even because the Kasner solution to which it is associated is an approximate regime obtained by neglecting the potential walls. This consideration, together with the instability of the BKL map, leads to think of u as a statistically distributed variable and it can be shown that it admits the following steady probability density [13]

$$w(u) = \frac{1}{\ln 2} \frac{1}{u(u+1)}. \quad (32)$$

In [27,28] it has been shown that, starting from a generic initial value u^0 , the situation of a very large u is always reached, at least one time, as the BKL map is iterated for a sufficiently long time. Actually, the BKL map has (especially when expressed in terms of the fractional part of the parameter u , see [34]) “strong mixing” properties, and therefore, starting from a generic irrational value of u , all the other irrational ones (including very large values) are, sooner or later, generated. This result ensures that, also from a statistical point of view, in each point of the space (enough smooth space region), the conditions for the system to deeply entering the corner are reached.

However, in [13] it has been argued how the iteration of the BKL map in two close space points gives uncorrelated values of the parameter u after some steps and thus is at the ground of the progressive increasing of the spatial gradients towards the initial singularity. As a consequence of this result, the proposed scenario takes place in different instant of time in dynamical independent space regions. Nevertheless, once the system enters the corner, the BKL map is no longer applicable, because two potential walls are simultaneously relevant. Furthermore, once our paradigm is implemented, the increasing behavior of the spatial gradients is naturally stopped. Each smooth space region is characterized by a nonsingular static space-time and the statistical properties of the BKL map are reflected only on the specific initial condition at which the corner dynamics is implemented.

V. CONCLUSIONS

We investigated the possibility to obtain a nonsingular generic cosmological solution as result of a quantum behavior of the small anisotropy β_- within a deep corner configuration. In other words, we separated the Universe dynamics into a classical nonsingular one, plus a quantum effect which manifests in a simple small oscillation of β_- according to a time-independent frequency.

In order to establish this configuration, we inferred that, for a sufficiently large value of the parameter u , the variable β_- is extremely small well inside the corner of the potential, so that it explores the uncertainty principle.

To characterize the generality of the proposed scheme, we made use of two complementary effects:

- (i) In each assigned space point, the iteration of the BKL map is associated to a significant probability for a very long era, i.e., a trapping of the system dynamics deeply in the corner, see [27,28].
- (ii) The existence of the so-called fragmentation process, i.e., the impossibility to exclude rational values of u in a continuous function representation $u(x^i)$, which generates on all the corresponding space surfaces exactly the limit $\beta_- \equiv 0$, with associated neighborhoods where a long era must take place [13].

This analysis completes and generalizes the consideration made in [33] about the WKB approach to the homogeneous case, see also [31,32] for related topics. The basic motivation for such a generalization consists in the natural character that the corner configuration acquires in the inhomogeneous picture, as effect of the fragmentation process. This means that few iterations of the map can be enough to generate very high values of u in correspondence of all the rational values of the initially assigned function $u^0(x^i)$.

In [35] it has been argued the possibility for a synchronization of the dynamics of different spatial regions of the

inhomogeneous mixmaster. Without entering in the discussion of such a proposal and its validity, we observe that such a synchronization would reduce the relevance of the spatial gradients, in favor of an homogeneouslike picture. The proposed feature would likely reduce the impact of the fragmentation process, but it would not prevent the realization of the present scenario, according to the point (i) above. The transition of the inhomogeneous mixmaster to a new regime of gravitational turbulence could instead be of different impact on the present scenario, as inferred in [14], see also [15]. In this case it would be clearly the applicability of the potential representation in a fully turbulent Universe. Finally, about the possible implications of the rotation of the vectors $\vec{7}^i$ in the presence of a matter source, like a perfect fluid (a question not yet fully explored in the inhomogeneous sector), see [36].

In conclusion, the present analysis revives, on a slightly different level, the original (pre-“BKL map”) Lifshitz and Khalatnikov idea that the anisotropy of the Universe could resolve its singularity.

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