

Barrow entropy corrections to Friedmann equations

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Recently, Barrow argued that quantum-gravitational effects may introduce intricate, fractal features on the black hole horizon [J. D. Barrow, , The area of a rough black hole, *Phys. Lett. B* **808**, 135643 (2020)]. In this viewpoint, black hole entropy no longer obeys the area law and instead it can be given by $S \sim A^{1+\delta/2}$, where the exponent δ ranges $0 \leq \delta \leq 1$, and indicates the amount of the quantum-gravitational deformation effects. Based on this, and using the deep connection between gravity and thermodynamics, we disclose the effects of Barrow entropy on the cosmological equations. For this purpose, we start from the first law of thermodynamics, $dE = TdS + WdV$, on the apparent horizon of the Friedmann-Robertson-Walker universe, and derive the corresponding modified Friedmann equations by assuming that the entropy associated with the apparent horizon has the form of Barrow entropy. We also examine the validity of the generalized second law of thermodynamics for the Universe enclosed by the apparent horizon. Finally, we employ the emergence scenario of gravity and extract the modified Friedmann equation in the presence of Barrow entropy which coincides with one obtained from the first law of thermodynamics. When $\delta = 0$, the results of standard cosmology are deduced.

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I. INTRODUCTION

The fast spreading of Covid-19 virus around the world in 2020 and its continuation until now in 2021 provide strong motivations for many scientists to consider the structure of this virus from different perspectives. Inspired by the fractal illustrations of this virus, recently Barrow proposed a new structure for the horizon geometry of black holes [1]. Assuming a infinite diminishing hierarchy of touching spheres around the event horizon, he suggested that black hole horizons might have intricate geometry down to arbitrary small scales. This fractal structure for the horizon geometry, leads to finite volume and infinite (or finite) area. Based on this modification to the area of the horizon, the entropy of the black holes no longer obeys the area law and will be increased due to the possible quantum-gravitational effects of spacetime foam. The modified entropy of the black hole is given by [1]

$$S_h = \left(\frac{A}{A_0} \right)^{1+\delta/2}, \quad (1)$$

where A is the black hole horizon area and A_0 is the Planck area. The exponent δ ranges as $0 \leq \delta \leq 1$ and stands for the amount of the quantum-gravitational deformation effects [1–3]. When $\delta = 0$, the area law is restored and $A_0 \rightarrow 4G$, while $\delta = 1$ represents the most intricate and fractal

structure of the horizon. Although the corrected entropy expression (1) resembles Tsallis entropy in the nonextensive statistical thermodynamics [4–8]; however, the origin and motivation of the correction, as well as the physical principles are completely different. In order to remind the origin of Tsallis entropy, let us note that for large-scale gravitational systems the usual Boltzmann-Gibbs additive entropy must be generalized to the nonadditive (nonextensive) entropy [4–6]. In this regards, Tsallis generalized standard thermodynamics to a nonextensive one, which can be applied in all cases, and still possessing the standard Boltzmann-Gibbs theory as a limit [4]. Based on this and using the statistical arguments, Tsallis and Cirto argued that the entropy of a black hole does not obey the area law and can be modified as $S \sim A^\beta$, where β is known as a nonextensive parameter [7]. Therefore, Tsallis entropy originates from the nonextensive thermodynamics, while the Barrow correction to entropy comes from the intricate and fractal geometry of the horizon due to the quantum-gravitational effects.

Some efforts have been done to disclose the influences of Barrow entropy in the cosmological setups. A holographic dark energy model based on entropy (1) was formulated in [9]. It was argued that this scenario can describe the history of the Universe, with the sequence of matter and dark energy eras. Observational constraints on Barrow holographic dark energy were performed in [10]. A modified cosmological scenario based on Barrow entropy was presented in [11], which modifies the cosmological field

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equations in such a way that contains new extra terms acting as the role of an effective dark-energy sector. The generalized second law of thermodynamics, when the entropy of the Universe is in the form of Barrow entropy, was investigated in [12]. Other cosmological consequences of the Barrow entropy can be followed in [13–20].

The profound connection between gravitational field equations and laws of thermodynamics has now been well established (see, e.g., [21–34] and references therein). It has been confirmed that gravity has a thermodynamical predisposition and the Einstein field equation of general relativity is just an equation of state for the spacetime. Considering the spacetime as a thermodynamic system, the laws of thermodynamics on the large scales, can be translated as the laws of gravity. According to “gravity-thermodynamics” conjecture, one can rewrite the Friedmann equations in the form of the first law of thermodynamics on the apparent horizon and vice versa [35–38]. When the entropy associated with the apparent horizon is in the form of the Tsallis entropy, $S \sim A^\beta$, one can reproduce the modified Friedmann equations by starting from the first law on the apparent horizon [39]. It was shown that, in the relativistic regime and for $\beta < 1/2$, the obtained modified Friedmann equation can reproduce the late time cosmic acceleration without invoking any kind of dark energy [40]. In the nonrelativistic regime, however, the modified Newton’s law of gravitation can explain the flat galactic rotation curves without invoking particle dark matter provided $\beta \lesssim 1/2$ [40]. In line with studies to understand the nature of gravity, Padmanabhan [41] argued that the spacial expansion of our Universe can be understood as the consequence of emergence of space. Equating the difference between the number of degrees of freedom in the bulk and on the boundary with the volume change, he extracted the Friedmann equation describing the evolution of the Universe [41]. The idea of emergence spacetime was also extended to Gauss-Bonnet, Lovelock, and braneworld scenarios [42–46].

In the present work, we are going to construct the cosmological field equations of the Friedmann-Robertson-Walker (FRW) universe with any special curvature, when the entropy associated with the apparent horizon is in the form of (1). Our work differs from [11] in that the author of [11] derived the modified Friedmann equations by applying the first law of thermodynamics, $TdS = -dE$, to the apparent horizon of a FRW universe with the assumption that the entropy is given by (1). Note that $-dE$ in [11] is just the energy flux crossing the apparent horizon, and the apparent horizon radius is kept fixed during an infinitesimal interval of time dt . However, in the present work, we assume the first law of thermodynamics on the apparent horizon in the form, $dE = TdS + WdV$, where dE is now the change in the energy inside the apparent horizon due to the volume change dV of the expanding Universe. This is consistent with the fact that in thermodynamics the work is done when the volume of the

system is changed. Besides, in [11], the author focuses on a flat FRW universe and modifies the total energy density in the Friedmann equations by considering the contribution of the Barrow entropy in the field equations as a dark-energy component. Here, we consider the FRW universe with any special curvature and modify the geometry (gravity) part (left-hand side) of the cosmological field equations based on Barrow entropy. The approach we present here is more reasonable, since the entropy expression basically depends on the geometry (gravity). For example, the entropy expressions differ in Einstein, Gauss-Bonnet, or $f(R)$ gravities. Any modifications to the entropy should change the gravity (geometry) sector of the field equations and vice versa. We shall also employ the emergence idea of [41] to derive the modified cosmological equations based on Barrow entropy. Again, we assume that the energy density (and hence the number of degrees of freedom in the bulk) is not affected by the Barrow entropy, while the horizon area (and hence the number of degrees of freedom on the boundary) get modified due to the change in the entropy expression. Throughout this paper we set $\kappa_B = 1 = c = \hbar$, for simplicity.

This paper is outlined as follows. In the next section, we derive the modified Friedmann equations, based on Barrow entropy, by applying the first law of thermodynamics of the apparent horizon. In Sec. III, we examine the validity of the generalized second law of thermodynamics for the universe filled with Barrow entropy. In Sec. IV, we derive the modified Friedmann equations by applying the emergence scenario for the cosmic space. We finish with conclusions in the last section.

II. MODIFIED FRIEDMAN EQUATIONS BASED ON BARROW ENTROPY

Our starting point is a spatially homogeneous and isotropic universe with the metric

$$ds^2 = h_{\mu\nu} dx^\mu dx^\nu + \tilde{r}^2 (d\theta^2 + \sin^2\theta d\phi^2), \quad (2)$$

where $\tilde{r} = a(t)r$, $x^0 = t$, $x^1 = r$, and $h_{\mu\nu} = \text{diag}(-1, a^2/(1 - kr^2))$ represents the two-dimensional metric. The open, flat, and closed universes correspond to $k = -1, 0, 1$, respectively. The boundary of the Universe is assumed to be the apparent horizon with radius [31]

$$\tilde{r}_A = \frac{1}{\sqrt{H^2 + k/a^2}}. \quad (3)$$

From the thermodynamical viewpoint the apparent horizon is a suitable horizon consistent with first and second law of thermodynamics. Also, using the definition of the surface gravity, κ , on the apparent horizon [36], we can associate with the apparent horizon a temperature, which is defined as [36,44]

$$T_h = \frac{\kappa}{2\pi} = -\frac{1}{2\pi\dot{\tilde{r}}_A} \left(1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A} \right). \quad (4)$$

For $\dot{\tilde{r}}_A \leq 2H\tilde{r}_A$, the temperature becomes $T \leq 0$. The negative temperature is not physically acceptable and hence we define $T = |\kappa|/2\pi$. Also, within an infinitesimal internal of time dt one may assume $\dot{\tilde{r}}_A \ll 2H\tilde{r}_A$, which physically means that the apparent horizon radius is fixed. Thus there is no volume change in it and one may define $T = 1/(2\pi\dot{\tilde{r}}_A)$ [35]. The profound connection between temperature on the apparent horizon and the Hawking radiation has been disclosed in [47], which further confirms the existence of the temperature associated with the apparent horizon.

The matter and energy content of the Universe is assumed to be in the form of perfect fluid with energy-momentum tensor

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}, \quad (5)$$

where ρ and p are the energy density and pressure, respectively. Independent of the dynamical field equations, we propose the total energy content of the Universe satisfies the conservation equation, namely, $\nabla_\mu T^{\mu\nu} = 0$. This implies that

$$\dot{\rho} + 3H(\rho + p) = 0, \quad (6)$$

where $H = \dot{a}/a$ is the Hubble parameter. Since our Universe is expanding, thus we have volume change. The work density associated with this volume change is defined as [48]

$$W = -\frac{1}{2}T^{\mu\nu}h_{\mu\nu}. \quad (7)$$

For FRW background with a stress-energy tensor (5), the work density is calculated,

$$W = \frac{1}{2}(\rho - p). \quad (8)$$

We further assume the first law of thermodynamics on the apparent horizon is satisfied and has the form

$$dE = T_h dS_h + WdV, \quad (9)$$

where $E = \rho V$ is the total energy of the Universe enclosed by the apparent horizon, and T_h and S_h are, respectively, the temperature and entropy associated with the apparent horizon. The last term in the first law is the work term due to change in the apparent horizon radius. Comparing with the usual first law of thermodynamics, $dE = TdS - pdV$, we see that the work term $-pdV$ is replaced by WdV ,

unless for a pure de Sitter space where $\rho = -p$, where the work term WdV reduces to the standard $-pdV$.

Taking differential form of the total matter and energy inside a three-dimensional sphere of radius \tilde{r}_A , we find

$$dE = 4\pi\tilde{r}_A^2\rho d\tilde{r}_A + \frac{4\pi}{3}\tilde{r}_A^3\dot{\rho}dt. \quad (10)$$

where we have assumed that $V = \frac{4\pi}{3}\tilde{r}_A^3$ is the volume enveloped by a three-dimensional sphere with the area of apparent horizon $A = 4\pi\tilde{r}_A^2$. Combining with the conservation equation (6), we arrive at

$$dE = 4\pi\tilde{r}_A^2\rho d\tilde{r}_A - 4\pi H\tilde{r}_A^3(\rho + p)dt. \quad (11)$$

The key assumption here is to take the entropy associated with the apparent horizon in the form of Barrow entropy (1). The only change needed is to replace the black hole horizon radius with the apparent horizon radius, $r_+ \rightarrow \tilde{r}_A$. If we take the differential form of the Barrow entropy (1), we get

$$\begin{aligned} dS_h &= d\left(\frac{A}{A_0}\right)^{1+\delta/2} = \left(\frac{4\pi}{A_0}\right)^{1+\delta/2} d(\tilde{r}_A^{2+\delta}) \\ &= (2+\delta)\left(\frac{4\pi}{A_0}\right)^{1+\delta/2} \tilde{r}_A^{1+\delta}\dot{\tilde{r}}_A dt. \end{aligned} \quad (12)$$

Inserting relation (8), (11), and (12) in the first law of thermodynamics (9) and using definition (4) for the temperature, after some calculations, we find the differential form of the Friedmann equation as

$$\frac{2+\delta}{2\pi A_0} \left(\frac{4\pi}{A_0}\right)^{\delta/2} \frac{d\tilde{r}_A}{\tilde{r}_A^{3-\delta}} = H(\rho + p)dt. \quad (13)$$

Combining with the continuity equation (6), we arrive at

$$-\frac{2+\delta}{2\pi A_0} \left(\frac{4\pi}{A_0}\right)^{\delta/2} \frac{d\tilde{r}_A}{\tilde{r}_A^{3-\delta}} = \frac{1}{3}d\rho. \quad (14)$$

Integration yields

$$-\frac{2+\delta}{2\pi A_0} \left(\frac{4\pi}{A_0}\right)^{\delta/2} \int \frac{d\tilde{r}_A}{\tilde{r}_A^{3-\delta}} = \frac{\rho}{3}, \quad (15)$$

which results

$$\frac{2+\delta}{2-\delta} \left(\frac{4\pi}{A_0}\right)^{\delta/2} \frac{1}{2\pi A_0} \frac{1}{\tilde{r}_A^{2-\delta}} = \frac{\rho}{3}, \quad (16)$$

where we have set the integration constant equal to zero. The integration constant may be also regarded as the energy density of the cosmological constant and hence it can be absorbed in the total energy density ρ . Substituting \tilde{r}_A from Eq. (3) we immediately arrive at

$$\frac{2 + \delta}{2 - \delta} \left(\frac{4\pi}{A_0} \right)^{\delta/2} \frac{1}{2\pi A_0} \left(H^2 + \frac{k}{a^2} \right)^{1-\delta/2} = \frac{\rho}{3}. \quad (17)$$

The above equation can be further rewritten as

$$\left(H^2 + \frac{k}{a^2} \right)^{1-\delta/2} = \frac{8\pi G_{\text{eff}}}{3} \rho, \quad (18)$$

where we have defined the effective Newtonian gravitational constant as

$$G_{\text{eff}} \equiv \frac{A_0}{4} \left(\frac{2 - \delta}{2 + \delta} \right) \left(\frac{A_0}{4\pi} \right)^{\delta/2}. \quad (19)$$

Equation (18) is the modified Friedmann equation based on the Barrow entropy. Thus, starting from the first law of thermodynamics at the apparent horizon of a FRW universe, and assuming that the apparent horizon area has fractal features, due to the quantum-gravitational effects, we derive the corresponding modified Friedmann equation of a FRW universe with any spatial curvature. It is important to note that in this case, there is no dark energy component emerged from the Barrow entropy that was derived in Ref. [11]. In the limiting case where $\delta = 0$, the area law of entropy is recovered and we have $A_0 \rightarrow 4G$. In this case, $G_{\text{eff}} \rightarrow G$, and Eq. (18) reduces to the standard Friedmann equation in Einstein gravity.

We can also derive the second Friedmann equation by combining the first Friedmann equation (18) with the continuity equation (6). If we take the derivative of the first Friedmann equation (18), we arrive at

$$2H \left(1 - \frac{\delta}{2} \right) \left(\dot{H} - \frac{k}{a^2} \right) \left(H^2 + \frac{k}{a^2} \right)^{-\delta/2} = \frac{8\pi G_{\text{eff}}}{3} \dot{\rho}. \quad (20)$$

Using the continuity equation (6), we arrive at

$$\begin{aligned} & \left(1 - \frac{\delta}{2} \right) \left(\dot{H} - \frac{k}{a^2} \right) \left(H^2 + \frac{k}{a^2} \right)^{-\delta/2} \\ & = -4\pi G_{\text{eff}} (\rho + p). \end{aligned} \quad (21)$$

Now using the fact that $\dot{H} = \ddot{a}/a - H^2$, and replacing ρ from the first Friedmann equation (18), we can rewrite the above equation as

$$\begin{aligned} & \left(1 - \frac{\delta}{2} \right) \left(\frac{\ddot{a}}{a} - H^2 - \frac{k}{a^2} \right) \left(H^2 + \frac{k}{a^2} \right)^{-\delta/2} \\ & = -4\pi G_{\text{eff}} p - \frac{3}{2} \left(H^2 + \frac{k}{a^2} \right)^{1-\delta/2}. \end{aligned} \quad (22)$$

After some simplification and rearranging terms, we find

$$\begin{aligned} & (2 - \delta) \frac{\ddot{a}}{a} \left(H^2 + \frac{k}{a^2} \right)^{-\delta/2} + (1 + \delta) \left(H^2 + \frac{k}{a^2} \right)^{1-\delta/2} \\ & = -8\pi G_{\text{eff}} p. \end{aligned} \quad (23)$$

This is the second modified Friedmann equation governing the evolution of the Universe based on Barrow entropy. For $\delta = 0$ ($G_{\text{eff}} \rightarrow G$), Eq. (23) reproduces the second Friedmann equation in standard cosmology

$$2 \frac{\ddot{a}}{a} + H^2 + \frac{k}{a^2} = -8\pi G p. \quad (24)$$

If we combine the first and second modified Friedmann equations (18) and (23), then we can obtain the equation for the second time derivative of the scale factor. We find

$$\begin{aligned} (2 - \delta) \frac{\ddot{a}}{a} \left(H^2 + \frac{k}{a^2} \right)^{-\delta/2} & = -\frac{8\pi G_{\text{eff}}}{3} [(1 + \delta)\rho + 3p] \\ & = -\frac{8\pi G_{\text{eff}}}{3} \rho [(1 + \delta) + 3w], \end{aligned} \quad (25)$$

where $w = p/\rho$ is the equation of state parameter. Taking into account the fact that $0 \leq \delta \leq 1$, the condition for the cosmic accelerated expansion ($\ddot{a} > 0$), implies

$$(1 + \delta) + 3w < 0 \rightarrow w < -\frac{(1 + \delta)}{3}. \quad (26)$$

When $\delta = 0$, which corresponds to the simplest horizon structure with area law of entropy, we arrive at the well-known inequality $w < -1/3$ in Friedmann cosmology, while for $\delta = 1$, which implies the most intricate and fractal structure, we find $w < -2/3$. This implies that, in an accelerating universe, the fractal structure of the apparent horizon enforces the equation of state parameter to become more negative.

In summary, in this section we derived the modified cosmological equations given by Eqs. (18) and (25) in Barrow cosmology. These equations describe the evolution of the Universe with any spacial curvature, when the entropy associated with the apparent horizon get modified due to the quantum-gravitational effects. We leave the cosmological consequences of the obtained Friedmann equations for future studies, and in the remanning part of this paper, we focus on the generalized second law of thermodynamics as well as a derivation of the Friedmann equation (18) from an emergence perspective.

III. GENERALIZED SECOND LAW OF THERMODYNAMICS

Our aim here is to investigate another law of thermodynamics, when the horizon area of the Universe has a fractal structure and the associated entropy is given by Barrow entropy (1). To do this, we consider the generalized

second law of thermodynamics for the Universe enclosed by the apparent horizon. Our approach here differs from the one presented in [12]. Indeed the authors of [12] modified the total energy density in the Friedmann equations based on Barrow entropy. The cosmological field equations given in relations (3) and (4) of [12] are nothing but the standard Friedmann equations, in a flat universe, with an additional energy component that acts as a dark energy sector [49]. However, as we mentioned in the introduction, here the effects of the Barrow entropy enter the gravity (geometry) part of the cosmological field equations. Thus, we assume the energy component of the Universe is not affected by the Barrow entropy. Besides we consider the FRW universe with any special curvature, while the authors of [12] only considered a flat universe. In the context of the accelerating Universe, the generalized second law of thermodynamics has been explored in [50–52].

Combining Eq. (14) with Eq. (6) and using (19), we get

$$\frac{1}{\tilde{r}_A^{3-\delta}}(2-\delta)\dot{\tilde{r}}_A = 8\pi G_{\text{eff}}H(\rho+p). \quad (27)$$

Solving for $\dot{\tilde{r}}_A$, we find

$$\dot{\tilde{r}}_A = \frac{8\pi G_{\text{eff}}}{2-\delta}H\tilde{r}_A^{3-\delta}(\rho+p). \quad (28)$$

Since $\delta \leq 1$, thus the sign of $\rho+p$ determines the sign of $\dot{\tilde{r}}_A$. In case where the dominant energy condition holds, $\rho+p \geq 0$, we have $\dot{\tilde{r}}_A \geq 0$. Our next step is to calculate $T_h\dot{S}_h$,

$$\begin{aligned} T_h\dot{S}_h &= \frac{1}{2\pi\tilde{r}_A} \left(1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A}\right) \frac{d}{dt} \left(\frac{A}{A_0}\right)^{1+\delta/2}, \\ &= \frac{2+\delta}{2\pi} \left(1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A}\right) \left(\frac{4\pi}{A_0}\right)^{1+\delta/2} \tilde{r}_A^\delta \dot{\tilde{r}}_A. \end{aligned} \quad (29)$$

Substituting $\dot{\tilde{r}}_A$ from Eq. (28) and using definition (19), we reach

$$T_h\dot{S}_h = 4\pi H\tilde{r}_A^3(\rho+p) \left(1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A}\right). \quad (30)$$

For an accelerating universe, the equation of state parameter can cross the phantom line ($w = p/\rho < -1$), which means that the dominant energy condition may violate $\rho+p < 0$. As a result, the second law of thermodynamics, $\dot{S}_h \geq 0$, no longer valid. In this case, one can consider the total entropy of the Universe as $S = S_h + S_m$, where S_m is the entropy of the matter field inside the apparent horizon. Therefore, one should study the time evolution of the total entropy S . If the generalized second law of thermodynamics holds, then we should have $\dot{S}_h + \dot{S}_m \geq 0$ for the total entropy.

From the Gibbs equation we have [53]

$$T_m dS_m = d(\rho V) + p dV = V d\rho + (\rho + p) dV, \quad (31)$$

where T_m stands for the temperature of the matter fields inside the apparent horizon. We further propose the thermal system bounded by the apparent horizon remains in equilibrium with the matter inside the Universe. This is indeed the local equilibrium hypothesis, which yields the temperature of the matter field inside the Universe must be uniform and the same as the temperature of its boundary, $T_m = T_h$ [53]. One may expect that, under thermal equilibrium condition, the temperature of the matter field inside the universe is estimated as the temperature of the cosmic microwave background radiation. In the absence of local equilibrium hypothesis, there will be spontaneous heat flow between the horizon and the bulk fluid that is not physically acceptable for our Universe. Thus, from the Gibbs equation (31) we have

$$T_h\dot{S}_m = 4\pi\tilde{r}_A^2\dot{\tilde{r}}_A(\rho+p) - 4\pi\tilde{r}_A^3H(\rho+p). \quad (32)$$

Next, we examine the generalized second law of thermodynamics, namely, we study the time evolution of the total entropy $S_h + S_m$. Adding Eqs. (30) and (32), we get

$$T_h(\dot{S}_h + \dot{S}_m) = 2\pi\tilde{r}_A^2(\rho+p)\dot{\tilde{r}}_A. \quad (33)$$

Substituting $\dot{\tilde{r}}_A$ from Eq. (28) into (33) we reach

$$T_h(\dot{S}_h + \dot{S}_m) = \frac{16\pi^2}{2-\delta}G_{\text{eff}}H\tilde{r}_A^{5-\delta}(\rho+p)^2 \geq 0, \quad (34)$$

which is clearly a non-negative function during the history of the Universe. This confirms the validity of the generalized second law of thermodynamics for a universe with a fractal boundary, namely when the associated entropy with the apparent horizon of the Universe is in the form of Barrow entropy (1).

IV. EMERGENCE OF MODIFIED FRIEDMANN EQUATION

In his proposal [41], Padmanabhan argued that gravity is an emergence phenomenon and that the cosmic space is emergent as the cosmic time progressed. He argued that the difference between the number of degrees of freedom on the holographic surface and the one in the emerged bulk, is proportional to the cosmic volume change. In this regards, he extracted successfully the Friedmann equation governing the evolution of the Universe with zero spacial curvature [41]. In this perspective the spatial expansion of our Universe can be regarded as the consequence of emergence of space and the cosmic space is emergent, following the progressing in the cosmic time. According to Padmanabhan's proposal, in an infinitesimal interval dt of

cosmic time, the increase dV of the cosmic volume is given by [41]

$$\frac{dV}{dt} = G(N_{\text{sur}} - N_{\text{bulk}}), \quad (35)$$

where G is the Newtonian gravitational constant. Here N_{sur} and N_{bulk} stand for the number of degrees of freedom on the boundary and in the bulk, respectively. Following Padmanabhan, the studies were generalized to Gauss-Bonnet and Lovelock gravity [42]. While the authors of [42] were able to derive the Friedmann equations with any spacial curvature in Einstein gravity, they failed to extract the Friedmann equations of a nonflat FRW universe in higher order gravity theories [42]. In [44], we modified Padmanabhan's proposal in such a way that it could produce the Friedmann equations in higher order gravity theories, such as Gauss-Bonnet and Lovelock gravities, with any spacial curvature. The modified version of relation (35) is given by [44]

$$\frac{dV}{dt} = G \frac{\tilde{r}_A}{H^{-1}} (N_{\text{sur}} - N_{\text{bulk}}). \quad (36)$$

Comparing with the original proposal of Padmanabhan in Eq. (35), we see that in a nonflat universe, the volume increase is still proportional to the difference between the number of degrees of freedom on the apparent horizon and in the bulk, but the function of proportionality is not just a constant, it is equal to the ratio of the apparent horizon and Hubble radius. For a flat universe, $\tilde{r}_A = H^{-1}$, and one recovers the proposal (35).

Our aim here is to derive the modified Friedmann equation based on Barrow entropy from emergence proposal of cosmic space. Inspired by Barrow entropy expression (1), let us define the effective area of the apparent horizon, which is our holographic screen, as

$$\tilde{A} = 4\pi\tilde{r}_A^{2+\delta}. \quad (37)$$

In order to choose the correct expression for the number of degrees of freedom on the apparent horizon, N_{sur} , we note that N_{sur} on a holographic screen should be proportional to the effective area. Hence we write down

$$N_{\text{sur}} = \frac{\tilde{A}}{G_{\text{eff}}} = \frac{4\pi\tilde{r}_A^{2+\delta}}{G_{\text{eff}}}, \quad (38)$$

Next, we calculate the increasing in the effective volume as

$$\frac{d\tilde{V}}{dt} = \frac{\tilde{r}_A}{2} \frac{d\tilde{A}}{dt} = 2\pi(2 + \delta)\tilde{r}_A^{2+\delta}\dot{\tilde{r}}_A. \quad (39)$$

We also assume the temperature associated with the apparent horizon is the Hawking temperature, which is given by [42]

$$T = \frac{1}{2\pi\tilde{r}_A}. \quad (40)$$

The reason for taking this expression for the temperature instead of relation (4) comes from the fact that here we would like to consider an equilibrium system, thus within an infinitesimal interval of time dt we propose $\dot{\tilde{r}}_A \ll 2H\tilde{r}_A$, which physically means that the apparent horizon radius is fixed during an infinitesimal interval of time dt , similar to the de Sitter universe. Note that, in Sec. II, one can also consider the temperature associated with the apparent horizon in the form of (40); however, in this case one should apply the first law as $-dE = TdS$, where $-dE$ is the energy flux crossing the horizon [36]. The energy contained inside the sphere with volume $V = 4\pi\tilde{r}_A^3/3$ is the Komar energy

$$E_{\text{Komar}} = |(\rho + 3p)|V. \quad (41)$$

We further assume the matter inside the universe remains in thermal equilibrium with the apparent horizon. This implies that the temperature of the matter and energy inside the bulk is equal to the temperature associated with the boundary. Under the equilibrium condition the equipartition law of energy for the matter and energy inside the Universe holds. Considering the equipartition law of energy, we can define the bulk degrees of freedom as

$$N_{\text{bulk}} = \frac{2|E_{\text{Komar}}|}{T}. \quad (42)$$

In order to have $N_{\text{bulk}} > 0$, we take $\rho + 3p < 0$, consistent with the condition of an accelerated universe [41]. Thus the number of degrees of freedom in the bulk is obtained as

$$N_{\text{bulk}} = -\frac{16\pi^2}{3}\tilde{r}_A^4(\rho + 3p). \quad (43)$$

The key assumption here is to take the correct form of expression (36). To write the correct proposal, we make replacement $G \rightarrow G_{\text{eff}}$ and $V \rightarrow \tilde{V}$ in the proposal (36) and rewrite it as

$$\alpha \frac{d\tilde{V}}{dt} = G_{\text{eff}} \frac{\tilde{r}_A}{H^{-1}} (N_{\text{sur}} - N_{\text{bulk}}). \quad (44)$$

Note that for latter convenience, we introduce a factor α on the left-hand side of Eq. (44). One may regard the assumption (44), and in particular the coefficient α , to the status of a postulate and verify whether it can lead to the correct Friedmann equation describing the evolution of the Universe. As we shall see below, this assumption allows us to extract the Friedmann equations from emergence scenario.

Substituting relations (38), (39), and (43) in Eq. (44), we arrive at

$$\frac{\alpha}{2}(2+\delta)\dot{\tilde{r}}_A^{2+\delta}\tilde{r}_A = \frac{G_{\text{eff}}\tilde{r}_A}{H^{-1}} \left[\frac{\dot{\tilde{r}}_A^{2+\delta}}{G_{\text{eff}}} + \frac{4\pi}{3}\tilde{r}_A^4(\rho+3p) \right]. \quad (45)$$

Simplifying yields

$$\alpha(2+\delta)\dot{\tilde{r}}_A^{\delta-3}\tilde{r}_A - 2\tilde{r}_A^{\delta-2} = \frac{8\pi G_{\text{eff}}}{3}(\rho+3p). \quad (46)$$

If we multiply the both sides of Eq. (46) by factor $\dot{a}a$, after using the continuity equation (6), we arrive at

$$-\alpha(2+\delta)a^2\dot{\tilde{r}}_A\tilde{r}_A^{\delta-3} + 2\dot{a}a\tilde{r}_A^{\delta-2} = \frac{8\pi G_{\text{eff}}}{3}\frac{d}{dt}(\rho a^2). \quad (47)$$

If we take $\alpha = (2-\delta)/(2+\delta)$, then the above equation can be written in the differential form of the Friedmann equation

$$\frac{d}{dt}(a^2\dot{\tilde{r}}_A^{\delta-2}) = \frac{8\pi G_{\text{eff}}}{3}\frac{d}{dt}(\rho a^2). \quad (48)$$

Integrating, yields

$$\left(H^2 + \frac{k}{a^2}\right)^{1-\delta/2} = \frac{8\pi G_{\text{eff}}}{3}\rho, \quad (49)$$

where in the last step we have used relation (3), and set the integration constant equal to zero. This is indeed the modified Friedmann equation derived from emergence of the cosmic space when the entropy associated with the apparent horizon is in the form of Barrow entropy (1). The result obtained here from the emergence approach coincides with the obtained modified Friedmann equation from the first law of thermodynamics in Sec. II. Our study indicates that the approach presented here is powerful enough and further supports the viability of the Padmanabhan's perspective of emergence gravity and its modification given by Eq. (44).

Let us note that without taking into account the coefficient α , one cannot arrive at the correct form of the Friedmann equation. Thus, in order to reproduce the desired result we need to take into account α in Eq. (44). When $\delta = 0$, we have $\alpha = 1$ and our assumption (44) restores the one given in (36). One may also argue that in the presence of Barrow entropy, the effective area should be defined as $\tilde{A} = (4\pi\tilde{r}_A^2)^{1+\delta/2}$, which differs from (37) by a factor of $(4\pi)^{\delta/2}$. However, one can easily check that, in this case, even by taking α into account, a coefficient $(4\pi)^{-\delta/2}$ appears on the right-hand side of Friedmann equation (49). As a result, the obtained Friedmann equation from emergence approach does not coincide with the one obtained from the first law of thermodynamics in Sec. II. This may justify the correctness of our assumption in (37) for the effective area.

V. CONCLUSIONS

Recently, and motivated by the Covid-19 virus structure, Barrow proposed a new expression for the black hole entropy [1]. He demonstrated that taking into account the quantum-gravitational effects may lead to intricate, fractal features of the black hole horizon. This complex structure implies a finite volume for the black hole but with an infinite/finite area for the horizon. In this viewpoint, the deformed entropy associated with the black hole horizon no longer obeys the area law and increases compared to the area law due to fractal structure of the horizon. The amount of increase in entropy depends on the amount of quantum-gravitational deformation of the horizon which is characterized by an exponent δ .

Based on Barrow's proposal for black hole entropy and assuming the entropy associated with the apparent horizon of the Universe has the same expression as black hole entropy, we investigated the corrections to the Friedmann equations of a FRW universe, with any spacial curvature. These corrections come due to the quantum-gravitational fractal intricate structure of the apparent horizon. To do this, and motivated by the "gravity-thermodynamics" conjecture, we proposed the first law of thermodynamics, $dE = T_h dS_h + WdV$, holds on the apparent horizon of FRW universe and the entropy associated with the apparent horizon is given by Barrow entropy (1). Starting from the first law of thermodynamics and taking the entropy in the form of Barrow entropy (1), we extracted modified Friedmann equations describing the evolution of the Universe. Then, we checked the validity of the generalized second law of thermodynamics by considering the time evolution of the matter entropy together with the Barrow entropy associated with the apparent horizon. We also employed the idea of emergence gravity suggested by Padmanabhan [41] and calculated the number of degrees of freedom in the bulk and on the boundary of the Universe. Then, we propose the number of degrees of freedom on the holographic surface get modified, due to the fractal structure of the horizon, and is given by (38). We revised the assumption (36) by replacing $G \rightarrow G_{\text{eff}}$ and write down it as Eq. (44). We also find out that one can deduce the Friedmann equation from this proposal provided a factor $\alpha = (2-\delta)/(2+\delta)$ is taken into account. Clearly for $\delta = 0$, we have $\alpha = 1$, $G_{\text{eff}} \rightarrow G$, and proposal (36) is restored. Under this assumption, the obtained Friedmann equation coincides with the one obtained from the first law of thermodynamics in Sec. II. Our studies further support the viability of the emergence gravity proposed in [41,44].

Many interesting topics remain for future considerations. The cosmological implications of the modified Friedmann equations and the evolution of the Universe can be addressed. The influences of the modified Friedmann equations on the gravitational collapse, structure formation

and galaxies evolution can be investigated. The effects of the fractal parameter δ on the thermal history of the Universe, as well as anisotropy of cosmic microwave background are also of great interest and deserve exploration. These studies belong beyond the scope of the present work and we leave them for the future projects.

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