# Is negative kinetic energy metastable?

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Local minima of the potential can be metastable up to cosmologically long times thanks to energy conservation. We explore the possibility that theories with negative kinetic energy (ghosts) can be metastable up to cosmologically long times. In classical mechanics, ghosts undergo spontaneous lockdown rather than run away if weakly coupled and nonresonant. Physical examples of this phenomenon are shown. In quantum mechanics, this leads to metastability similar to vacuum decay. In classical field theory, lockdown is broken by resonances and ghosts behave statistically, drifting toward infinite entropy as no thermal equilibrium exists. We analytically and numerically compute the runaway rate finding that it is cosmologically slow in four-derivative gravity, where ghosts have gravitational interactions only. In quantum field theory, the ghost runaway rate is naively infinite in perturbation theory, analogously to what is found in early attempts to compute vacuum tunnelling; we do not know the true rate.

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## I. INTRODUCTION

A tentative quantum theory of gravity and matter is obtained writing the most generic action with renormalizable terms, taking into account that the graviton $g_{\mu\nu}$ has mass dimension 0. Such action is [1],

$$S = \int d^4x \sqrt{|\det g|}\left[\frac{R^2}{6f_0^2} + \frac{\frac{1}{3}R^2 - R_{\mu\nu}^2}{f_2^2} - \frac{1}{2}\bar{M}_{\rm Pl}^2 R + \mathcal{L}_{\rm matter}\right] \tag{1}$$

where $R_{\mu\nu}$ is the Ricci tensor, $R$ is the curvature, and $\mathcal{L}_{\rm matter}$ contains scalars, fermions, and vectors. The first two terms, suppressed by the dimensionless gravitational couplings $f_0$ and $f_2$ (in the notation of [2]), are graviton kinetic terms with four derivatives.

However, a classical degree of freedom with four derivatives can be rewritten as 2 degrees of freedom with two derivatives, and one of the two (dubbed ghost) has negative kinetic energy [3]. Gravity is no exception. The four-derivative graviton splits into the massless graviton and a ghost graviton with mass $M_2 = f_2\bar{M}_{\rm Pl}/\sqrt{2}$. The full action in split form can be found in [4], and the negative kinetic energy can be seen through the following simple argument. Omitting Lorentz indices, the propagator of the four-derivative graviton is

$$\frac{1}{M_2^2 p^2 - p^4} = \frac{1}{M_2^2}\left[\frac{1}{p^2} - \frac{1}{p^2 - M_2^2}\right], \tag{2}$$

where the minus sign indicates negative kinetic energy.[1] It makes the theory renormalizable, canceling the graviton propagator at large energy $p \gg M_2$. We explore the possibility that the degrees of freedom with negative kinetic energy are physical, unlike what happens in gauge theories, where similar states are unphysical, introduced as mathematical tools to deal with gauge redundancies.

A classical degree of freedom with positive kinetic energy interacting with negative kinetic energy has runaway solutions, where the total energy is conserved while individual energies diverge. Thereby, negative *kinetic* energy is dubbed "ghost," meaning an unphysical object to be excluded from sensible theories. However, theories with negative and even unbounded-from-below *potential* energy can give sensible metastable physics around a false vacuum. Can unbounded-from-below kinetic energy similarly give rise to metastability?

To explore this issue, we will consider theories featuring some positive-energy degree of freedom $q_1(t)$ interacting with a ghost $q_2(t)$ as described by Lagrangians, such as

$$L = m_1\left(\frac{\dot{q}_1^2}{2} - \omega_1^2\frac{q_1^2}{2}\right) \pm m_2\left(\frac{\dot{q}_2^2}{2} - \omega_2^2\frac{q_2^2}{2}\right) - \frac{\lambda}{2}q_1^2 q_2^2, \tag{3}$$

[1]Thereby, many authors searched for a positive-energy quantization [5–13], analogously to what happens for fermions (classically their kinetic energy is undefined, but a sensible positive-energy quantum theory exists). It is unclear what is their large-action limit that possibly modifies classical physics into some positive-energy version.

as well as the analogous relativistic theory of fields $\varphi_{1,2}(\vec{x},t)$ (scalars, for simplicity) with Lagrangian density,

$$\mathcal{L} = \frac{(\partial_\mu\varphi_1)^2 - m_1^2\varphi_1^2}{2} \pm \frac{(\partial_\mu\varphi_2)^2 - m_2^2\varphi_2^2}{2} - \frac{\lambda}{2}\varphi_1^2\varphi_2^2. \quad (4)$$

In both cases, the ghost is obtained for $\pm = -1$.

We preliminarily need to address the concerns of those authors who, at this point, dismiss the study with the motivation that an unbounded-from-below Hamiltonian is inconsistent, for example, because it allows for classical solutions that hit singularities. These authors also view as inconsistent positive kinetic energies but with unbounded-from-below potentials.

What we want to study is how long the physical system can stay around a "false vacuum" before falling to other regions. In the case of potential metastability, the WKB approximation in quantum mechanics shows that the metastability time is determined only by the potential barrier, irrespectively of the fate beyond the barrier. The potential beyond the barrier might be unbounded from below (giving rise to singular solutions) or have a true minimum; this does not affect the metastability time. The fate beyond the barrier depends on possibly unknown high-energy theory. In effective quantum field theories (QFT), one considers extra nonrenormalizable terms that stabilize an unbounded-from-below potential. As such operators have negligible impact at low field values, the metastability time is computable in terms of low-energy physics.

Returning back from the analogy to the argument of the present study, we want to explore if a theory with negative kinetic energy might similarly be metastable up to cosmologically large times. Let us consider, for example, the model in Eq. (3). Its Hamiltonian is unbounded from below but can be modified, for example, into

$$H = \left(\frac{p_1^2}{2m_1} + m_1\omega_1^2\frac{q_1^2}{2}\right) - \left(\frac{p_2^2}{2m_2} + m_2\omega_2^2\frac{q_2^2}{2}\right) + \frac{1}{2E_0}\left(\frac{p_2^2}{2m_2} + m_2\omega_2^2\frac{q_2^2}{2}\right)^2 + \frac{\lambda}{2}q_1^2q_2^2, \quad (5)$$

which is bounded from below and negligibly differs from the original theory at energies $E \ll E_0$. The energy of the 2nd degree of freedom has a Mexican-hat form that avoids singularities, replacing them with a generalization of "ghost condensation" [14], such that $q_2$ reaches a constant but finite velocity. The critical energy $E_0$ plays a role analogous to coefficients of nonrenormalizable operators: In the limit where it is much higher than the energies available around the false vacuum, it plays no role until the escape event happens. In the following, we can thereby study the metastability issue in the simpler model of Eq. (3) where energy is unbounded from below.

In order to see if a ghost is really excluded, we start studying the problem in the simplest limit, classical mechanics.

It has been noticed that, in classical mechanics, some theories containing an interacting ghost have stable classical solutions with appropriate initial conditions dubbed "islands of stability" [15–24]. This happens even when interactions are generic enough that no constant of motion forbids interacting ghosts to evolve toward catastrophic runaway instabilities. Rather, ghosts undergo spontaneous lockdown, with energies that vary but remain in a nontrivial restricted range. Studies based on numerical computations of classical time evolution cannot reach cosmological metastability times, so an analytic understanding is needed. Extending earlier works [16], we will show that the needed mathematics had been already developed to understand a related problem: Why is the solar system metastable, despite that no constant of motion forbids planets to escape? Oversimplifying, it has been shown that classical systems that can be approximated as oscillators plus *small* interactions tend to undergo ordered epicyclelike motions, while large interactions lead to chaos. We will see that this implies that ghosts with large interactions run away, but ghosts with generic *small* interactions are stable. Weakly coupled theories contain hidden quasi-constants of motion. Since this might appear exotic, in Appendix A, we recall that known physical systems exhibit this behavior: Asteroids around the Lagrangian point $L_4$ and electrons in magnetic fields plus repulsive potentials are described by a ghost degree of freedom, and yet they are metastable.

Since classical mechanics does not exclude ghosts, in Sec. III, we study quantum mechanics, finding that metastability persists: A ghost (negative kinetic energy, $K$ instability) is not qualitatively less metastable than a negative potential energy ($V$ instability).

However, resonances [as $\omega_1 = \omega_2$ in Eq. (3)] can lead to ghost runaway even at small coupling, depending on the specific form of the interaction. Studying in Sec. IV classical field theory, we encounter an infinite number of resonances by expanding a field in Fourier modes. While local field theories can give resonances of benign type, the infinite number of resonances removes the hidden constants of motion. We then perform a statistical analysis, showing that systems containing ghosts do not have a thermal state: Heat keeps flowing from ghost fields to positive-energy fields because this increases entropy. We compute the rate of this instability through Boltzmann equations, finding a rate not exponentially suppressed by small couplings. Nevertheless, in the special case of four-derivative gravity, the graviton ghost has Planck-suppressed interactions, which are small enough that the ghost runaway rate is not problematic in cosmology. We validate this analytic understanding through classical lattice simulations.

In Sec. V, we finally consider relativistic quantum field theory, which is the relevant but most difficult theory. By performing the zero-temperature limit of Boltzmann equations, we find a divergent tree level ghost runaway rate. Such divergence arises because the initial vacuum state is Lorentz invariant, giving rise to an integral over the noncompact Lorentz group that describes a boost of the final state. The same Lorentz integral arose in earlier computations of $V$-instability tunneling, but Coleman later argued that vacuum decay can be computed in terms of a Lorentz-invariant instanton, the "bounce," and its rate is exponentially suppressed at small coupling. We don't know if something similar holds for $K$ instability.

Conclusions are presented in Sec. VI.

## II. GHOST METASTABILITY IN CLASSICAL MECHANICS

We consider a degree of freedom $q(t)$ in $0+1$ dimensions with four-derivative kinetic term,

$$L = -\frac{1}{2}q\left(\frac{\partial^2}{\partial t^2}+\omega_1^2\right)\left(\frac{\partial^2}{\partial t^2}+\omega_2^2\right)q - V_I(q,\ddot{q}), \quad (6)$$

where the first term is quadratic in $q$, and $V_I$ contains interactions. We add zero as a perfect square containing an auxiliary degree of freedom $\tilde{q}$ with no kinetic term:

$$L = \frac{1}{2}[-\ddot{q}^2 + (\omega_1^2+\omega_2^2)\dot{q}^2 - \omega_1^2\omega_2^2 q^2] + \frac{1}{2}\left[\ddot{q} + (\omega_1^2+\omega_2^2)\frac{q}{2} - \frac{\tilde{q}}{2}\right]^2 - V_I. \quad (7)$$

Expanding the square cancels both the second-order and the fourth-order kinetic terms, leaving

$$L = -\frac{\tilde{q}\,\ddot{q}}{2} + (\omega_1^2-\omega_2^2)^2\frac{q^2}{8} - (\omega_1^2+\omega_2^2)\frac{\tilde{q}q}{4} + \frac{\tilde{q}^2}{8} - V_I. \quad (8)$$

The kinetic and mass terms are diagonalized, performing the field redefinition,

$$\begin{cases} \tilde{q} = \sqrt{\omega_2^2-\omega_1^2}(q_1 - q_2) \\ q = (q_1+q_2)/\sqrt{\omega_2^2-\omega_1^2}, \end{cases} \quad (9)$$

obtaining, after an integration by parts,

$$L = \frac{\dot{q}_1^2 - \omega_1^2 q_1^2}{2} - \frac{\dot{q}_2^2 - \omega_2^2 q_2^2}{2} - V_I\left(\frac{q_1+q_2}{\sqrt{\omega_2^2-\omega_1^2}}, -\frac{\omega_1^2 q_1 + \omega_2^2 q_2}{\sqrt{\omega_2^2-\omega_1^2}}\right). \quad (10)$$

We can thereby focus on the toy model of Eq. (3) that captures the relevant physics. This classical theory only has one free physical parameter, $\omega_1/\omega_2$, plus the initial conditions for its time evolution. Indeed, without loss of generality, we can rescale $q_1$ and $q_2$ to set $m_1 = m_2 = 1$. By rescaling $t$, we can set $\omega_1 = 1$. Furthermore, classical physics is invariant under a multiplicative rescaling of $L$ so that we could set $\lambda = 1$. To improve readability, we keep $\omega_1$, $\omega_2$, and $\lambda$ as apparent parameters, but it should be clear that our following analysis is general.

The classical equations of motion are

$$\ddot{q}_1 + \omega_1^2 q_1 + \lambda q_1 q_2^2 = 0, \qquad \ddot{q}_2 + \omega_2^2 q_2 - \lambda q_2 q_1^2 = 0. \quad (11)$$

The only constant of motion is the total energy $E = E_1 - E_2 + V_I$, which is conserved, where

$$E_i = \frac{\dot{q}_i^2}{2} + \omega_i^2\frac{q_i^2}{2} > 0, \qquad V_I = \frac{\lambda}{2}q_1^2 q_2^2, \quad (12)$$

while $E_1$ and $E_2$ are not conserved; e.g., $\dot{E}_1 = -\lambda q_2^2 d(q_1^2)/dt$. No conservation law prevents rapid ghost runaway to $E_1, E_2 \to \infty$. Numerical evolution shows that solutions starting from $|E_1 - E_2| \gtrsim V_I$ quickly undergo runaway. On the other hand, for solutions starting from small enough initial energies $E_1, E_2 \ll V_I$, $E_1(t)$ and $E_2(t)$ evolve, remaining confined to a small range, for a time longer than what can be numerically computed.[2] Analytic work is needed to understand this surprising phenomenon.

### A. Action-angle variables

A technique used to study perturbed quasiperiodic motions in celestial mechanics is useful. Considering one pair $(q,p)$ of Hamiltonian variables, it is useful to pass to canonical action-angle variables $(\Theta, J)$, such that the Hamiltonian only depends on $J$, and motion is immediately solved.

In the simplest case of an harmonic oscillator, this gives

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}q^2 = \omega J, \quad (13)$$

where $m > 0$ ($m < 0$) for a normal particle (a ghost). The canonical transformation is

$$q = \sqrt{\frac{2J}{m\omega}}\sin\Theta, \qquad p = \sqrt{2m\omega J}\cos\Theta, \quad (14)$$

and its inverse is

$$\Theta = \arccos\frac{p}{\sqrt{p^2+(m\omega q)^2}}, \qquad J = \frac{p^2+(m\omega q)^2}{2m\omega}. \quad (15)$$

[2] In agravity, this kind of initial conditions correspond to small gradients, which might be selected by inflationary cosmology [23].

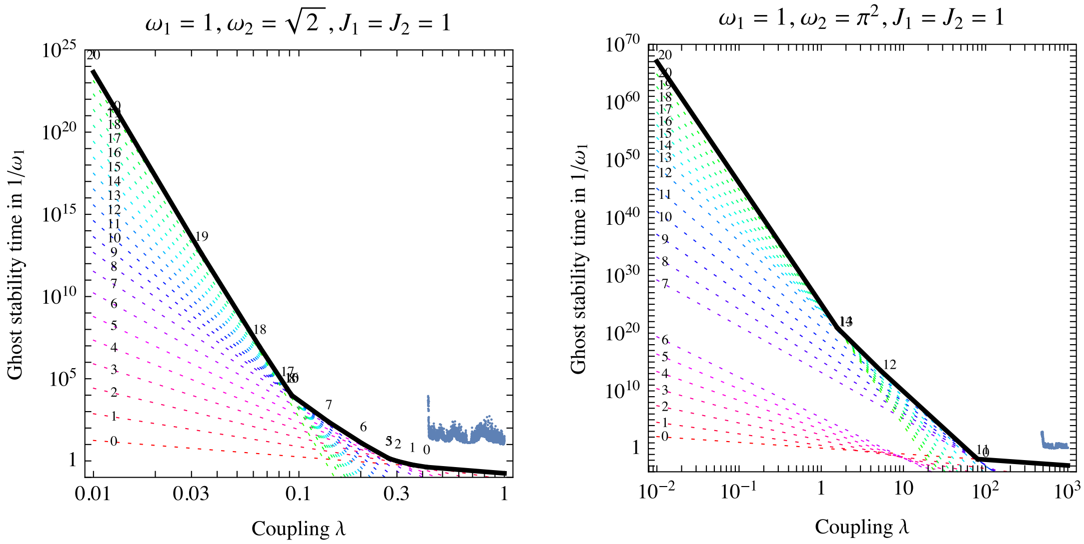


FIG. 1. We consider the ghost model of Eq. (3) with $n = 2$ degrees of freedom and quartic coupling $\lambda$. The dots are numerical results of observed ghost instability. The black curve is the analytic lower bound on the ghost stability time, computed up to 20th order in $\lambda$.

One can verify that $[\Theta, J] = (\partial\Theta/\partial q)(\partial J/\partial p) - (\partial Q/\partial p)(\partial J/\partial q) = 1$ or, more formally, write the generator of the canonical transformation,

$$W(q, J) = \int p dq = \frac{1}{2} q \sqrt{m\omega(2J - m\omega q^2)} + J \arccos \sqrt{1 - \frac{m\omega q^2}{2J}}. \tag{16}$$

In action-angle variables, $H = \omega J$ so that motion of a harmonic oscillator is trivially solved by $\Theta = \Theta_0 + \omega t$, $J = E/\omega$. For a generic anharmonic oscillator, the transformation to action-angle variables such that $H$ depends only on $J_i$ cannot be written analytically.

Going to action-angle variables for the two free harmonic oscillators, our toy ghost model of Eq. (3) becomes

$$H = \omega_1 J_1 - \omega_2 J_2 + \epsilon J_1 J_2 \sin^2\Theta_1 \sin^2\Theta_2, \quad \text{where } \epsilon = \frac{2\lambda}{\omega_1\omega_2}, \tag{17}$$

and $E_i = \omega_i J_i \geq 0$. The $-$ signals a ghost. The change of variables makes numerics stable up to longer time scales. Starting from $t = 0$, Fig. 1 shows the time $t_{\text{end}}$ at which the

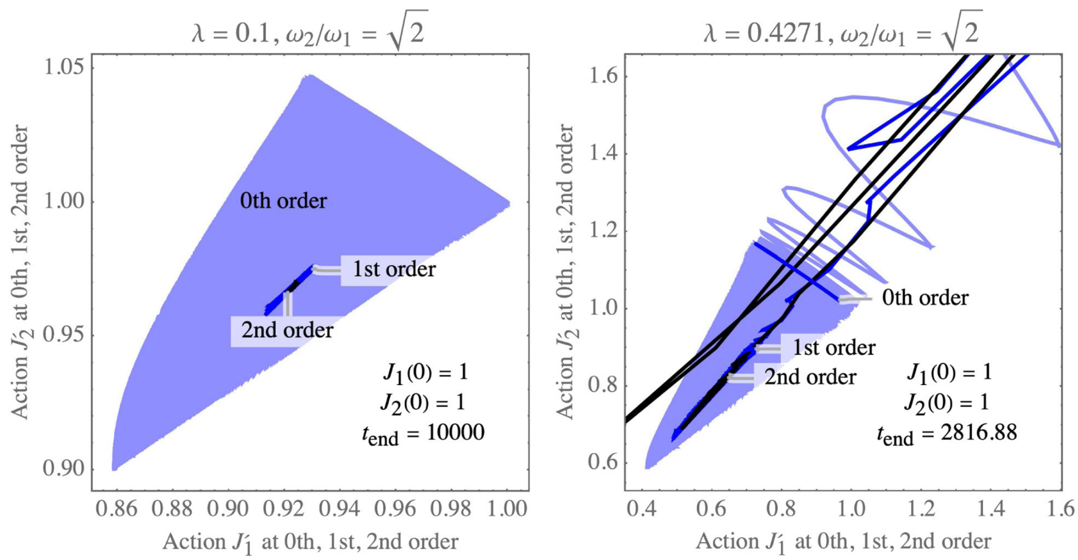


FIG. 2. Time evolution of the two quasiconserved energies $(J_1', J_2')$ computed at 0th order (lighter, $J_i' = J_i$), 1st order (medium), 2nd order (darker) in $\lambda$. For small $\lambda$ (left plot), time evolutions remain in a confined region that gets smaller and smaller as higher orders are included. Metastability is lost above some critical value of the coupling $\lambda$ (right plot), when the Birkhoff series in $\lambda$ stops converging.

ghost runaway happens as function of $\lambda$ for some fixed initial conditions and given $\omega_2/\omega_1$. We see a chaotic behavior at larger $\lambda$ that sharply starts above some critical value.

Figure 2(a) shows that, for small $\lambda$, $J_1$ and $J_2$ remain confined in a well-defined region up to long times, while $\Theta_1$ and $\Theta_2$ evolve almost linearly in time. Analytic work is needed to know if smaller $\lambda$ leads to metastability or to absolute stability. The region in the $(J_1, J_2)$ plane extends with increasing $\lambda$ until suddenly chaos and ghost runaway take over.

This behavior is characteristic of near-integrable system. Integrable systems (such as $n$ independent oscillators) are those for which any trajectory evolves along tori in phase space, rather than filling higher-dimensional subspaces up to the whole phase space. Adding small interactions, a near-ordered behavior persists because the system can be computed perturbatively. In the case of ghosts, this implies their metastability. For large coupling, the perturbative expansion fails, and the system becomes chaotic. If the system contains ghosts, this leads to runaways.

For small $\epsilon$, we can analytically solve the equations of motion as power series in $\epsilon$. At 0th order in $\epsilon = 2\lambda/\omega_1\omega_2$, the equations of motion are solved by

$$J_i(t) = J_{i0}, \qquad \Theta_1(t) = \Theta_1(0) + \omega_1 t, \qquad \Theta_2(t) = \Theta_2(0) - \omega_2 t. \tag{18}$$

We see that $J_i(t) = J_{i0}$ are constant, for both $i = \{1, 2\}$. Their equations of motion at 1st order,

$$J_1' = -\epsilon J_{10} J_{20} \sin(2\omega_1 t)\sin(\omega_2 t)^2, \qquad J_2' = \epsilon J_{10} J_{20} \sin(2\omega_2 t)\sin(\omega_1 t)^2, \tag{19}$$

are solved by

$$J_1(t) = J_{10} + \epsilon J_{10} J_{20} \frac{\omega_1\omega_{2-1}\cos(2t\omega_{1+2}) + \omega_{1+2}(2\omega_{1-2}\cos(2t\omega_1) - \omega_1\cos(2t\omega_{1-2})) + 2\omega_2^2}{8\omega_1(\omega_1 - \omega_2)(\omega_1 + \omega_2)}, \tag{20}$$

having defined $\omega_{1-2} = \omega_1 - \omega_2$ etc. The dimensionless expansion parameter is $\sim\epsilon J/\omega$, which describes the energy in the interaction term divided by the energy in the free quadratic part of the Hamiltonian. This 1st order approximation fails after some oscillations; nevertheless, for small $\epsilon$, it approximates well the range of $(J_1, J_2)$ covered by the full numerical solution. The 1st order perturbation diverges if $\omega_1 = \pm\omega_2$. More in general, higher orders diverge if $\omega_1$ and $\omega_2$ are "commensurable," namely if the resonance condition $N_1\omega_1 + N_2\omega_2 = 0$ is satisfied for some integers $N_{1,2}$.[3]

A more important problem is that the perturbative series in $\epsilon$ (or $\lambda$) is not convergent but at most asymptotic. Thereby, its existence does not imply absolute stability, and a more complicated analysis is needed, yielding stability over exponentially long times.

### B. Perturbative Birkhoff series

We consider the toy model described by the Hamiltonian of Eq. (17). Rather than finding solutions perturbatively in $\epsilon$, we follow a more general, equivalent, approach. We seek to "diagonalize" the classical Hamiltonian. Namely, we search for a canonical transformation $J_i \to J_i'$ and $\Theta_i \to \Theta_i'$ such that the Hamiltonian does not depend on $\Theta_i'$:

$$H(J_i, \Theta_i) = H'(J_i'). \tag{21}$$

We perform a generic canonical transformation with generator,

$$J_i'\Theta_i + W(J', \Theta); \quad \text{i.e.,} \quad J = J' + \partial_{\Theta_i} W, \quad \Theta' = \Theta + \partial_{J'} W. \tag{22}$$

So, defining $f = \sin^2\Theta_1 \sin^2\Theta_2$, one gets

$$H'(J') = H(J) = \omega_1(J_1' + \partial_{\Theta_1} W) - \omega_2(J_2' + \partial_{\Theta_2} W) + \epsilon f (J_1' + \partial_{\Theta_1} W)(J_2' + \partial_{\Theta_2} W). \tag{23}$$

If we could solve this equation, all $J_i'$ would be exact constants of motion, and the system would be integrable. However, we can only expand and perturbatively solve Eq. (23) in powers of $\epsilon$,

$$W = \epsilon W^{(1)} + \epsilon^2 W^{(2)} + \cdots, \qquad H' = H + \epsilon H^{(1)} + \epsilon^2 H^{(2)} + \cdots. \tag{24}$$

Since the system is not integrable, the Birkhoff series is only asymptotic, and $J_i'$ are the approximated constants of motion observed in numerics. Because of the periodicity in $\vec{\Theta} \equiv (\Theta_1, \Theta_2)$, we expand each term in Fourier series, e.g.,

$$W^{(n)}(\Theta_1, \Theta_2) = -i \sum_{N_1, N_2 = -\infty}^{\infty} e^{i\vec{N}\cdot\vec{\Theta}} W_{\vec{N}}^{(n)}, \qquad \vec{N} = (N_1, N_2). \tag{25}$$

[3] These resonances correspond to what in field theory are on-shell scattering and decay processes; in zero spatial dimensions, $\omega_i$ do not depend on momenta so that on-shell processes are only possible among appropriate integer numbers $N_i$ of modes.

The only nonzero coefficient of the Fourier series of $f = \sum_{\vec{N}} e^{iN_i\Theta_i} f_{\vec{N}}$ are $f_{00} = 1/4$, $f_{\pm2,\pm2} = f_{\pm2,\mp2} = 1/16$, $f_{\pm2,0} = f_{0,\pm2} = -1/8$.

#### 1. First order in the coupling

Expanding Eq. (23) at first order gives

$$H^{(1)} = \omega_1 \frac{\partial W^{(1)}}{\partial \Theta_1} - \omega_2 \frac{\partial W^{(1)}}{\partial \Theta_2} + J'_1 J'_2 f(\Theta_1, \Theta_2). \tag{26}$$

The first term involves derivatives of a periodic function with period $2\pi$, by the very definition of the angle variables $\Theta_i$. Therefore, its average over a period is zero. Averaging over $\Theta_i$ we get

$$H^{(1)} = \frac{J'_1 J'_2}{4} \quad \text{i.e.} \quad H' = \omega_1 J'_1 - \omega_2 J'_2 + \epsilon \frac{J'_1 J'_2}{4} + \mathcal{O}(\epsilon^2). \tag{27}$$

We next compute the canonical transformation $W^{(1)}$ through the Fourier expansion. We get $W^{(1)}_{00} = 0$ and

$$W^{(1)}_{N_1 N_2} = \frac{J'_1 J'_2 f_{N_1 N_2}}{N_2\omega_2 - N_1\omega_1} \tag{28}$$

for $N_1, N_2 \neq 0$. Summing over the nonvanishing $\vec{N}$, this means

$$W^{(1)} = \frac{J'_1 J'_2}{8(\omega_2^2 - \omega_1^2)} \left[ \left( \omega_1 \cos(2\Theta_2) - \omega_1 + \frac{\omega_2^2}{\omega_1} \right) \sin(2\Theta_1) + \left( \omega_2 \cos(2\Theta_1) - \omega_2 + \frac{\omega_1^2}{\omega_2} \right) \sin(2\Theta_2) \right], \tag{29}$$

and thereby,

$$J'_1 = J_1 + \epsilon \frac{J_1 J_2}{4\omega_1(\omega_1^2 - \omega_2^2)} [\cos 2\Theta_1 (\omega_2^2 - \omega_1^2 + \omega_1^2 \cos 2\Theta_2) - \omega_1\omega_2 \sin 2\Theta_1 \sin 2\Theta_2] + \mathcal{O}(\epsilon^2), \tag{30}$$

which gives the extra approximate integral of motion (in addition to energy, an exact constant). At this order, the only resonance is $\omega_1 = \pm\omega_2$. The perturbative expansion fails close to the resonance. The numerical solution shows that $J'_1$ is an approximate pseudointegral of motion for small $\epsilon$, unless $\omega_1 \approx \omega_2$.

#### 2. Generic order in the coupling

Equation (23) expanded at order $n > 1$ ($n = 1$ is special) is

$$H^{(n)}(J'_1, J'_2) = \omega_1 \frac{\partial W^{(n)}}{\partial \Theta_1} - \omega_2 \frac{\partial W^{(n)}}{\partial \Theta_2} + f(\Theta_1, \Theta_2) \times \left[ J'_1 \frac{\partial W^{(n-1)}}{\partial \Theta_2} + J'_2 \frac{\partial W^{(n-1)}}{\partial \Theta_1} + \sum_{m=1}^{n-2} \frac{\partial W^{(m)}}{\partial \Theta_1} \frac{\partial W^{(n-1-m)}}{\partial \Theta_2} \right]. \tag{31}$$

At each order $n$, only a finite set of coefficients of $W^{(n)}_{N_1 N_2}$ are non-zero, since $f$ only has few nonzero Fourier coefficients. The constant term ($p_1 = p_2 = 0$) allows one to find explicitly the Hamiltonian, whereas the other terms give the canonical transformation. We may fix the freedom of performing $\Theta$-only transformations by choosing $W^{(n)}_{00} = 0$, finding

$$H^{(n)} = \sum_{\vec{q}+\vec{r}=\vec{0}} (r_2 J'_1 + r_1 J'_2) f_{\vec{q}} W^{(n-1)}_{\vec{r}} + \sum_{m=1}^{n-2} \sum_{\vec{q}+\vec{r}+\vec{s}=\vec{0}} r_1 s_2 f_{\vec{q}} W^{(m)}_{\vec{r}} W^{(n-1-m)}_{\vec{s}}, \tag{32}$$

$$W^{(n)}_{\vec{N}} = \frac{1}{N_2\omega_2 - N_1\omega_1} \left[ \sum_{\vec{q}+\vec{r}=\vec{p}} (r_2 J'_1 + r_1 J'_2) f_{\vec{q}} W^{(n-1)}_{\vec{r}} + \sum_{m=1}^{n-2} \sum_{\vec{q}+\vec{r}+\vec{s}=\vec{p}} r_1 s_2 f_{\vec{q}} W^{(m)}_{\vec{r}} W^{(n-1-m)}_{\vec{s}} \right], \tag{33}$$

which are explicit equations for the Hamiltonian and the canonical transformation at order $n$ in terms of the lower orders. $H^{(n)}$ is a polynomial of degree $n+1$ in $J'_{1,2}$ with coefficients that depend on $\omega_i$.[4]

### C. Stability estimates

For typical interacting systems, frequencies vary depending on initial conditions and can thereby hit resonances, invalidating the Birkhoff series that guarantees stability. Kolmogorov proved that instability only happens for a subset of values of initial conditions that are as rare as rational numbers within real numbers; most initial conditions lead to stable motion. For systems with 2 degrees of freedom and conserved energy, this is enough to guarantee exact stability because there is only one quasiconstant of motion, say $J'_1$. Any initial condition is "surrounded" by nearby values so that stability holds. On the other hand, with more than 2 degrees of freedom, there are two or more

[4]Relations such as $W^{(n)}_{-N_1,N_2}(\omega_1, \omega_2) = (-1)^n W^{(n)}_{N_1,N_2}(-\omega_1, \omega_2)$ allow one to compute only for positive $N_{1,2} \geq 0$, if $\omega_i$ are left generic. However, this produces cumbersome expressions, and computations are more efficiently performed setting $\omega_i$ to numerical values, such that each term is a short polynomial in $J'_i$.

quasiconstants $J'_i$ so that they can undergo Arnold diffusion; their values slowly drift through the rare instabilities, not being surrounded by stable values. This drift is not visible in perturbation theory because it takes place for "rational" values of $\omega_i$, such that perturbation theory fails. Nekhoroshev estimated that the drift is nonperturbatively slow, giving rise to an exponentially large instability time [25].

In concrete systems, the metastability time can be computed as follows. The perturbative Birkhoff series allows one to remove interactions up to an arbitrarily large power $\epsilon^k$ so that the remaining small interaction can destroy stability on long enough time-scales, of order $\epsilon^{-k}$. As the Birkhoff series is only asymptotic, stability estimates are obtained by computing up to some high optimal order in the asymptotic expansion. For example, [26] computed the metastability time of asteroids around the Lagrangian point L4, which contain a ghost degree of freedom.

In our model, we can compute the time $\tau_n(J_{\max}^{\rm in}\to J_{\max})$, for which, we are guaranteed that any evolution starting from $J'_i\le J_{\max}^{\rm in}$ remains within $J'_i\le J_{\max}>J_{\max}^{\rm in}$. We maximize over $J_{\max}$, when possible, having in mind Lyapunov stability so that $\tau_n(J_{\max}^{\rm in})\equiv \max_{J_{\max}}\tau_n(J_{\max}^{\rm in}\to J_{\max})$.

Computing at different orders $n$ in the expansion give different $J'_i$ and different times $\tau_n$; because of the asymptotic character of the Birkhoff series, stability is guaranteed up to the largest $\tau_n$. Nonconservation of $J'_i$ happens because interactions $\delta H$ remain at higher order:

$$H(J_i,\Theta_i)=H^{(\le n)}(J'_i)+\delta H(J'_i,\Theta_i), \tag{34}$$

where $H^{(\le n)}=\sum_{k=0}^n\epsilon^k H^{(k)}$ includes terms up to order $n$. The leading-order contribution to the residual is

$$\delta H=-\epsilon^{n+1}\left[\omega_1\frac{\partial W^{(n+1)}}{\partial\Theta_1}-\omega_2\frac{\partial W^{(n+1)}}{\partial\Theta_2}\right]+\mathcal{O}(\epsilon^{n+2}). \tag{35}$$

Such term can be computed from its Fourier coefficients,

$$\delta H_{\vec N}^{(n+1)}=\begin{cases}H^{(n+1)} & \text{for}\,\vec n=\vec 0\\ (N_2\omega_2-N_1\omega_1)W_{\vec N}^{(n+1)} & \text{for}\,\vec n\neq\vec 0\end{cases}. \tag{36}$$

The residual time evolution of $J'_i$ is given by its Hamiltonian equation of motion,

$$\dot J'_i=-\frac{\partial}{\partial\Theta'_i}\delta H, \tag{37}$$

where, at leading order in the residual, we can approximate $\partial/\partial\Theta'_i\simeq\partial/\partial\Theta_i$ and thereby avoid reexpressing $\Theta$ in terms of $\Theta'$ in $\delta H$. A lower bound on the stability time is obtained by substituting $\dot J'_i$ with its maximal value. Neglecting higher orders in $\epsilon$,

$$\left|\frac{\partial}{\partial\Theta'_i}\delta H\right|\le\sum_{\vec p}\left|\frac{\partial}{\partial\Theta'_i}\delta H_{\vec p}^{(n)}\right|=\sum_{N_1,N_2}|N_i(N_2\omega_2-N_1\omega_1)W_{\vec N}^{(n)}|, \tag{38}$$

having used the triangular inequality. Higher orders in $\epsilon$ weaken the bound in Eq. (38) by a factor of 2 [26].

### 1. Stability at lowest order

To start, we outline the procedure at lowest order, such that the approximately conserved quantities are simply $J'_i=J_i$, and the remainder in the Birkhoff series simply is the whole interaction,

$$\delta H^{(0)}=2\lambda\frac{J_1J_2}{\omega_1\omega_2}\sin^2\Theta_1\sin^2\Theta_2. \tag{39}$$

To compute the stability time, we use the inequality,

$$|J'_i(t)-J'_i(0)|\le t\max_{J'_i\le J_{\max}}|\dot J'_i|\le t\frac{2\lambda J_{\max}^2}{\omega_1\omega_2}. \tag{40}$$

The region can be abandoned only after a time,

$$t\ge\tau_0(J_{\max}^{\rm in}\to J_{\max})=\omega_1\omega_2\frac{J_{\max}-J_{\max}^{\rm in}}{2\lambda J_{\max}^2}. \tag{41}$$

Its maximal value, achieved for $J_{\max}=2J_{\max}^{\rm in}$, is the Lyapunov stability time:

$$\tau_0(J_{\max}^{\rm in})=\frac{\omega_1\omega_2}{8\lambda J_{\max}^{\rm in}}. \tag{42}$$

### 2. Stability at generic order

The above discussion is easily generalized at order $n$. The residual time evolution is bounded by

$$\begin{aligned}&\max_{i,J'_i\le J_{\max}}\left|\frac{\partial}{\partial\Theta_i}\delta H\right|\\&\le 2\epsilon^{n+1}\max_{i,J'_i\le J_{\max}}\sum_{N_1,N_2}|N_i(N_2\omega_2-N_1\omega_1)W_{\vec N}^{(n+1)}|\\&\equiv\epsilon^{n+1}J_{\max}^{n+2}\beta_n,\end{aligned} \tag{43}$$

where we included the factor of 2 due to higher orders, maximized over the free index $i=1,2$, and used the fact that the remainder is a homogeneous polynomial in $J'_i$ of order $n+2$. The function $\beta_n(\omega_1,\omega_2)$ can be computed numerically and diverges close to resonances:

$$\tau_n(J_{\max}^{\rm in}\to J_{\max})=\frac{J_{\max}-J_{\max}^{\rm in}}{\epsilon^{n+1}J_{\max}^{n+2}\beta_n}. \tag{44}$$

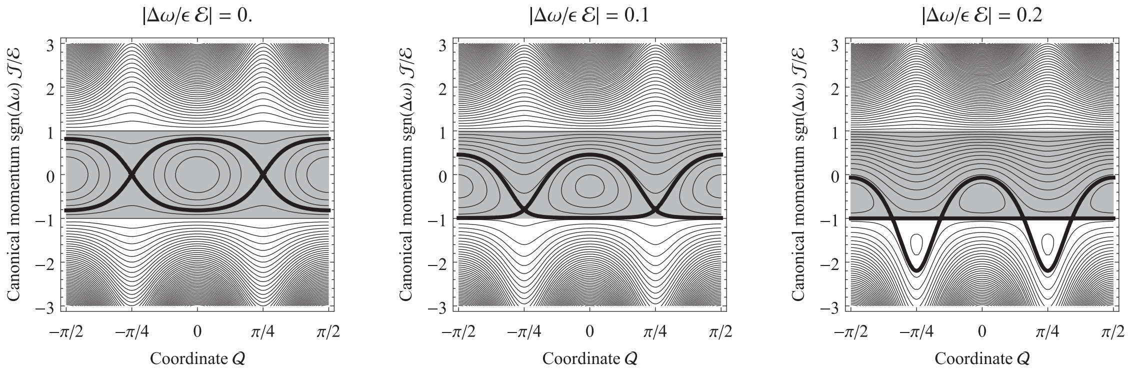


FIG. 3. Phase portrait of the auxiliary system close to the resonance. The thick line is the separatrix between the different kinds of motion. The shaded gray region in phase space cannot be accessed with $J'_{1,2} \geq 0$.

The Lyapunov stability time is

$$\tau_n(J^{\rm in}_{\rm max}) = \frac{1}{\beta_n}\frac{(n+1)^{n+1}}{(n+2)^{n+2}}\left(\frac{\omega_1\omega_2}{2\lambda J^{\rm in}_{\rm max}}\right)^{n+1}. \qquad (45)$$

In view of the asymptotic character of the Birkhoff series, for each value of $\rho_0$, there is an optimal order $n$ that gives the strongest bound.

As an example, in Fig. 1(a), we show the stability bound computed for $\omega_2/\omega_1 = \sqrt{2}$. The numbers on the curve indicate the optimal order. Some order dominates for a larger range when it contains enhanced denominators. Our example contains enhanced denominators at 7th order $(1/(7\omega_1 - 5\omega_2))$ and 17th order. Something similar happens in Fig. 1(b), where we consider $\omega_2/\omega_1 = \pi^2$. In both cases, we keep fixed $J'_{1,2} = 1$ at any given order, which approximatively means $J_{1,2} = 1$ for values of $\lambda$, small enough that the series converges.

For small enough couplings, we proved ghost metastability up to cosmological times that cannot be probed by numerical studies. We consider a specific model that contains no special features; a similar analysis can be performed for any other model.

### D. Resonances *i.e.*, on-shell processes

The previous perturbative approximation becomes less accurate close to resonances. The most dangerous resonance corresponds to $\omega_1 = \omega_2$, as $1/(\omega_1 - \omega_2)$ enhancements occur at leading order in the coupling. As a result, the Birkhoff series already fails for $E_{\rm int}/E_{\rm free} \gtrsim (\omega_1 - \omega_2)/\omega_{1,2}$, instead of holding, as usual, when the energy in the interaction terms is smaller than the free energy. Numerical solutions in our model with resonant $\omega_1 \approx \omega_2$ (and very small $\lambda$ such that interactions negligibly modify frequencies) show that a linear combination of $J_{1,2}$ fails to be quasiconstant of motion but remains bounded so that runaways remain avoided.

We extend analytic techniques to study resonances as they will be important in our subsequent study of classical and quantum field theories.[5]

As described in advanced books about analytic mechanics [27], resonant processes can be analytically studied by modifying the Birkhoff normal form into a "*resonant* normal form" that avoids the enhanced terms by selectively downgrading the goal of canceling all dependence on the angle variables. One needs to keep those that give resonant combinations, obtaining a more complex but still manageable partially diagonalized Hamiltonian. Some combinations of $J'$ remain quasiconserved, whereas others evolve as governed by the resonant form.

#### 1. Example: ghost that remains stable close to resonance

To clarify with a worked example, we reconsider our model of Eq. (17) in the resonant case $\omega_2 \to \omega_1$. We perform a canonical transformation analogous to Eq. (28) (at leading order) but omitting the singular Fourier modes with $N_1 = N_2 \equiv \bar{N}$, which multiply $\Theta_1 + \Theta_2$. A straightforward but tedious change of variables gives

$$H' = \omega_1 J'_1 - \omega_2 J'_2 + \epsilon\frac{J'_1 J'_2}{4}\left[1 + \frac{1}{2}\cos 2(\Theta'_1 + \Theta'_2)\right] + \cdots. \qquad (46)$$

The same result can be reobtained by expanding Eq. (26) in Fourier modes and taking into account that off-diagonal elements of $W^{(1)}_{N_1 N_2}$ cancel the contribution from $f_{N_1 N_2}$, while diagonal elements $W^{(1)}_{\bar{N}\bar{N}}$ vanish, leaving the Hamiltonian Fourier coefficients $H^{(1)}_{\bar{N}} = J'_1 J'_2 f_{\bar{N}\bar{N}}$ so that

[5]By expanding fields into Fourier modes, one gets an infinite number of interactions, which always contain resonances $\sum_i \omega_i^{\rm in} = \sum_j \omega_j^{\rm out}$, giving rise to decays and other on-shell process, using the standard terminology of quantum field theory (when $E = \hbar\omega$ the resonance condition becomes conservation of energy and momentum).

$$H^{(1)} = \sum_{\bar{N}} e^{i\bar{N}(\Theta_1+\Theta_2)} H^{(1)}_{\bar{N}} \tag{47}$$

gives again Eq. (46), after taking into account that $\Theta_i \simeq \Theta'_i$. The series expansion is no longer singular at the resonance so that its first order is accurate at small coupling. We can use it to study the dynamics close to the resonance finding that, since $1 + \frac{1}{2}\cos 2(\Theta_1 + \Theta_2) > 0$, motion remains bounded. This can be better seen by performing the canonical transformation,

$$\mathcal{Q} \equiv \frac{\Theta'_1 + \Theta'_2}{2}, \quad \mathcal{J} \equiv J'_1 + J'_2, \quad \mathcal{E} \equiv J'_1 - J'_2, \tag{48}$$

such that, writing $\omega \equiv (\omega_1 + \omega_2)/2$, $\Delta\omega \equiv \omega_1 - \omega_2$, the Hamiltonian becomes

$$H' \simeq \omega\mathcal{E} + \Delta\omega\frac{\mathcal{J}}{2} + \frac{\epsilon}{16}(\mathcal{J}^2 - \mathcal{E}^2)\left(1 + \frac{1}{2}\cos 4\mathcal{Q}\right). \tag{49}$$

$H'$ and $\mathcal{E}$ are constants of motion,[6] while $\mathcal{J}$ is no longer conserved and forms, together with $\mathcal{Q}$, a system with 1 degree of freedom, simple enough that it can be analytically studied. The key point is that its Hamiltonian is bounded so that $\mathcal{J}$, despite not being constant, is bounded, and the action variables $J'_{1,2}$ are bounded too. The possible motions are shown in Fig. 3. Typical trajectories move away from the resonance and then go back to it. $\mathcal{J}_{\max}/\mathcal{J}_{\min}$ is generically of order one, with the maximal variation $\sqrt{3}$ obtained for $\Delta\omega = \mathcal{E} = 0$. For $\Delta\omega$ sufficiently large, some of the trajectories in phase space oscillate. All trajectories are bounded.

In conclusion, the ghost system with quartic interaction $q_1^2 q_2^2$ is stable when perturbed around the noninteracting equilibrium point. Away from resonances, stability follows from the Birkhoff expansion and the KAM theorem [27,28]; the latter states that away from resonances, most trajectories in phase space are still confined to be toroidal, even in the presence of small interactions. Close to the $\omega_1 \simeq \omega_2$ resonance, stability follows because the extra system is not a ghost, so its motion is bounded; higher-order resonances are not dangerous because their resonant normal forms remain dominated by leading-order nonresonant terms.

[6] In terms of original variables, the "resonant" constant of motion $\mathcal{E} \equiv J'_1 - J'_2$ is

$$\mathcal{E} = J_1 - J_2 + \frac{\lambda J_1 J_2}{2\omega_1\omega_2}\left[\frac{\cos 2(\Theta_1 - \Theta_2)}{\omega_1 + \omega_2} - \frac{\cos 2\Theta_1}{\omega_1} - \frac{\cos 2\Theta_2}{\omega_2}\right] + \mathcal{O}(\lambda^2). \tag{50}$$

### 2. Example: ghost that undergoes runaway close to resonance

The safe situation found in the previous model is not generic. In other models, a ghost can become unstable close to resonances. This happens when the auxiliary dynamics that approximates the system close to a resonance is ghostlike, and the resonant surface in phase space extending to $J' \to \infty$ (at fixed energy/approximate integrals of motion) is attractive.

This happens, for example, replacing the quartic interaction $q_1^2 q_2^2$ with a cubic interaction $q_1^2 q_2$. The Hamiltonian in action-angle variables is

$$H = \omega_1 J_1 - \omega_2 J_2 + \epsilon J_1 \sqrt{J_2}\sin^2\Theta_1 \sin\Theta_2, \tag{51}$$

and the dangerous resonance is $\omega_2 \approx 2\omega_1$ that (loosely speaking) allows for a $q_1 \to q_1 + q_2$ decay. The resonant Birkhoff form at first order is

$$H' = \omega_1 J'_1 - \omega_2 J'_2 - \frac{\epsilon}{4} J'_1 \sqrt{J'_2}\sin(2\Theta'_1 + \Theta'_2). \tag{52}$$

The sign of $\sin(2\Theta'_1 + \Theta'_2)$ now qualitatively impacts the system. This can be seen performing the canonical transformation, $\mathcal{E} \equiv J'_1 - 2J'_2$, $\mathcal{J} \equiv (J'_1 + 2J'_2)/4$, $\mathcal{Q} \equiv 2\Theta'_1 + \Theta'_2$, such that

$$H' = \tilde{\omega}\mathcal{E} + \Delta\omega\mathcal{J} - \frac{\epsilon}{4}\left(\frac{\mathcal{E}}{2} + 2\mathcal{J}\right)\sqrt{\mathcal{J} - \frac{\mathcal{E}}{4}}\sin\mathcal{Q}, \tag{53}$$

with $\tilde{\omega} = (2\omega_1 + \omega_2)/4$, $\Delta\omega = 2\omega_1 - \omega_2$. The auxiliary system is now a ghost: The resonant ($\Delta\omega = 0$) trajectories at fixed $\mathcal{E}$ extends to $\mathcal{J} \to \infty$, e.g., the trajectory with $\mathcal{Q} = 0$. Moreover, these trajectories are attractive. At the resonance, all trajectories are unbounded. Moving away from the resonance, some stable KAM tori appear "on one side" for $\mathcal{J}$ small enough, but nothing protects stability on the other side (large $\mathcal{J}$).

Notice that the condition of ghost safety is independent from the condition of bounded-from-below potential. For example, consider a model with quartic interactions $H \supset \lambda(q_1^2 q_2^2 + \kappa q_1^3 q_2)/2$. Close to the resonance $\omega_2 \simeq 3\omega_1$, we find that the ghost is safe for $|\kappa| < 2/\sqrt{3}$, despite the potential is unstable for any $\kappa \neq 0$ (for instance, along the line $q_2 = 1$, $q_1 \to -\infty$). Conversely, the potential with quartic interactions $H \supset \lambda'(q_1^4 + \kappa q_1^3 q_2)$ is stable for any finite value of $\kappa$, but the ghost causes runaway for $|\kappa| > 3\sqrt{3}$.

The above considerations generalize to systems with more degrees of freedom. For instance, let us consider a system of 3 degrees of freedom with interaction $q_1 q_2 q_3$, where $q_2$ is a ghost. The Hamiltonian in action-angle variables is

$$H = \omega_1 J_1 - \omega_2 J_2 + \omega_3 J_3 + \epsilon\sqrt{J_1 J_2 J_3}\sin\Theta_1 \sin\Theta_2 \sin\Theta_3. \tag{54}$$

The first-order resonant form close to the dangerous resonance $\omega_1 - \omega_2 + \omega_3 \equiv \Delta\omega \simeq 0$ is

$$H \simeq -\omega_2 \mathcal{E}_2 + \omega_3 \mathcal{E}_3 + \Delta\omega \frac{\mathcal{J}}{3} - \frac{\epsilon}{4}\sqrt{\frac{\mathcal{J}}{3}\left(\frac{\mathcal{J}}{3} + \mathcal{E}_2\right)\left(\frac{\mathcal{J}}{3} + \mathcal{E}_3\right)} \sin 3\mathcal{Q}, \tag{55}$$

where $\mathcal{E}_i \equiv J'_i - J'_1$, $\mathcal{Q} = (\Theta'_1 + \Theta'_2 + \Theta'_3)/3$, and $\mathcal{J} = 3J'_1$. The extra-system Hamiltonian is unbounded and as a consequence the system, on resonance, undergoes ghost runaway.

The discussion of various examples allows one to identify a useful general property; only the part of the Hamiltonian at most quadratic in $J'$ is typically relevant for stability, since close enough to the origin, cubic and quartic interactions dominate over higher orders. In the presence of both cubics and quartics, quartic interactions generically stabilize the otherwise unsafe behavior of cubic-only interactions. This can be seen by noticing that resonant normal forms of quartic interactions contain stabilizing terms $\sim J'^2$ [as in Eq. (46)], which dominate with respect to the dangerous dynamical terms $\sim J'^{3/2} f(\Theta)$ for sufficiently large $J$.

In conclusion, ghost stability in classical mechanics is generic at small coupling away from resonances. In most models, resonances do not lead to ghost runaway but only to partial energy flow.

## III. GHOST METASTABILITY IN QUANTUM MECHANICS

Moving from classical to quantum mechanics, we again consider the prototype model of Eq. (3), described by the Hamiltonian,

$$H = \frac{p_1^2}{2} - \frac{p_2^2}{2} + V, \quad V = \omega_1^2 \frac{q_1^2}{2} - \omega_2^2 \frac{q_2^2}{2} + \frac{\lambda}{2} q_1^2 q_2^2, \tag{56}$$

which leads to the Schroedinger equation for the wave-function $\psi(q_1, q_2)$,

$$-\frac{\hbar^2}{2}\frac{\partial^2 \psi}{\partial q_1^2} + \frac{\hbar^2}{2}\frac{\partial^2 \psi}{\partial q_2^2} = (E - V)\psi. \tag{57}$$

We remind the following features of the Schroedinger equation in the absence of ghosts and relevant for computing vacuum tunneling through a potential barrier:

(1) the sign of $E - V$ tells one in which regions $\psi$ oscillates or gets exponentially suppressed;

(2) the vanishing of $E - V$ determines the "release point" $q_*$ on the other side of the potential barrier after which classical motion is unstable;

(3) the tunneling rate is exponentially suppressed by the WKB bounce action $W = \min \int_0^{q_*} dq\sqrt{2V}/\hbar$, where the integral is along the path in multidimensional field space that minimizes $W$.

These features are now lost because the ghost appears with an opposite sign in Eq. (57). So, the classically metastable ghost $q_2$ might become unstable if the wave function $\psi(q_1, q_2)$ of any state extends along the classically allowed region $q_1 \approx q_2$, reaching the large values where classical motion leads to runaway.

### A. Model computation

In the presence of a ghost, an infinite numbers of states have $E = 0$, or any other value. The same happens, without ghosts, in the presence of a potential like $V = \omega^2 q^2/2 + \lambda q^4/2$ with negative $\lambda$: Despite that $V$ is unbounded from below, the lowest-energy bound state is special. We focus on the analogous of this state for the ghost system. In the free theory, such groundlike bound state has minimal positive energy and maximal negative energy. Thanks to this property, it might be selected by cosmological evolution. We now show that the groundlike state is metastable.

We start by numerically computing the ghost model described by the Hamiltonian of Eq. (56). If the coupling $\lambda$ vanishes, it reduces to two decoupled harmonic oscillators, with the usual eigenstates $|n_1, n_2\rangle$. The groundlike state is $|0,0\rangle$ with wave function $\psi_{00}(q_1, q_2) = \psi_0(q_1)\psi_0(q_2)$ with $\psi_0(q_i) \propto e^{-q_i^2 \omega_i/2\hbar}$. For $\lambda \neq 0$, the groundlike state is the one that tends to $|0,0\rangle$ as $\lambda \to 0$, and that thereby, at small $\lambda$, has a maximal projection along $|0,0\rangle$. Its wave function $\psi(q_1, q_2)$ has no nodes around $q_1 \sim q_2 \sim 0$ and can be computed either numerically solving the Schroedinger Eq. (57) or by writing the Hamiltonian $H$ of Eq. (56) as a matrix in the $|n_1, n_2\rangle$ basis and diagonalizing it. Matrix elements of the interaction term $\lambda q_1^2 q_2^2/2$ are computed, using

$$\langle q_i^2 \rangle_{n_i m_i} = \frac{\hbar}{2\omega_i}\begin{cases} \sqrt{(m_i+1)(m_i+2)} & n_i = m_i + 2 \\ \sqrt{(n_i+1)(n_i+2)} & m_i = n_i + 2 \\ 2n_i + 1 & n_i = m_i \\ 0 & \text{otherwise.} \end{cases} \tag{58}$$

Figure 4 shows examples of numerical results in a nonresonant case $\omega_1 \neq \omega_2$: The ghost model gives a $|\psi(q_1, q_2)|^2$ qualitatively similar to what is obtained in a model with two positive energy $q_{1,2}$ and an unbounded-from-below potential with $\lambda < 0$. Inside the barrier at $q_1 \sim q_2 \sim 0$, the wave function is the usual Gaussian; outside, it has an oscillatory pattern with exponentially suppressed amplitude. In our approximation, the wave

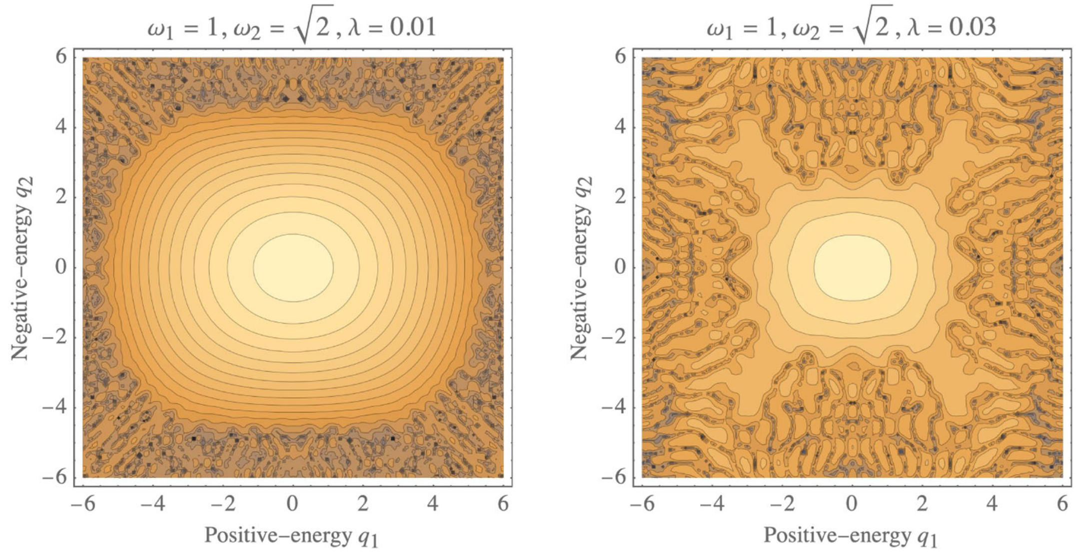


FIG. 4. Isocurves of the groundlike state wave function $|\psi(q_1, q_2)|^2$ for different values of the quartic coupling $\lambda$ between the positive-energy $q_1$ and the negative-energy $q_2$. Contour curves are separated by 1 order of magnitude.

function is real, but one can compute a more accurate bound state with complex wave function, such that the exponentially suppressed probability current is outflowing only. Its flux equals the vacuum decay rate, and the energy eigenvalue acquires a correspondingly exponentially suppressed imaginary part (see e.g., [29]).

The ghost case qualitatively differs from the negative-potential case only in the resonant situation $\omega_1 = \omega_2$; the ghost groundlike state does not reduce to $|0,0\rangle$ as $\lambda \to 0$.

### B. The WKB approximation

Ghost metastability can be understood more in general, taking into account that tunneling can be approximated *a la* WKB. Writing the wave function as $\psi = e^{iS/\hbar}$, the Schroedinger equation reduces to the classical Hamilton-Jacobi (HJ) equation,

$$\frac{\partial S}{\partial t} = -H\left(q_i, p_i = \frac{\partial S}{\partial q_i}\right), \tag{59}$$

plus extra terms $\frac{1}{2} i\hbar \partial^2 S/\partial q_i^2$ neglected at leading order in the semiclassical expansion, which is enough to approximate vacuum decay at weak coupling.

In Hamiltonian mechanics, Eq. (59) is obtained by demanding that $S$ generates a classical canonical transformation, such that the transformed Hamiltonian vanishes. Its solution is the classical action $S(q,t) = \int_{0,0}^{q,t} L(q_{\rm cl}) dt$ computed along the classical particle trajectory going from $q = 0$ at time $t = 0$ to $q$ at time $t$. Thereby, the HJ wave equation provides a bridge between waves and particles: $S$ respects the good hidden properties of a classical ghost discussed in Sec. II. To make better contact with the formalism of Sec. II, we consider a Hamiltonian $H$ that does not depend on time. Then, Eq. (59) can also be solved by separating variables as $S(q,t) = W(q) - Et$, where $E = H$ is the constant energy, and $W$ generates a canonical transformation to action-angle variables $(\Theta_i, J_i)$, such that $H$ only depends on $J_i$. The "reduced action" $W$ satisfies the wave equation,

$$E = H\left(q_i, p_i = \frac{\partial W}{\partial q_i}\right) \Rightarrow W = \int p_i dq_i. \tag{60}$$

The classical change of variables to action-angle coordinates essentially is a "diagonalization" of the classical Hamiltonian. Equation (59) [Eq. (60)] approximates the time-dependent (time-independent) Schroedinger equation Eq. (57), with the first (second) form being more useful for computing the propagator (energy eigenstates).

The hidden constants of motion that in the classical theory forbid motion into the dangerous region $q_1 \approx q_2$ still play a role in the semiclassical approximation. No new dramatically fast ghost instabilities appear in the quantum theory as, going away from the origin $q_1 \sim q_2 \sim 0$, the wave function gets exponentially suppressed by the semiclassical WKB factor $W$. Having a quantum Hamiltonian in action-angle variables, $H = \omega(J)J$, its eigenstates are the $|J\rangle$ states with eigenvalues $E = H(J)$ and wave function $\langle\Theta|J\rangle = e^{iJ\Theta/\hbar}$ so that its periodicity demands $J = n\hbar$ with $n$ an integer.

To obtain tunneling rates, we need to compute how the wave function extends into the classically forbidden region: As well-known it is useful to perform an analytic continuation to Euclidean time, $t_{\rm E} = it$ and solve the Euclidean HJ equation with $L_{\rm E} = \frac{1}{2}(d\vec{q}/dt_{\rm E})^2 - V_{\rm E}$ and inverted potential $V_{\rm E} = -V$. A well-known computational simplification allows one to approximate potential tunneling in the absence of ghosts; the vacuum decay rate is approximated by $e^{-B}$, where the bounce action $B = \min W_{\rm E}$ is computed along the classical Euclidean trajectory in field space that connects the false vacuum to the other side of the potential barrier with minimal $W_{\rm E}$. For example,

$$B = \min W_{\rm E} = \min S_{\rm E} = \min \lim_{t_{\rm E}\to+\infty} \int_{0,0}^{\vec{q}_*, t_{\rm E}} L_{\rm E} dt_{\rm E} = \min \int_0^{\vec{q}_*} dq \sqrt{2V_{\rm E}}, \qquad (61)$$

for the ground state with $E \to 0^+$. This simplification holds in the presence of multiple degrees of freedom and thereby, allows to compute vacuum decay in quantum field theory [30].

A similar result holds in the presence of ghosts only, with the only difference that boundary conditions (normalizable wave function) now demand picking the opposite-sign solution to the HJ equation. The sign of $W$ is not fixed because $H$ contains $p^2 = (\partial W/\partial q)^2$. For the ground state $E \to 0^-$, the bounce action is similar to Eq. (61) but with $t_{\rm E} \to -\infty$. Equivalently, an opposite-sign Wick rotation is needed to make the Euclidean ghost action positive.

In the presence of positive-energy particles that interact with ghosts, the desired solution to the HJ equation can be found numerically or perturbatively up to $q^2 \lesssim \omega/\lambda$,

$$W_{\rm E}(q_1, q_2)|_{E=0} = \frac{1}{2} q_1^2 \omega_1 + \frac{1}{2} q_2^2 \omega_2 + \frac{\lambda q_1^2 q_2^2}{4(\omega_1 - \omega_2)} + \frac{\lambda^2 (q_2^2 q_1^4 \omega_1 - 2 q_2^2 q_1^4 \omega_2 - 2 q_2^4 q_1^2 \omega_1 + q_2^4 q_1^2 \omega_2)}{16(\omega_1 - \omega_2)^2 (2\omega_1 - \omega_2)(\omega_1 - 2\omega_2)} + \cdots \qquad (62)$$

but we don't know how to compute vacuum decay bypassing a full solution to the HJ equation [31]. Physically, the new complication arises because we are interested in the groundlike state, which is neither the lowest nor the highest energy state so that selecting it gets more complicated.

## IV. GHOST METASTABILITY IN CLASSICAL FIELD THEORY

A field $\varphi(\vec{x}, t)$ can be decomposed as an infinite number of Fourier modes $q_{\vec{n}}(t)$. An infinite numbers of degrees of freedom allows for new phenomena. Some of them make any interacting classical field theory problematic; others are a problem for theories containing ghosts. As ghosts are at most a comorbidity of the theory, one needs to address and disentangle the new intertwined issues.

(1) In order to compute numerically one has to regularize the theory by introducing a cut off on the number of degrees of freedom, usually realized by a minimal length $a$, such as a lattice discretisation of space-time. Typical discretised field equations do no conserve energy and can lead to fake runaway behaviors when evolving configurations with excited modes near the cut off (the ones where energy conservation is badly violated). We will define special discretized classical equations that exactly conserve total energy, but hidden pseudoconstants of motion can be violated by the regularization.

(2) At some moment and in some region of space, some modes can acquire a higher energy density and overcome the energy barrier between stability and instability. In thermal field theories with local minima in the potential, this is the well-known thermal tunneling, characterized by a space-time tunneling probability density.[7] The same mechanism contributes to ghost instabilities.

(3) General initial field configurations tend to thermalize. However, a thermal state is impossible in classical field theory, as each one of the infinite modes should have the same energy $\sim T$. In electromagnetism, this is the well known black-body problem. An interacting field theory gives rise to a cascade of energy toward higher-frequency modes, and the temperature evolves toward $T \to 0$. On a lattice, this cascade stops when the problematic modes at the cut off thermalize.

(4) The above issue is solved by quantum mechanics. For a thermal state, classical field theory only holds for modes with $E \lesssim T$ and is replaced by quantum field theory for modes with $E \gtrsim T$ that get suppressed energy density:

$$f = \frac{1}{e^{E/T} - 1}, \quad fE \simeq \begin{cases} T & E \ll T \\ E e^{-E/T} & E \gg T \end{cases}. \qquad (63)$$

(5) Finally, the main new point. Field theory contains an infinite number of modes $q_{\vec{n}}(t)$ with frequencies $\omega_n$, so resonances are always possible. These resonances are the usual on-shell processes such as decays and scatterings. In the presence of ghosts, resonances can lead to partial or total loss of hidden constants of motion as discussed in Sec. II D.

In Sec. IV A, we decompose fields $\varphi(x, t)$ into modes $q_n(t)$, and in Sec. IV B, we perform a stability analysis of the resonances: Hidden constant of motion persist up to $\mathcal{O}(1)$, but the number of resonance is so large that dangerous

[7] Some authors claim that they can approximate quantum vacuum decay rate by classically evolving a field starting from quantum-like initial conditions [32,33] and waiting for a large enough energy fluctuation that goes over the potential barrier. However, this can only be a rough approximation because an interacting classical field theory tends to evolve toward a thermal state where energy is equipartitioned among all modes.

energy transfer between normal fields and ghosts can take place. As a consequence, assuming no protection, in Sec. IV D, we use statistical methods to compute the energy transfer between normal fields and ghost fields. Finally, in Sec. IV E, we compare analytic results to numerical classical lattice simulations (using the convenient discretized field equations described in Appendix C).

### A. Classical equations of motion in momentum space

We consider a scalar field $\varphi(x,t)$ in $1+1$ dimensions. In a box $0 \le x \le L$ with periodic boundary condition, the scalar field is expanded in normal modes $q_n$ as

$$\varphi(x,t) = \frac{1}{\sqrt{L}} \sum_{n=-\infty}^{\infty} q_n(t) e^{ik_n x} \quad k_n = \frac{2\pi n}{L}. \tag{64}$$

We consider a real scalar field, so $q_{-n} = q_n^*$. The Lagrangian density $\mathcal{L}_\varphi = (\partial_\mu \varphi)^2/2 - m^2\varphi^2/2 + \mathcal{L}_I$ gives the Lagrangian,

$$\mathcal{L} = \int_0^L dx \mathcal{L}_\varphi = \frac{\dot{q}_0^2 - m^2 q_0^2}{2} + \sum_{n=1}^{\infty} (|\dot{q}_n|^2 - \omega_n^2 |q_n|^2) + \mathcal{L}_I,$$
$$\omega_n^2 = m^2 + k_n^2. \tag{65}$$

The $dx$ integral is simply given by $L$ times the expansion of $\mathcal{L}$, keeping only those terms such that their $e^{ikx}$ factors multiply to 1. The classical equations of motion are

$$\ddot{q}_n + \omega_n^2 q_n = \frac{\partial \mathcal{L}_I}{\partial q_n}. \tag{66}$$

Classical evolution can be restricted to real $q_n$, which means zero momentum for each mode. The averaged free classical Hamiltonian is

$$\langle H \rangle = \int dx \frac{1}{2} \langle \dot{\varphi}^2 + \varphi'^2 + m^2 \varphi^2 \rangle = \sum_{n=-\infty}^{+\infty} \omega_n^2 \langle q_n q_{-n} \rangle \tag{67}$$

so that the classical thermal state with equipartition of relativistic energy corresponds to $q_n = \sqrt{T}/\omega_n$, which is the (UV divergent) classical limit of the Bose-Einstein distribution, $\langle q_n q_{-n} \rangle = \hbar(1/2 + f_n)/\omega_n$ with $f = 1/(e^{E/T} - 1) \to T/E \gg 1$ at $E \ll T$. The extra $1/2$ is the purely quantum fluctuation. $\langle H \rangle$ is UV divergent both in classical physics at finite temperature $T$ and in quantum physics.

### B. Analytic study of one ghost resonance in field theory

As a prototypical field theory containing a normal field $\varphi_1$ interacting with a ghost field $\varphi_2$, we consider the Lagrangian of Eq. (4) where the ghost is obtained setting $\pm = -1$. For simplicity, we here compute in $1+1$ dimensions, as this is enough to encounter the new key phenomena. The two fields $\varphi_{1,2}$ have positive and negative kinetic energy, respectively. We expand each of them in normal modes $q_{n_1}$ and $q_{n_2}$ as outlined in the previous section. The interactions among momentum modes $q_{n_i}$ are complicated because locality is not manifest. Let us focus on four generic modes: $n_1$ and $n_1'$ for $\varphi_1$ and $n_2$ and $n_2'$ for $\varphi_2$. We assume that $k_{n_1} + k_{n_1'} + k_{n_2} + k_{n_2'} = 0$. Then, their interaction term is

$$\int dx \varphi_1^2 \varphi_2^2 = \frac{4}{L} (q_{n_1} q_{n_1'} q_{n_2} q_{n_2'} + q_{-n_1} q_{-n_1'} q_{-n_2} q_{-n_2'} + q_{n_1} q_{-n_1} q_{n_2} q_{-n_2}$$
$$+ + q_{n_1'} q_{-n_1'} q_{n_2} q_{-n_2} + q_{n_1} q_{-n_1} q_{n_2'} q_{-n_2'} + q_{n_1'} q_{-n_1'} q_{n_2'} q_{-n_2'} + \cdots). \tag{68}$$

The frequencies are generically off resonance, but for some choice of momenta, they satisfy resonant conditions such as $N_1 \omega_{n_1} + N_2 \omega_{n_1'} - N_3 \omega_{n_2} - N_4 \omega_{n_2'}$ even for $N_i = \pm 1$, giving rise to on-shell processes.

We isolate a subsystem of four such degrees of freedom $q_{n_i}$. For simplicity, we can assume that their initial conditions are real so that they remain real and we can treat $q_n = q_{-n}$ as a single degree of freedom. Moving to action-angle variables and simplifying the notation, we write their pulsations as $\omega_{1,2,3,4}$ and their actions as $J_{1,2}$ (positive energy) and $J_{3,4}$ (negative energy). The Hamiltonian of the subsystem is

$$H = \omega_1 J_1 + \omega_2 J_2 - \omega_3 J_3 - \omega_4 J_4 + \epsilon \left( \frac{J_1}{\omega_1} \frac{J_3}{\omega_3} \sin^2 \Theta_1 \sin^2 \Theta_3 + \frac{J_1}{\omega_1} \frac{J_4}{\omega_4} \sin^2 \Theta_1 \sin^2 \Theta_4 \right.$$
$$\left. + + \frac{J_2}{\omega_2} \frac{J_3}{\omega_3} \sin^2 \Theta_2 \sin^2 \Theta_3 + \frac{J_2}{\omega_2} \frac{J_4}{\omega_4} \sin^2 \Theta_2 \sin^2 \Theta_4 + 2 \sqrt{\frac{J_1}{\omega_1} \frac{J_2}{\omega_2} \frac{J_3}{\omega_3} \frac{J_4}{\omega_4}} \sin \Theta_1 \sin \Theta_2 \sin \Theta_3 \sin \Theta_4 \right), \tag{69}$$

where $\epsilon = 8\lambda/L$. Off resonance, the system is stable, and we now study the possibly dangerous resonant case, assuming $\omega_1 + \omega_2 - \omega_3 - \omega_4 \equiv \Delta\omega \simeq 0$.[8] Close to resonance, the normal resonant form at leading order is

$$H \simeq \omega_1 J_1{}' + \omega_2 J_2{}' - \omega_3 J_3{}' - \omega_4 J_4{}' + \frac{\epsilon}{4}\left(\frac{J_1{}'}{\omega_1}\frac{J_3{}'}{\omega_3} + \frac{J_1{}'}{\omega_1}\frac{J_4{}'}{\omega_4} + \frac{J_2{}'}{\omega_2}\frac{J_3{}'}{\omega_3} + \frac{J_2{}'}{\omega_2}\frac{J_4{}'}{\omega_4} + +2\sqrt{\frac{J_1{}'}{\omega_1}\frac{J_2{}'}{\omega_2}\frac{J_3{}'}{\omega_3}\frac{J_4{}'}{\omega_4}}\cos(\Theta_1{}' + \Theta_2{}' + \Theta_3{}' + \Theta_4{}')\right). \tag{70}$$

We isolate the auxiliary system by the canonical change of variables generated by

$$W = \mathcal{J}(\Theta_1' + \Theta_2' + \Theta_3' + \Theta_4')/4 + \mathcal{E}_2\Theta_2' + \mathcal{E}_3\Theta_3' + \mathcal{E}_4\Theta_4'; \tag{71}$$

i.e., $4\mathcal{Q} = \Theta_1' + \Theta_2' + \Theta_3' + \Theta_4'$ and $J_1' = \mathcal{J}/4$, $J_i' = \mathcal{J}/4 + \mathcal{E}_i$. The resonant form becomes

$$H \simeq \omega_2\mathcal{E}_2 - \omega_3\mathcal{E}_3 - \omega_4\mathcal{E}_4 + \Delta\omega\frac{\mathcal{J}}{4} + \frac{\epsilon}{4}\left[\frac{1}{\omega_1\omega_3}\frac{\mathcal{J}}{4}\left(\frac{\mathcal{J}}{4} + \mathcal{E}_3\right) + \frac{1}{\omega_1\omega_4}\frac{\mathcal{J}}{4}\left(\frac{\mathcal{J}}{4} + \mathcal{E}_4\right) + + \frac{1}{\omega_2\omega_3}\left(\frac{\mathcal{J}}{4} + \mathcal{E}_2\right)\left(\frac{\mathcal{J}}{4} + \mathcal{E}_3\right) + \frac{1}{\omega_2\omega_4}\left(\frac{\mathcal{J}}{4} + \mathcal{E}_2\right)\left(\frac{\mathcal{J}}{4} + \mathcal{E}_4\right) + \frac{1}{\sqrt{\omega_1\omega_2\omega_3\omega_4}}\sqrt{\frac{\mathcal{J}}{4}\left(\frac{\mathcal{J}}{4} + \mathcal{E}_2\right)\left(\frac{\mathcal{J}}{4} + \mathcal{E}_3\right)\left(\frac{\mathcal{J}}{4} + \mathcal{E}_4\right)}\cos 4\mathcal{Q}\right] \tag{72}$$

so that $\mathcal{E}_{1,2,3}$ are constant of motion; i.e., all $J_i'$ vary by a common amount $\mathcal{J}/4$. The important result is that $\cos 4\mathcal{Q}$ cannot dominate over the sum of other terms so that this resonance does not lead to ghost runaway, but only to a partial violation up to $\mathcal{O}(1)$ factors of the hidden conservation law. This means that the local interaction $\varphi_1^2\varphi_2^2$ of field theory gives, when expanded in normal modes, a specific set of interactions among them such that each on-shell resonance allows an order one energy transfer among the modes, but no ghost runaway.

### C. Analytic study of multiple ghost resonances in field theory

We next need to study what is the collective effect of the infinite number of such resonances present in the continuum limit: the number of modes $N = L/a$ diverges when the lattice cut off $a$ becomes infinitesimally small, or the box size $L$ infinitely large. The Hamiltonian in action-angle variables is an infinite sum of terms like those discussed in the previous section,

$$H = \sum_{n_1=-\infty}^{+\infty}\omega_{n_1}J_{n_1} - \sum_{n_2=-\infty}^{+\infty}\omega_{n_2}J_{n_2} + \epsilon\sum_{n_1,n_1',n_2,n_2'}\delta_{0,n_1+n_1'+n_2+n_2'}\sqrt{J_{n_1}J_{n_1'}J_{n_2}J_{n_2'}} \times \sin\Theta_{n_1}\sin\Theta_{n_1'}\sin\Theta_{n_2}\sin\Theta_{n_2'}, \tag{73}$$

with $\epsilon = 2\lambda/L(\omega_{n_1}\omega_{n_1'}\omega_{n_2}\omega_{n_2})^{1/2}$. The rough argument goes as follows. At small coupling, the theory contains $2N$ quasi-integral of motion: one for each degree of freedom. In the continuum limit, the number of resonances scales as $N^2$ (out of the 4 momenta, two combinations are fixed by momentum conservation and resonance condition, i.e., energy conservation). Each resonance produces the partial loss of a quasi-integral of motion $\mathcal{E}$. Asymptotically, all quasi-integrals of motion are lost, and the available phase space is filled up, allowing for ghost runaway.

The argument above can be made more precise. A combination is resonant if the detuning $\Delta\omega \equiv \omega_{n_1} + \omega_{n_1'} - \omega_{n_2} - \omega_{n_2}$ is smaller than the expansion parameter $\epsilon J$, where $J$ is the typical value of the actions, e.g., $J = T/\omega$ for a thermal state. For finite $L$, the resonance is not exactly satisfied and the expansion parameter is finite. Both quantities go to zero in the continuum limit, so a careful analysis is needed. Let us consider modes up to an UV cut off $k \lesssim k_{\max}$. A resonance that would be perfect in the continuum acquires, in view of the discreteness $\delta k = 2\pi/L$,

[8] We assume for now that no other combinations are vanishing so that resonances do not "overlap". In Appendix B, we show that the case of all frequencies close to each other leads to similar conclusions as the ones discussed here.

a typical detuning $\Delta\omega \approx (8\pi/L)(k_{\max}/\omega_{\max})$. Here, $\omega_{\max}$ is the frequency corresponding to $k_{\max}$ having ignored, for simplicity, that it differs for fields $\varphi_1$ and $\varphi_2$ if $m_1 \neq m_2$. The fraction of such interactions that are resonant for finite $L$ is $f = \epsilon J/\Delta\omega$. This stays finite in the continuum limit, as both $\epsilon$ and $\Delta\omega$ scale as $1/L$. So, the ghost is not protected when $fN^2 \gtrsim N$ i.e., $N = L/a \gtrsim 1/f \sim \omega^3/\lambda T$. Then, the action $J_n$ of one typical microscopic mode can change by order one on a time-scale $\Gamma \sim \lambda T/\omega^2$, linear in $\lambda$ at leading order. As discussed in the next section, the macroscopic properties of the system evolve on a slower time-scale $1/\tau = \Gamma/N \sim \lambda^2 T^2/\omega^5$. As we will see, this is the scale of the instability time. If, instead, there were no microscopic protection for the single modes, the instability time would have been much faster, linear in $\lambda$.

### D. The ghost runaway rate

Based on the previous discussion, we assume that the extra quasiconserved energies get violated in field theory by resonances. Then, the system evolves statistically toward the direction that increases total entropy $S = S_1 + S_2$, where 1 is the positive-energy sector and 2 is the ghost. We define the ghost temperature $T_2$ as the average ghost energy $E_2 \leq 0$ per degree of freedom, $T_2 = E_2/N \leq 0$. Let us compute $S_2$. The volume in phase space is easily found in action-angle variables:

$$\mathcal{V}_2 = (2\pi)^N \frac{N^N |T_2|^N}{N!}. \tag{74}$$

The factor of $(2\pi)^N$ is the contribution of the angle variables, whereas the remaining factor is the volume of the simplex $\sum \omega_n J_n \leq |E_2|$. Therefore, the ghost entropy is

$$S_2 = N \log |T_2| \tag{75}$$

up to a $T_2$-independent constant. The total entropy $S = S_1 + S_2$ of the system at fixed total energy $E_1 + E_2$ is maximal when

$$\delta S = \frac{\partial S}{\partial E_1}\delta E_1 + \frac{\partial S}{\partial E_2}\delta E_2 = \delta E_1 \left(\frac{1}{T_1} - \frac{1}{T_2}\right) = 0, \tag{76}$$

which can only occur for $T_1 \to \infty$ and $T_2 \to -\infty$. Heat flows from the ghost to the positive-energy system, and the thermodynamic evolution eventually causes the runaway on a time-scale $\tau$, which we now compute.

We consider a theory in $d$ spatial dimensions with the Lagrangian of Eq. (4). To set the formalism, we first assume that both fields $\varphi_{1,2}$ have positive kinetic energy. Then, starting from temperatures $T_{1,2} \geq 0$, they thermalize toward the equilibrium state with a common temperature $T = (T_1 + T_2)/2$ via the $\lambda\varphi_1^2\varphi_2^2/2$ interaction. The thermalization process can be computed using Boltzmann equations. We consider their well-known quantum expression and perform its classical limit, to later compare with numerical classical evolution on a lattice. In order to keep $\hbar$ factors explicit, it is convenient to express quadrimomenta $P_\mu$ in terms of wave vectors, $P_\mu = (E, \vec{p}) = \hbar K_\mu = \hbar(\omega, \vec{k})$. The Lagrangian $\mathcal{L}$ contains no $\hbar$ factors, so the mass parameters $m_{1,2}$ have dimension 1/time. The contribution of $12 \leftrightarrow 1'2'$ scatterings to the Boltzmann equation for the energy density $\rho_1$ (assumed to be spatially homogeneous) of $\varphi_1$ at leading order in the interaction $\lambda$ is

$$\dot{\rho}_1 = -\int d\vec{k}_1 d\vec{k}_2 d\vec{k}'_1 d\vec{k}'_2 E_1 (2\pi)^{d+1} \delta(K_1 + K_2 - K'_1 - K'_2) |\mathcal{A}|^2 F, \tag{77}$$

where $\mathcal{A} = 2\hbar\lambda$ is the amplitude; $d\vec{k} = d^d k/2\omega(2\pi)^3$ is the usual relativistic phase space; one can symmetrize $E_1 \to (E_1 - E'_1)/2$. Finally, $F$ depends on particle number densities $dn_i = f_i d^d k_i/(2\pi)^d$:

$$F = f_1(E'_1) f_2(E'_2)[1 + f_1(E_1)][1 + f_2(E_2)] - f_1(E_1) f_2(E_2)[1 + f_1(E'_1)][1 + f_2(E'_2)]. \tag{78}$$

It vanishes when Bose-Einstein distributions $f(E) = 1/(e^{E/T} - 1)$ realize thermal equilibrium. Total energy is conserved, so $\dot{\rho}_2 = -\dot{\rho}_1$. The quantum Boltzmann Eq. (77) has two classical limits: particle and wave. The particle limit corresponds to small occupation numbers $f \ll 1$ such that $1 + f \simeq 1$ and $f \simeq e^{-E/T}$. We are here interested in the wave classical limit, which corresponds to large occupation numbers $f \simeq T/E \gg 1$. The classical wave term arises at leading order $f^3$ [34,35], where

$$F \simeq f_1(E_1) f_2(E_2)[f_1(E'_1) + f_2(E'_2)] - f_1(E'_1) f_2(E'_2)[f_1(E_1) + f_2(E_2)]. \tag{79}$$

In this limit, $\hbar$ factors cancel leaving the classical Boltzmann equation,

$$\dot{\rho}_1 = -4\lambda^2 \int d\vec{k}_1 d\vec{k}_2 d\vec{k}'_1 d\vec{k}'_2 \omega_1 (2\pi)^{d+1} \delta(K_1 + K_2 - K'_1 - K'_2) \times \frac{\omega_1 - \omega'_1}{\omega_1 \omega'_1 \omega_2 \omega'_2} T_1 T_2 (T_1 - T_2), \tag{80}$$

where the latter term is $\hbar^3 F$. One can similarly compute the contribution to $\dot{\rho}_1$ from $11' \leftrightarrow 22'$ scatterings. Furthermore, a $g\varphi_1^2\varphi_2/2$ interaction among positive-energy fields $\varphi_{1,2}$ gives rise to $2 \leftrightarrow 11'$ decays for $m_2 > 2m_1$ such that

$$\dot{\rho}_1 = -\int d\vec{k}_1 d\vec{k}_1' d\vec{k}_2' \omega_1 (2\pi)^{d+1} \delta(K_1 + K_1' - K_2) |\mathcal{A}|^2 F, \tag{81}$$

with $\mathcal{A} = g\hbar^{1/2}$ and

$$F = f_1(E_1) f_2(E_1')[1 + f_2(E_2)] - f_2(E_2)[1 + f_1(E_1')][1 + f_2(E_1')] \simeq \frac{T_1(T_1 - T_2)}{E_1 E_1'} \tag{82}$$

in the classical limit.

We can now repeat the computation assuming that $\varphi_2$ is a ghost. Boltzmann equations again involve a sum over on-shell processes, and the resonance condition among $\omega$'s now has an extra $-$ sign when a ghost is involved; see, e.g., Eq. (28). This is equivalent to telling that ghosts appear with negative energy in the quantum Boltzmann equations. One can reexpress the unusual (negative-energy) kinematical integrals in terms of usual (positive-energy) ones by rewriting each ghost wave vector as $K_\mu = -\tilde{K}_\mu$ so that a negative-energy particle in the initial (final) state becomes a positive-energy particle in the final (initial) state. In the limit where each field is thermal, the Bose-Einstein distribution satisfies the identity $f(E/T) = -(1 + f(-E/T))$, so statistical factors too match those of the positive-energy process, up to an overall $-$ sign when an odd number of ghosts is flipped. Let us consider some examples:

(i) A $\varphi_1^2 \varphi_2^2$ ghost interaction allows the kinematically open on-shell processes $12 \leftrightarrow 1'2'$ and $11'22' \leftrightarrow \varnothing$, which become $1\tilde{2}' \leftrightarrow 1\tilde{2}$ and $11' \leftrightarrow \tilde{2}\tilde{2}'$. In the classical limit, one then has $\dot{\rho}_1 \propto +T_1 T_2 (T_2 - T_1)$ both in the ghost and the nonghost cases.

(ii) A $\varphi_1 \varphi_2^2$ ghost interaction allows the kinematically open on-shell process $122' \leftrightarrow \varnothing$, which becomes a $1 \leftrightarrow \tilde{2}\tilde{2}'$ decay. In the classical limit one then has $\dot{\rho}_1 \propto +T_2(T_2 - T_1)$ both in the ghost and the nonghost cases.

(iii) A $\varphi_1^2 \varphi_2$ ghost interaction allows the kinematically open on-shell process $\varnothing \leftrightarrow 11'2$ that becomes a $\tilde{2} \leftrightarrow 11'$ decay. In the classical limit, one then has $\dot{\rho}_1 \propto +T_1(T_2 - T_1)$ in the nonghost case, which becomes $\dot{\rho}_1 \propto -T_1(T_2 - T_1)$ in the ghost case.

The factors $F$ vanish in the thermal limit with a common temperature, $f(E) = 1/(e^{E/T} - 1)$. However, ghosts have $E_2 < 0$, so that a physical $f(E_2) \geq 0$ is obtained for $T_2 \leq 0$: ghosts must have a negative temperature.[9] We now see the key difference that arises in the presence of a ghost; there is no thermal equilibrium at common $T$ such that the factor $F$ vanishes thanks to detailed balance, because the two systems have opposite-sign energies and thereby temperatures. In all cases listed above, this means that the nonghost system heats up, $\dot{\rho}_1 > 0$. This sign of the heat flow agrees with our earlier considerations about increase of entropy $\dot{S} \geq 0$: both $|T_1|$ and $|T_2|$ increase, as higher temperature allows for more states. Boltzmann equations add that the energy flow rate is proportional to the coupling squared.

The purely quantum effect will be studied in Sec V. We here study the classical effect, which can be isolated as long as the low-frequency modes excited classically $\omega \lesssim \omega_{\max}$ are separated from the high-frequency modes at which the divergent quantum effect starts giving a larger contribution to $\dot{\rho}_1$. In such a case, the quantum contribution is smaller than the classical contribution assuming a cut off $\Lambda_{\rm UV} \gtrsim \omega_{\max}$.

We next compare these analytic results with numerical classical simulations in toy models and finally provide estimates for situations of physical interests.

### E. Results

First, we simulate the classical thermalization among two positive-energy fields, finding that the simulated rate agrees with the rate obtained from Boltzmann equations, such as Eq. (77).

We next consider a positive-energy field $\varphi_1$ interacting with a ghost field $\varphi_2$. We numerically simulate their time evolution for $m_{1,2} = 1$ and $\lambda = 0.01$ in $1 + 1$ dimensions on a lattice with spacing $a = 0.1$ and size $L = 200$. We start from a thermal-like distribution with $T_{1,2} = 1$ cut at the maximal momentum $k_{\max} = 200 \cdot 2\pi/L$. This means that each excited mode has an initial amplitude as extracted from the thermal distribution and a random phase. Figure 5(a) shows that the system undergoes ghost runaway. Figure 6 shows the energy spectra of $\varphi_1$ (left) and $\varphi_2$ (right) at some selected times. We see that modes at higher $k$ get progressively excited: Energy cascades toward the UV giving rise to the usual black-body instability of interacting field theories (see e.g., [40,41]). In order to disentangle this phenomenon (that lowers $T_1$) from ghost runaway (that increases $T_1$), we choose a small enough $k_{\max}$ such that modes around the cut off are still negligibly excited when ghost runaway happens.

[9]We verified that the nonequilibrium Kadanoff-Baym formalism (see e.g., [36]) gives the same Boltzmann equations. In particular, for a ghost, the form of its two thermal Wightman propagators is exchanged with respect to positive-energy fields so that initial-state ghosts are equivalent to final-state normal particles. In this formalism, $f \geq 0$ because it is the expectation value of a positive number operator.
Previous literature studied possible thermal equilibrium thermodynamics for Lee-Wick resonances with negative classical energy [37–39] finding contradictory results. We now see that there is no thermal equilibrium.

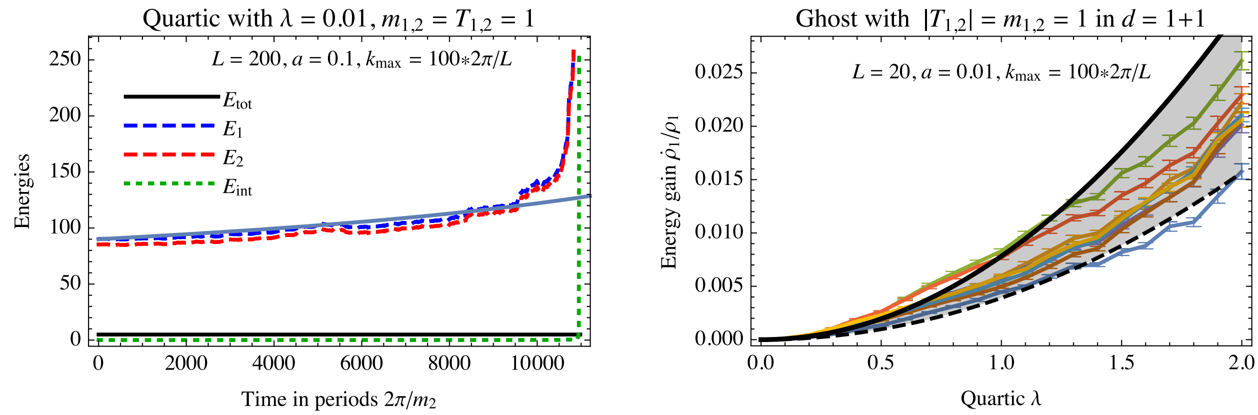


FIG. 5. Left: time evolution of the total energies of the normal field, of the ghost field, of their interaction energy, of the total conserved energy. The continuous curve is the analytic approximation. Right: heat flow $d\rho_1/dt$ as function of the coupling. The data point are from lattice simulations, for different small values of $dt$. The black curve is the analytic result; we also show the analytic result without the IR-divergent diagram that might contribute in the numerics on longer time-scales (dashed curve).

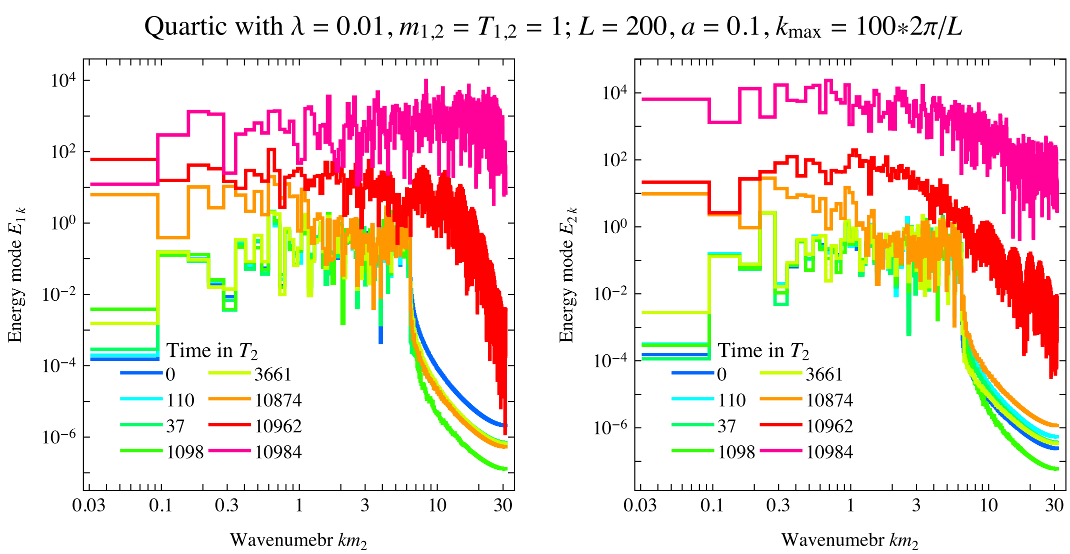


FIG. 6. Energy spectrum of the normal field (left) and of the ghost field (right) at some fixed times.

Each random initial condition with fixed temperatures produces final runaway times that differ by order one. In order to better compare with the analytic approach, which predicts the average energy flow $\dot{\rho}_1$ between the two fields, we run for a short time many different simulations with the same initial temperatures and average over them. Having assumed one spatial dimension and $m_1 = m_2$, we can analytically perform the integrals in the Boltzmann Eq. (80),

$$\dot{\rho}_1 = \frac{\lambda^2 T_1 T_2 (T_2 - T_1)}{4\pi^2 m_{1,2}^4}\left[(\ln 4 - 1) + \frac{1}{2}\left(1 + \ln\frac{Lm}{8\pi}\right)\right], \tag{83}$$

where we added, in the second term, the contribution of $11' \leftrightarrow 22'$ scatterings. This process contains a logarithmic IR divergence at vanishing relative velocity between the particles, which is typical of field theory in 1 spatial dimension.[10] Despite this aside issue, Fig. 5(b) shows that the analytic rate agrees with the numerical rate. We can next compute $\rho_1$ in terms of $T_1$,

[10]In order to isolate the IR divergence, it is useful to put the $11' \leftrightarrow 22'$ contribution to $\dot{\rho}_1$ into the form,

$$T_1 T_2 (T_2 - T_1)\frac{2\lambda^2}{\pi^2}\int_{4m^2}^{\infty} ds \int_{\sqrt{s}}^{\infty} dK_0 \times \frac{s^2 K_0^2}{s\sqrt{K_0^2 - s}(s - 4m^2)(s^2 + 4m^2(K_0^2 - s))^2}. \tag{84}$$

In lattice simulations, the IR divergence gets regulated by the finite box size $L$ so that the lower limit of $s$ integration changes into $(2m + 2\pi/L)^2 \simeq 4m^2 + 8\pi m/L$.

$$\rho_1 = \int E_1 dn_1 = T_1 \int_{2\pi/L}^{k_{\max}} \frac{dk_1}{2\pi},$$

and obtain a differential equation $\dot{T}_1 = \gamma T_1 T_2 (T_2 - T_1) = -\dot{T}_2$ that can be solved,

$$T_1(t) = \frac{T_{10} + T_{20}}{2} \times \left[1 + \left(1 + \frac{4T_{10}T_{20}}{(T_{10}+T_{20})^2} e^{\gamma t (T_{10}+T_{20})^2/2}\right)^{-1/2}\right], \tag{85}$$

obtaining the average time evolution [one example is plotted in Fig. 5(a)].

We next vary the lattice spacing, box size, and number of digits used in the numerics, finding consistent results. We also run for different values of the physical parameters; additional IR divergences arise when a field is massless. Running for special initial conditions, such as starting from a single excited mode, $f(E) \propto \delta(E - E_0)$ blocks or delays the ghost runaway until when enough modes can get excited so that many resonances can happen.

Based on the above experience, we can now consider the more complicated theory of possible physical interest: four-derivative gravity. First, the resonances caused by cubic interactions present in four-derivative gravity, while potentially unsafe, are stabilized by quartic and higher interactions, as argued at the end of Sec. II D. Then, each resonance causes an $\mathcal{O}(1)$ energy flow variation and the system as a whole evolves statistically, as described above. The massive ghost present in four-derivative gravity only has Planck-suppressed nonrenormalizable interactions. Thereby, its runaway rate $\Gamma \equiv \dot{\rho}/\rho \sim T^3/M_{\rm Pl}^2$ is smaller than the Hubble cooling rate $H \sim T^2/M_{\rm Pl}$. As usual, gravitational short-range interactions give negligible effects in big-bang cosmology.

Furthermore, inflation with Hubble constant $H$ roughly behaves as a thermal bath with temperature $T \sim H$, producing a spectrum of primordial inflationary fluctuations for the graviton, its ghost, and the other fields.

In conclusion, a ghost undergoes runaway in classical field theory, but in four-derivative gravity, ghost runaway is negligibly slow on cosmological time-scales.

## V. GHOST METASTABILITY IN QUANTUM FIELD THEORY

We again consider a field theory with two scalars $\varphi_1$ (positive energy) and $\varphi_2$ (negative-energy ghost) in $d$ space dimensions. We want to compute the purely quantum rate for the qualitatively new processes where particles are emitted from the Lorentz-symmetric vacuum. For example, a $g\varphi_1^2\varphi_2/2$ interaction allows for the three-body process $\varnothing \leftrightarrow 11'2$.

The rates of such processes can be obtained from the finite-temperature rates discussed in the previous section in the limit $T_1 \to 0^+$, $T_2 \to 0^-$ and thereby $f_{1,2} \to 0$, $1 + f_{1,2} \to 1$. Following the discussion in Sec. IV D, it is convenient to rewrite the Boltzmann equation in terms of positive energies $\tilde{2} \leftrightarrow 11'$ by defining $\tilde{K}_2 = -K_2$. Since $f_2(-E_2/T_2) \to -1$, the statistical factor at zero temperature is $F \to -1$, while it would be $F = 0$ for a usual process involving only positive-energy particles. The resulting quantum rate for the three-body process $\varnothing \leftrightarrow 11'2$,

$$\dot{\rho}_1 = \frac{\hbar^2 g^2}{2^{3d-1}\pi^{d-1}\Gamma(d/2)^2} \frac{(m_2^2 - 4m_1^2)^{\frac{d}{2}-1}}{m_2} \times \int_{m_2}^{\infty} dK_0 K_0 (K_0^2 - s)^{\frac{d}{2}-1}, \tag{86}$$

contains a UV-divergent integral over $K_0$.

Similarly, an interaction $\lambda\varphi_1^2\varphi_2^2/2$ allows for the four-body process $\varnothing \leftrightarrow 11'22'$ that leads to the energy flow rate,

$$\dot{\rho}_1 = \int d\vec{k}_1 d\vec{k}'_1 d\vec{k}_2 d\vec{k}'_2 E_1 (2\pi)^{d+1} \times \delta(K_1 + K'_1 - \tilde{K}_2 - \tilde{K}'_2) \frac{1}{2} |\mathcal{A}|^2. \tag{87}$$

By introducing $K \equiv K_1 + K'_1 = \tilde{K}_2 + \tilde{K}'_2$ and $s \equiv K^2$, it becomes

$$\dot{\rho}_1 = \frac{\hbar^3 \lambda^2}{2^{5d-3}\pi^{\frac{3d}{2}-1}\Gamma(d/2)^3} \int_{4m^2}^{\infty} ds \frac{(s - 4m^2)^{d-2}}{s} \times \int_{\sqrt{s}}^{\infty} dK_0 K_0 (K_0^2 - s)^{\frac{d}{2}-1}. \tag{88}$$

Again, the integral over $K_0$ is UV divergent.

This new divergence arises because, unlike in the thermal case, the vacuum initial state $\varnothing$ is now Lorentz-invariant so that the final state too must be the same in all frames. This is why the rate contains a $dK_0$ integral over the noncompact Lorentz group.

This is the same divergent boost integral discussed by [42,43] (and, more recently, by [44]). These early studies of vacuum decay considered a theory containing a scalar with positive kinetic energy (no ghost) and assumed that its potential $V$ contains a local minimum e.g., with $V = 0$ and a deeper minimum with $V < 0$. The vacuum decay bubble with mass $m = 0$ can appear with any initial velocity, giving rise to the divergent Lorentz integral [42,43]. Furthermore, by, e.g., increasing its radius, one obtains field configurations with generic $m_2 < 0$ that thereby have negative energy with $K_2 = (m_2, \vec{0})$. Such ghost configurations can be emitted from the vacuum together with one particle with positive energy $K_1 = (m_1, \vec{0})$, for $m_1 + m_2 = 0$. Due to relativistic invariance, this process happens with the same amplitude for arbitrarily boosted $K_2$ and $K_1$, giving rise to a divergent $dK_0$ integral over boosts [44].

One then wonders if both $K$ instability (ghosts) and $V$ instability (vacuum tunneling) proceed with infinite rate, in contradiction with our usual understanding of vacuum tunneling as exponentially slow [44].

In the case of $V$ instability, Coleman [30] and more recently [45] interpreted the Lorentz boost divergence as emission of lots of extra quanta, i.e., that the naive perturbative computation is not expanding the path integral around the right saddle point.[11] These authors argue that vacuum tunneling must instead be computed expanding around a Lorentz-invariant bounce configuration, such that an integral over the Lorentz group is not needed because it would be an overcounting of the same configuration. Accepting this argument, the WKB approximation allows one to find the desired configuration as the bounce instanton that minimizes an effective Euclidean action. The bounce is the solution to the scalar field equations that only depends on the Euclidean $r_{\rm E}^2 = x^2 + y^2 + z^2 + (it)^2$ (the Euclidean Lorentz group is compact) and has the desired boundary conditions: false vacuum at $r \to \infty$ and over the barrier at $r \to 0$:

$$\begin{cases} \varphi_i(r) = 0 & \text{as } r \to \infty, \text{false vacuum} \\ \dot{\varphi}_i(r) = 0 & \text{as } r \to 0, \text{true vacuum} \end{cases}. \tag{89}$$

The resulting vacuum decay rate is exponentially suppressed by the coupling, $e^{-\mathcal{O}(1)/\lambda}$.

In the ghost case, we do not have a similarly simple formulation nor a positive Euclidean action. Unless a suitable continuation is found, a brute-force computation is needed to establish if the ghost decay rate is exponentially suppressed (restricting the action to Lorentz-invariant field configurations removes field-theory resonances but leads to $r$-dependent frequencies). We speculate that, if the vacuum decay rate will turn out to be exponentially suppressed, the difficulties that seem to hinder unitarity and/or renormalizability of Minkowskian theories with ghosts (see e.g., [49–52]) will turn out to be similarly suppressed by similar factors.

## VI. CONCLUSIONS

Systems containing positive kinetic energy $K_1$ interacting with negative kinetic energy $K_2$ can undergo a runaway where the total energy $E = K_1 + K_2 + V$ is constant while $|K_i| \to \infty$. Thereby, negative kinetic energy is considered as unphysical and dubbed ghost. We explored the possibility that negative kinetic energy can be physically acceptable because metastable up to cosmological times, similarly to negative potential energy. In order to exclude this possibility, we started from the simplest limit (classical mechanics), but we found that a weakly-interacting ghost behaves almost as well as a free ghost:

(i) In Sec. II, we found that *ghosts are metastable in classical mechanics*. Recent numerical studies rediscovered that, in some cases, energies of individual degrees of freedom surprisingly remain confined to a finite region despite that no constant of motion imposes such lock-down. Ghost metastability is understood using the same mathematical techniques developed in the past centuries to study if multibody systems like the solar system are stable up to cosmological times despite that individual planets can acquire enough energy to escape. One diagonalizes the classical Hamiltonian by performing a perturbative expansion around the limit where each degree of freedom undergoes periodic motion with pulsation $\omega_i$. Technically, this means finding a canonical transformation to action-angle variables such that the Hamiltonian does not depend on angle variables. If interactions are strong, outside the convergence radius of the perturbative series, motion is chaotic, planets escape, and ghosts runaway. If interactions are weak, the perturbative series is convergent; planets undergo quasiperiodic motion with epicycles, and ghosts are stable. The dimensionless expansion parameter is the energy in the interaction term divided by the energy in the free quadratic part of the Hamiltonian. Ghost lockdown within finite regions of phase space is understood as being due to hidden quasiconstants of motion present in almost generic theories at weak coupling. Extending toward infinite time reveals an exponentially suppressed runaway rate, which we controlled in some model.

Actually, some physical systems are metastable ghosts, such as asteroids around the Lagrangian point 4 (Appendix A 1) or electrons in magnetic fields plus a destabilizing radial force (Appendix A 2).

(ii) However, the perturbative series contains terms proportional to $1/(N_1\omega_1 - N_2\omega_2)$ where $N_i$ are integers that grow at higher orders. One can thereby encounter resonances where such terms are large or divergent. The most dangerous case arises at leading order $N_{1,2} = 1$ when $\omega_1 = \omega_2$. We studied what happens using *resonant normal forms*: Some interactions lead to ghost runaway, and others only to order-one violations of hidden quasiconstant of motion. We argued that the latter situation seems quite generic in the presence of multiple interactions.

In order to exclude a ghost, we then moved to less simple limits:

(i) In Sec. III, we argued that *ghosts are metastable in quantum mechanics*. We first performed a brute-force computation in our toy model. Wave functions with no nodes (the groundlike state with lowest positive energy and highest negative energy) get

[11] Other authors regulate the boost divergence through cosmology adding a Lorentz-breaking or nonlocal cutoff [46–48].

exponentially suppressed away from the origin even into the dangerous new region that leads to ghost runaway (large $|K_i|$ and small $K_1 + K_2$). The ghost runaway time is thereby exponentially suppressed at small coupling analogously to usual tunneling. In general, tunneling can be approximated in the semi-classical limit, which inherits the good properties of ghosts in classical mechanics. We could however not generalise the WKB simple formula to the ghost case.

(ii) In Sec. IV, we studied *classical field theory*. The infinite number of degrees of freedom give rise to new phenomena. One is the black-body problem of interacting classical field theories, which complicates our study. More relevant for us is the presence of an infinite number of modes with different frequencies and thereby an infinite number of resonances, which correspond to the usual on-shell decays and scatterings. Each resonance is potentially deadly in the presence of ghosts. By expanding examples of local interactions in terms of momentum modes, we found specific resonances that do not immediately lead to runaways, but only to partial loss of hidden constants of motion. Nevertheless, we argued that the infinite number of resonances makes ghosts unprotected in the continuum limit. Based on general entropy arguments, we found that there is no thermal state when a system with positive temperature $T_1 > 0$ interacts with a ghost system with negative $T_2 < 0$; heat keeps flowing such that both $|T_{1,2}|$ increase up to infinity. By writing Boltzmann equations in specific models, we computed the rate of such process, finding that it is quadratic in the couplings, rather than nonperturbatively suppressed. We validated this finding by evolving classical field theories on appropriate lattice discretizations. In principle, both our analytic understanding and the numerics might have missed hidden properties that keep ghosts stable, but various checks do not find evidence in this sense.

We next considered the case of four-derivative gravity—a renormalizable theory of gravity containing a spin-2 field with negative kinetic energy and gravitational interactions only—finding that the ghost runaway time is negligible on cosmological time-scales.

In order to exclude such ghost, we finally considered the theory currently considered as fundamental.

(i) In Sec. V, we considered relativistic quantum field theory in the presence of a ghost. Since the initial vacuum state is Lorentz invariant (unlike a thermal state), the naive tree-level vacuum decay rate contains a divergent integral over the noncompact Lorentz group, which describes an arbitrary boost of the same final state. We recalled that this same problem was encountered in early computations of vacuum decay due to potential instability; even in the absence of a ghost, negative potential energy gives rise to field configurations that behave as a ghost. Using WKB Euclidean techniques, Coleman argued that the vacuum decay rate is finite and exponentially suppressed. We could not extend such tecniques to the case of ghost instability, so we do not know if it is fast (thereby ruling out theories containing ghosts) or exponentially suppressed at small couplings.

It will be important to fully clarify if negative kinetic energy can be metastable up to cosmological time-scales, as the negative-energy quantization of four-derivative gravity would provide a renormalizable theory of quantum gravity.

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## APPENDIX A: PHYSICAL SYSTEMS DESCRIBED BY GHOSTS

### 1. Asteroids around the Lagrangian point $\mathcal{L}_4$

Let us consider an asteroid with negligible mass around Lagrangian point $\mathcal{L}_4$ of the Sun and Jupiter system. The quadratic part of the asteroid Hamiltonian contains a negative frequency (see e.g., [26]); we next show that it is a ghost degree of freedom (negative kinetic energy).

The Hamiltonian of a free particle with mass $m$ in a reference frame rotating with angular velocity $\omega$ around the $z$ axis is $H_{\rm free} = \vec{p}^2/2m + \omega(yp_x - xp_y)$. We compute the Hamiltonian of an asteroid in the center-of-mass frame of the Sun and Jupiter system, where the Sun is fixed at $\vec{x}_S = (-\mu, 0, 0)$ and Jupiter at $\vec{x}_J = (1-\mu, 0, 0)$. In suitable units, their masses are $M_J = \mu$ and $M_S = 1-\mu$. The asteroid Hamiltonian in the $x$, $y$ plane is

$$H = \frac{\vec{p}^2}{2} + yp_x - xp_y - \frac{M_S}{|\vec{x}-\vec{x}_S|} - \frac{M_J}{|\vec{x}-\vec{x}_J|}. \quad \text{(A1)}$$

The momentum $p$ has a possible stationary point at $z = 0$, $p_x = -y$ and $p_y = x$. Inserting this in $H$ gives an effective potential with stationary points along the $x$ axis, as well as at the $\mathcal{L}_4$ points $x = \frac{1}{2} - \mu$ and $y = \pm\sqrt{3}/2$. Interesting motion happens along the $xy$ plane, and we can ignore motion along the $z$ axis.

Expanding $H$ around $\mathcal{L}_4$ gives, at quadratic order,

$$H_2 = \frac{p_x^2 + p_y^2}{2} + yp_x - xp_y + \frac{x^2}{8} - \frac{5y^2}{8} + \frac{\sqrt{27}}{4}(2\mu - 1)xy. \quad \text{(A2)}$$

Writing such quadratic part of the Hamiltonian as $H_2 = \frac{1}{2}v_i \hat{H}_{ij} v_j$ where $v \equiv (x, y, p_x, p_y)$, the Hamilton equations are $\dot{v} = \hat{J}\,\hat{H}\,v$, where

$$\hat{J} = \begin{pmatrix} 0_{2\times 2} & \mathbb{I}_{2\times 2} \\ -\mathbb{I}_{2\times 2} & 0_{2\times 2} \end{pmatrix} \tag{A3}$$

is the symplectic invariant tensor. The eigenvalues of $\hat{J}\,\hat{H}$ give the frequencies of the normal modes. Since $\hat{H}$ is real and symmetric, if $\lambda$ is an eigenvalue, then $-\lambda, \lambda^*, -\lambda^*$ too are eigenvalues. Thus, we can write the four eigenvalues as $(i\omega_1, -i\omega_1, i\omega_2, -i\omega_2)$. We are interested in the case where $\omega_{1,2}$ are real so that the solutions to the equations of motions for the linearized Hamiltonian $H_2$ are stable oscillations rather than exponential tachyonic solutions (a free $2\times 2$ Hamiltonian has eigenvalues $\pm i\omega$, such that $e^{\pm i\omega t}$ solutions give sine and cosine). Restricting without loss of generality to the interval $0 < \mu < 1/2$, the eigenvalues are imaginary for $0 < \mu < \mu_{\rm Routh}$ where $\mu_{\rm Routh} = \frac{1}{2}(1 - \sqrt{23/27}) \approx 3.9\times 10^{-2}$ (the Jupiter-Sun system corresponds to $\mu \approx 0.95\times 10^{-3}$). One finds the frequencies

$$\omega_{1,2} = \sqrt{\frac{1\pm r}{2}} \quad \text{where } r = \sqrt{1 - 27\mu(1-\mu)}. \tag{A4}$$

$H_2$ is not positive definite, signaling the presence of a tachyon (negative potential energy) and/or of a ghost (negative kinetic energy). To clarify, we identify the normal modes by bringing $H_2$ to normal form through a linear change of variables $v = \hat{N} v'$, where $\hat{N}$ must be real and symplectic (i.e., $\hat{N}^T \hat{J}\,\hat{N} = \hat{J}$) in order to preserve the Hamiltonian structure of the equations of motion. The needed Sp(4) rotation is [53]

$$\hat{N} = \left(\frac{\mathrm{Re}(z_1)}{\sqrt{|c_1|}}, \frac{\mathrm{Re}(z_2)}{\sqrt{|c_2|}}, \mathrm{sign}(c_1)\frac{\mathrm{Im}(z_1)}{\sqrt{|c_1|}}, \mathrm{sign}(c_2)\frac{\mathrm{Im}(z_2)}{\sqrt{|c_2|}}\right), \tag{A5}$$

where $z_j$ are the complex eigenvectors of $\hat{J}\,\hat{H}$ corresponding to the eigenvalues $+i\omega_j$ (the opposite convention is also applicable), and $c_j = \mathrm{Re}(z_j)^T \hat{J} \mathrm{Im}(z_j)$. Writing $v' = (q_1, q_2, p_1, p_2)$, the diagonalized Hamiltonian is

$$H_2 = \omega_1 \frac{p_1^2 + q_1^2}{2} - \omega_2 \frac{p_2^2 + q_2^2}{2}. \tag{A6}$$

As expected, $H_2$ is not positive definite, and the ghost is $q_2$, $p_2$. At linear order the system is stable, because the two oscillators do not interact. At higher order, the ghost couples to the normal oscillator, and one might expect quick runaway. Still, asteroids remain close to $\mathcal{L}_4$ for exponentially long time [26].

One can maybe more intuitively see how a positive-energy particle written in a rotating frame becomes a ghost in the Lagrangian formalism. A free particle is described by $L = (\dot{x}^2 + \dot{y}^2)/2 + \omega(x\dot{y} - y\dot{x}) + \omega^2(x^2+y^2)/2$. The second term is the Coriolis force. The third term is the centrifugal force; kinetic energy becomes a potential term. Thereby, extra potential terms (such as gravity) can modify the kinetic term, giving rise to a ghost.

#### 2. Charged particle in a magnetic field

The Hamiltonian of a nonrelativistic particle with mass $m$ and electric charge $e$ in a constant magnetic field $\vec{B} = (0, 0, B_z)$ described by the vector potential $\vec{A} = \vec{B}\times\vec{r}/2$ is

$$H_0 = \frac{(\vec{p} - e\vec{A})^2}{2m} + e\varphi = \frac{\vec{p}^2}{2m} + \omega_B(yp_x - xp_y) + \frac{m}{2}\omega_B^2(x^2+y^2). \tag{A7}$$

The first two terms are equal to the Hamiltonian of a free particle written in a frame rotating with cyclotron frequency $\omega_B = eB_z/2m$. The equations of motion give $m\dot{\vec{x}} = \vec{p} - e\vec{A}$, showing that the magnetic force does not affect energy. We add to $H_0$ a *de*stabilizing potential $\delta H = -m\omega_0^2(x^2+y^2)/2$, $H = H_0 + \delta H$. The eigenvalues of $\hat{J}\,\hat{H}$ are $\pm i\omega_\pm$, with

$$\omega_\pm = \omega_B \pm \delta\omega \quad \text{where } \delta\omega = \sqrt{\omega_B^2 - \omega_0^2}. \tag{A8}$$

For $0 < \omega_0^2 < \omega_B^2$, one has $\omega_+ > \omega_- > 0$ and, diagonalizing $H$ via a canonical transformation,

$$H = \omega_+ \frac{p_+^2 + q_+^2}{2} - \omega_- \frac{p_-^2 + q_-^2}{2} \tag{A9}$$

shows that the $-$ mode is a ghost. The two pulsations $\omega_\pm$ become degenerate for $\omega_0^2 = \omega_B^2$ (in this limit one has the same $H$ as a free particle seen from a rotating frame), and tachyons appear for $\omega_0^2 > \omega_B^2$.

## APPENDIX B: RESONANT FORM FOR OVERLAPPING RESONANCES

In this Appendix, we repeat the argument of Sec. IV B for the case of multiple resonances. For the system considered in Sec. IV B, this can happen if and only if all frequencies are approximately equal, $\omega$. Therefore, three resonant combinations are now present:

$$\begin{aligned} 4\Theta_s &\equiv \Theta'_1 + \Theta'_2 + \Theta'_3 + \Theta'_4, \\ 4\Theta_t &\equiv \Theta'_1 - \Theta'_2 - \Theta'_3 + \Theta'_4, \\ 4\Theta_u &\equiv \Theta'_1 - \Theta'_2 + \Theta'_3 - \Theta'_4. \end{aligned} \tag{B1}$$

The corresponding resonant form is

$$H \simeq \omega(J'_1 + J'_2 - J'_3 - J'_4) + \frac{\epsilon}{4}\Big[J'_1J'_3 + J'_1J'_4 + J'_2J'_3 + J'_2J'_4 + \sqrt{J'_1J'_2J'_3J'_4}(\cos 4\Theta_s + \cos 4\Theta_t + \cos 4\Theta_u)\Big], \tag{B2}$$

The only quasi-integral of motion (in addition to $H$) is $\mathcal{E} \equiv J'_1 + J'_2 - J'_3 - J'_4$. The Hamiltonian of the extra-system can be easily obtained from Eq. (B2) and, recalling that the combination $\mathcal{E}$ is approximately constant, is found to be bounded (this can be seen by noticing that the absolute value of the oscillatory term in the square brackets is smaller than $4\sqrt{J_1'J_2'J_3'J_4'}$ and using twice the inequality $2\sqrt{xy} < x + y$ between arithmetic and geometric means).

## APPENDIX C: CLASSICAL LATTICE SIMULATIONS

We consider the Lagrangian of Eq. (4) with a $\lambda\varphi_1^2\varphi_2^2/2$ interaction in $1+1$ dimensions with coordinates $(x_0, x_1)$. We express all dimensionful quantities in units of the ghost mass $m_2$ by introducing the dimensionless coordinates $t \equiv m_2 x_0$ and $x \equiv m_2 x_1$, as well as the dimensionless parameters $\kappa \equiv m_1^2/m_2^2$ and $\bar{\lambda} \equiv \lambda/m_2^2$. Then, we obtain the dimensionless Lagrangian,

$$\frac{\mathcal{L}}{m_2^2} \equiv \bar{\mathcal{L}} = \frac{1}{2}[(\dot{\varphi_1}^2 - \varphi_1'^2 - \kappa\varphi_1^2) - (\dot{\varphi_2}^2 - \varphi_2'^2 - \varphi_2^2) - \bar{\lambda}\varphi_1^2\varphi_2^2]. \tag{C1}$$

The equations of motion are

$$\begin{cases} \ddot{\varphi}_1 - \varphi_1'' + \varphi_1(\kappa + \bar{\lambda}\varphi_2^2) = 0 \\ \ddot{\varphi}_2 - \varphi_2'' + \varphi_2(1 - \bar{\lambda}\varphi_1^2) = 0. \end{cases}$$

These nonlinear 2nd-order hyperbolic partial differential equations can be solved with finite-difference lattice methods. For a $\varphi^4$ theory, this has been done in $1+1$ [40] and $3+1$ [41] dimensions using a light cone lattice (namely, a square lattice in $x \pm t$ coordinates) and an exactly conserved energy on the lattice. We generalize this procedure to two fields. This is nontrivial, as one needs to achieve energy conservation around cut off scales while avoiding choices that lead to impractically complicated discretized field equations.

The continuum Hamilton density is $\bar{\mathcal{H}} = \frac{1}{2}[(\pi_1^2 + \varphi_1'^2 + \kappa\varphi_1^2) - (\pi_2^2 + \varphi_2'^2 + \varphi_2^2) + \bar{\lambda}\varphi_1^2\varphi_2^2]$ where $\pi_i = \dot{\varphi}_i$. We introduce two lattice Hamilton densities,

$$\mathcal{H}_\pm = \frac{1}{2}[(\pi_{1\pm}^2 + \varphi_{1\pm}'^2 + \kappa\varphi_{1\pm}^2) - (\pi_{2\pm}^2 + \varphi_{2\pm}'^2 + \varphi_{2\pm}^2) + \bar{\lambda}[\varphi_1^2\varphi_2^2]_\pm], \tag{C2}$$

where we defined

$$\begin{aligned} \pi_{i\pm}^2 &= \left(\frac{2\varphi_i(x,t_\pm) - \varphi_i(x_-,t) - \varphi_i(x_+,t)}{2a}\right)^2 \\ \varphi_{i\pm}'^2 &= \left(\frac{\varphi_i(x_-,t) - \varphi_i(x_+,t)}{2a}\right)^2 \\ \varphi_{i\pm}^2 &= \frac{2\varphi_i(x,t_\pm)^2 + \varphi_i(x_-,t)^2 + \varphi_i(x_+,t)^2}{4} \\ [\varphi_1^2\varphi_2^2]_\pm &= \frac{\varphi_1(x_-,t)\varphi_2(x_-,t) + \varphi_1(x_+,t)\varphi_2(x_+,t)}{2}\varphi_1(x,t_\pm)\varphi_2(x,t_\pm). \end{aligned} \tag{C3}$$

Here, $a$ is the dimensionless lattice distance and we abbreviated $x_\pm = x \pm a$ and $t_\pm = t \pm a$. In the continuum limit, $\pi_{i\pm} \to \dot{\varphi}_i$. The definition $[\varphi_1^2\varphi_2^2]_\pm$ of the lattice interaction term significantly simplifies equations compared to the naive interaction term $\varphi_{1\pm}^2\varphi_{2\pm}^2$. In the continuum limit, $\mathcal{H}_+$ and $\mathcal{H}_-$ both approach the continuum Hamilton density: $\lim_{a\to 0}\mathcal{H}_\pm = \bar{\mathcal{H}}$. Their difference can be expressed as

$$\mathcal{H}_+ - \mathcal{H}_- = \frac{\varphi_1(x,t_+) - \varphi_1(x,t_-)}{2a^2}Q_1 - \frac{\varphi_2(x,t_+) - \varphi_2(x,t_-)}{2a^2}Q_2, \tag{C4}$$

where

$$\begin{aligned} Q_1 &= [\varphi_1(x,t_+) + \varphi_1(x,t_-)](1 + \kappa a^2/2) - [\varphi_1]_a(x,t) + \frac{\bar{\lambda}a^2}{4}[\varphi_2(x,t_+) + \varphi_2(x,t_-)][\varphi_1\varphi_2]_a(x,t) \\ Q_2 &= [\varphi_2(x,t_+) + \varphi_2(x,t_-)](1 + a^2/2) - [\varphi_2]_a(x,t) - \frac{\bar{\lambda}a^2}{4}[\varphi_1(x,t_+) + \varphi_1(x,t_-)][\varphi_1\varphi_2]_a(x,t), \end{aligned} \tag{C5}$$

and we defined

$$\begin{aligned} [\varphi_i]_a(x,t) &= \varphi_i(x_-,t) + \varphi_i(x_+,t) \\ [\varphi_1\varphi_2]_a(x,t) &= \varphi_1(x_-,t)\varphi_2(x_-,t) + \varphi_1(x_+,t)\varphi_2(x_+,t). \end{aligned} \tag{C6}$$

Energy is exactly conserved on the lattice if $Q_1 = Q_2 = 0$. In the continuum limit, this condition becomes the equations of motion in Eq. (C2):

$$\ddot{\varphi}_1 - \varphi_1'' + \varphi_1(\kappa + \bar{\lambda}\varphi_2^2) = -a^2\left[\frac{\kappa}{2}\ddot{\varphi}_1 + \frac{\ddddot{\varphi}_1 - \varphi_1''''}{12} + \frac{\bar{\lambda}}{2}\varphi_2((\varphi_1\varphi_2)'' + \varphi_1\ddot{\varphi}_2)\right] + \mathcal{O}(a^4)$$
$$\ddot{\varphi}_2 - \varphi_2'' + \varphi_2(1 - \bar{\lambda}\varphi_1^2) = -a^2\left[\frac{1}{2}\ddot{\varphi}_2 + \frac{\ddddot{\varphi}_2 - \varphi_2''''}{12} - \frac{\bar{\lambda}}{2}\varphi_1((\varphi_1\varphi_2)'' + \ddot{\varphi}_1\varphi_2)\right] + \mathcal{O}(a^4). \tag{C7}$$

So, by imposing $Q_1 = Q_2 = 0$ and solving for $\varphi_1(x, t_+)$ and $\varphi_2(x, t_+)$, we get discretized equations of motion that exactly conserve energy:

$$\varphi_1(x, t_+) = -\varphi_1(x, t_-) + \frac{(1 + a^2/2)[\varphi_1]_a(x, t) - (\bar{\lambda}a^2/4)[\varphi_2]_a(x, t)[\varphi_1\varphi_2]_a(x, t)}{(1 + a^2/2)(1 + \kappa a^2/2) + (\bar{\lambda}a^2/4)^2[\varphi_1\varphi_2]_a^2(x, t)}$$
$$\varphi_2(x, t_+) = -\varphi_2(x, t_-) + \frac{(1 + \kappa a^2/2)[\varphi_2]_a(x, t) + (\bar{\lambda}a^2/4)[\varphi_1]_a(x, t)[\varphi_1\varphi_2]_a(x, t)}{(1 + a^2/2)(1 + \kappa a^2/2) + (\bar{\lambda}a^2/4)^2[\varphi_1\varphi_2]_a^2(x, t)}. \tag{C8}$$

For zero interaction $\lambda = 0$, the energies of $\varphi_1$ and $\varphi_2$ are separately exactly conserved. The method can be extended to cubic interactions.


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