Revisiting the P-wave charmonium radiative decays $h_c \to \gamma\eta^{(\prime)}$ with relativistic corrections

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The P-wave charmonium decays $h_c \to \gamma\eta^{(\prime)}$ are revisited by taking into account relativistic corrections. The decay amplitudes are derived in the Bethe-Salpeter formalism, in which the involved one-loop integrals are evaluated analytically. Intriguingly, from both the quark-antiquark content and the gluonic content of $\eta^{(\prime)}$, the relativistic corrections make significant contributions to the decay rates of $h_c \to \gamma\eta^{(\prime)}$. By comparison with the leading-order contributions from the quark-antiquark content (one-loop level), the ones from the gluonic content (tree level) are also important, which is compatible with the conclusion obtained without relativistic corrections. Usually, for the η production processes, the predicted branching ratios are sensitive to the angle of $\eta-\eta'$ mixing. As an illustration, using the Feldmann-Kroll-Stech result about the mixing angle $\phi = 39.3° \pm 1.0°$ as input, we find that the predicted ratio $R_{h_c} = \mathcal{B}(h_c \to \gamma\eta)/\mathcal{B}(h_c \to \gamma\eta')$ is much smaller than the experiment measurement, while, with $\phi = 33.5° \pm 0.9°$ extracted from the asymptotic limit of the $\gamma^*\gamma-\eta'$ transition form factor, we obtain $R_{h_c} = 30.3\%$, consistent with $R_{h_c}^{\exp} = (30.7 \pm 11.3 \pm 8.7)\%$. As a cross-check, the mixing angle $\phi = 33.8° \pm 2.5°$ is extracted by employing the ratio R_{h_c}, and a brief discussion on the difference in the determinations of ϕ is given.

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I. INTRODUCTION

The hadronic decays of charmonia have played important roles for our understanding of QCD, especially the interplay of perturbative QCD and nonperturbative QCD [1–4], since the first charmonium-state J/ψ was observed [5,6]. One of the interesting topics is the Okubo-Zweig-Iizuka (OZI-)suppressed [7–10] radiative decays of charmonia to the light mesons $\eta^{(\prime)}$. On the one hand, these decays are closely related to the issue of $\eta-\eta'$ mixing, which could shed light on the $U(1)_A$ anomaly [11–19] and the $SU(3)_F$ breaking [17–21]. On the other hand, these decays provide a relatively clean environment to study the gluonic content of $\eta^{(\prime)}$, since there is no complication of interactions between the final light hadrons.

In recent years, there have been more and more experimental measurements on the radiative decays of charmonia to $\eta^{(\prime)}$, such as $J/\psi \to \gamma\eta^{(\prime)}$ [22–25], $\psi' \to \gamma\eta^{(\prime)}$ [24,26,27], $\psi(3770) \to \gamma\eta^{(\prime)}$ [24], and $h_c \to \gamma\eta^{(\prime)}$ [28]. In the theoretical aspect, the S-wave charmonium decays $J/\psi(\psi') \to \gamma\eta^{(\prime)}$ have been investigated in various approaches, such as QCD sum rules [29], the chiral and large N_C approach [30–32], QCD multipole expansion [33], the effective Lagrangian approach [34], perturbative QCD [35–40], and phenomenological models [41,42], and predictions of the branching ratios are compatible with experimental data. Furthermore, by the ratio $R_{J/\psi} = \mathcal{B}(J/\psi \to \gamma\eta')/\mathcal{B}(J/\psi \to \gamma\eta)$, the angle of $\eta-\eta'$ mixing was obtained at $\phi = 39.0° \pm 1.6°$ [17] with nonperturbative matrix elements $\langle 0|G^a_{\mu\nu}\tilde{G}^{a,\mu\nu}|\eta^{(\prime)}\rangle$ and $\phi = 33.9° \pm 0.6°$ [40] with perturbative QCD. It is worth noting that the recent lattice calculation of the ETM Collaboration gives the mixing angle $\phi = 46° \pm 1° \pm 3°$ [43,44], while the UKQCD Collaboration obtains $\phi = 34° \pm 3°$ [45]. The discrepancies in these determinations of the mixing angle might indicate that our understanding of the $\eta-\eta'$ mixing scheme [17,18,21,46–51] is incomplete, and further experimental and theoretical investigations are needed to make sense of the $\eta-\eta'$ mixing.

Turning to P-wave charmonia decays, the physical picture seems more complex, since the higher Fock-state contributions and the relativistic corrections may become important. It is well known that IR divergences are

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encountered in the color-singlet state contributions for the inclusive P-wave charmonia decays with the zero-binding approximation [52–55]. Although these IR divergences can be removed by considering the higher Fock-state contributions from the point of view of nonrelativistic QCD (NRQCD) [56,57], they may imply that the effects beyond those contained in the derivative of the nonrelativistic wave function at the origin $R'(0)$ may play a key role. Generally, it should be noted that similar IR divergences do not appear in exclusive P-wave charmonia decays [58–61]. Nevertheless, as pointed out in Refs. [58–64], the higher-order contributions, such as the higher Fock-state contributions [58–61,64] and the relativistic corrections [62–64], are still important to exclusive P-wave charmonia decays. For the exclusive P-wave charmonium decays $h_c \to \gamma\eta^{(\prime)}$, there have been a few studies [65–67] in the theoretical aspect ever since the branching ratios $\mathcal{B}(h_c \to \gamma\eta')$ and $\mathcal{B}(h_c \to \gamma\eta)$ were first measured to be, respectively, $(1.52 \pm 0.27 \pm 0.29) \times 10^{-3}$ and $(4.7 \pm 1.5 \pm 1.4) \times 10^{-4}$ by the BESIII Collaboration [28]. In the nonrelativistic limit [65,67], the relativistic corrections related to the internal momentum of the P-wave charmonium h_c have been neglected in the calculation of the decay rates, and all the nonperturbative effects are absorbed in $R'_{h_c}(0)$ with the Taylor expansion of the hard-scattering amplitudes up to the linear terms. Then, it is found that the calculations are IR safe and the predicted branching ratios $\mathcal{B}(h_c \to \gamma\eta^{(\prime)})$ are much smaller than the experimental measurements. Obviously, this indicates that the relativistic corrections or/and the contributions from the higher Fock-state of h_c are highly significant, while from the point of view of NRQCD, the next-to-leading-order Fock-state contributions are suppressed by a relative factor $v_{c\bar{c}}^2\alpha_s$ in the decays $h_c \to \gamma\eta^{(\prime)}$ [65]. So it means that the relativistic corrections are needed in the exclusive P-wave charmonium decays $h_c \to \gamma\eta^{(\prime)}$.

One of the major concerns of this paper is to study the relativistic corrections in the exclusive P-wave charmonium decays $h_c \to \gamma\eta^{(\prime)}$ by performing an explicit calculation. To make these relativistic corrections clear, the Bethe-Salpeter (B-S) framework [68–70] is used to calculate the wave function of h_c and the decay amplitudes of $h_c \to \gamma\eta^{(\prime)}$, where the internal momentum of h_c is retained in both the soft bound-state wave function and the hard-scattering amplitude. Here, it is worth noting that there are at least two sources of the relativistic corrections. One is from the kinematical corrections which appear in the annihilation amplitudes, and the other is from the dynamical corrections of bound-state wave function itself. For the final light mesons $\eta^{(\prime)}$, light-cone distribution amplitudes (DAs) are adopted because of the large momentum transfer. And the contributions of the quark-antiquark content and those of the gluonic content of $\eta^{(\prime)}$ are both taken into account in our calculations.

In this paper, with the technique of the helicity projector, we evaluate analytically the involved one-loop integrals with the internal momentum of h_c kept. For the contributions from the quark-antiquark content of $\eta^{(\prime)}$ in the decays $h_c \to \gamma\eta^{(\prime)}$, the relativistic effects mainly originate from the kinematic part of the annihilation amplitudes, especially when the internal momentum of h_c makes the propagator near on shell. For the contributions of the gluonic content of $\eta^{(\prime)}$ in the decays $h_c \to \gamma\eta^{(\prime)}$, the next-to-leading-order effects related to the internal momentum are not substantially suppressed in the major region of the wave function of h_c, and therefore the corresponding relativistic corrections are extremely important. Furthermore, we find out that the gluonic contributions and the quark-antiquark contributions are comparable, unlike the situation in the heavy vector quarkonium decays $V \to \gamma\eta^{(\prime)}$ [36,40,71] where the gluonic contributions are strongly suppressed due to the special form of the spin structure of their amplitudes. In addition, it is also unlike the phenomenological fits [72–74] where the gluonic content of η can be neglected. This signifies that the decays $h_c \to \gamma\eta^{(\prime)}$ can be used to test the gluonic content of the $\eta^{(\prime)}$ more efficiently than the decay processes $V \to \gamma\eta^{(\prime)}$. It is worthwhile to point out that the decay rates of $h_c \to \gamma\eta^{(\prime)}$ are insensitive to the light quark masses and the shapes of the $\eta^{(\prime)}$ DAs [67], so the theoretical uncertainties from the $\eta^{(\prime)}$ DAs are negligible, and the mixing angle of the $\eta - \eta'$ system could be reliably extracted in our calculations.

The paper is organized as follows. The formalism for the decays $h_c \to \gamma\eta^{(\prime)}$ is presented in Sec. II. In Sec. III, we obtain our numerical results, and the final section is our summary. The expressions of the numerators involved in Sec. II are given in the Appendix.

II. FORMALISM FOR RADIATIVE DECAYS $h_c \to \gamma\eta^{(\prime)}$

A. Bethe-Salpeter equation

It is generally known that the B-S equation [75,76] is a relativistic equation describing a bound state and has a solid basis in quantum field theory. So, it is a conventional approach to treat various relativistic bound-state problems. In this subsection, we briefly review the formulation of the B-S framework. For charmonia, the B-S equation has the form [77–80]

$$S_F^{-1}(f)\Psi(K,q)S_F^{-1}(-\bar{f}) = \int \frac{\mathrm{d}^4 q'}{(2\pi)^4}[-i\mathcal{K}(K,q,q')\Psi(K,q')], \tag{2.1}$$

where $\mathcal{K}(K,q,q')$ represents the interaction kernel between the internal quark and antiquark, and $S_F(p) = i/(\not{p} - \hat{m}_c + i\epsilon)$ represents the propagator with the effective mass of c quark $\hat{m}_c$. The momenta of the quark and antiquark can be written as

$$f = \frac{K}{2} + q, \qquad \bar{f} = \frac{K}{2} - q, \tag{2.2}$$

where q and K represent the internal momentum and the total momentum of the charmonia, respectively.

For convenience, one can divide the internal momentum q into two parts. One part is the transverse component $\hat{q}$ with $\hat{q} \cdot K = 0$, and the other is the longitudinal component $q_\parallel$ which is parallel to the total momentum K:

$$\begin{aligned} q^\mu &= q_\parallel^\mu + \hat{q}^\mu, \\ q_\parallel^\mu &= \frac{q_K}{M} K^\mu. \end{aligned} \tag{2.3}$$

Here, both $q_K = \frac{q \cdot K}{M}$ and $\hat{q}^2 = q^2 - q_K^2$ are Lorentz invariant variables, and M is the mass of the charmonia. From Eq. (2.3), one can know that the variable $\hat{q}$ involves 3 degrees of freedom orthogonal to the total momentum K, and the remaining 1 degree of freedom is contained in the variable q_K, which represents the component q^0 in the rest frame of the charmonia. So, the volume element of the internal momentum can be written in the form

$$\mathrm{d}^4 q = \mathrm{d}^3 \hat{q} \mathrm{d} q_K. \tag{2.4}$$

Under the covariant instantaneous ansatz (CIA) [68–70], the interaction kernel $\mathcal{K}(K, q, q')$ is taken to be dependent only on the momentum $\hat{q}$,

$$\mathcal{K}(K, q, q') \approx \mathcal{K}(K, \hat{q}, \hat{q}'), \tag{2.5}$$

and we employ the shorthand $V(\hat{q}, \hat{q}') \equiv \mathcal{K}(K, \hat{q}, \hat{q}')$, in which the dependence of the total momentum K is hidden. Generally, the interaction kernel $V(\hat{q}, \hat{q}')$ includes both the long-ranged confinement and the short-ranged one-gluon exchange interactions [80–86]. Then, similar abbreviations are also adopted for the functions $\Gamma(\hat{q})$ and $\psi(\hat{q})$.

Under CIA, the B-S wave function reads

$$\Psi(K, q) = -S_F(f)\Gamma(\hat{q})S_F(-\bar{f}) \tag{2.6}$$

with the hadron-quark vertex function

$$\begin{aligned} \Gamma(\hat{q}) &= i \int \frac{\mathrm{d}^3 \hat{q}' \mathrm{d} q_K'}{(2\pi)^4} V(\hat{q}, \hat{q}') \Psi(K, q') \\ &= \int \frac{\mathrm{d}^3 \hat{q}'}{(2\pi)^3} V(\hat{q}, \hat{q}') \psi(\hat{q}'). \end{aligned} \tag{2.7}$$

Here, the Salpeter wave function is defined as

$$\psi(\hat{q}) = \frac{i}{2\pi} \int \mathrm{d} q_K \Psi(K, q). \tag{2.8}$$

Obviously, one can find that the vertex function and the interaction kernel are independent on the variable q_K. So, in the following contour integration of q_K, one just needs to keep the residues from the quark propagator poles.

Using the operators [78–80,86]

$$\Lambda_i^\pm(\hat{q}) = \frac{1}{2\omega}\left[\frac{\not{K}}{M}\omega \pm (-1)^{(i+1)}(\hat{m}_c + \hat{\not{q}})\right], \tag{2.9}$$

the propagators can be decomposed as

$$\begin{aligned} \frac{1}{\not{f} - \hat{m}_c + i\epsilon} &= \frac{\Lambda_1^+(\hat{q})}{q_K + \frac{M}{2} - \omega + i\epsilon} + \frac{\Lambda_1^-(\hat{q})}{q_K + \frac{M}{2} + \omega - i\epsilon}, \\ \frac{1}{\bar{\not{f}} + \hat{m}_c - i\epsilon} &= \frac{\Lambda_2^+(\hat{q})}{-q_K + \frac{M}{2} - \omega + i\epsilon} + \frac{\Lambda_2^-(\hat{q})}{-q_K + \frac{M}{2} + \omega - i\epsilon} \end{aligned} \tag{2.10}$$

with $\omega = \sqrt{\hat{m}_c^2 - \hat{q}^2}$. Performing the contour integration of q_K on both sides of Eq. (2.6) by the residue theorem, one can obtain [78–80,86]

$$\begin{aligned} (M - 2\omega)\psi^{++}(\hat{q}) &= -\Lambda_1^+(\hat{q})\Gamma(\hat{q})\Lambda_2^+(\hat{q}), \\ (M + 2\omega)\psi^{--}(\hat{q}) &= \Lambda_1^-(\hat{q})\Gamma(\hat{q})\Lambda_2^-(\hat{q}), \\ \psi^{+-}(\hat{q}) &= 0, \\ \psi^{-+}(\hat{q}) &= 0 \end{aligned} \tag{2.11}$$

with $\psi^{\pm\pm}(\hat{q}) = \Lambda_1^\pm(\hat{q})\frac{\not{P}}{M}\psi(\hat{q})\frac{\not{P}}{M}\Lambda_2^\pm(\hat{q})$ and $\psi(\hat{q}) = \psi^{++}(\hat{q}) + \psi^{+-}(\hat{q}) + \psi^{-+}(\hat{q}) + \psi^{--}(\hat{q})$. It is worth noting that the three-dimensional reduction of the B-S equation can also been done in a fashionable way, namely, covariant spectator theory [87], and more details about this method can be found in Ref. [88].

For the axial vector meson h_c, the Salpeter wave function can be approximately written as [79,80,89–91]

$$\psi(\hat{q}) = \hat{q} \cdot \varepsilon(K)\left[1 + \frac{\not{K}}{M} + \frac{\hat{\not{q}}\not{K}}{\hat{m}_c M}\right]\gamma^5 f(\hat{q}^2), \tag{2.12}$$

where M and $\varepsilon(K)$ are the mass and the polarization vector of h_c, respectively, and the front factor $\hat{q} \cdot \varepsilon(K)$ indicates that the wave function is of P-wave nature mainly and $f(\hat{q}^2)$ is a scalar function of $\hat{q}^2$. In the rest frame of h_c, the momenta K and $\hat{q}$ have the form

$$K^\mu = (M, \mathbf{0}), \qquad \hat{q}^\mu = (0, \hat{q}) = (0, \mathbf{q}), \tag{2.13}$$

and the scalar function $f(\hat{q}^2)$ satisfies the harmonic oscillator equation (more details and discussions could be found in Refs. [80,86]). The expression of $f(\hat{q}^2)$ reads

$$f(\hat{q}^2) = N_A \left(\frac{2}{3}\right)^{\frac{1}{2}} \frac{1}{\pi^{\frac{3}{4}}\beta_A^{\frac{5}{2}}} |\mathbf{q}| e^{-\frac{\mathbf{q}^2}{2\beta_A^2}}, \tag{2.14}$$

where N_A is the normalization constant and β_A is the harmonic oscillator parameter [80,86]. The normalization equation of $f(\hat{q}^2)$ reads

$$\int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{4\omega \mathbf{q}^2}{3\hat{m}_c M} f^2(\hat{q}^2) = 1. \tag{2.15}$$

B. Contributions of the quark-antiquark content of $\eta^{(\prime)}$

For the quark-antiquark content of $\eta^{(\prime)}$, one of the leading-order Feynman diagrams for the radiative decays $h_c \to \gamma\eta^{(\prime)}$ is depicted in Fig. 1. The other five diagrams arise from permutations of the photon and gluon legs. And it is convenient to divide the amplitude of $h_c \to \gamma\eta^{(\prime)}$ into two parts [67]. One part describes the effective coupling between h_c, a real photon, and two virtual gluons, and the other part describes the effective coupling between $\eta^{(\prime)}$ and two virtual gluons. Because of the large momentum transfer in these decays, the internal quark and antiquark of final light mesons are almost collinear, and this means the momenta of the internal quark and antiquark are both nearly parallel to the total momentum p. As conventions, the variables u and $\bar{u}$ represent the momentum fractions carried by the quark and antiquark with $\bar{u} = 1 - u$ in Fig. 1.

In the rest frame of h_c, the amplitude of $h_c \to \gamma g^* g^*$ has the form [35,67,92–94]

$$\begin{aligned}
&\mathcal{A}^{\alpha\beta\mu\nu} \varepsilon_\alpha(K)\epsilon^*_\beta(k)\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2) \\
&\quad = \sqrt{3} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \mathrm{Tr}[\Psi(K,q)\mathcal{O}(f,\bar{f})],
\end{aligned} \tag{2.16}$$

where k, k_1, k_2 and $\epsilon(k), \epsilon(k_1), \epsilon(k_2)$ stand for the momenta and polarization vectors of the photon and the gluons, respectively; the factor $\sqrt{3}$ is included to account for the color properties of the quark-antiquark content; $\mathcal{O}(f,\bar{f})$ is the hard-scattering amplitude; and the momenta of the quark and antiquark read

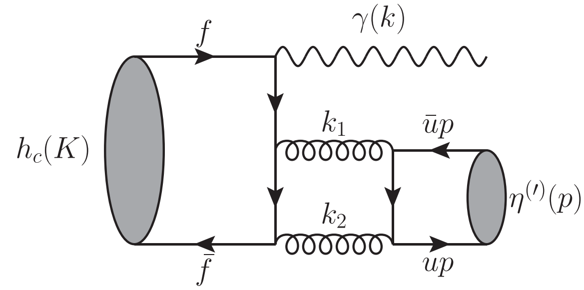

FIG. 1. One typical Feynman diagram for $h_c \to \gamma\eta^{(\prime)}$ with the quark-antiquark content of $\eta^{(\prime)}$. Here, the kinematic variables are labeled.

$$\begin{aligned}
f^\mu &= \frac{K^\mu}{2} + q^\mu = \left(\frac{M}{2} + q^0, \mathbf{q}\right), \\
\bar{f}^\mu &= \frac{K^\mu}{2} - q^\mu = \left(\frac{M}{2} - q^0, -\mathbf{q}\right).
\end{aligned} \tag{2.17}$$

Under the CIA, a more relevant treatment is to take $q^0 \ll M$, so one can obtain the momenta

$$\begin{aligned}
f^\mu &\approx \left(\frac{M}{2}, \mathbf{q}\right) = \frac{K^\mu}{2} + \hat{q}^\mu, \\
\bar{f}^\mu &\approx \left(\frac{M}{2}, -\mathbf{q}\right) = \frac{K^\mu}{2} - \hat{q}^\mu
\end{aligned} \tag{2.18}$$

and the hard-scattering amplitude

$$\mathcal{O}(f,\bar{f}) \approx \mathcal{O}(\hat{q}). \tag{2.19}$$

From another point of view [82], this treatment can be connected with the on-shell condition, which maintains the gauge invariance of the hard-scattering amplitude.

Then, the amplitude of $h_c \to \gamma g^* g^*$ can be written as

$$\begin{aligned}
\mathcal{A}^{\alpha\beta\mu\nu} \varepsilon_\alpha(K)\epsilon^*_\beta(k)\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2) &= \sqrt{3} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \mathrm{Tr}[\Psi(K,q)\mathcal{O}(\hat{q})] \\
&= -i\sqrt{3} \int \frac{\mathrm{d}^3 \hat{q}}{(2\pi)^3} \mathrm{Tr}[\psi(\hat{q})\mathcal{O}(\hat{q})],
\end{aligned} \tag{2.20}$$

where the hard-scattering amplitude $\mathcal{O}(\hat{q})$ reads

$$\begin{aligned}
\mathcal{O}(\hat{q}) = iQ_c e g_s^2 \frac{\delta_{ab}}{6} \Bigg[&\not{\epsilon}^*(k_2) \frac{\frac{\not{k}_2 - \not{k} - \not{k}_1}{2} + \hat{\not{q}} + m_c}{(\frac{k_2 - k - k_1}{2} + \hat{q})^2 - m_c^2} \not{\epsilon}^*(k) \frac{\frac{\not{k}_2 + \not{k} - \not{k}_1}{2} + \hat{\not{q}} + m_c}{(\frac{k_2 + k - k_1}{2} + \hat{q})^2 - m_c^2} \not{\epsilon}^*(k_1) \\
&+ \not{\epsilon}^*(k_1) \frac{\frac{\not{k}_1 - \not{k} - \not{k}_2}{2} + \hat{\not{q}} + m_c}{(\frac{k_1 - k - k_2}{2} + \hat{q})^2 - m_c^2} \not{\epsilon}^*(k) \frac{\frac{\not{k}_1 + \not{k} - \not{k}_2}{2} + \hat{\not{q}} + m_c}{(\frac{k_1 + k - k_2}{2} + \hat{q})^2 - m_c^2} \not{\epsilon}^*(k_2)
\end{aligned}$$

$$+ \not\epsilon^*(k_2)\frac{\frac{\not k_2-\not k_1-\not k}{2}+\hat{\not q}+m_c}{(\frac{k_2-k_1-k}{2}+\hat{q})^2-m_c^2}\not\epsilon^*(k_1)\frac{\frac{\not k_2+\not k_1-\not k}{2}+\hat{\not q}+m_c}{(\frac{k_2+k_1-k}{2}+\hat{q})^2-m_c^2}\not\epsilon^*(k)$$
$$+ \not\epsilon^*(k)\frac{\frac{\not k-\not k_2-\not k_1}{2}+\hat{\not q}+m_c}{(\frac{k-k_2-k_1}{2}+\hat{q})^2-m_c^2}\not\epsilon^*(k_2)\frac{\frac{\not k+\not k_2-\not k_1}{2}+\hat{\not q}+m_c}{(\frac{k+k_2-k_1}{2}+\hat{q})^2-m_c^2}\not\epsilon^*(k_1)$$
$$+ \not\epsilon^*(k_1)\frac{\frac{\not k_1-\not k_2-\not k}{2}+\hat{\not q}+m_c}{(\frac{k_1-k_2-k}{2}+\hat{q})^2-m_c^2}\not\epsilon^*(k_2)\frac{\frac{\not k_1+\not k_2-\not k}{2}+\hat{\not q}+m_c}{(\frac{k_1+k_2-k}{2}+\hat{q})^2-m_c^2}\not\epsilon^*(k)$$
$$\left.+ \not\epsilon^*(k)\frac{\frac{\not k-\not k_1-\not k_2}{2}+\hat{\not q}+m_c}{(\frac{k-k_1-k_2}{2}+\hat{q})^2-m_c^2}\not\epsilon^*(k_1)\frac{\frac{\not k+\not k_1-\not k_2}{2}+\hat{\not q}+m_c}{(\frac{k+k_1-k_2}{2}+\hat{q})^2-m_c^2}\not\epsilon^*(k_2)\right] \tag{2.21}$$

with the c quark mass m_c.

For the final light mesons $\eta^{(\prime)}$, the internal quark q and antiquark $\bar{q}$ are almost collinear because of the large momentum transfer. Under the collinear limit, the light-cone expansion of the matrix elements of $\eta^{(\prime)}$ over quark and antiquark fields has been given in Refs. [3,95,96], and one can obtain the coupling of $g^*g^*-\eta^{(\prime)}$ up to twist-3 level [67,97–99],

$$\mathcal{M}^{\mu\nu}\epsilon_\mu(k_1)\epsilon_\nu(k_2) = -i(4\pi\alpha_s)\delta_{ab}\epsilon^{\mu\nu\rho\sigma}\epsilon_\mu(k_1)\epsilon_\nu(k_2)k_{1\rho}k_{2\sigma}\sum_{q=u,d,s}\frac{f^q_{\eta^{(\prime)}}}{6}\int_0^1 du\phi^q(u)\left(\frac{1}{\bar{u}k_1^2+uk_2^2-u\bar{u}p^2-m_q^2}+(u\leftrightarrow\bar{u})\right), \tag{2.22}$$

where the superscript (subscript) $q=u$, d, s denotes the flavor of the light quarks, $f^q_{\eta^{(\prime)}}$ is the decay constant, and m_q is the mass of the light quark. For the light-cone DA $\phi^q(u)$, we take three models listed in Table 1 of Ref. [67]. As pointed out in Refs. [40,67], the decay rates of $h_c\to\gamma\eta^{(\prime)}$ barely depend on the shapes of $\eta^{(\prime)}$ DAs (we will estimate them below). It means that the mixing angle of $\eta-\eta'$ system could be reliably extracted in our calculations due to the negligible uncertainties from $\eta^{(\prime)}$ DAs.

To proceed, the decay amplitude of $h_c\to\gamma\eta^{(\prime)}$ can be obtained directly by contracting the two couplings $\mathcal{A}^{\alpha\beta\mu\nu}$ and $\mathcal{M}_{\mu\nu}$, inserting the gluon propagators and integrating over the loop momentum,

$$\begin{aligned}M_T &= T^{\alpha\beta}\varepsilon_\alpha(K)\epsilon^*_\beta(k)\\ &= \frac{1}{2}\int\frac{d^4k_1}{(2\pi)^4}\mathcal{A}^{\alpha\beta\mu\nu}\mathcal{M}_{\mu\nu}\frac{i}{k_1^2+i\epsilon}\frac{i}{k_2^2+i\epsilon}\varepsilon_\alpha(K)\epsilon^*_\beta(k),\end{aligned} \tag{2.23}$$

where the factor $1/2$ takes into account that the two gluons have already been interchanged in both $A^{\alpha\beta\mu\nu}$ and $M_{\mu\nu}$. By Lorentz invariance, parity conservation, and gauge invariance, one can obtain the helicity amplitude [67]

$$H^q_{\rm QCD} = \frac{2Q_c}{3\sqrt{3}}\sqrt{4\pi\alpha}(4\pi\alpha_s)^2\sum_{q=u,d,s}f^q_{\eta^{(\prime)}}H_q, \tag{2.24}$$

where the dimensionless function H_q reads

$$H_q = \int\frac{d^3\hat{q}}{(2\pi)^3}f(\hat{q}^2)\int du\phi^q(u)I_q(u,\hat{q}). \tag{2.25}$$

$I_q(u,\hat{q})$ represents the sum of the loop integrals of all the Feynman diagrams

$$\begin{aligned}I_q(u,\hat{q}) = \int\frac{d^4l}{(2\pi)^4}&\left(\frac{N_1}{D_1D_2D_3D_4D_5}+\frac{N_2}{C_1D_1D_3D_4D_5}\right.\\ &\left.+\frac{N_3}{C_2D_1D_2D_4D_5}\right)+(u\leftrightarrow\bar{u})\end{aligned} \tag{2.26}$$

with $l=k_1-k_2$ and the denominators of the propagators

$$\begin{aligned}C_1 &= (p-k+2\hat{q})^2-4m_c^2+i\epsilon,\\ C_2 &= (p-k-2\hat{q})^2-4m_c^2+i\epsilon,\\ D_1 &= [l+(\bar{u}-u)p]^2-4m_q^2+i\epsilon,\\ D_2 &= (l-k-2\hat{q})^2-4m_c^2+i\epsilon,\\ D_3 &= (l+k-2\hat{q})^2-4m_c^2+i\epsilon,\\ D_4 &= (l+p)^2+i\epsilon,\\ D_5 &= (l-p)^2+i\epsilon.\end{aligned} \tag{2.27}$$

As shown in Eq. (2.25), the spin structures of the bound-state wave function are absorbed into the loop function $I_q(u,\hat{q})$, and the expressions of the numerators N_1, N_2, and

N_3 are presented in the Appendix. Since the loop function $I_q(u,\hat{q})$ has no soft singularities and the dimensionless function H_q is very insensitive to the light quark mass m_q [40,67], one can take the simplicity safely,

$$I_0(u,\hat{q}) = \lim_{m_q\to 0} I_q(u,\hat{q}),$$
$$H_0 = \int \frac{\mathrm{d}^3\hat{q}}{(2\pi)^3} f(\hat{q}^2) \int \mathrm{d}u\phi^q(u) I_0(u,\hat{q}); \tag{2.28}$$

i.e., $H_q(q=u,d,s) = H_0$. Then, the helicity amplitude in Eq. (2.24) can be rewritten as

$$H^q_{\mathrm{QCD}} = \frac{2Q_c}{3\sqrt{3}} \sqrt{4\pi\alpha}(4\pi\alpha_s)^2 f_{\eta^{(\prime)}} H_0 \tag{2.29}$$

with the effective decay constants

$$f_{\eta'} = f^u_{\eta'} + f^d_{\eta'} + f^s_{\eta'}, \qquad f_\eta = f^u_\eta + f^d_\eta + f^s_\eta. \tag{2.30}$$

By using the algebraic identity ($\xi \neq \pm 1$)

$$\frac{1}{m^2(\xi^2-1)} D_1 - \frac{1}{2m^2(\xi-1)} D_4 + \frac{1}{2m^2(\xi+1)} D_5 = 1 \tag{2.31}$$

with $\xi = 1$–$2u$ and the mass of $\eta^{(\prime)}$ meson m, the loop function $I_0(u,\hat{q})$ can be decomposed into a sum of four-point one-loop integrals

$$I_0(u,\hat{q}) = \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \left(\frac{N_1}{m^2(\xi^2-1) D_2 D_3 D_4 D_5} - \frac{N_1}{2m^2(\xi-1) D_1 D_2 D_3 D_5} + \frac{N_1}{2m^2(\xi+1) D_1 D_2 D_3 D_4} + \frac{N_2}{C_1 D_1 D_3 D_4 D_5} + \frac{N_3}{C_2 D_1 D_2 D_4 D_5} \right) + (u \leftrightarrow \bar{u}). \tag{2.32}$$

When $\xi = 1$, the denominators of the propagators have the relation $D_1 = D_4$, and the loop function $I_0(u,\hat{q})$ becomes

$$I_0(u,\hat{q}) = \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \left[\frac{N_1}{D_2 D_3 D_4^2 D_5} + \frac{N_2}{C_1 D_3 D_4^2 D_5} + \frac{N_3}{C_2 D_2 D_4^2 D_5} \right] + (u \leftrightarrow \bar{u}). \tag{2.33}$$

And when $\xi = -1$, the denominators of the propagators have the relation $D_1 = D_5$; then, the loop function $I_0(u,\hat{q})$ becomes

$$I_0(u,\hat{q}) = \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \left[\frac{N_1}{D_2 D_3 D_4 D_5^2} + \frac{N_2}{C_1 D_3 D_4 D_5^2} + \frac{N_3}{C_2 D_2 D_4 D_5^2} \right] + (u \leftrightarrow \bar{u}). \tag{2.34}$$

With the program PACKAGE-X [100,101], one can evaluate the above one-loop integrals analytically. Similarly to the situations without considering the internal momentum of charmonium [40,67], we find that the loop function $I_0(u,\hat{q})$ is also almost unchanged over the most region of the momentum fraction u, and this results in the dimensionless function H_0 being very insensitive to the shapes of the $\eta^{(\prime)}$ DAs. Numerically, our results show that the change among the dimensionless function H_0 with the different models of the DAs is less than 1%. Therefore, the theoretical uncertainties from the DAs are ignorable in our calculations of the branching ratios $\mathcal{B}(h_c \to \gamma\eta^{(\prime)})$. In addition, because the internal momentum $\hat{q}$ could make the propagator on shell in the region of the wave function of h_c unsuppressed strongly, the convolution of the loop function $I_0(u,\hat{q})$ and the wave function $f(\hat{q}^2)$ (i.e., the dimensionless function H_0) would gain substantial kinematical corrections. Specifically, there is a significant enhancement in the absorptive part of H_0. Accordingly, the relativistic effects are important in the quark-antiquark contributions.

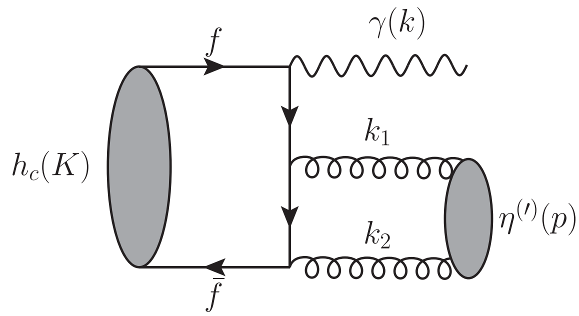

FIG. 2. One typical Feynman diagram for $h_c \to \gamma\eta^{(\prime)}$ with the gluonic content of $\eta^{(\prime)}$. Here, the kinematic variables are labeled.

C. Contributions of the gluonic content of $\eta^{(\prime)}$

The gluonic content of $\eta^{(\prime)}$ can directly contribute to the decay processes $h_c \to \gamma\eta^{(\prime)}$ from tree level. One typical Feynman diagram is exhibited in Fig. 2, and there are other two diagrams from permutations of the photon and the gluon legs. Generally, contributions of gluonic content are supposed to be small [40,72], since gluonic content can be seen as the higher-order effects from the point of view of the QCD evolution of the two-gluon DA, which vanishes in the asymptotic limit. However, as we have pointed out in Ref. [67], these contributions may become important in the $\eta^{(\prime)}$ production because the two-gluon DA of $\eta^{(\prime)}$ can mix with their quark-antiquark DA due to the $U(1)_A$ anomaly. Furthermore, from Figs. 1 and 2, one can easily find that the leading-order contributions (one-loop level) from the quark-antiquark content of $\eta^{(\prime)}$ are suppressed by a factor of α_s as compared with the contributions from the gluonic content of $\eta^{(\prime)}$. Therefore, there is an interesting question: which kind of contributions is dominant in the decays $h_c \to \gamma\eta^{(\prime)}$, especially with considering the relativistic effects in the two decay processes? The answer is given below.

The matrix elements of the mesons $\eta^{(\prime)}$ over two-gluon fields in the light-cone expansion at the leading-twist level read [96,102,103]

$$\langle \eta^{(\prime)}(p)|A_\alpha^a(x)A_\beta^b(y)|0\rangle = \frac{1}{4}\epsilon_{\alpha\beta\mu\nu}\frac{k^\mu p^\nu}{p\cdot k}\frac{C_F}{\sqrt{3}}\frac{\delta^{ab}}{8}f^1_{\eta^{(\prime)}} \times \int \mathrm{d}u e^{i(up\cdot x+\bar{u}p\cdot y)}\frac{\phi^g(u)}{u(1-u)} \tag{2.35}$$

with the effective decay constant $f^1_{\eta^{(\prime)}} = \frac{1}{\sqrt{3}}(f^u_{\eta^{(\prime)}} + f^d_{\eta^{(\prime)}} + f^s_{\eta^{(\prime)}})$ and the gluonic twist-2 DA [96,103,104]

$$\phi^g(u) = 30u^2(1-u)^2\sum_{n=2,4\cdots} c_n^g(\mu)C_{n-1}^{\frac{5}{2}}(2u-1). \tag{2.36}$$

One can obtain the corresponding helicity amplitude

$$H^g_{\rm QCD} = \frac{2Q_c}{9}\sqrt{4\pi\alpha}(4\pi\alpha_s)f^1_{\eta^{(\prime)}}H_g, \tag{2.37}$$

where the dimensionless function H_g has the form

$$H_g = \int\frac{\mathrm{d}^3\hat{q}}{(2\pi)^3}f(\hat{q}^2)\int \mathrm{d}u\frac{\phi^g(u)}{u(1-u)} \times \left(\frac{N_4}{C_1C_4}+\frac{N_5}{C_2C_3}+\frac{N_6}{C_3C_4}\right). \tag{2.38}$$

Here, the denominators of the propagators C_3 and C_4 read

$$\begin{aligned} C_3 &= (\xi p + k + 2\hat{q})^2 - 4m_c^2 + i\epsilon, \\ C_4 &= (\xi p - k + 2\hat{q})^2 - 4m_c^2 + i\epsilon, \end{aligned} \tag{2.39}$$

and the expressions of the numerators N_4, N_5, and N_6 are given in the Appendix.

In the remaining part of this section, we present a brief discussion about the relativistic effects. To the $\mathcal{O}(\hat{q}^2)$-order corrections related to the internal momentum from the numerators N_4, N_5 and N_6, if we take $\hat{m}_c \approx m_c \approx M/2$ and $m^2/M^2 \approx 0$, they exhibit the following behavior:

$$\begin{aligned} N_4 &\propto \left(1-\frac{\xi}{1+\xi}\frac{\hat{q}^2}{k\cdot\hat{q}}+\mathcal{O}(\hat{q}^2)\right), \\ N_5 &\propto \left(1-\frac{\xi}{1-\xi}\frac{\hat{q}^2}{k\cdot\hat{q}}+\mathcal{O}(\hat{q}^2)\right), \\ N_6 &\propto \left(1+\frac{\xi}{1-\xi^2}\frac{\hat{q}^2}{k\cdot\hat{q}}+\mathcal{O}(\hat{q}^2)\right). \end{aligned} \tag{2.40}$$

Obviously, the next-to-leading-order contributions are not suppressed enough in the major region[1] of the integration variable $\hat{q}$. Therefore, in the decay processes $h_c \to \gamma\eta^{(\prime)}$, the relativistic corrections should be taken into account.

III. NUMERICAL RESULTS

The decay widths of $h_c \to \gamma\eta^{(\prime)}$ can be expressed as

$$\Gamma(h_c \to \gamma\eta^{(\prime)}) = \frac{2}{3}\frac{1-x}{16\pi M}|H^q_{\rm QCD} + H^g_{\rm QCD}|^2 \tag{3.1}$$

with $x = m^2/M^2$. In the following numerical calculations, we take the parameters $M = 3525$ MeV, $m_\eta = 548$ MeV, $m_{\eta'} = 958$ MeV, $m_c = 1270$ MeV, $\Gamma_{h_c} = (0.70 \pm 0.28 \pm 0.22)$ MeV, and $f_\pi = 130.2$ MeV, which are quoted from the particle data group [105]. The QCD running coupling constant is adopted, $\alpha_s(m_c) = 0.38$, which is calculated through the two-loop renormalization group equation. The effective masses of the c quark and the harmonic oscillator parameter appearing in the bound-state wave function, which contains the long-distance nonperturbative dynamical effect of quark-antiquark interaction, are respectively taken as $\hat{m}_c = 1490$ MeV and $\beta_A = 590$ MeV, and more discussions can be found in Refs. [79,80,86]. As we have already mentioned, the theoretical uncertainties from $\eta^{(\prime)}$ DAs are negligible. So, in our calculations, we choose model I of the meson DA in Table 1 of Ref. [67].

For the mixing of the $\eta - \eta'$ system, we take the single-mixing-angle scheme in the quark-flavor basis [17,18,46,47,51], and then the effective decay constants can be parametrized: as

$$\begin{aligned} f^{u(d)}_\eta &= \frac{f_q}{\sqrt{2}}\cos\phi, & f^s_\eta &= -f_s\sin\phi, \\ f^{u(d)}_{\eta'} &= \frac{f_q}{\sqrt{2}}\sin\phi, & f^s_{\eta'} &= f_s\cos\phi. \end{aligned} \tag{3.2}$$

Here, the mixing angle ϕ and the decay constants f_q, f_s are three phenomenological parameters which can been

[1]Of course, the contributions from the large integration variable $\hat{q}$ would be strongly suppressed by the bound-state wave function [see Eqs. (2.38) and (2.14)]. Empirically, the major region of the wave function of the charmonia may be supposed near or below 1 GeV [80].

TABLE I. The quark-antiquark contributions obtained with the zero-binding approximation and those obtained with the B-S formalism, respectively.

Ratios	Zero binding [67]	This work	Experiment [28]
$\mathcal{B}(h_c \to \gamma\eta)$	0.1×10^{-4}	0.3×10^{-4}	$(4.7 \pm 1.5 \pm 1.4) \times 10^{-4}$
$\mathcal{B}(h_c \to \gamma\eta')$	0.38×10^{-3}	0.87×10^{-3}	$(1.52 \pm 0.27 \pm 0.29) \times 10^{-3}$
R_{h_c}	3.4%	4.0%	$(30.7 \pm 11.3 \pm 8.7)\%$

TABLE II. The gluonic contributions obtained with the zero-binding approximation and those obtained with the B-S formalism, respectively.

Ratios	Zero binding [67]	This work	Experiment [28]
$\mathcal{B}(h_c \to \gamma\eta)$	0.1×10^{-4}	0.1×10^{-4}	$(4.7 \pm 1.5 \pm 1.4) \times 10^{-4}$
$\mathcal{B}(h_c \to \gamma\eta')$	0.27×10^{-3}	0.45×10^{-3}	$(1.52 \pm 0.27 \pm 0.29) \times 10^{-3}$
R_{h_c}	3.0%	2.9%	$(30.7 \pm 11.3 \pm 8.7)\%$

TABLE III. Both the quark-antiquark and gluonic contributions obtained with the zero-binding approximation and the B-S formalism, respectively.

Ratios	Zero binding [67]	This work	Experiment [28]
$\mathcal{B}(h_c \to \gamma\eta)$	0.3×10^{-4}	0.9×10^{-4}	$(4.7 \pm 1.5 \pm 1.4) \times 10^{-4}$
$\mathcal{B}(h_c \to \gamma\eta')$	0.92×10^{-3}	2.29×10^{-3}	$(1.52 \pm 0.27 \pm 0.29) \times 10^{-3}$
R_{h_c}	3.8%	3.8%	$(30.7 \pm 11.3 \pm 8.7)\%$

determined in different methods (see Refs. [17,19,34, 43,72,106–111] and references therein).

A. Branching ratios

The phenomenological parameters ϕ, f_q, and f_s have been determined in Ref. [17] as

$$\phi = 39.3° \pm 1.0°, \qquad f_q = (1.07 \pm 0.02) f_\pi, \\ f_s = (1.34 \pm 0.06) f_\pi, \tag{3.3}$$

which are the known Feldmann-Kroll-Stech (FKS) results. With the set of parameter values and $\Gamma_{h_c} = 0.70$ MeV, we obtain our numerical results of the contributions from different contents of $\eta^{(\prime)}$ in Tables I–III, where the branching ratios $\mathcal{B}(h_c \to \gamma\eta)$ and $\mathcal{B}(h_c \to \gamma\eta')$ and their ratio $R_{h_c} = \mathcal{B}(h_c \to \gamma\eta)/\mathcal{B}(h_c \to \gamma\eta')$ are presented in the first, second, and third lines of these tables, respectively. The contributions from the quark-antiquark content of $\eta^{(\prime)}$ and those from the gluonic content of $\eta^{(\prime)}$ are presented in Tables I and II, respectively. The total contributions from both the quark-antiquark content and the gluonic content of $\eta^{(\prime)}$ are presented in Table III. To show the contributions from the relativistic effects more clearly, we present the results with the zero-binding approximation[2] (i.e., without relativistic corrections) [67] and those with the relativistic corrections in the first and second columns of these tables, respectively.

From the Tables I and II, one can find that the gluonic contributions and the quark-antiquark contributions are comparable with each other, whether the relativistic corrections are taken into account or not. It is unlike the situation where the gluonic contributions are strongly suppressed in the heavy vector quarkonium decays $V \to \gamma\eta^{(\prime)}$ [36,40,71] because of an additional suppression factor (i.e., m^2/M^2) from the spin structure of their amplitudes. Besides, our results are also unlike the phenomenological fits where the gluonic content can be neglected [72–74], especially for the meson η. By comparing with the results of the zero-binding approximation, we find out that the relativistic corrections are significant for both the quark-antiquark contributions and the gluonic contributions. Intriguingly, the importance of the gluonic contributions in the decay processes $h_c \to \gamma\eta^{(\prime)}$ indicates that these two decay processes can test the gluonic content of $\eta^{(\prime)}$ more efficiently than the decay processes $V \to \gamma\eta^{(\prime)}$.

Comparing the results listed in Tables I and II with those listed in Table III, we find that both the branching ratios $\mathcal{B}(h_c \to \gamma\eta)$ and $\mathcal{B}(h_c \to \gamma\eta')$ are greatly enhanced with the constructive interference of the quark-antiquark contributions and the gluonic contributions, whether the relativistic corrections are taken into account or not. Unfortunately, the branching ratio $\mathcal{B}(h_c \to \gamma\eta)$ is still much smaller than its

[2]We update the previous results [67] with the QCD running coupling constant $\alpha_s(m_c) = 0.38$.

TABLE IV. The quark-antiquark contributions obtained with a smaller value of the mixing angle.

Ratios	Zero binding [67]	This work	Experiment [28]
$\mathcal{B}(h_c \to \gamma\eta)$	0.7×10^{-4}	1.9×10^{-4}	$(4.7 \pm 1.5 \pm 1.4) \times 10^{-4}$
$\mathcal{B}(h_c \to \gamma\eta')$	0.26×10^{-3}	0.6×10^{-3}	$(1.52 \pm 0.27 \pm 0.29) \times 10^{-3}$
R_{h_c}	27.5%	31.7%	$(30.7 \pm 11.3 \pm 8.7)\%$

TABLE V. The gluonic contributions obtained with a smaller value of the mixing angle.

Ratios	Zero binding [67]	This work	Experiment [28]
$\mathcal{B}(h_c \to \gamma\eta)$	0.4×10^{-4}	0.7×10^{-4}	$(4.7 \pm 1.5 \pm 1.4) \times 10^{-4}$
$\mathcal{B}(h_c \to \gamma\eta')$	0.19×10^{-3}	0.31×10^{-3}	$(1.52 \pm 0.27 \pm 0.29) \times 10^{-3}$
R_{h_c}	23.8%	23.4%	$(30.7 \pm 11.3 \pm 8.7)\%$

TABLE VI. Both the quark-antiquark and gluonic contributions obtained with a smaller value of the mixing angle.

Ratios	Zero binding [67]	This work	Experiment [28]
$\mathcal{B}(h_c \to \gamma\eta)$	1.9×10^{-4}	4.7×10^{-4}	$(4.7 \pm 1.5 \pm 1.4) \times 10^{-4}$
$\mathcal{B}(h_c \to \gamma\eta')$	0.63×10^{-3}	1.57×10^{-3}	$(1.52 \pm 0.27 \pm 0.29) \times 10^{-3}$
R_{h_c}	30.2%	30.3%	$(30.7 \pm 11.3 \pm 8.7)\%$

experimental value [28], even though it is substantially enhanced by the relativistic corrections. Moreover, it is thought provoking that the ratio R_{h_c}, containing more dynamical corrections from the initial meson h_c, hardly changes[3] and is still smaller than the experimental value [28]. These might imply that the set of parameters of FKS results is not unquestionable.

As we have already mentioned, there are some obvious discrepancies in the determinations of the mixing angle [17,34,40,43–45,106,108,109]. Besides the value of the mixing angle $\phi \sim 40°$ (see, e.g., Refs [17,43,44,106, 108,109]), a smaller value of the mixing angle $\phi \sim 34°$ is also usually obtained in many methods [34,40,45,106]. Therefore, it would be interesting to show the results of the branching ratios $\mathcal{B}(h_c \to \gamma\eta)$, $\mathcal{B}(h_c \to \gamma\eta')$ and their ratio R_{h_c} with the different sets of the phenomenological parameters.

With the set of the parameter values [106]

$$\phi = 33.5° \pm 0.9°, \qquad f_q = (1.09 \pm 0.02) f_\pi,$$
$$f_s = (0.96 \pm 0.04) f_\pi, \tag{3.4}$$

extracted from the transition form factor (TFF) $F_{\gamma^*\gamma\eta'}(+\infty)$, which is in accord with the *BABAR* measurement in the timelike region at $q^2 = 112\ \mathrm{GeV}^2$ [114], we present the numerical results in Tables IV–VI with $\Gamma_{h_c} = 0.70$ MeV.

As we have already mentioned, theoretical calculation of the branching ratio $\mathcal{B}(h_c \to \gamma\eta)$ is certainly sensitive to the mixing angle of $\eta - \eta'$ mixing, and the main reason is that the decay amplitude is proportional to the factor $(\sqrt{2} f_q \cos\phi - f_s \sin\phi)$, which lead to large cancellations in the matrix elements. More interestingly, after taking into account the relativistic corrections, not only the ratio R_{h_c} but also the individual branching ratios $\mathcal{B}(h_c \to \gamma\eta)$ and $\mathcal{B}(h_c \to \gamma\eta')$ are in very nice agreement with their experiment data [28]. Furthermore, we find that relativistic corrections increase the individual branching ratios $\mathcal{B}(h_c \to \gamma\eta)$ and $\mathcal{B}(h_c \to \gamma\eta')$ by about a factor of 2, which is independent of the choice of the phenomenological parameters. This may imply that relativistic corrections are also important in other decay processes of the P-wave h_c. In addition, the gluonic contributions are comparable with the quark-antiquark contributions, and this is also independent of the choice of the phenomenological parameters.

B. $\eta - \eta'$ mixing

For a cross-check, we give a prediction of the mixing angle ϕ in the B-S formalism. By the ratio

$$R_{h_c} = \frac{M^2 - m_\eta^2}{M^2 - m_{\eta'}^2} \frac{|H^q_{\mathrm{QCD}} + H^g_{\mathrm{QCD}}|^2_{m=m_\eta}}{|H^q_{\mathrm{QCD}} + H^g_{\mathrm{QCD}}|^2_{m=m_{\eta'}}}; \tag{3.5}$$

the ratio

[3]The main reason is that there is no node in the wave function of h_c [80,112,113], so the relativistic corrections to a large extent could cancel in the ratio R_{h_c}.

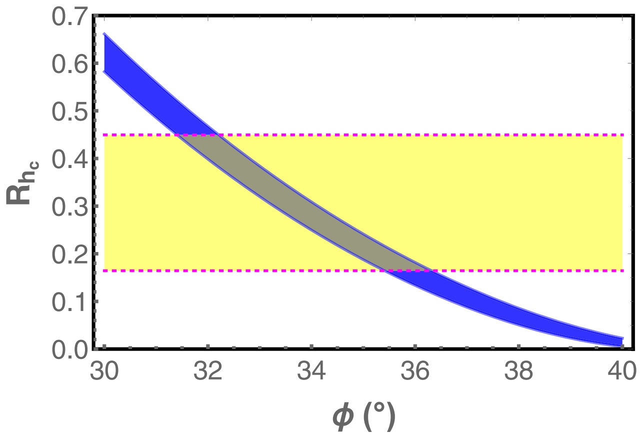

FIG. 3. The dependence of the ratio R_{h_c} on the mixing angle ϕ. The blue band is our calculated results with the uncertainties from the $\Gamma^{\rm exp}(\eta^{(\prime)} \to \gamma\gamma)$. The yellow band denotes the experimental value of R_{h_c} with 1σ uncertainty.

$$\frac{\Gamma(\eta \to \gamma\gamma)}{\Gamma(\eta' \to \gamma\gamma)} = \frac{m_\eta^3}{m_{\eta'}^3}\left(\frac{5\sqrt{2}\frac{f_s}{f_q} - 2\tan\phi}{5\sqrt{2}\frac{f_s}{f_q}\tan\phi + 2}\right)^2; \qquad (3.6)$$

and the experimental measurements [28,105,115]

$$\begin{aligned} R_{h_c}^{\rm exp} &= (30.7 \pm 11.3 \pm 8.7)\%, \\ \Gamma^{\rm exp}(\eta' \to \gamma\gamma) &= 4.34(14)\ \text{keV}, \\ \Gamma^{\rm exp}(\eta \to \gamma\gamma) &= 0.516(18)\ \text{keV}, \end{aligned} \qquad (3.7)$$

one can obtain the mixing angle

$$\phi = 33.8° \pm 2.5°, \qquad (3.8)$$

where the uncertainty comes mainly from the $R_{h_c}^{\rm exp}$. It is worth noting that the recent lattice QCD calculations give the following values: $\phi = 34° \pm 3°$ from the UKQCD Collaboration [45] and $\phi = 38.8° \pm 3.3°$ from the ETM Collaboration [110]. Schematically, we show the dependence of the ratio R_{h_c} on the mixing angle ϕ in Fig. 3. Obviously, the prediction of the mixing angle in the B-S framework is consistent with the value $\phi = 33.5° \pm 0.9°$ extracted from η' TFF [106]. Moreover, it is also in good agreement with our previous determinations $\phi = 33.9° \pm 0.6°$ [40] and $\phi = 33.8° \pm 2.5°$ [67], which are obtained by the ratios $R_{J/\psi}$ and R_{h_c} without considering the relativistic corrections, respectively.

It is worth noting that there are discrepancies in the determinations of the mixing angle ϕ, and it may imply an incomplete understanding of $\eta - \eta'$ mixing. Lastly but interestingly, in the same framework of perturbative QCD, we obtain a consistency check about the mixing angle by the ratio $R_{J/\psi}$ and the ratio R_{h_c}. No matter what it is, the physics associated with the $\eta - \eta'$ mixing is very important and certainly worth further investigations to catch more features about it.

IV. CONCLUSIONS

In this work, we have revisited the P-wave charmonium radiative decays $h_c \to \gamma\eta^{(\prime)}$ in the B-S formalism, where the internal momentum of h_c has been retained in both the soft wave function $\psi(\hat{q})$ and the hard-scattering amplitude $\mathcal{O}(\hat{q})$. The B-S wave function is employed to describe the bound-state properties of the P-wave charmonium h_c, while the light-cone DAs are adopted for the light mesons $\eta^{(\prime)}$. And then, the involved one-loop integrals are carried out analytically. It is found that the relativistic corrections from both the quark-antiquark content and the gluonic content of $\eta^{(\prime)}$ make significant contributions to the decay rates of $h_c \to \gamma\eta^{(\prime)}$. In addition, the predicted branching ratios $\mathcal{B}(h_c \to \gamma\eta^{(\prime)})$ are insensitive to the shapes of the $\eta^{(\prime)}$ DAs, and the gluonic contributions as well as the quark-antiquark contributions are both important in these two decay processes. What is more, by the ratio R_{h_c}, the widths $\Gamma(\eta^{(\prime)} \to \gamma\gamma)$, and their experimental values, we have obtained a consistency check about the mixing angle ϕ in the framework of perturbative QCD.

As mentioned before in the discussion, since the decay amplitude of the η channel is proportional to the factor $(\sqrt{2}f_q\cos\phi - f_s\sin\phi)$, which could lead to large cancellations in the matrix elements, the predicted branching ratio $\mathcal{B}(h_c \to \gamma\eta)$ is sensitive to the angle of $\eta - \eta'$ mixing. This means that the branching ratios are generally hard to predict precisely but would be more efficient to determine the mixing angle in η production and decay processes. On the other hand, by the comparison between the results without the relativistic corrections and these obtained in this work, we find that the relativistic corrections are rather significant in the exclusive P-wave charmonium decays $h_c \to \gamma\eta^{(\prime)}$. This may imply that the relativistic effects should be taken into account in the production and decay processes of the higher excited charmonia, especially for the radially excited states with nodes contained in their wave functions. Further investigations about these issues are certainly deserved.

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APPENDIX: THE EXPRESSIONS OF THE SIX NUMERATORS

The expressions of the numerators N_i $(i = 1 \sim 6)$ read

$$\begin{aligned}
N_1 = {}& \frac{64}{1-x}(M\hat{q}^2 l^2(x^2-1) - 2m_c k\cdot\hat{q}(k\cdot l - p\cdot l + xK\cdot l)) + 16Ml\cdot\hat{q}l^2(x+1) \\
&+ \frac{128k\cdot\hat{q}K\cdot l^2}{M^3\hat{m}_c(1-x)}(\hat{m}_c(2k\cdot\hat{q} + p\cdot l) - 2m_c k\cdot\hat{q}) + \frac{64K\cdot ll\cdot p(2\hat{q}^2 - l\cdot\hat{q})}{M} \\
&- \frac{128k\cdot\hat{q}K\cdot ll\cdot\hat{q}(2m_c x - \hat{m}_c x - \hat{m}_c)}{M\hat{m}_c(1-x)} + \frac{64Mm_c l\cdot\hat{q}^2(x+1)}{\hat{m}_c} \\
&+ \frac{64Ml\cdot\hat{q}}{\hat{m}_c}(Mm_c\hat{m}_c(x-1) + (x+1)(m_c^2\hat{m}_c - 2m_c\hat{q}^2 + \hat{m}_c\hat{q}^2)) \\
&+ \frac{128k\cdot\hat{q}K\cdot l}{M\hat{m}_c}(m_c^2\hat{m}_c - 2m_c\hat{q}^2 + \hat{m}_c\hat{q}^2) + \frac{32k\cdot\hat{q}K\cdot ll^2(x+3)}{M(x-1)},
\end{aligned}$$

$$\begin{aligned}
N_2 = {}& \frac{16Mx}{\hat{m}_c}(M^2 x\hat{m}_c l\cdot\hat{q} + 2\hat{q}^2(m_c - \hat{m}_c)(2k\cdot l + l\cdot p) - \hat{m}_c l\cdot\hat{q}((M-2m_c)^2 - 4\hat{q}^2)) \\
&+ \frac{32Mk\cdot\hat{q}}{\hat{m}_c(x-1)}(2\hat{m}_c l\cdot p - 2\hat{m}_c(x^2-2x-1)l\cdot\hat{q} + \hat{m}_c(x^2-3x)K\cdot l - 4m_c xl\cdot\hat{q}) \\
&- \frac{32l^2 M}{\hat{m}_c(x-1)}(m_c(x^2-1)\hat{q}^2 + 2x\hat{m}_c k\cdot\hat{q} + \hat{m}_c(x^2-1)\hat{q}^2) + \frac{64(x-2)k\cdot\hat{q}l\cdot p^2}{M(1-x)} \\
&+ \frac{128k\cdot\hat{q}^2}{M\hat{m}_c(1-x)}(m_c l\cdot p + \hat{m}_c(x-2)l\cdot p - \hat{m}_c(1-x)k\cdot l) + \frac{64xk\cdot\hat{q}k\cdot ll\cdot p}{M(1-x)} \\
&- \frac{32M}{\hat{m}_c}(2m_c^2\hat{m}_c l\cdot\hat{q} + m_c\hat{q}^2 l\cdot p - \hat{m}_c\hat{q}^2(l\cdot p + 2l\cdot\hat{q})) - \frac{128(m_c^2 - \hat{q}^2)k\cdot\hat{q}K\cdot l}{M} \\
&+ \frac{32l\cdot pl\cdot\hat{q}}{M\hat{m}_c(x-1)}(M^2\hat{m}_c(x-1)x - 2Mm_c\hat{m}_c(x-1) + 4(m_c - \hat{m}_c)k\cdot\hat{q}) \\
&+ \frac{64K\cdot ll\cdot p}{M^2\hat{m}_c(x-1)}(Mm_c(x-1)\hat{q}^2 + 2m_c\hat{m}_c k\cdot\hat{q} + M\hat{m}_c(x-1)\hat{q}^2) \\
&- \frac{128l^2(Mm_c\hat{m}_c k\cdot\hat{q} + k\cdot\hat{q}^2(m_c - \hat{m}_c))}{M\hat{m}_c(x-1)} + \frac{128m_c k\cdot\hat{q}(xK\cdot l - l\cdot p)}{(x-1)},
\end{aligned}$$

$$
\begin{aligned}
N_3 = {} & \frac{32Mm_c}{\hat{m}_c}(2M\hat{m}_c x l\cdot\hat{q} + l\cdot p\hat{q}^2 - 2xk\cdot l\hat{q}^2 + xl\cdot p\hat{q}^2 - 2m_c\hat{m}_c(x+1)l\cdot\hat{q}) \\
& + \frac{32l\cdot pl\cdot\hat{q}}{M\hat{m}_c(x-1)}(M^2\hat{m}_c(x^2-x) - 2Mm_c\hat{m}_c(x-1) + 4(\hat{m}_c - m_c)k\cdot\hat{q}) \\
& + \frac{64xk\cdot\hat{q}k\cdot ll\cdot p}{M(x-1)} + \frac{128m_c k\cdot\hat{q}(xK\cdot l - l\cdot p)}{(x-1)} - \frac{128(m_c^2-\hat{q}^2)k\cdot\hat{q}K\cdot l}{M} \\
& + 16M(M^2(x^2-x)l\cdot\hat{q} + 2\hat{q}^2(2xk\cdot l + 2xl\cdot\hat{q} + 2l\cdot\hat{q} + (x-1)l\cdot p)) \\
& + \frac{32Mk\cdot\hat{q}}{(x-1)}((x^2-3x)k\cdot l + (x^2-3x+2)l\cdot p + 2(x^2-2x-1)l\cdot\hat{q}) \\
& + \frac{128l^2k\cdot\hat{q}}{M\hat{m}_c(x-1)}(Mm_c\hat{m}_c - (m_c-\hat{m}_c)k\cdot\hat{q}) + \frac{64(x-2)k\cdot\hat{q}l\cdot p^2}{M(x-1)} \\
& + \frac{64K\cdot ll\cdot p}{M^2\hat{m}_c(x-1)}(Mm_c(x-1)\hat{q}^2 - 2m_c\hat{m}_c k\cdot\hat{q} + M\hat{m}_c(x-1)\hat{q}^2) \\
& - \frac{32l^2M}{\hat{m}_c(x-1)}(m_c(x^2-1)\hat{q}^2 + \hat{m}_c(x^2-1)\hat{q}^2 - 2\hat{m}_c xk\cdot\hat{q}) \\
& + \frac{128k\cdot\hat{q}^2}{M\hat{m}_c(x-1)}(m_c l\cdot p - \hat{m}_c(k\cdot l + 2l\cdot p) + x\hat{m}_c K\cdot l) \\
& + \frac{128Mm_c xk\cdot\hat{q}l\cdot\hat{q}}{\hat{m}_c(x-1)},
\end{aligned}
$$

$$
\begin{aligned}
N_4 = {} & \frac{8i(M^2(x\xi+x-1) - 2Mm_c(\xi-1) - 4m_c^2)}{M(x-1)}k\cdot\hat{q} + \frac{32i}{M(x-1)}k\cdot\hat{q}\hat{q}^2 \\
& + \frac{16i(m_c(x+1)(\xi-1) - \hat{m}_c(x\xi+x+\xi-3))}{M\hat{m}_c(x-1)^2}(k\cdot\hat{q})^2 \\
& + \frac{4iM(x-1)(m_c(\xi-1) + \hat{m}_c(\xi+1))}{\hat{m}_c}\hat{q}^2,
\end{aligned}
$$

$$
\begin{aligned}
N_5 = {} & -\frac{8i(M^2(x(\xi-1)+1) - 2Mm_c(\xi+1) + 4m_c^2)}{M(x-1)}k\cdot\hat{q} + \frac{32i}{M(x-1)}k\cdot\hat{q}\hat{q}^2 \\
& + \frac{16i(m_c(x+1)(\xi+1) - \hat{m}_c(x(\xi-1)+\xi+3))}{M\hat{m}_c(x-1)^2}(k\cdot\hat{q})^2 \\
& + \frac{4iM(x-1)(m_c(\xi+1) + \hat{m}_c(\xi-1))}{\hat{m}_c}\hat{q}^2,
\end{aligned}
$$

$$
\begin{aligned}
N_6 = {} & -\frac{8i(M^2\xi^2 - 4m_c^2)}{M(x-1)}k\cdot\hat{q} - \frac{32i(2m_c-\hat{m}_c)}{M\hat{m}_c(x-1)}k\cdot\hat{q}\hat{q}^2 \\
& - \frac{32i\xi(2m_c - \hat{m}_c(x+1))}{M\hat{m}_c(x-1)^2}(k\cdot\hat{q})^2 + 8iM(x-1)\xi\hat{q}^2
\end{aligned}
$$

with $x = m^2/M^2$.

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