Correlations between the strange quark condensate, strange quark mass, and kaon PCAC relation

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Correlations between the strange quark mass, strange quark condensate $\langle\bar{s}s\rangle$, and the kaon partially conserved axial current (PCAC) relation are developed. The key dimensionless and renormalization-group invariant quantities in these correlations are the ratio of the strange to nonstrange quark mass $r_m = m_s/m_q$, the condensate ratio $r_c = \langle\bar{s}s\rangle/\langle\bar{q}q\rangle$, and the kaon PCAC deviation parameter $r_p = -m_s\langle\bar{s}s + \bar{q}q\rangle/2f_K^2 m_K^2$. The correlations define a self-consistent trajectory in the $\{r_m, r_c, r_p\}$ parameter space constraining strange quark parameters that can be used to assess the compatibility of different predictions of these parameters. Combining the constraint with Particle Data Group (PDG) values of r_m results in $\{r_c, r_p\}$ constraint trajectories that are used to assess the self-consistency of various theoretical determinations of $\{r_c, r_p\}$. The most precise determinations of r_c and r_p are shown to be mutually consistent with the constraint trajectories and provide improved bounds on r_p. In general, the constraint trajectories combined with r_c determinations tend to provide more accurate bounds on r_p than direct determinations. The $\{r_c, r_p\}$ correlations provide a natural identification of a self-consistent set of strange quark mass and strange quark condensate parameters.

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QCD sum-rules techniques probe hadronic properties via QCD composite operators with appropriate quantum numbers and quark/gluonic valence content to serve as interpolating fields for hadronic states [1,2] (for reviews, see e.g., Refs. [3–5]). Correlation functions of these composite operators are calculated from QCD using the operator-product expansion and are then related to hadronic properties through dispersion relations, which are converted to a QCD sum rule through application of an appropriate transform. Important features of these correlation functions are the power-law corrections induced by QCD condensates (vacuum expectations values) that parametrize nonperturbative aspects of the QCD vacuum.

In QCD sum-rules analyses of hadronic systems containing strange quarks, the strange quark mass m_s, and the strange quark condensate $\langle\bar{s}s\rangle$ are important parameters. Depending upon the system, the condensate may emerge as $\langle\bar{s}s\rangle$ or be accompanied with quark mass factors (e.g., $m_s\langle\bar{s}s\rangle$, $m_c\langle\bar{s}s\rangle$). However, $\langle\bar{s}s\rangle$ is also used within determinations of higher-dimension condensates, including the vacuum-saturation approximation for dimension-six quark condensates and the dimension-five mixed condensate [6,7]

$$\langle g\bar{s}\sigma_{\mu\nu}\lambda^a G^a_{\mu\nu}s\rangle = \langle g\bar{s}\sigma Gs\rangle = ((0.8 \pm 0.1)\ \mathrm{GeV}^2)\langle\bar{s}s\rangle. \quad (1)$$

Because of these multiple roles in determining different QCD condensates, the strange quark condensate is an essential parameter in QCD sum rules and provides insight into $SU(3)$ flavor symmetry of the QCD vacuum.

QCD sum-rules studies of the strange quark condensate are based upon pseudoscalar and scalar correlation functions combined with low-energy theorems (see e.g., Refs. [8–10])

$$\Psi_5(q^2) = i\int d^4x\langle\Omega|T(J_5(x)J_5^\dagger(0)|\Omega\rangle,$$
$$J_5(x) = i(m_u + m_s)\bar{s}(x)\gamma_5 u(x), \quad (2)$$

$$\Psi_5(0) = -(m_s + m_u)\langle\bar{s}s + \bar{q}q\rangle, \quad (3)$$

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$$\Psi(q^2) = i\int d^4x\langle\Omega|T(J(x)J^\dagger(0)|\Omega\rangle,$$
$$J(x) = i(m_s - m_u)\bar{s}(x)u(x), \tag{4}$$

$$\Psi(0) = -(m_s - m_u)\langle\bar{s}s - \bar{q}q\rangle, \tag{5}$$

where $SU(2)$ isospin symmetry of the vacuum implies $\langle\bar{q}q\rangle = \langle\bar{u}u\rangle = \langle\bar{d}d\rangle$.

QCD sum-rules determinations of the low-energy constants $\Psi_5(0)$ and $\Psi(0)$ contain $\langle\bar{s}s\rangle$ dependence and thus provide a natural means to extract the strange quark condensate (see e.g., Refs. [8,9]). For example, the ratio $\Psi_5(0)/\Psi(0)$ can be used to reference the strange quark condensate to the nonstrange condensate through the renormalization-group (RG) invariant ratio and the RG invariant strange quark mass ratio $\xi = m_u/m_s = 0.024 \pm 0.006 \ll 1$ [11]

$$\frac{\Psi_5(0)}{\Psi(0)} = \left(\frac{1+\xi}{1-\xi}\right)\left(\frac{r_c+1}{r_c-1}\right) \approx \frac{r_c+1}{r_c-1}, \tag{6}$$

$$\frac{\langle\bar{s}s\rangle}{\langle\bar{q}q\rangle} = r_c, \qquad \langle\bar{q}q\rangle = \langle\bar{u}u\rangle = \langle\bar{d}d\rangle. \tag{7}$$

Furthermore, as $\langle\bar{q}q\rangle < 0$, if $r_c < 1$ then $\Psi(0) < 0$, so the sign of the low-energy constant $\Psi(0)$ provides valuable information on qualitative aspects of $SU(3)$ flavor symmetry breaking by the vacuum.

There is a wide range of r_c theoretical determinations. QCD sum-rules analyses of light scalar and pseudoscalar meson correlation functions have been used to determine r_c [9,12–14], and a combined estimate from these results is $r_c = 0.57 \pm 0.12$ [4]. QCD sum rules for baryon mass splittings tend to yield larger values ranging from the earliest determinations $r_c = 0.8 \pm 0.1$ [15,16] to the updated values $r_c = 0.75 \pm 0.08$ [17] and $r_c = 0.74 \pm 0.03$ [18]. Because the QCD sum rules for heavy baryon mass splittings have better perturbative convergence compared to the light meson analyses, the most precise sum-rule value is $r_c = 0.74 \pm 0.03$ [18]. A more conservative sum-rule value obtained from an average of light meson and baryon systems is $r_c = 0.66 \pm 0.10$ [4]. The lattice QCD determination $r_c = 1.08 \pm 0.16$ [19] (multiple sources of uncertainty have been combined) is considerably larger than the sum-rules determinations. An analysis combining aspects of QCD sum-rules and lattice QCD results yields $r_c = 0.8 \pm 0.3$ [20]. The various theoretical determinations of r_c are shown in Fig. 1 and Table I.

The partially conserved axial current (PCAC) Gell-Mann-Oakes-Renner relation relates the RG invariant combination of the nonstrange quark condensate and mass parameter to pion properties [22]

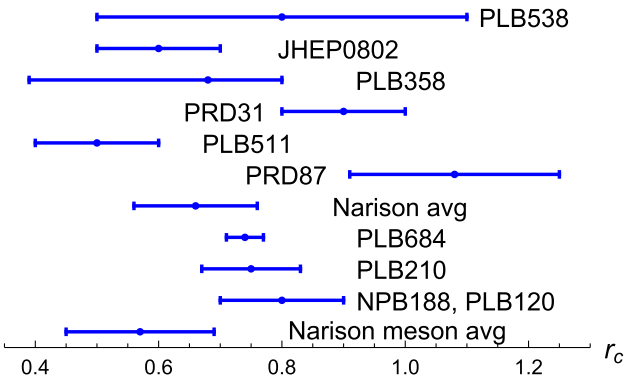

FIG. 1. Summary of r_c theoretical determinations. See Table I for key to references.

$$2m_q\langle\bar{q}q\rangle = -f_\pi^2 m_\pi^2, \qquad m_q = \frac{m_u + m_d}{2}, \tag{8}$$

where in our conventions $f_\pi = 130/\sqrt{2}$ MeV [11]. Deviations from the pion PCAC relation are bounded by approximately 5% [10,19,20] and are thus a small numerical effect. The RG invariant quark mass ratio [11]

$$r_m = \frac{m_s}{m_q} = 27.3 \pm 0.7 \tag{9}$$

can then be combined with r_c to obtain the following strange quark condensate $m_s\langle\bar{s}s\rangle$ in terms of (8)

$$m_s\langle\bar{s}s\rangle = r_m r_c m_q\langle\bar{q}q\rangle. \tag{10}$$

A complementary approach to determinations of $m_s\langle\bar{s}s\rangle$ is through the deviation from the kaon PCAC relation as parametrized by r_p

$$-(m_s + m_u)\langle\bar{q}q + \bar{s}s\rangle \approx -m_s\langle\bar{q}q + \bar{s}s\rangle = r_p 2f_K^2 m_K^2, \tag{11}$$

where $r_p = 1$ corresponds to the kaon PCAC result (in conventions where $f_K = 156/\sqrt{2}$ MeV [11]) and (9) implies that neglecting m_u is a small numerical effect [see e.g., Eq. (6)]. The PCAC deviation parameter r_p can be determined in QCD sum rules by using the low-energy theorem for the pseudoscalar correlation function (3) (see e.g., Ref. [8]). A sum-rule evaluation of $\Psi_5(0)$ thus allows determination of r_p via Eqs. (11) and (3). Significant deviations from the kaon PCAC result have been found in this approach ranging from the earliest values $r_p = 0.63 \pm 0.08$ [8] and $r_p = 0.5 \pm 0.17$ [21], to later determinations from Laplace sum rules $r_p = 0.57 \pm 0.19$ [13],[1] QCD sum rules $r_p = 0.66^{+0.23}_{-0.17}$ [12], lattice QCD $r_p = 0.74 \pm 0.16$ [19], and combined approaches (merging Laplace sum

[1]The result of [13] has been augmented with an estimated truncation error [4].

TABLE I. Summary of $\{r_c, r_p\}$ theoretical determinations shown in Figs. 1, 2, and 4. In some entries, multiple sources of uncertainty have been combined or ranges have been converted to an uncertainty. FESR denotes finite-energy sum-rules, LSR denotes Laplace sum rules, ChPT denotes chiral perturbation theory, QCDSR denotes QCD sum rules, AQCD denotes analytic continuation by duality, and LQCD denotes lattice QCD. The PLB511 entry includes a truncation uncertainty from Ref. [4].

r_p	r_c	Reference	Methodology
0.56 ± 0.16	0.5 ± 0.1	[13] (PLB511)	LSR (four loops)
$0.66^{+0.23}_{-0.17}$	$0.68^{+0.15}_{-0.29}$	[12] (PLB358)	QCDSR
0.57 ± 0.19	0.66 ± 0.1	[4] (Narison avg)	QCDSR average
0.66 ± 0.05	0.6 ± 0.1	[14] (JHEP0802)	FESR five loop
0.39 ± 0.22	0.8 ± 0.3	[20] (PLB538)	ChPT, LSR, and LQCD input
0.74 ± 0.16	1.08 ± 0.16	[19] (PRD87)	LQCD
…	0.9 ± 0.1	[9] (PRD31)	LSR
…	0.74 ± 0.03	[18] (PLB684)	LSR heavy baryons
…	0.75 ± 0.08	[17] (PLB210)	QCDSR heavy mesons
…	0.8 ± 0.1	[15,16] (NPB188, PLB120)	QCDSR baryons
…	0.57 ± 0.12	[4] (Narison meson avg)	QCDSR meson average
0.5 ± 0.17	…	[21] (ZPC27)	LSR, AQCD
0.63 ± 0.08	…	[8] (PLB104)	QCDSR

rules, chiral perturbation theory, and lattice QCD input) $r_p = 0.39 \pm 0.22$ [20]. The most precise determination $r_p = 0.66 \pm 0.05$ emerges from finite-energy sum rules [14]. The various determinations of r_p are shown in Fig. 2.

The RG invariant combination of the strange quark mass and condensate emerging from the kaon PCAC relation (11) is

$$m_s\langle\bar{s}s\rangle = -r_p 2 f_K^2 m_K^2 - r_m m_q \langle\bar{q}q\rangle. \tag{12}$$

The two expressions (10) and (12) for $m_s\langle\bar{s}s\rangle$ are thus self-consistent if the following constraint is satisfied:

$$r_c = \frac{r_p}{\sigma_m} - 1, \tag{13}$$

$$\sigma_m = r_m\left(\frac{f_\pi^2 m_\pi^2}{4 f_K^2 m_K^2}\right), \tag{14}$$

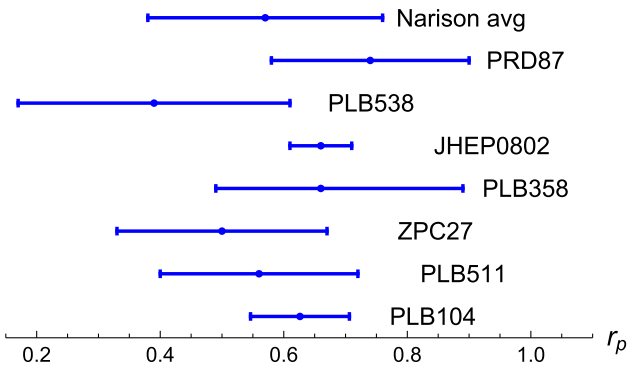

FIG. 2. Summary of r_p theoretical determinations. See Table I for key to references.

providing a correlation in the $\{r_m, r_c, r_p\}$ parameter space. The $\{r_c, r_p\}$ linear trajectories resulting from the PDG r_m range (9) are shown in Fig. 3. Determinations of r_c and r_p that lie along these trajectories will thus be self-consistent for the PDG range of the strange quark mass. The correlation trajectories provide a relatively more stringent constraint on r_p compared to r_c. For example, the conservative range $0.6 < r_c < 1.2$ leads to a relatively narrow range $0.57 < r_p < 0.86$ corresponding to a significant deviation from the kaon PCAC relation.

The most interesting analyses from the literature are those which simultaneously allow determination of both r_c and r_p because they map out a region in the $\{r_c, r_p\}$ parameter space that can be compared with the constraint trajectories. In Fig. 4, the simultaneous determinations of $\{r_c, r_p\}$ are compared with the linear constraint trajectories. The determinations from Refs. [4,12–14,19] show

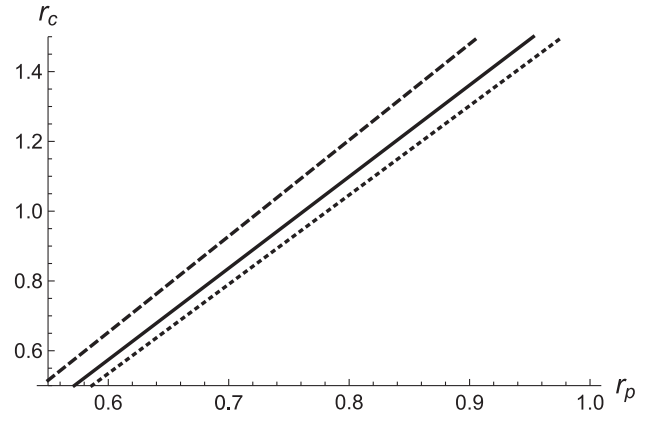

FIG. 3. Correlation (13) between r_p and r_c for PDG values [11] of r_m (solid black, dashed black, and dotted black lines respectively representing central, upper, and lower PDG values).

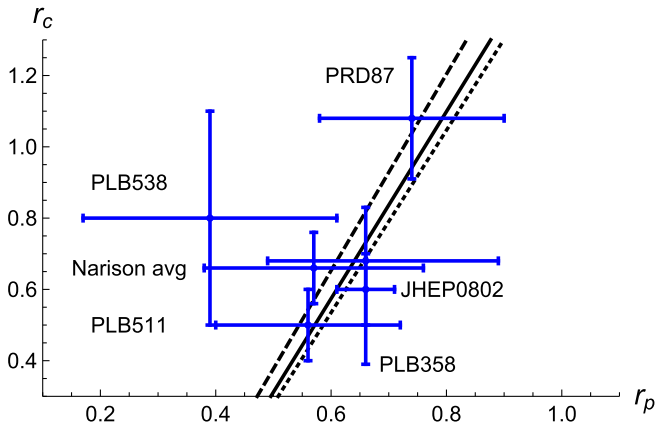

FIG. 4. Comparison of simultaneous $\{r_c, r_p\}$ determinations with the $\{r_p, r_c\}$ correlation trajectories (13) respectively representing central, upper, and lower PDG r_m values (solid black, dashed black, and dotted black lines). See Table I for key to references.

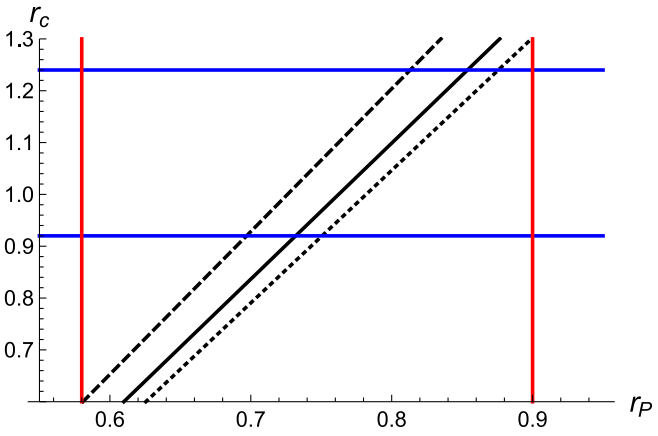

FIG. 6. Comparison of the Ref. [19] r_c bounds (blue lines) and r_p bounds (red vertical lines) with the r_p, r_c correlation trajectories (13) respectively representing central, upper, and lower PDG r_m values (solid black, dashed black, and dotted black lines).

good agreement with the trajectories, but Ref. [20], which has the smallest determination of r_p, does not intersect the trajectories. However, Ref. [20] is somewhat different than the other simultaneous determinations in Fig. 4 because it combines different methodologies (the r_p determination in Ref. [20] involves chiral perturbation theory whereas r_c involves QCD sum rules).

As a final consideration, Fig. 5 assesses the most precise individual determinations $r_c = 0.74 \pm 0.03$ [18] and $r_p = 0.66 \pm 0.05$ [14] against the constraint trajectories. The two determinations delineate a compatible region of $\{r_c, r_p\}$ parameter space, and as discussed above, the r_c determination provides a tighter bound $0.62 < r_p < 0.69$ completely contained within the Ref. [14] determination.

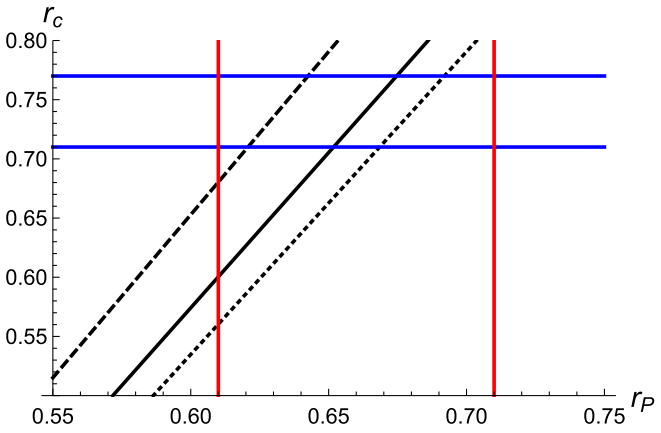

FIG. 5. Comparison of the Ref. [18] r_c bounds (blue lines) and Ref. [14] r_p bounds (red vertical lines) with the r_p, r_c correlation trajectories (13) respectively representing central, upper, and lower PDG r_m values (solid black, dashed black, and dotted black lines).

A similar feature for the Ref. [19] lattice determinations is shown in Fig. 6; the constraint trajectories combined with the r_c determination again provide a tighter bound $0.69 < r_p < 0.88$ completely contained within the Ref. [19] determination. Thus the constraint trajectories combined with r_c determinations tend to provide more accurate bounds on r_p than direct determinations.

Based on a small set of input parameters, we can generate a collection of self-consistent numerical results for some QCD condensates containing strange quarks. The strange quark condensate $\langle \bar{s}s \rangle$ is often multiplied by a quark mass, *i.e.*, $M\langle \bar{s}s \rangle$, an RG invariant quantity. Using $r_c = 0.74 \pm 0.03$ and (8)–(10), we find

$$m_s\langle \bar{s}s \rangle = (-1.7 \times 10^{-3})\ \text{GeV}^4. \tag{15}$$

Then, using

$$m_c\langle \bar{s}s \rangle = \left(\frac{m_c}{m_s}\right) m_s\langle \bar{s}s \rangle, \tag{16}$$

$$m_b\langle \bar{s}s \rangle = \left(\frac{m_b}{m_s}\right) m_s\langle \bar{s}s \rangle \tag{17}$$

with quark mass ratios [11]

$$\frac{m_c}{m_s} = 11.72 \pm 0.25, \tag{18}$$

$$\frac{m_b}{m_s} = 53.94 \pm 0.12, \tag{19}$$

we find

$$m_c\langle \bar{s}s \rangle = (-1.9 \times 10^{-2})\ \text{GeV}^4, \tag{20}$$

$$m_b\langle\bar{s}s\rangle = (-9.0\times10^{-2})\ \mathrm{GeV}^4. \quad (21)$$

Uncertainties in (15), (20), and (21) are at most 5% and stem mainly from deviations from pion PCAC in (8). Like $\langle\bar{s}s\rangle$, the dimension-five mixed strange quark condensate (1) is often multiplied by a quark mass, i.e., $M\langle g\bar{s}\sigma Gs\rangle$. Equations (1) and (15) give

$$m_s\langle g\bar{s}\sigma Gs\rangle = (-1.3\times10^{-3})\ \mathrm{GeV}^6. \quad (22)$$

Then, again using the quark mass ratios (18) and (19), we find

$$m_c\langle g\bar{s}\sigma Gs\rangle = (-1.6\times10^{-2})\ \mathrm{GeV}^6, \quad (23)$$

$$m_b\langle g\bar{s}\sigma Gs\rangle = (-7.2\times10^{-2})\ \mathrm{GeV}^6. \quad (24)$$

Uncertainties in (22)–(24) are roughly 18% and stem from deviations from pion PCAC (8) and from the ratio of the strange quark condensate to the dimension-five mixed strange quark condensate in (1). Dimension-six quark condensates are often multiplied by a factor of α_s. In addition, the vacuum saturation hypothesis is generally used to express dimension-six quark condensates as products of dimension-three quark condensates [1]. Using

$$\alpha_s\langle\bar{q}q\rangle^2 = 1.8\times10^{-4}\ \mathrm{GeV}^6 \quad (25)$$

from Ref. [2], we find with $r_c = 0.74$ that

$$\alpha_s\langle\bar{s}s\rangle\langle\bar{q}q\rangle = 1.3\times10^{-4}\ \mathrm{GeV}^6, \quad (26)$$

$$\alpha_s\langle\bar{s}s\rangle^2 = 9.9\times10^{-5}\ \mathrm{GeV}^6. \quad (27)$$

Deviations from vacuum saturation can be applied by multiplying the results (25)–(26) by a vacuum saturation parameter κ where $1 \le \kappa \lesssim 4$ (e.g., [23,24]). Note that the result (25) is remarkably consistent with (8) and the PDG [11] values $m_q(2\ \mathrm{GeV}) = 3.45$ MeV and $\alpha_s(2\ \mathrm{GeV})$.

In summary, we have developed the constraint (13) relating the strange quark parameters $r_m = m_s/m_q$, $r_c = \langle\bar{s}s\rangle/\langle\bar{q}q\rangle$ and the kaon PCAC deviation parameter $r_p = -m_s\langle\bar{s}s+\bar{q}q\rangle/2f_K^2m_K^2$. Using r_m PDG [11] values, $\{r_c, r_p\}$ theoretical determinations (see Table I) are compared with the constraint trajectories (see Fig. 4). Theoretical predictions corresponding to the smallest value of r_p show poor agreement with the constraint trajectories. However, Fig. 5 demonstrates that the most precise values $r_c = 0.74 \pm 0.03$ [18] and $r_p = 0.66 \pm 0.05$ [14] are mutually consistent with the $\{r_c, r_p\}$ constraint trajectories and provide an improved determination $0.62 < r_p < 0.69$. The combination of $\{r_c, r_p\}$ constraint trajectories with r_c determinations to obtain improved r_p bounds is also observed in Fig. 6 for the lattice values [19]. Thus the $\{r_c, r_p\}$ constraint trajectories provide a valuable methodology for assessing self-consistency or improving accuracy of determinations of the condensate ratio $r_c = \langle\bar{s}s\rangle/\langle\bar{q}q\rangle$ and the kaon PCAC deviation parameter $r_p = -m_s\langle\bar{s}s+\bar{q}q\rangle/2f_K^2m_K^2$.

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[1] M. A. Shifman, A. Vainshtein, and V. I. Zakharov, Nucl. Phys. **B147**, 385 (1979).
[2] M. A. Shifman, A. Vainshtein, and V. I. Zakharov, Nucl. Phys. **B147**, 448 (1979).
[3] L. J. Reinders, H. Rubinstein, and S. Yazaki, Phys. Rep. **127**, 1 (1985).
[4] S. Narison, *QCD as a Theory of Hadrons From Partons to Confinement* (Cambridge University Press, Cambridge, England, 2007), https://doi.org/10.1017/CBO9780511535000.
[5] P. Gubler and D. Satow, Prog. Part. Nucl. Phys. **106**, 1 (2019).
[6] M. Beneke and H. G. Dosch, Phys. Lett. B **284**, 116 (1992).
[7] V. Belyaev and B. Ioffe, Sov. Phys. JETP **56**, 493 (1982).
[8] S. Narison, Phys. Lett. **104B**, 485 (1981).
[9] C. A. Dominguez and M. Loewe, Phys. Rev. D **31**, 2930 (1985).
[10] C. A. Dominguez, *Quantum Chromodynamics Sum Rules* (Springer International Publishing, New York, 2018), https://doi.org/10.1007/978-3-319-97722-5.
[11] P. A. Zyla *et al.* (Particle Data Group), Prog. Theor. Exp. Phys. (**2020**), 083C01.
[12] S. Narison, Phys. Lett. B **358**, 113 (1995).
[13] C. A. Dominguez, A. Ramlakan, and K. Schilcher, Phys. Lett. B **511**, 59 (2001).
[14] C. A. Dominguez, N. F. Nasrallah, and K. Schilcher, J. High Energy Phys. 02 (2008) 072.
[15] B. L. Ioffe, Nucl. Phys. **B188**, 317 (1981); **B191**, 591(E) (1981).
[16] L. J. Reinders, H. R. Rubinstein, and S. Yazaki, Phys. Lett. **120B**, 209 (1983); **122B**, 487(E) (1983).
[17] S. Narison, Phys. Lett. B **210**, 238 (1988).
[18] R. M. Albuquerque, S. Narison, and M. Nielsen, Phys. Lett. B **684**, 236 (2010).

[19] C. McNeile, A. Bazavov, C. T. H. Davies, R. J. Dowdall, K. Hornbostel, G. P. Lepage, and H. D. Trottier, Phys. Rev. D **87**, 034503 (2013).
[20] M. Jamin, Phys. Lett. B **538**, 71 (2002).
[21] C. A. Dominguez, M. Kremer, N. A. Papadopoulos, and K. Schilcher, Z. Phys. C **27**, 481 (1985).
[22] M. Gell-Mann, R. Oakes, and B. Renner, Phys. Rev. **175**, 2195 (1968).
[23] S. Narison, Phys. Lett. B **361**, 121 (1995).
[24] R. A. Bertlmann, C. A. Dominguez, M. Loewe, M. Perrottet, and E. de Rafael, Z. Phys. C **39**, 231 (1988).