κ-deformed complex fields and discrete symmetries

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We present a construction of κ-deformed complex scalar field theory with the objective of shedding light on the way discrete symmetries and CPT invariance are affected by the deformation. Our starting point is the observation that, in order to have an appropriate action of Lorentz symmetries on antiparticle states, these should be described by four-momenta living on the complement of the portion of the de Sitter group manifold to which κ-deformed particle four-momenta belong. Once the equations of motions are properly worked out from the deformed action, we obtain that the particle and antiparticle are characterized by different mass-shell constraints, leading to a subtle form of departure from CPT invariance. The remaining part of our work is dedicated to a detailed description of the action of deformed Poincaré and discrete symmetries on the complex field.

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I. INTRODUCTION

It is commonly expected that the usual description of space-time as a smooth manifold is no longer reliable as we approach the Planck scale when quantum effects of the geometry can no longer be neglected. Since the prehistory of research on quantum gravity,[1] noncommutativity of space-time has been advocated as a possible way to effectively model quantum gravitational effects in regimes of negligible curvature. A widely studied incarnation of this idea suggests that the scale of noncommutativity should be seen as an observer-independent length scale [3] and that, in order to accommodate such a fundamental scale, ordinary relativistic symmetries should be *deformed* into nontrivial Hopf algebras which, in the limit of vanishing noncommutativity, should reproduce the usual Poincaré algebra.

The κ-Poincaré algebra is an example of such deformations which has been intensively investigated for almost 30 years. Such algebra was originally derived by contracting the quantum anti–de Sitter algebra [4,5]. It was brought to its modern form a few years later in Refs. [6,7], where, in particular, the role of noncommutative κ-Minkowski space-time was discovered and investigated. The deformation parameter κ has dimensions of mass, and, in light of the possible role of the κ-Poincaré algebra in describing the symmetries of a flat-space-time limit of quantum gravity, it is usually identified with the Planck energy. Such a putative relationship with a semiclassical limit of quantum gravity renders this model especially relevant for the search of possible experimental signatures of Planck-scale physics [8,10]. So far, most of the proposed observational frameworks having sufficient sensitivity to capture effects of quantum gravity origin [9,10] were based on purely kinematical models, like, for example, the well-known case of measuring the time of flight of gamma-ray-burst photons of different energies [11,12]. It has, however, been argued that κ deformations may have a subtle, and, in principle, measurable, effect on elementary particles, linked to the deformation of CPT symmetry [13]. For these reasons, we believe that developing a comprehensive theory of deformed quantum fields will be beneficial for better understanding known phenomena related to κ deformation and possibly shed light on some new ones that might be of phenomenological relevance (besides, of course, its relevance at a purely theoretical level).

In this series of papers, of which the present one is the first, we will formulate the theory of a free, complex κ-deformed scalar field. The next paper in the series will be devoted to free scalar field propagator and n-point

[1] According to Jackiw [1], the idea of noncommuting space-time coordinates was first suggested by Heisenberg back in the 1930s. He then discussed it with Peierls, who, in turn, told Pauli, who told Oppenheimer, who asked his student Snyder to work it out in detail, and, thus, the first paper on noncommutative space-time was published in 1947 [2].

functions. We will consider next massive higher-spin fields and then the quantum deformed Abelian gauge fields. We will discuss interacting fields in the final, fifth paper of the series.

The present paper has its roots in the work in Ref. [14], from which we borrow the notation and most of conventions. However, there are important differences. In particular, the definition of the scalar field is different here. This change of definition is a consequence of the assumed nice behavior of the field with respect to the discrete *CPT* transformations and leads to one of the major results of this paper, that the mass-shell relations of particles and antiparticles differ from each other, although as a manifold the mass shell in both cases is the same hyperboloid in momentum space, as anticipated in Refs. [13,15]. Thanks to this new definition of fields, also the creation-annihilation operator algebra becomes particularly simple. In the present paper, we also consistently use the star product formalism instead of the equivalent formalism of noncommutative space-time used in Ref. [14].

Various aspects of the theory of κ-deformed fields were discussed in the past. Here, we mention papers that influenced us [16–27] in working on this project, but we stress that the crucial aspects of the present construction, like the doubling of momentum space and insistence on the proper action of discrete symmetries, are new.

II. PRELIMINARIES

As is well known, there are two complementary pictures of κ deformation. One deals with the presence of noncommutative space-time with Lie-type noncommutativity, called κ-Minkowski space [6,7], where the commutator of coordinates $\hat{x}^\mu$ form the $\mathfrak{an}(3)$ Lie algebra

$$[\hat{x}^0, \hat{x}^i] = \frac{i}{\kappa}\hat{x}^i, \tag{1}$$

with the parameter κ defining the "strength" of noncommutativity. Another concerns the momentum space picture, in which the momentum space is curved and is a submanifold of de Sitter space with curvature $1/\kappa^2$ [28,29], which is constructed as follows.

Let us consider the following five-dimensional matrix representation of the Lie algebra (1):

$$\hat{x}^0 = -\frac{i}{\kappa}\begin{pmatrix} 0 & \mathbf{0} & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 1 & \mathbf{0} & 0 \end{pmatrix}, \qquad \hat{\mathbf{x}} = -\frac{i}{\kappa}\begin{pmatrix} 0 & \boldsymbol{\epsilon}^T & 0 \\ \boldsymbol{\epsilon} & \mathbf{0} & \boldsymbol{\epsilon} \\ 0 & -\boldsymbol{\epsilon}^T & 0 \end{pmatrix}, \tag{2}$$

where bold fonts are used to denote space components of a 4-vector (with the exception of the central $\mathbf{0}$, which is a 3×3 matrix) and $\boldsymbol{\epsilon}$ is a three-dimensional vector with a single unit entry, e.g., $\epsilon^1 = (1,0,0)$.

Let us now consider an element $\hat{e}_k$ of the Lie group $AN(3)$, which, as we will see in a moment, represents a group-valued momentum:

$$\hat{e}_k = e^{ik_i\hat{x}^i}e^{ik_0\hat{x}^0}. \tag{3}$$

In the representation (2), this group element is represented by a 5×5 matrix which acts on five-dimensional Minkowski space as a linear transformation. One finds

$$\exp(ik_0\hat{x}^0) = \begin{pmatrix} \cosh\frac{k_0}{\kappa} & \mathbf{0} & \sinh\frac{k_0}{\kappa} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \sinh\frac{k_0}{\kappa} & \mathbf{0} & \cosh\frac{k_0}{\kappa} \end{pmatrix},$$

$$\exp(ik_i\hat{x}^i) = \begin{pmatrix} 1+\frac{\mathbf{k}^2}{2\kappa^2} & \frac{\mathbf{k}}{\kappa} & \frac{\mathbf{k}^2}{2\kappa^2} \\ \frac{\mathbf{k}}{\kappa} & \mathbf{1} & \frac{\mathbf{k}}{\kappa} \\ -\frac{\mathbf{k}^2}{2\kappa^2} & -\frac{\mathbf{k}}{\kappa} & 1-\frac{\mathbf{k}^2}{2\kappa^2} \end{pmatrix},$$

where $\mathbf{1}$ is the unit 3×3 matrix, and $\hat{e}_k$ can be written in schematic form

$$\hat{e}_k = \begin{pmatrix} \frac{\bar{p}_4}{\kappa} & \frac{\mathbf{k}}{\kappa} & \frac{p_0}{\kappa} \\ \frac{\mathbf{p}}{\kappa} & \mathbf{1} & \frac{\mathbf{p}}{\kappa} \\ \frac{\bar{p}_0}{\kappa} & -\frac{\mathbf{k}}{\kappa} & \frac{p_4}{\kappa} \end{pmatrix}, \tag{4}$$

where p_0, p_i, and p_4 are defined below, while $\bar{p}_0 = \kappa\sinh\frac{k_0}{\kappa} - \frac{\mathbf{k}^2}{2\kappa}$ and $\bar{p}_4 = \kappa\cosh\frac{k_0}{\kappa} + \frac{\mathbf{k}^2}{2\kappa}e^{k_0/\kappa}$.

To describe the manifold of the group $AN(3)$, we choose a point in five-dimensional Minkowski space, which becomes the momentum space origin $\mathcal{O}$ with coordinates $(0,\ldots,0,\kappa)$, and act on it with the matrix $\hat{e}_k$ (4), obtaining

$$(p_0, p_i, p_4) = \hat{e}_k\mathcal{O}.$$

On the left-hand side, we have coordinates of a point in the five-dimensional Minkowski space, being in one-to-one correspondence with the group element $\hat{e}_k$. The coordinates (p_0, p_i, p_4) are related to the original parametrization (k_0, k_i) of the group element as follows:

$$\begin{aligned} p_0(k_0,\mathbf{k}) &= \kappa\sinh\frac{k_0}{\kappa} + \frac{\mathbf{k}^2}{2\kappa}e^{k_0/\kappa}, \\ p_i(k_0,\mathbf{k}) &= k_i e^{k_0/\kappa}, \\ p_4(k_0,\mathbf{k}) &= \kappa\cosh\frac{k_0}{\kappa} - \frac{\mathbf{k}^2}{2\kappa}e^{k_0/\kappa}. \end{aligned} \tag{5}$$

There is a natural action of the four-dimensional Lorentz group on the five-dimensional Minkowski space, which takes the form

$$\delta_\lambda p_0 = \lambda^i p_i, \qquad \delta_\lambda p_i = \lambda_i p_0, \qquad \delta_\lambda p_4 = 0,$$
$$\delta_\rho p_0 = 0, \qquad \delta_\rho p_i = \epsilon_{ijk}\rho^j p^k, \qquad \delta_\rho p_4 = 0$$

for infinitesimal boosts and rotation parameters λ_i and ρ_i. Since the Lorentzian momenta components p_0 and $\mathbf{p}$, transform as a vector, $p_0^2 - \mathbf{p}^2$ is Lorentz invariant and, as usual, the representations of the Lorentz group, in the spinless case that we consider here, are labeled by values of the mass m^2 and sign of energy p_0. Therefore, the representations of the Poincaré algebra are characterized by mass-shell condition $p_0^2 - \mathbf{p}^2 = m^2$.

It is easy to check that[2]

$$-p_0^2 + \mathbf{p}^2 + p_4^2 = \kappa^2, \qquad p_4 > 0. \tag{6}$$

It follows that the group $AN(3)$ is isomorphic, as a manifold, to a submanifold of the four-dimensional de Sitter space. This submanifold is defined by the conditions

$$p_0 + p_4 = \kappa e^{k_0/\kappa} > 0, \quad p_4 \equiv \sqrt{\kappa^2 + p_0^2 - \mathbf{p}^2} > 0. \tag{7}$$

On shell, $p_0^2 - \mathbf{p}^2 = m^2$, and the condition (7) takes the form

$$p_0 + \sqrt{m^2 + \kappa^2} > 0. \tag{8}$$

Observe that this condition does not impose any restrictions on positive energy states but provides a lower bound on the negative energy ones: $0 > p_0 > -\sqrt{m^2 + \kappa^2}$. This condition seemed first to be Lorentz invariance violating [30], because by acting with the Lorentz boost we can make p_0 acquire an arbitrary negative value, but was later shown to preserve Lorentz symmetry in a nontrivial way [31]. To understand how it comes about, let us introduce the antipodal map $S(p)$ defined as

$$S(p_0) = -p_0 + \frac{\mathbf{p}^2}{p_0 + p_4} = \frac{\kappa^2}{p_0 + p_4} - p_4,$$
$$S(\mathbf{p}) = -\frac{\kappa \mathbf{p}}{p_0 + p_4}, \qquad S(p_4) = p_4. \tag{9}$$

Notice that on shell $S(\omega_p) = S(\sqrt{m^2 + \mathbf{p}^2})$ is always negative.

It is worth mentioning in passing that if $p_0^2 - \mathbf{p}^2 = m^2$, then $S(p_0)^2 - S(\mathbf{p})^2 = m^2$ and vice versa, so the former serves as an alternative form of mass-shell relation. As we will see, both these mass-shell conditions will arise in the theory of a deformed scalar field.

[2]There are two solutions of the first equation in (6), but, since the point $\mathcal{O}$ for which $p_4 = 1$ belongs to the solution we are interested in, we choose p_4 positive.

One checks that this map provides a one-to-one correspondence between the "positive energy" submanifold $p_0 > 0$ and the negative energy one, satisfying the constraint (8). Indeed, take a positive energy state with energy $p_0 > 0$ and momentum $\mathbf{p}$ and apply the antipode to it. We find

$$S(p_0) + p_4 = -p_0 + \frac{\mathbf{p}^2}{p_0 + p_4} + p_4 = \frac{\kappa^2}{p_0 + p_4} > 0.$$

We define the action of Lorentz symmetry on negative energy states by applying it to the corresponding positive energy one and taking the antipode of the result, schematically:

$$L \rhd S(p) \equiv S(L \rhd p), \qquad p_0 > 0. \tag{10}$$

With this definition, the orbits of the Lorentz group for both positive and negative energies belong to the momentum space. We will describe the Lorentz transformations of the antipode in Appendix A.

The coordinates p_A (5) cover only half of de Sitter momentum space. It turns out (see below) that, in order to construct a field with well-defined properties under discrete space-time symmetries, we have to introduce another, dual, momentum space defined as an orbit of $AN(3)$ group emanating from the point $\mathcal{O}^*$ with coordinates $(0, \ldots, 0, -\kappa)$. These coordinates can be constructed with the help of a special element $\mathfrak{z}$ [14] that maps $(0, \ldots, 0, \kappa)$ to $(0, \ldots, 0, -\kappa)$:

$$\mathfrak{z} = e^{\pi\kappa \hat{X}^0} = \begin{pmatrix} -1 & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 0 & \mathbf{0} & -1 \end{pmatrix} \tag{11}$$

[or $\hat{e}_k$ in Eq. (3) with $k_i = 0$, $k_0 = -i\pi\kappa$].

We define

$$\hat{e}_k^* = \hat{e}_k \mathfrak{z} = e^{ik_i \hat{x}^i} e^{ik_0 \hat{x}^0} \mathfrak{z}, \tag{12}$$

and acting with this group element on $(0, \ldots, 0, \kappa)$, instead of Eq. (5) we get

$$p_0{}^*(k_0, \mathbf{k}) = -\kappa \sinh\frac{k_0}{\kappa} - \frac{\mathbf{k}^2}{2\kappa} e^{k_0/\kappa},$$
$$p_i^*(k_0, \mathbf{k}) = -k_i e^{k_0/\kappa},$$
$$p_4{}^*(k_0, \mathbf{k}) = -\kappa \cosh\frac{k_0}{\kappa} + \frac{\mathbf{k}^2}{2\kappa} e^{k_0/\kappa} \tag{13}$$

with

$$p_0^* + p_4^* = -\kappa e^{k_0/\kappa} < 0,$$
$$p_4^* \equiv -\sqrt{\kappa^2 + (p_0^*)^2 - (\mathbf{p}^*)^2} < 0. \tag{14}$$

On shell, the condition (14) takes the form

$$p_0^* - \sqrt{m^2 + \kappa^2} < 0 \tag{15}$$

so that this time it does not impose any restrictions on negative energy states but provides an upper bound on the positive energy ones: $0 < p_0^* < \sqrt{m^2 + \kappa^2}$. Again, one solves the apparent problem with Lorentz symmetry with the help of the antipode, which has the form

$$S(p_0^*) = -p_0^* + \frac{\mathbf{p}^{*2}}{p_0^* + p_4^*} = \frac{\kappa^2}{p_0^* + p_4^*} - p_4^*,$$
$$S(\mathbf{p}^*) = \frac{\kappa \mathbf{p}^*}{p_0^* + p_4^*}, \qquad S(p_4^*) = p_4^*. \tag{16}$$

On shell, $S(\omega_p^*) = S(-\sqrt{m^2 + \mathbf{p}^{*2}})$ is always positive.

To formulate the field theory, we must first describe the algebra of plane waves and differential calculus. We start with the group elements (also called "noncommutative" plane waves) $\hat{e}_k$ (3) (associated with the submanifold $p_0 + p_4 > 0$) and $\hat{e}_k^*$ (12) (for the submanifold $p_0 + p_4 < 0$). We use the five-dimensional Lorentz covariant differential calculus; see Ref. [14] and references therein for details. To this end, we introduce the spacetime derivatives $\hat{\partial}_\mu$ and an additional derivative in fourth direction $\hat{\partial}_4$ defined by their action on the plane waves:

$$\hat{\partial}_\mu \hat{e}_k = i p_\mu(k) \hat{e}_k, \qquad \hat{\partial}_4 \hat{e}_k = i(\kappa - p_4(k)) \hat{e}_k,$$
$$\hat{\partial}_\mu \hat{e}_k^* = i p_\mu^*(k) \hat{e}_k^*, \qquad \hat{\partial}_4 \hat{e}_k^* = i(\kappa - p_4^*(k)) \hat{e}_k^*. \tag{17}$$

Following Ref. [14], we define the Weyl map[3] $\mathcal{W}$ that maps group elements (plane waves on noncommutative κ-Minkowski space-time) to ordinary plane waves on commutative space-time manifold with coordinates x, as

$$\mathcal{W}(\hat{e}_k(\hat{x})) = e_p(x) \tag{18}$$

defined by the action of the derivatives

$$\mathcal{W}(\hat{\partial}_\mu \hat{e}_k)(\hat{x}) = \partial_\mu e_p(x), \quad \mathcal{W}(\hat{\partial}_\mu \hat{e}_k^*)(\hat{x}) = \partial_\mu e_p^*(x) \tag{19}$$

with ∂_μ being the standard partial derivative.[4] The star product presented here coincides with the one proposed in [34] and further discussed in Refs. [35–37]. It follows that

$$e_p(x) = e^{i p_\mu x^\mu} = e^{-i(\omega_{\mathbf{p}} t - \mathbf{p}\mathbf{x})},$$
$$e_p^*(x) = e^{i p_\mu^* x^\mu} = e^{-i(\omega_{\mathbf{p}}^* t - \mathbf{p}^* \mathbf{x})} \tag{20}$$

with the on-shell relations

$$\omega_{\mathbf{p}} = \sqrt{m^2 + p^2}, \qquad \omega_{\mathbf{p}}^* = -\sqrt{m^2 + p^{*2}},$$
$$p_4 = \sqrt{m^2 + \kappa^2}, \qquad p_4^* = -\sqrt{m^2 + \kappa^2}. \tag{21}$$

The Weyl map makes it possible to construct the star product of two commuting plane waves from the product of two group elements:

$$\mathcal{W}(\hat{e}_k \hat{e}_l) \equiv e_{p(k)} \star e_{q(l)} = e_{p \oplus q}. \tag{22}$$

In the case of two positive energy plane waves, we have

$$\hat{e}_k \hat{e}_l = \hat{e}_{k \oplus l} \tag{23}$$

with

$$(k \oplus l)_0 = k_0 + l_0, \qquad (k \oplus l)_i = k_i + e^{-k_0/\kappa} l_i. \tag{24}$$

Then, acting with the group element (23) on the reference vector $(0, \ldots, 0, \kappa)$, we get

$$(p \oplus q)_0 = \frac{1}{\kappa} p_0 (q_0 + q_4) + \frac{\mathbf{p}\mathbf{q}}{p_0 + p_4} + \frac{\kappa}{p_0 + p_4} q_0,$$
$$(p \oplus q)_i = \frac{1}{\kappa} p_i (q_0 + q_4) + q_i,$$
$$(p \oplus q)_4 = \frac{1}{\kappa} p_4 (q_0 + q_4) - \frac{\mathbf{p}\mathbf{q}}{p_0 + p_4} - \frac{\kappa}{p_0 + p_4} q_0. \tag{25}$$

Let us use the same construction in the case of the negative energy plane waves. To this end, we must first compute the product

$$\mathfrak{z} e_{(p_0, \mathbf{p})} = e_{(p_0, -\mathbf{p})} \mathfrak{z}. \tag{26}$$

From

$$\mathcal{W}(\hat{e}_k^* \hat{e}_l) \equiv e_{p(k)}^* \star e_{q(l)} = e_{p^* \oplus q}, \tag{27}$$

we find

$$(p^* \oplus q)_0 = \frac{1}{\kappa} p_0^* (q_0 + q_4) + \frac{\mathbf{p}^* \mathbf{q}}{p_0^* + p_4^*} + \frac{\kappa}{p_0^* + p_4^*} q_0,$$
$$(p^* \oplus q)_i = \frac{1}{\kappa} p_i^* (q_0 + q_4) + q_i,$$
$$(p^* \oplus q)_4 = \frac{1}{\kappa} p_4^* (q_0 + q_4) - \frac{\mathbf{p}^* \mathbf{q}}{p_0^* + p_4^*} - \frac{\kappa}{p_0^* + p_4^*} q_0. \tag{28}$$

[3]Notice that the choice of Weyl map is not unique (see, for instance, [16] for a different choice and the discussion in Ref. [32]) and from this choice depend also the star product structures. In this paper, we choose to adopt the Weyl map introduced in Ref. [14], mapping "time-to-the-right" ordered noncommutative plane waves to standard exponentials of commutative coordinates, expressed in terms of "embedding" momenta $p_A(k)$ $(A = 0, 1, \ldots, 4)$.

[4]An explicit realization of this star product was presented in Ref. [33].

[To compute this, one starts with Eq. (25), changes the overall sign, then changes the sign of p replacing it by p^*, and finally changes the sign of $\mathbf{q}$ according to Eq. (26).]

Similarly,

$$\begin{aligned}(p\oplus q^*)_0 &= \frac{1}{\kappa}p_0(q_0^*+q_4^*)+\frac{\mathbf{p}\mathbf{q}^*}{p_0+p_4}+\frac{\kappa}{p_0+p_4}q_0^*,\\ (p\oplus q^*)_i &= \frac{1}{\kappa}p_i(q_0^*+q_4^*)+q_i^*,\\ (p\oplus q^*)_4 &= \frac{1}{\kappa}p_4(q_0^*+q_4^*)-\frac{\mathbf{p}\mathbf{q}^*}{p_0+p_4}-\frac{\kappa}{p_0+p_4}q_0^*.\end{aligned} \tag{29}$$

Finally, we consider the composition of two negative energy plane waves (in this case, after moving through the Q plane wave, we get $\mathfrak{z}^2=1$):

$$\begin{aligned}(p^*\oplus q^*)_0 &= \frac{1}{\kappa}p_0^*(q_0^*+q_4^*)+\frac{\mathbf{p}^*\mathbf{q}^*}{p_0^*+p_4^*}+\frac{\kappa}{p_0^*+p_4^*}q_0^*,\\ (p^*\oplus q^*)_i &= \frac{1}{\kappa}p_i^*(q_0^*+q_4^*)+q_i^*,\\ (p^*\oplus q^*)_4 &= \frac{1}{\kappa}p_4^*(q_0^*+q_4^*)-\frac{\mathbf{p}\mathbf{q}^*}{p_0^*+p_4^*}-\frac{\kappa}{p_0^*+p_4^*}q_0^*.\end{aligned} \tag{30}$$

Notice that, remarkably, all the composition laws (25)–(30) have exactly the same form, so there is no need to distinguish between them.

Let us finish this section with the definition of an adjoint of the plane wave. For the noncommutative plane wave $\hat{e}_k$, its adjoint $\hat{e}_k^\dagger$ is defined by the condition

$$\hat{e}_k\hat{e}_k^\dagger=\hat{e}_k^\dagger\hat{e}_k=1, \tag{31}$$

from which it follows that

$$\hat{e}_k^\dagger=\hat{e}_{S(k)}. \tag{32}$$

Accordingly, in the star product formalism, we express these equations as

$$e_p\star e_q^\dagger=e_q^\dagger\star e_p=1, \tag{33}$$

from which it follows that

$$e_p^\dagger=e_{S(p)}. \tag{34}$$

The analogous expressions for p_A^* coordinates are easy to obtain.

III. ACTION AND FIELD EQUATIONS

Having discussed all the necessary technical tools in the preceding section, we can now turn to the construction of the theory of free complex scalar field. As customary in noncommutative field theories, we define a notion of integral on noncommutative space-time via the Weyl (or quantization) map (18). In particular, we set

$$\hat{\int}\hat{e}_k(\hat{x})\coloneqq\int_{\mathbb{R}^4}d^4x\mathcal{W}(\hat{e}_k(\hat{x}))=\int_{\mathbb{R}^4}d^4xe^{ipx}. \tag{35}$$

Fields on κ-Minkowski can be defined in terms of a suitable "noncommutative" (or, for some authors, quantum group) Fourier transform [14,32,38–41]. In accordance with our choice of Weyl map, we adopt the noncommutative Fourier transform introduced in Ref. [14]:

$$\hat{\phi}(\hat{x})=\int_{AN(3)}d\mu(p)\tilde{\phi}(p)\hat{e}_k(\hat{x}) \tag{36}$$

and its inverse

$$\tilde{\phi}(p)=\hat{\int}\hat{e}_k^\dagger(\hat{x})\phi(\hat{x}), \tag{37}$$

where the measure $d\mu(p)$ is the $AN(3)$ left-invariant measure

$$d\mu(p)=\frac{d^4p}{p_4/\kappa}\bigg|_{p_+>0\ \&\ p_4=\sqrt{\kappa^2+p_0^2-\mathbf{p}^2}}, \tag{38}$$

and the coordinates p are intended as the "embedding" coordinates $p(k)$ given by Eq. (5). The definition can be thus extended to fields of commutative coordinates through Weyl map

$$\phi(x)\coloneqq\mathcal{W}(\hat{\phi}(\hat{x})). \tag{39}$$

Explicitly,

$$\phi(x)=\int_{AN(3)}d\mu(p)\tilde{\phi}(p)e_p(x). \tag{40}$$

Notice that the $\phi(x)$ defined by Eqs. (39) and (40) depend on the choice of Weyl map. In the explicit expression (40), the dependence is encoded in both the measure of integration, expressed in terms of embedding momenta p restricted to the $AN(3)$ manifold, and on the Fourier "coefficients" $\tilde{\phi}(p)$.

From Eqs. (22) and (35), it follows that the inverse noncommutative Fourier transform can be expressed as

$$\tilde{\phi}(p)=\int_{\mathbb{R}^4}e_p^\dagger(x)\star\phi(x) \tag{41}$$

and that the noncommutative product extends to a star product of fields of commutative coordinates

$$\mathcal{W}(\hat{\phi}(\hat{x})\hat{\psi}(\hat{x})) = \phi(x)\star\psi(x). \tag{42}$$

In particular, we have the following useful identity:

$$\hat{\int} \hat{\phi}(\hat{x})\hat{\psi}(\hat{x}) = \int_{\mathbb{R}^4} \phi(x)\star\psi(x). \tag{43}$$

The star product here coincides with the one defined in Ref. [34] (generalized to 4D), which can be checked by calculating that it gives the identical result for the coordinate functions x^μ. However, it is not clear if the construction of the integral or twisted trace presented in that paper coincides with our definition of the integral.

Using the noncommutative Fourier transform and the star product, we can formulate the action of free fields on κ-Minkowski space-time as a standard integral action in terms of (properly defined as above) fields of commutative coordinates. In particular, we define the action to be an integral of the bilinear Hermitian expression, in fields and derivatives, obtained with the help of the star product. The integral satisfies the exchange properties for the plane waves [14]:

$$\int_{\mathbb{R}^4} d^4x e_p^\dagger \star e_q = \int_{\mathbb{R}^4} d^4x e_q^\dagger \star e_p \tag{44}$$

and the most general expression for the Hermitian action is

$$S = \frac{1}{2}\int_{\mathbb{R}^4} d^4x[(\partial_\mu\phi)^\dagger\star\partial^\mu\phi + (\partial_\mu\phi)\star(\partial^\mu\phi)^\dagger - m^2(\phi^\dagger\star\phi + \phi\star\phi^\dagger)]. \tag{45}$$

In order to compute the variation of the action and to derive field equations, we have to make use of the $\star$ integration by parts, which is described in detail in Appendix B. Writing $S = \frac{1}{2}(S_1 + S_2)$, where

$$S_1 = \int_{\mathbb{R}^4} d^4x(\partial^\mu\phi)^\dagger\star(\partial_\mu\phi) - m^2\phi^\dagger\star\phi \tag{46}$$

and

$$S_2 = \int_{\mathbb{R}^4} d^4x(\partial_\mu\phi)\star(\partial^\mu\phi)^\dagger - m^2\phi\star\phi^\dagger, \tag{47}$$

we find

$$\delta S_1 = \frac{1}{2}\int_{\mathbb{R}^4} d^4x(\partial_\mu\delta\phi)^\dagger\star\partial^\mu\phi + (\partial_\mu\phi)^\dagger\star\partial^\mu\delta\phi - m^2\delta\phi^\dagger\star\phi - m^2\phi^\dagger\star\delta\phi, \tag{48}$$

which can be rewritten as

$$\delta S_1 = \frac{1}{2}\int_{\mathbb{R}^4} d^4x\left\{-\frac{\Delta_+}{\kappa}[(\partial_\mu^\dagger(\partial^\mu)^\dagger - m^2)\phi^\dagger\star\delta\phi] + \partial_A(\Pi^A\star\delta\phi) - \frac{\kappa}{\Delta_+}[\delta\phi^\dagger\star(\partial_\mu\partial^\mu - m^2)\phi] + \partial_A^\dagger(\delta\phi^\dagger\star(\Pi^A)^\dagger)\right\}, \tag{49}$$

where

$$\Pi_1^0 = (\Pi_0)_1 = \frac{1}{\kappa}(\Delta_+\partial_0^\dagger + im^2)\phi^\dagger, \tag{50}$$

$$\Pi_1^i = -(\Pi_i)_1 = (-\partial_i(1 + i\Delta_+^{-1}\partial_0))\phi^\dagger, \tag{51}$$

$$\Pi_1^4 = (\Pi_4)_1 = -i\frac{m^2\phi^\dagger}{\kappa} \tag{52}$$

and, analogously,

$$\delta S_2 = \frac{1}{2}\int_{\mathbb{R}^4} d^4x\partial^\mu\phi\star(\partial_\mu\delta\phi)^\dagger + \partial^\mu\delta\phi\star(\partial_\mu\phi)^\dagger - m^2\phi\star\delta\phi^\dagger - m^2\delta\phi\star\phi^\dagger, \tag{53}$$

which can be rewritten as

$$\delta S_2 = \frac{1}{2}\int_{\mathbb{R}^4} d^4x\{-[\delta\phi\star(\partial_\mu^\dagger(\partial^\mu)^\dagger - m^2)\phi^\dagger] + \partial_A(\delta\phi\star\Pi^A) - [(\partial_\mu\partial^\mu - m^2)\phi\star\delta\phi^\dagger] + \partial_A^\dagger((\Pi^A)^\dagger\star\delta\phi^\dagger)\}, \tag{54}$$

where

$$\Pi_2^0 = (\Pi_0)_2 = \left(\frac{\kappa}{\Delta_+}\partial_0^\dagger + \frac{i}{\kappa}(\partial_0^\dagger)^2\right)\phi^\dagger, \tag{55}$$

$$\Pi_2^i = -(\Pi_i)_2 = -\frac{\kappa}{\Delta_+}(\partial_i^\dagger + i\partial_i\partial_0^\dagger)\phi^\dagger, \tag{56}$$

$$\Pi_2^4 = (\Pi_4)_2 = +i\frac{(\partial_0^\dagger)^2}{\kappa}\phi^\dagger. \tag{57}$$

Therefore, the field equations have the form

$$(\partial_\mu\partial^\mu - m^2)\phi = 0, \qquad (\partial_\mu^\dagger(\partial^\mu)^\dagger - m^2)\phi^\dagger = 0, \tag{58}$$

which, as we will see below, lead to two nontrivially related mass-shell conditions, describing the same orbit of the Lorentz group on the momentum manifold.

IV. THE COMPLEX SCALAR FIELD

Now we are in position to formulate the theory of the deformed free complex scalar field. In what follows, we will use the strategy adopted in Ref. [14] of developing the noncommutative field theory in terms of fields on commutative Minkowski space-time equipped with a noncommutative star product. Using the identity [cf. (31)–(34)]

$$e^{-ipx}\star e^{-iS(p)x}=e^{-iS(p)x}\star e^{-ipx}=1$$

to define the adjoint of the plane wave

$$(e^{-ipx})^{\dagger}=e^{-iS(p)x}, \tag{59}$$

we can write the adjoint field as

$$\phi^{\dagger}(\hat{x})=\int d\mu(p(k))\tilde{\phi}^{\dagger}(p)\mathcal{W}(e^{-iS(p(k))x}), \tag{60}$$

and one can define

$$\phi^{\dagger}(x)=\mathcal{W}^{-1}(\phi^{\dagger}(\hat{x}))=\int d\mu(p)\tilde{\phi}^{\dagger}(p)e^{-iS(p)x}. \tag{61}$$

Changing integration variables in the last expression and using that $S(S(p))=p$, we can rewrite it as

$$\phi^{\dagger}(x)=\kappa^{3}\int d\mu(p)p_{+}^{-3}\tilde{\phi}^{\dagger}(S(p))e^{-ipx}, \tag{62}$$

where we used[5]

$$d\mu(S(p))=\frac{\kappa^{3}}{p_{+}^{3}}d\mu(p), \tag{63}$$

as one can easily check.

It follows, by comparing (62) with (40), that the condition for $\phi(x)$ to be real is[6]

$$\tilde{\phi}^{\dagger}(p)=\kappa^{-3}S^{3}(p_{+})\tilde{\phi}(S(p)) \tag{64}$$

or, equivalently,

$$\tilde{\phi}^{\dagger}(S(p))=\kappa^{-3}p_{+}^{3}\tilde{\phi}(p), \tag{65}$$

where we considered that $S(p_{+})=\kappa^{2}p_{+}^{-1}$. We will discuss real fields in the forthcoming paper, and here we will concentrate on the complex fields only.

According to the properties of the momentum space manifold described in Sec. II [see especially Eq. (7)], the left-invariant Haar measure on $AN(3)$ can be rewritten as the ordinary Lebesgue measure on a restricted five-dimensional momentum space with (the factor 2κ here is included is for dimensional reasons)

$$d\mu(p)=2\kappa d^{5}p\delta(p_{0}^{2}-\mathbf{p}^{2}-p_{4}^{2}+\kappa^{2})\theta(p_{+})\theta(p_{4}). \tag{66}$$

Let us now consider a field on the mass shell defined by m, that we can write as ($A=0,1,\ldots,4$)

$$\phi(x)=\int d^{5}p2\kappa\delta(p_{A}p^{A}+\kappa^{2})\theta(p_{+})\theta(p_{4})\times\delta(p_{\mu}p^{\mu}-m^{2})\tilde{\phi}(p)e^{-ipx}. \tag{67}$$

One way of splitting the $\delta(p_{\mu}p^{\mu}-m^{2})$ into "positive and negative energy" solutions is to rewrite it as

$$\delta(p_{\mu}p^{\mu}-m^{2})=\delta(p_{\mu}p^{\mu}-m^{2})\theta(p_{0}-m)+\delta(p_{\mu}p^{\mu}-m^{2})\theta(-p_{0}-m). \tag{68}$$

Using this, we can rewrite the field as

$$\begin{aligned}\phi(x)&=\phi_{+}(x)+\phi_{-}(x)\\&=\int d^{5}p2\kappa\delta(p_{A}p^{A}+\kappa^{2})\theta(p_{+})\theta(p_{4})\delta(p_{\mu}p^{\mu}-m^{2})\\&\quad\times\theta(p_{0}-m)\tilde{\phi}(p)e^{-ipx}\\&\quad+\int d^{5}p2\kappa\delta(p_{A}p^{A}+\kappa^{2})\theta(p_{+})\theta(p_{4})\delta(p_{\mu}p^{\mu}-m^{2})\\&\quad\times\theta(-p_{0}-m)\tilde{\phi}(p)e^{-ipx},\end{aligned} \tag{69}$$

where $\phi_{+}(x)$ and $\phi_{-}(x)$ denote the positive and negative energy components, respectively, of the on-shell field. Consider the negative energy part $\phi_{-}(x)$. From the properties of the antipode map

$$S(p_{\mu})S(p^{\mu})=p_{\mu}p^{\mu},\qquad S(p_{4})=p_{4}, \tag{70}$$

that imply also $S(p_{A})S(p^{A})=p_{A}p^{A}$, if we change the integration variables as $p\to S(p)$ and use that $S(S(p))=p$ and Eq. (63), we can rewrite $\phi_{-}(x)$ as

$$\begin{aligned}\phi_{-}(x)&=\int d^{5}S(p)2\kappa\delta(S(p_{A})S(p^{A})+\kappa^{2})\\&\quad\times\theta(S(p_{+}))\theta(S(p_{4}))\delta(S(p_{\mu})S(p^{\mu})-m^{2})\\&\quad\times\theta(-S(p_{0})-m)\tilde{\phi}(S(p))e^{-iS(p)x}\\&=\int d^{5}p2\kappa\delta(p_{A}p^{A}+\kappa^{2})\theta(p_{+})\theta(p_{4})\\&\quad\times\delta(p_{\mu}p^{\mu}-m^{2})\theta(-S(p_{0})-m)\\&\quad\times S(p_{+}^{3})\tilde{\phi}(S(p))e^{-iS(p)x},\end{aligned} \tag{71}$$

where we take into account the property

[5]Notice in passing that the rhs of Eq. (63) coincides with the right invariant on AN_{3}, as one can check from the multiplication of two group elements. If we denote the left-invariant measure we are using as $d\mu_{L}(p)$, one thus has the property that, under antipode, $d\mu_{L}(S(p))=d\mu_{R}(p)$. This property is indeed a manifestation of the fact that the antipode map on the manifold corresponds to the inversion on the group elements.

[6]The same result was obtained in Ref. [42] working with the k parametrization.

$$\theta(S(p_+)) = \theta(p_+^{-1}) = \theta(p_+). \tag{72}$$

Now, notice that (accordingly to the discussion of Sec. II)

$$\text{if } p_4 > 0 \,\&\, p_+ > 0 \,\&\, p_\mu p^\mu = m^2, \quad \Rightarrow S(p)_0 < -m \Leftrightarrow p_0 > m. \tag{73}$$

The proof is straightforward, since, on the mass shell,

$$S(p)_0 = \frac{-p_0^2 + \mathbf{p}^2 - p_0 p_4}{p_+} = \frac{-m^2 - p_0 p_4}{p_+} \tag{74}$$

and, thus,

$$S(p)_0 < -m \Rightarrow -m^2 - p_0 p_4 < -m p_+ = -m(p_0 + p_4) \Rightarrow p_0 > m. \tag{75}$$

The proof that $p_0 > m$ implies $S(p)_0 < -m$ is also straightforward. This shows that, for $p_+ > 0$ and $p_4 > 0$, i.e., on the $AN(3)$ submanifold we are interested in [i.e., on that section of the de Sitter hyperboloid selected by the measure $d\mu(p)$], the antipode acts indeed as a bijective map that splits the positive and negative energy parts of the manifold belonging to the same mass shell, as argued in Sec. II and in agreement with the observations reported in Ref. [31]. Since the map is bijective (one to one), we can then interchange the $\theta(-S(p)_0 - m)$ with the $\theta(p_0 - m)$ in the integral and rewrite finally $\phi_-(x)$ as

$$\phi_-(x) = \kappa^{-3} \int d^5 p 2\kappa \delta(p_A p^A + \kappa^2)\theta(p_+)\theta(p_4) \times \delta(p_\mu p^\mu - m^2)\theta(p_0 - m) S(p_+^3)\tilde{\phi}(S(p)) e^{-iS(p)x}. \tag{76}$$

If the field is real, condition (65) holds, and we have obtained the following result: On the $AN(3)$ measure, the on-shellness condition naturally splits the field into positive and negative energy components that are conjugate with each other, with the antipode playing the role of conjugation for the plane wave, i.e.,

$$\begin{aligned}\phi(x) &= \int d\mu(p)\delta(p_\mu p^\mu - m^2)\theta(p_0 - m)[\tilde{\phi}(p)e^{-ipx} + \tilde{\phi}^\dagger(p)e^{-iS(p)x}] \\ &= \int \frac{d^3 p}{2\omega_{\mathbf{p}} p_4/\kappa}[\tilde{\phi}(\omega_{\mathbf{p}}, \mathbf{p})e^{-i(\omega_{\mathbf{p}} t - \mathbf{p}\cdot\mathbf{x})} + \tilde{\phi}^\dagger(\omega_{\mathbf{p}}, \mathbf{p})e^{-i(S(\omega_{\mathbf{p}})t - S(\mathbf{p})\cdot\mathbf{x})}],\end{aligned} \tag{77}$$

where in the last row p_4 is "on shell": $p_4 = \sqrt{m^2 + \kappa^2}$.

For a complex field, it will be convenient to define the antiparticle states, i.e., the ones associated to the negative energy part of the field, as the ones associated to the dual (starred) copy of momentum space. We first substitute, for $\phi_-(x)$, $p \to -p = p^*$, so that [since $S(p^*) = S(-p) = -S(p)$] it becomes

$$\begin{aligned}\phi_-(x) &= \kappa^{-3} \int d^5 p^* 2\kappa \delta(p_A^* p_*^A + \kappa^2)\theta(-p_+^*)\theta(-p_4^*) \times \delta(p_\mu^* p_*^\mu - m^2)\theta(-p_0^* - m)[-S(p_+^{*3})\tilde{\phi}(-S(p^*))]e^{iS(p^*)x} \\ &= \kappa^{-3} \int \frac{d^3 p}{2|\omega_{\mathbf{p}}^*| p_4^*/\kappa} S(p_+^{*3})\tilde{\phi}(-S(\omega_{\mathbf{p}}^*), -S(\mathbf{p}^*))e^{i(S(\omega_{\mathbf{p}}^*)t - S(\mathbf{p}^*)\cdot\mathbf{x})},\end{aligned} \tag{78}$$

where $p_4^* = -\sqrt{m^2 + \kappa^2}$. Thus, using Eq. (69), we have the expansion

$$\phi(x) = \int \frac{d^3 p}{2\omega_{\mathbf{p}} p_4/\kappa}\tilde{\phi}(\omega_{\mathbf{p}}, \mathbf{p})e^{-i(\omega_{\mathbf{p}} t - \mathbf{p}\cdot\mathbf{x})} + \kappa^{-3} \int \frac{d^3 p}{2|\omega_{\mathbf{p}}^*| p_4^*/\kappa} S(p_+^{*3})\tilde{\phi}(-S(\omega_{\mathbf{p}}^*), -S(\mathbf{p}^*))e^{i(S(\omega_{\mathbf{p}}^*)t - S(\mathbf{p}^*)\cdot\mathbf{x})}. \tag{79}$$

Since the mass-shell condition

$$p_0^2 - \mathbf{p}^2 = m^2 \quad \text{or} \quad S(p_0)^2 - S(\mathbf{p})^2 = m^2 \tag{80}$$

has the standard classical form, we would like to define the Fourier components of the complex field as close as possible as the classical expression [43] in terms of creation and annihilation operators

$$\begin{aligned}\phi_+(x) &\sim \int \frac{d^3 p}{\sqrt{2\omega_{\mathbf{p}}}} a_{\mathbf{p}} e^{-i(\omega_p t - \mathbf{p}\mathbf{x})}, \\ \phi_-(x) &\sim \int \frac{d^3 p}{\sqrt{2\omega_{\mathbf{p}}}} b_{\mathbf{p}}^\dagger e^{i(\omega_p t - \mathbf{p}\mathbf{x})}.\end{aligned} \tag{81}$$

We postulate

$$a_{\mathbf{p}} = \frac{\xi^{-1}(p)}{\sqrt{2\omega_{\mathbf{p}} p_4/\kappa}} \tilde{\phi}(\omega_{\mathbf{p}}, \mathbf{p}),$$
$$b_{\mathbf{p}^*} = \kappa^{-2} \frac{\xi^{-1}(p^*)}{\sqrt{2|\omega_{\mathbf{p}}| p_4^*}} S(p_+^{*3}) \tilde{\phi}^\dagger(-S(\omega_{\mathbf{p}}^*), -S(\mathbf{p}^*)), \quad (82)$$

where we include an additional factor (p_+ has to be considered on shell: $p_+ = \sqrt{\mathbf{p}^2 + m^2} + \sqrt{\kappa^2 + m^2}$)

$$\xi(p) = \left(1 + \frac{|p_+|^3}{\kappa^3}\right)^{-1/2}, \quad (83)$$

that makes the form of the momentum space action, which we will make use of later, particularly simple.

Finally, we have, for the on-shell complex field and its adjoint, the expansions

$$\phi(x) = \int \frac{d^3p}{\sqrt{2\omega_p}} \left[1 + \frac{|p_+|^3}{\kappa^3}\right]^{-1/2} a_{\mathbf{p}} e^{-i(\omega_p t - \mathbf{p}\mathbf{x})} + \int \frac{d^3p^*}{\sqrt{2|\omega_p^*|}} \left[1 + \frac{|p_+^*|^3}{\kappa^3}\right]^{-1/2} b_{\mathbf{p}^*}^\dagger e^{i(S(\omega_p^*)t - S(\mathbf{p}^*)\mathbf{x})} \equiv \phi_{(+)}(x) + \phi_{(-)}(x), \quad (84)$$

$$\phi^\dagger(x) = \int \frac{d^3p}{\sqrt{2\omega_p}} \left[1 + \frac{|p_+|^3}{\kappa^3}\right]^{-1/2} a_{\mathbf{p}}^\dagger e^{-i(S(\omega_p)t - S(\mathbf{p})\mathbf{x})} + \int \frac{d^3p^*}{\sqrt{2|\omega_p^*|}} \left[1 + \frac{|p_+^*|^3}{\kappa^3}\right]^{-1/2} b_{\mathbf{p}^*} e^{i(\omega_p^* t - \mathbf{p}^*\mathbf{x})} \equiv \phi_{(+)}^\dagger(x) + \phi_{(-)}^\dagger(x). \quad (85)$$

Since $\omega_p > 0$ and $S(\omega_p^*) > 0$, the field (84) is a combination of positive energy particle states and negative energy antiparticle ones, while in Eq. (85) we have the opposite arrangement, as it should be. This particular definition of the field and its adjoint, contrary to earlier approaches where to define the field and its adjoint only one portion of de Sitter space was used, allows for simple action of discrete symmetries; see Sec. VI below for the details.

From Eq. (49), one sees that the equations of motion (EOM) for the field ϕ are indeed the expected ones. Furthermore, one can get the EOM also for the $a_{\mathbf{p}}$, $a_{\mathbf{p}}^\dagger$, $b_{\mathbf{p}^*}$, and $b_{\mathbf{p}^*}^\dagger$ by applying the EOM to the fields in Eqs. (84) and (85). We get

$$(\partial_\mu \partial^\mu - m^2)\phi = \int \frac{d^3p}{\sqrt{2\omega_p}} \left[1 + \frac{|p_+|^3}{\kappa^3}\right]^{-1/2} (p_\mu p^\mu - m^2) a_{\mathbf{p}} e^{-i(\omega_p t - \mathbf{p}\mathbf{x})} \quad (86)$$

$$+ \int \frac{d^3p^*}{\sqrt{2|\omega_p^*|}} \left[1 + \frac{|p_+|^3}{\kappa^3}\right]^{-1/2} (S(p)_\mu S(p)^\mu - m^2) b_{\mathbf{p}^*}^\dagger e^{i(S(\omega_p^*)t - S(\mathbf{p}^*)\mathbf{x})}, \quad (87)$$

$$(\partial_\mu^\dagger (\partial^\mu)^\dagger - m^2)\phi^\dagger = \int \frac{d^3p}{\sqrt{2\omega_p}} \left[1 + \frac{|p_+|^3}{\kappa^3}\right]^{-1/2} (S(S(p))_\mu S(S(p))^\mu - m^2) a_{\mathbf{p}}^\dagger e^{-i(S(\omega_p)t - S(\mathbf{p})\mathbf{x})} \quad (88)$$

$$+ \int \frac{d^3p^*}{\sqrt{2|\omega_p^*|}} \left[1 + \frac{|p_+|^3}{\kappa^3}\right]^{-1/2} (S(p^*)_\mu S(p^*)^\mu - m^2) b_{\mathbf{p}^*} e^{i(\omega_p^* t - \mathbf{p}^*\mathbf{x})}. \quad (89)$$

Notice that Eq. (86) is equivalent to Eq. (88) because one can show that $S(S(p))_\mu S(S(p))^\mu = p_\mu p^\mu$, and analogously Eq. (89) is equivalent to Eq. (87) because $S(p^*)_\mu = -S(p)_\mu$.

We find that with the definition of the fields (84) and (85) the particle, characterized by creation (annihilation) operator $a_{\mathbf{p}}$ ($a_{\mathbf{p}}^\dagger$), has the mass-shell condition $p^2 - m^2 = 0$, while the antiparticle characterized by creation (annihilation) operator $b_{\mathbf{p}}$ ($b_{\mathbf{p}}^\dagger$) follows the mass-shell condition $S(p)^2 - m^2 = 0$. These mass shells are identical, so that both the particle and the antiparticle have the same rest mass, and the mass-shell manifold is in both cases the same, but when we apply a Lorentz boost to a particle and an antiparticle at rest with the same boost parameter, they would end up carrying different momenta and energies. This leads to subtle deformation of *CPT* symmetry, discussed in Refs. [13,15].

V. SYMMETRIES OF THE ACTION

Let us now check that the above-defined fields transform properly under Poincaré and discrete symmetries, rendering the action (45) invariant.

A. Poincaré symmetry of the action

In order to check the Poincaré invariance of the complex scalar field action[7] (45), it is convenient to rewrite it in the momentum space where such invariance can be easily checked. As for the space-time action, the procedure is much more involved, and it is reported in Appendix C.

Let us note that, in order to turn the space-time action (45) to a momentum space one, we cannot use the on-shell field decomposition (84) and (85), because the resulting momentum space action would contain the mass-shell conditions as coefficients, which will make the action identically equal to zero. Therefore, we use as a starting point the off-shell field decomposition

$$\phi^{\rm off}(x) = \int_{\mathfrak{F}^+} \frac{d^4p}{p_4/\kappa}\xi(p)a_p e^{-i(p_0t-\mathbf{px})} + \int_{\mathfrak{F}^-} \frac{d^4p^*}{|p_4^*|/\kappa}\xi(p^*)b^\dagger_{p^*}e^{i(S(p_0^*)t-S(\mathbf{p}^*)\mathbf{x})}, \quad (90)$$

where we include the additional factor (83) to make the momentum space action as simple as possible. In Eq. (90), we used the left-invariant measure (38) on the group manifold AN(3), and we restricted the range of integration in the first term to the positive energy $p_0 > 0$ subspace $\mathfrak{F}^+$ and to the energy $p_0^* < 0$ subspace $\mathfrak{F}^-$ in the second term. This arrangement is analogous to the introduction of the θ's in Eq. (69) but without the mass-shell restriction. The decomposition (90) can be further simplified observing that since p^* is a dummy variable we can instead use the variables $p = -p^*$ in the second integral, so that we have

$$\phi^{\rm off}(x) = \int_{\mathfrak{F}^+} \frac{d^4p}{p_4/\kappa}\left(1+\left(\frac{|p_+|}{\kappa}\right)^3\right)^{-1/2} \times (a_p e^{-i(p_0t-\mathbf{px})} + b^\dagger_{-p}e^{-i(S(p_0)t-S(\mathbf{p})\mathbf{x})}). \quad (91)$$

The adjoint field has the form

$$(\phi^{\rm off})^\dagger(x) = \int_{\mathfrak{F}^+} \frac{d^4p}{p_4/\kappa}\left(1+\left(\frac{|p_+|}{\kappa}\right)^3\right)^{-1/2} \times (a^\dagger_p e^{-i(S(p_0)t-S(\mathbf{p})\mathbf{x})} + b_{-p}e^{-i(p_0t-\mathbf{px})}). \quad (92)$$

Plugging these expressions to the action integral (45) after tedious computations, adjusting the free functions, we obtain the momentum space action in the form

$$S = \frac{1}{2}\int_{\mathfrak{F}^+} \frac{d^4p}{p_4/\kappa}(p_\mu p^\mu - m^2)a^\dagger_{\mathbf{p}}a_{\mathbf{p}} + (S(p)_\mu S(p)^\mu - m^2)b_{\mathbf{p}}b^\dagger_{\mathbf{p}}. \quad (93)$$

It is clear from the action in the form (93) above that the mass shell of the "particle" is $p^2 = m^2$, while for the "antiparticle" it has the form $S(p)^2 = m^2$, as discussed above.

Moreover, it is straightforward to check its Poincaré invariance. The translations act on a_p and b_p as phases; for the translation parameter $\boldsymbol{\varepsilon}$, we have

$$a_p \mapsto e^{i\varepsilon p}a_p, \qquad b_p \mapsto e^{i\varepsilon p}b_p. \quad (94)$$

Next, the action is clearly rotational invariant, if we assume that a_p and b_p are scalar functions of the spacial momenta $\mathbf{p}$. It therefore remains to check the Lorentz invariance of the action. But since the action (93) has the form of the standard undeformed momentum space action, the transformation properties of the creation and annihilation "operators" are just the standard ones $a_p \mapsto U(\Lambda)a_pU^{-1}(\Lambda) = a_{\Lambda p}$, where Λp is the Lorentz transformed four-vector p. Indeed,

$$\begin{aligned} U(\Lambda)SU^{-1}(\Lambda) &= \frac{1}{2}\int_{\mathfrak{F}^+} d^4p(p_\mu p^\mu - m^2)U(\Lambda)a^\dagger_p a_p U^{-1}(\Lambda) \\ &\quad + \frac{1}{2}\int_{\mathfrak{F}^+} d^4p(S(p)_\mu S(p)^\mu - m^2)U(\Lambda)b^\dagger_p b_p U^{-1}(\Lambda) \\ &= \frac{1}{2}\int_{\mathfrak{F}^+} d^4(\Lambda p)((\Lambda p)_\mu(\Lambda p)^\mu - m^2)a^\dagger_{\Lambda p}a_{\Lambda p} \\ &\quad + \frac{1}{2}\int_{\mathfrak{F}^+} d^4(\Lambda p)(S(\Lambda p)_\mu S(\Lambda p)^\mu - m^2)b^\dagger_{\Lambda p}b_{\Lambda p} \\ &= S. \end{aligned}$$

This completes the proof of Poincaré invariance of the action (93).

VI. DISCRETE SYMMETRIES

There are three discrete symmetries: parity $\mathcal{P}$, time reversal $\mathcal{T}$, and charge conjugation $\mathcal{C}$. In each case, we will first shortly recall their action on the undeformed field with decomposition

$$\phi(t,\mathbf{x}) = \int \frac{d^3p}{\sqrt{2\omega_p}} a_{\mathbf{p}}e^{-i(\omega_pt-i\mathbf{px})} + b^\dagger_{\mathbf{p}}e^{i(\omega_pt-\mathbf{px})} \quad (95)$$

and then generalize it to the case of the deformed fields (84) and (85). For parity and time reversal, we have space-time concepts to guide us, and, therefore, we consider these two first.

[7]Reference [34] provides a general abstract proof of Poincaré invariance of the κ-deformed complex scalar field action in two space-time dimensions; here, we show explicitly that the same holds in the particular of the theory considered here, in four dimensions.

A. Parity

The parity operator $\mathcal{P}$ acts on space coordinates as an inversion $x = (t, \mathbf{x}) \to x' = (t, -\mathbf{x})$. For the complex scalar quantum field, we define the parity operator as

$$\mathcal{P}\phi(t,\mathbf{x})\mathcal{P}^{-1} = \int \frac{d^3p}{\sqrt{2\omega_p}} \mathcal{P}a_{\mathbf{p}}\mathcal{P}^{-1}e^{-i(\omega_p t - \mathbf{px})} + \mathcal{P}b^\dagger_{\mathbf{p}}\mathcal{P}^{-1}e^{i(\omega_p t - \mathbf{px})} \equiv \phi(t, -\mathbf{x}), \tag{96}$$

and using (95) we see that for the creation or annihilation operators[8]

$$\mathcal{P}a_{\mathbf{p}}\mathcal{P}^{-1} = a_{-\mathbf{p}}, \qquad \mathcal{P}b_{\mathbf{p}}\mathcal{P}^{-1} = b_{-\mathbf{p}}. \tag{97}$$

Turning to the deformed case, we notice first that the space-time transformation $\hat{\mathbf{x}} \mapsto -\hat{\mathbf{x}}$ leaves the defining commutator (1) invariant and, therefore, is compatible with the form of κ-Minkowski noncommutativity. Furthermore, the positive and negative energy fields $\phi_{(\pm)}(x)$ can be considered separately. For the positive energy part, we can use exactly the same considerations as in the case of the undeformed field above. Since

$$S(p_0, -\mathbf{p})_0 = S(p_0, \mathbf{p})_0, \qquad S(p_0, -\mathbf{p})_i = -S(p_0, \mathbf{p})_i,$$

this is also true for the negative energy fields, and, thus, we can readily define

$$\mathcal{P}a_{\mathbf{p}}\mathcal{P}^{-1} = a_{-\mathbf{p}}, \qquad \mathcal{P}b_{\mathbf{p}^*}\mathcal{P}^{-1} = b_{-\mathbf{p}^*} \tag{98}$$

and

$$\mathcal{P}a^\dagger_{\mathbf{p}}\mathcal{P}^{-1} = a^\dagger_{-\mathbf{p}}, \qquad \mathcal{P}b^\dagger_{\mathbf{p}^*}\mathcal{P}^{-1} = b^\dagger_{-\mathbf{p}^*}. \tag{99}$$

B. Time reversal

Next, we consider the time reversal $\mathcal{T}$, which changes the time direction $x = (t, \mathbf{x}) \to x' = (-t, \mathbf{x})$ and

$$\mathcal{T}\phi(t,\mathbf{x})\mathcal{T}^{-1} = \phi(-t, \mathbf{x}). \tag{100}$$

It should be remembered that the operator $\mathcal{T}$ is anti-Hermitian $\mathcal{T}i\mathcal{T}^{-1} = -i$, and we have

$$\begin{aligned}\mathcal{T}\phi(t,\mathbf{x})\mathcal{T}^{-1} &= \int \frac{d^3p}{\sqrt{2\omega_p}} \mathcal{T}a_{\mathbf{p}}e^{-i(\omega_p t - i\mathbf{px})}\mathcal{T}^{-1} + \mathcal{T}b^\dagger_{\mathbf{p}}e^{i(\omega_p t - \mathbf{px})}\mathcal{T}^{-1} \\ &= \int \frac{d^3p}{\sqrt{2\omega_p}} \mathcal{T}a_{\mathbf{p}}\mathcal{T}^{-1}e^{i(\omega_p t - i\mathbf{px})} + \mathcal{T}b^\dagger_{\mathbf{p}}\mathcal{T}^{-1}e^{-i(\omega_p t - \mathbf{px})} = \phi(-t, \mathbf{x}).\end{aligned} \tag{101}$$

We find that

$$\mathcal{T}a_{\mathbf{p}}\mathcal{T}^{-1} = a_{-\mathbf{p}}, \qquad \mathcal{T}b_{\mathbf{p}}\mathcal{T}^{-1} = b_{-\mathbf{p}}. \tag{102}$$

Let us now discuss the deformed case. We start by noticing that as a consequence of anti-Hermiticity of $\mathcal{T}$ the defining algebra (1) is again invariant, so that we see that κ-Minkowski space is both parity and time reversal invariant. Turning to fields, we again see that the classical reasoning can be verbatim repeated in the case of time reversal as well, and we end up with

$$\mathcal{T}a_{\mathbf{p}}\mathcal{T}^{-1} = a_{-\mathbf{p}}, \qquad \mathcal{T}b_{\mathbf{p}^*}\mathcal{T}^{-1} = b_{-\mathbf{p}^*} \tag{103}$$

and

$$\mathcal{T}a^\dagger_{\mathbf{p}}\mathcal{T}^{-1} = a^\dagger_{-\mathbf{p}}, \qquad \mathcal{T}b^\dagger_{\mathbf{p}^*}\mathcal{T}^{-1} = b^\dagger_{-\mathbf{p}^*}. \tag{104}$$

C. Charge conjugation

The symmetry that exchanges particles with antiparticles does not have any space-time counterparts, and since it changes the charge it is called charge conjugation. The charge conjugation operator $\mathcal{C}$ acting on the field produces its conjugation

$$\mathcal{C}\phi(t,\mathbf{x})\mathcal{C}^{-1} = \phi^\dagger(t, \mathbf{x}) \tag{105}$$

and, therefore,

$$\mathcal{C}\phi(t,\mathbf{x})\mathcal{C}^{-1} = \int \frac{d^3p}{\sqrt{2\omega_p}} \mathcal{C}a_{\mathbf{p}}\mathcal{C}^{-1}e^{-i(\omega_p t - i\mathbf{px})} + \mathcal{C}b^\dagger_{\mathbf{p}}\mathcal{C}^{-1}e^{i(\omega_p t - \mathbf{px})} = \phi^\dagger(t, \mathbf{x}), \tag{106}$$

and we have

$$\mathcal{C}a_{\mathbf{p}}\mathcal{C}^{-1} = b_{\mathbf{p}}. \tag{107}$$

Let us now consider the deformed field. Take the $\phi_{(+)}$ component first:

$$\mathcal{C}\phi_{(+)}(t,\mathbf{x})\mathcal{C}^{-1} = \int \frac{d^3p}{\sqrt{2\omega_p}} \left[1 + \frac{|p_+|^3}{\kappa^3}\right]^{-1/2} \mathcal{C}a_{\mathbf{p}}\mathcal{C}^{-1}e^{-i(\omega_p t - i\mathbf{px})}. \tag{108}$$

[8] Here and below, we ignore a possible phase factor that may be present in the definition.

On the other hand, we have

$$\phi^{\dagger}_{(-)}(x) = \int \frac{d^3p^*}{\sqrt{2|\omega^*_p|}}\left[1+\frac{|p_+|^3}{\kappa^3}\right]^{-1/2} b_{\mathbf{p}^*}e^{i(\omega^*_p t - \mathbf{p}^*\mathbf{x})}, \tag{109}$$

so that we can conclude that

$$\mathcal{C}a_{\mathbf{p}}\mathcal{C}^{-1} = b_{\mathbf{p}^*}. \tag{110}$$

Analogously, for the $\phi_{(-)}$ component, we have

$$\mathcal{C}\phi_{(-)}(x)\mathcal{C}^{-1} = \int \frac{d^3p^*}{\sqrt{2|\omega^*_p|}}\left[1+\frac{|p_+|^3}{\kappa^3}\right]^{-1/2} \mathcal{C}b^{\dagger}_{\mathbf{p}^*}\mathcal{C}^{-1}e^{i(S(\omega^*_p)t - S(\mathbf{p}^*)\mathbf{x})} \tag{111}$$

and

$$\phi^{\dagger}_{(+)}(x) = \int \frac{d^3p}{\sqrt{2\omega_p}}\left[1+\frac{|p_+|^3}{\kappa^3}\right]^{-1/2} a^{\dagger}_{\mathbf{p}}e^{-i(S(\omega_p)t - S(\mathbf{p})\mathbf{x})} \tag{112}$$

so that

$$\mathcal{C}b^{\dagger}_{\mathbf{p}^*}\mathcal{C}^{-1} = a^{\dagger}_{\mathbf{p}}. \tag{113}$$

It should be stressed that this simple transformation rule of the field ϕ with respect to charge conjugation is a result of the use of the second (starred) copy of momentum space and of the particular arrangement of the components $\phi_{(\pm)}(x)$ and $\phi^{\dagger}_{(\pm)}(x)$. In particular, the field constructed in Ref. [14] and many other papers on this topic does not transform nicely under charge conjugation. It should be added also that the deformed action of discrete symmetries $\mathcal{P}$, $\mathcal{T}$, and $\mathcal{C}$ leads to the form of the $\mathcal{CPT}$ operator Θ anticipated in Ref. [13], although the action of parity and time reversal differ from that proposed in Ref. [44].

VII. CONSERVED CHARGES AND SYMPLECTIC STRUCTURE

In this section, we derive the conserved charges and symplectic structure associated with our free complex scalar field theory defined by the action

$$S = \frac{1}{2}(S_1+S_2) = \frac{1}{2}\int_{\mathbb{R}^4} d^4x (\partial^\mu\phi)^{\dagger}\star(\partial_\mu\phi) - m^2\phi^{\dagger}\star\phi + \frac{1}{2}\int_{\mathbb{R}^4} d^4x (\partial_\mu\phi)\star(\partial^\mu\phi)^{\dagger} - m^2\phi\star\phi^{\dagger}. \tag{114}$$

Both are given in terms of the appropriate boundary integrals and reflect, respectively, the symmetries of the theory (charges) and its kinematics (symplectic structure). Our starting point here will be the variations of the actions computed above, Eqs. (48)–(57). Assuming field equations in the bulk, these variations are just the boundary terms, which become conserved charges in the case of field variations corresponding to symmetries of the action and Liouville form, for generic variations.

A. Conserved charges

On shell, the variation of the action reduces to the boundary term, and we define the conserved charges associated with the field transformation that leaves the action invariant $\delta_S\phi$, $\delta_S\phi^{\dagger}$ as usual as an integral over the constant time surface

$$\mathcal{P}_S = \frac{1}{2}\int d^3x \Pi^0_1\star\delta_S\phi + \delta_S\phi^{\dagger}\star(\Pi^0_1)^{\dagger} + \delta_S\phi\star\Pi^0_2 + (\Pi^0_2)^{\dagger}\star\delta_S\phi^{\dagger}. \tag{115}$$

In the case of translational symmetry, for which

$$\delta_S\phi = d\phi = \epsilon^A\partial_A\phi, \tag{116}$$

we find

$$\mathcal{P}_A = \frac{1}{2}\int d^3x T_1{}^0{}_A + T_2{}^0{}_A, \tag{117}$$

where the relevant components of the energy-momentum tensor are

$$T_1{}^0{}_A = -\partial_A\Pi^0_1\star\phi + \partial_A\phi^{\dagger}\star\Pi^{\dagger 0}_1 \tag{118}$$

and

$$T_2{}^0{}_A = -\phi\star\partial_A\Pi^0_2 + \Pi^{\dagger 0}_2\star\partial_A\phi^{\dagger}. \tag{119}$$

Now we use the field decomposition (84) and (85) to find the expression for conserved translational charges $\mathcal{P}_A$ (117) in momentum space. After tedious computation, one finds that the time-dependent terms cancel as they should and the conserved charges have the form

$$\mathcal{P}_0 = -\frac{1}{2}\int d^3p a^{\dagger}_{\mathbf{p}}a_{\mathbf{p}}S(\omega_p)\left[1-\frac{\xi(p)^2\mathbf{p}^2}{\omega_p p_+}\right]\frac{p_4}{\kappa} - b_{\mathbf{p}^*}b^{\dagger}_{\mathbf{p}^*}\omega_p\left[1-\frac{\xi(p)^2\mathbf{p}^2}{\omega_p p_+}\right]\frac{p_4}{\kappa}, \tag{120}$$

$$\mathcal{P}_i = \frac{1}{2}\int d^3p a^{\dagger}_{\mathbf{p}}a_{\mathbf{p}}S(\mathbf{p})_i\left[1-\frac{\xi(p)^2\mathbf{p}^2}{\omega_p p_+}\right]\frac{p_4}{\kappa} - b_{\mathbf{p}^*}b^{\dagger}_{\mathbf{p}^*}\mathbf{p}_i\left[1-\frac{\xi(p)^2\mathbf{p}^2}{\omega_p p_+}\right]\frac{p_4}{\kappa}, \tag{121}$$

$$\mathcal{P}_4 = -\frac{1}{2}\int d^3p(p_4-\kappa)\left\{a^\dagger_{\mathbf{p}}a_{\mathbf{p}}\left[1-\frac{\xi(p)^2\mathbf{p}^2}{\omega_p p_+}\right]\frac{p_4}{\kappa} - b_{\mathbf{p}^*}b^\dagger_{\mathbf{p}^*}\left[1-\frac{\xi(p)^2\mathbf{p}^2}{\omega_p p_+}\right]\frac{p_4}{\kappa}\right\}. \tag{122}$$

B. Symplectic structure

To compute the symplectic structure of our theory we use the covariant phase space approach [45–47], which makes it possible to straightforwardly derive it from the action preserving all the relevant symmetries. To compute the symplectic structure, we must return to Eqs. (49) and (54). Defining the Liouville form θ as a boundary term in the variation of the action on shell, for generic variation of the field $\delta\phi$, $\delta\phi^\dagger$ we find

$$\theta = \theta_1+\theta_2 = -\frac{1}{2}\int_{\mathbb{R}^3} d^3x(\Pi^0_1\star\delta\phi + \delta\phi^\dagger\star(\Pi^0_1)^\dagger + \delta\phi\star\Pi^0_2 + (\Pi^0_2)^\dagger\star\delta\phi^\dagger). \tag{123}$$

To find the symplectic form, which will lead to the Poisson bracket of field coefficients a and b and, in turn, to the creation and annihilation operator commutators, we have to compute $\delta\theta$ and express the result using the momentum space decomposition (84) and (85). We find

$$\delta\theta_1+\delta\theta_2 = -\frac{i}{2}\int d^3p\, a_{\mathbf{p}}\wedge a^\dagger_{\mathbf{p}}\left[1-\frac{\xi(p)^2\mathbf{p}^2}{\omega_p p_+}\right]\frac{p_4}{\kappa} - \xi(p)^2 b^\dagger_{\mathbf{p}^*}\wedge b_{\mathbf{p}^*}\left[1-\frac{\xi(p)^2\mathbf{p}^2}{\omega_p p_+}\right]\frac{p_4}{\kappa}, \tag{124}$$

which implies the following Poisson brackets:

$$\{a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}\} = i\frac{\kappa}{p_4}\frac{2}{1-\frac{\xi(p)^2\mathbf{p}^2}{\omega_p p_+}}\delta(\mathbf{p}-\mathbf{q}), \tag{125}$$

$$\{b_{\mathbf{p}^*}, b^\dagger_{\mathbf{q}^*}\} = i\frac{\kappa}{p_4}\frac{2}{1-\frac{\xi(p)^2\mathbf{p}^2}{\omega_p p_+}}\delta(\mathbf{p}-\mathbf{q}). \tag{126}$$

VIII. TOWARD QUANTUM THEORY

In this section, we will construct the one-particle states in quantum field theory. At this stage, we cannot go any further; in particular, we cannot construct many-particle states and investigate their properties, because this would require knowing details of the coproduct properties of creation and annihilation operators, i.e., how they act on tensor product of states.

In quantum theory, the Poisson brackets (125) and (126) become commutators (from now on, we stop distinguishing **p** from $\mathbf{p}^*$):

$$[a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] = \frac{\kappa}{p_4}\frac{2}{1-\frac{\xi(p)^2\mathbf{p}^2}{\omega_p p_+}}\delta(\mathbf{p}-\mathbf{q}), \tag{127}$$

$$[b_{\mathbf{p}}, b^\dagger_{\mathbf{q}}] = \frac{\kappa}{p_4}\frac{2}{1-\frac{\xi(p)^2\mathbf{p}^2}{\omega_p p_+}}\delta(\mathbf{p}-\mathbf{q}). \tag{128}$$

We define the vacuum $|0\rangle$ that satisfies the condition

$$a_{\mathbf{p}}|0\rangle = b_{\mathbf{p}}|0\rangle = 0. \tag{129}$$

Then we define the one-particle and one-antiparticle states

$$|\mathbf{p}\rangle_a \equiv a^\dagger_{\mathbf{p}}|0\rangle, \tag{130}$$

$$|\mathbf{p}\rangle_b \equiv b^\dagger_{\mathbf{p}}|0\rangle. \tag{131}$$

Now we are ready to present the most important result of this investigations. Consider the state $|\mathbf{p}\rangle_a$ [Eq. (130)]. Its momentum can be computed by acting with the momentum operator $\mathcal{P}_i$ [Eq. (121)] on it. Using the commutational relation (127), we find

$$\mathcal{P}_i|\mathbf{p}\rangle_a = -S(p)_i|\mathbf{p}\rangle_a. \tag{132}$$

Analogously, for the one-antiparticle state $|\mathbf{p}\rangle_b$ [Eq. (130)], using the commutational (128) we get

$$\mathcal{P}_i|\mathbf{p}\rangle_b = p_i|\mathbf{p}\rangle_b. \tag{133}$$

In exactly the same manner, we can use the Hamiltonian (120) to compute the energy of the one-particle states, obtaining

$$\mathcal{P}_0|\mathbf{p}\rangle_a = -S(\omega_p)|\mathbf{p}\rangle_a \tag{134}$$

and

$$\mathcal{P}_0|\mathbf{p}\rangle_b = \omega_p|\mathbf{p}\rangle_b. \tag{135}$$

Therefore, one-particle and one-antiparticle states belong to the same mass-shell manifold, since

$$\omega_p^2 - \mathbf{p}^2 = m^2 = S(\omega_p)^2 - S(\mathbf{p})^2, \tag{136}$$

but **p** and $S(\mathbf{p})$ are, in general, different points on this manifold, with a single exception being the case $\mathbf{p} = S(\mathbf{p}) = 0$, $\omega_p = -S(\omega_p) = m$.

Finally, the momentum $\mathcal{P}_4$ measures, essentially, the deformed charge of the state

$$\mathcal{P}_4|\mathbf{p}\rangle_a = \left(\sqrt{\kappa^2+m^2}-\kappa\right)|\mathbf{p}\rangle_a \tag{137}$$

and

$$\mathcal{P}_4|\mathbf{p}\rangle_b = -(\sqrt{\kappa^2+m^2}-\kappa)|\mathbf{p}\rangle_b. \qquad (138)$$

Therefore, the one-particle state carries the momentum $-S(p)_i$, while the one-antiparticle state has the momentum p_i. But according to Eq. (113) the latter is the $\mathcal{C}$ (and also $\mathcal{CPT}$) of the former

$$\mathcal{C}|\mathbf{p}\rangle_b = \mathcal{C}b^\dagger_{\mathbf{p}}\mathcal{C}^{-1}\mathcal{C}|0\rangle = \mathcal{C}b^\dagger_{\mathbf{p}}\mathcal{C}^{-1}|0\rangle = a^\dagger_{\mathbf{p}}|0\rangle = |\mathbf{p}\rangle_a. \quad (139)$$

Therefore, as anticipated in Sec. IV, the charge conjugation (and $\mathcal{CPT}$) transforms a particle into an antiparticle with different momentum. This transformation has the remarkable property that the rest mass of the particle and antiparticle is the same. The phenomenological consequences of this have been recently discussed in Refs. [13,15].

IX. SUMMARY AND CONCLUSIONS

We laid down the basic ingredients for the construction of a complex field theory on κ-Minkowski space covariant under the action of deformed relativistic symmetries described by the κ-Poincaré algebra. The guiding principle which we followed in the definition of the field and its action was the requirement of an appropriate transformation of the former under the action of discrete symmetries. The main upshot of our construction is that the four-momenta of particle and antiparticles states related by charge conjugation $\mathcal{C}$ are not identical and given by Eqs. (132)–(138). After deriving the equations of motions from the deformed action, we worked the action of Poincaré symmetries on the field from both a coordinate and momentum space perspective and then moved onto the description of the action of discrete symmetries. The last part of our work was devoted to the analysis of the symplectic structure of the theory, which allowed us to derive the conserved charges associated to the deformed translation symmetries. This also made it possible to write down the Poisson brackets of the expansion coefficients of the field which upon quantization become creation and annihilation operators. With these, we were able to characterize the energy and momentum of one-particle and -antiparticle states and write down the action of discrete symmetries on them which showed that the CPT operator maps particle states into antiparticle states with a different momentum. This important result could have nontrivial phenomenological consequences which might be relevant for experimental searches of Planck-scale effects [13,15].

There are several open issues that we are going to address in the future publications. First, it does seem that the particle state and its associated charge conjugated antiparticle one have different momenta, and it is not trivial to define the real scalar field. We will return to it in the forthcoming publications. The main open issue at the quantum level concerns the construction of a Fock space on which the commutators that we derived for creation and annihilation operators can act, mapping multiparticle states given by appropriately symmetrized tensor products of one-particle states consistent with the nontrivial coproduct and covariant under the action of the κ-Poincaré algebra. This is notoriously a thorny issue which has not yet found a satisfactory answer [18,21,24,48–50] and which we hope we will be able to successfully address within the approach to field theory proposed in this work. The satisfactory solution of this problem is the major prerequisite for the construction of the interacting κ-deformed quantum field theory and κ-deformed standard model, which is our ultimate goal in the research project of which the present paper is the first step.

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APPENDIX A: LORENTZ TRANSFORMATIONS OF ANTIPODE

The antipodes were defined in Eq. (9) and are given by the following expressions:

$$S(p_0) = -p_0 + \frac{\mathbf{p}^2}{p_0+p_4} = \frac{\kappa^2}{p_0+p_4} - p_4,$$
$$S(\mathbf{p}) = -\frac{\kappa\mathbf{p}}{p_0+p_4}, \qquad S(p_4) = p_4 \qquad (A1)$$

The action of Lorentz boost transformation on the antipode is defined as [Eq. (10)]

$$L\triangleright S(p) \equiv S(L\triangleright p), \qquad p_0 > 0. \qquad (A2)$$

Let us investigate properties of this transformation in the case of a infinitesimal Lorentz transformation with parameter ξ^i:

$$\delta_\xi p_i = \xi_i p_0, \qquad \delta_\xi p_0 = \xi^i p_i. \qquad (A3)$$

Remembering that p_4 is Lorentz invariant using Eq. (A2), we find

$$\delta_\xi S(p_0) = -\xi^i p_i + \frac{2\xi^i p_i p_0}{p_0+p_4} - \frac{\mathbf{p}^2}{(p_0+p_4)^2}\xi^i p_i = \zeta^i S(p_i), \qquad (A4)$$

where we introduce a momentum-dependent infinitesimal parameter

$$\zeta^i = \xi^i \frac{\kappa}{p_0+p_4}. \qquad (A5)$$

Thus, the Lorentz transformation of the zero component of the antipode is an ordinary Lorentz transformation, with parameter ζ^i.

For the spatial component, we have a more complicated expression:

$$\delta_\xi S(p_i) = -\frac{\kappa\xi_i p_0}{p_0+p_4} + \frac{\kappa p_i}{(p_0+p_4)^2}\xi^j p_j = \zeta_i S(p_0) + \frac{\kappa}{(p_0+p_4)^2}(p_i\xi^j p_j - \xi_i \mathbf{p}^2). \tag{A6}$$

The first term here is again the standard Lorentz transformation with parameter ζ^i. The second term is an infinitesimal rotation of $S(p_i)$ with the parameter

$$\rho^j = \epsilon^j{}_{kl}\xi^k p^l$$

so that, finally,

$$\delta_\xi S(p_i) = \zeta_i S(p_0) + \epsilon_i{}^{jk}\rho_j S(p_k). \tag{A7}$$

Since under Lorentz boost transformation of momenta the components of the antipode transform under a combination of boost and rotation, it is clear that the components of the antipode satisfy the same mass-shell condition as the components of the original momenta.

APPENDIX B: INTEGRATION BY PARTS

In this Appendix, we derive the $\star$ integration by parts formula, which is necessary do derive field equations from the action (45).

The starting point is provided by the coproduct rules for the κ-Poincaré algebra in the classical basis (p_0, p_i, p_4) [14,51]:

$$\Delta p_i = \frac{1}{\kappa}p_i \otimes (p_0+p_4) + 1\otimes p_i, \tag{B1}$$

$$\Delta p_0 = \frac{1}{\kappa}p_0 \otimes (p_0+p_4) + \sum p_k(p_0+p_4)^{-1} \otimes p_k + \kappa(p_0+p_4)^{-1}\otimes p_0, \tag{B2}$$

$$\Delta p_4 = \frac{1}{\kappa}p_4 \otimes (p_0+p_4) - \sum p_k(p_0+p_4)^{-1} \otimes p_k - \kappa(p_0+p_4)^{-1}\otimes p_0. \tag{B3}$$

Notice that the coproduct relations are an immediate consequence of Eq. (25). The coproducts tell us how the momentum operators act on star products of two functions. Since momenta are space-time derivatives $p_0 = i\partial_0$ and $p_i = i\partial_i$, these equations tell us how derivatives act on the star products of functions on Minkowski space, defining in this way the modified Leibniz rules. In the calculation below, we use the shorthand notation $p_+ \to \Delta_+ = i\partial_0 + p_4 = i\partial_0 + (\kappa + i\partial_4)$, where the nonlocal operator p_4 is expressed in terms of the corresponding derivatives as $p_4 = \sqrt{\kappa^2 - \partial_0^2 + \partial_i^2}$. Equations (B1)–(B3) then imply

$$\partial_0(\phi\star\psi) = \frac{1}{\kappa}(\partial_0\phi)\star(\Delta_+\psi) + \kappa(\Delta_+^{-1}\phi)\star(\partial_0\psi) + i(\Delta_+^{-1}\partial_i\phi)\star(\partial_i\psi), \tag{B4}$$

$$\partial_i(\phi\star\psi) = \frac{1}{\kappa}(\partial_i\phi)\star(\Delta_+\psi) + \phi\star(\partial_i\psi), \tag{B5}$$

$$\Delta_+(\phi\star\psi) = \frac{1}{\kappa}(\Delta_+\phi)\star(\Delta_+\psi). \tag{B6}$$

Furthermore, defining the adjoint derivative

$$(\partial_A\phi)^\dagger \equiv \partial_A^\dagger\phi^\dagger, \qquad A = (\mu, 4, +) \tag{B7}$$

and using Eq. (9), we have

$$\partial_i^\dagger = \kappa\Delta_+^{-1}\partial_i, \qquad \partial_0^\dagger = \partial_0 - i\Delta_+^{-1}\boldsymbol{\partial}^2, \qquad \partial_4^\dagger = -\partial_4, \qquad \Delta_+^\dagger = \kappa^2\Delta_+^{-1}. \tag{B8}$$

We now use Eqs. (B4)–(B6) and (B8) to obtain the expressions needed for the integration by parts of expressions of the form $(\partial_\mu\phi)^\dagger\star\partial^\mu\psi$ and $(\partial_\mu\psi)\star(\partial^\mu\phi)^\dagger$. With some algebra, we find

$$(\partial_i\phi)^\dagger\star(\partial_i\psi) = \partial_i[(\partial_i\phi)^\dagger\star\psi] - \frac{\Delta_+}{\kappa}[(\boldsymbol{\partial}^2\phi)^\dagger\star\psi]. \tag{B9}$$

Similarly,

$$(\partial_0\phi)^\dagger\star(\partial_0\psi) = \frac{\partial_0}{\kappa}[(\Delta_+(\partial_0\phi)^\dagger)\star\psi] - i\partial_i[(\Delta_+^{-1}\partial_i\partial_0\phi)^\dagger\star\psi] - \frac{\Delta_+}{\kappa}[(\partial_0^2\phi)^\dagger\star\psi]. \tag{B10}$$

Notice that, using this convention, Eqs. (B9) and (B10) are still fine substituting $\phi^\dagger$ with any other quantity (because the above derivations do not use in any way the presence of the † over ϕ) and, therefore, can be used regardless of the combination of fields to which they can be applied.

The Hermitian conjugates of Eqs. (B9) and (B10) take the form

$$(\partial_i\psi)^\dagger\star(\partial_i\phi) = \partial_i^\dagger[\psi^\dagger\star(\partial_i\phi)] - \frac{\kappa}{\Delta_+}[\psi^\dagger\star(\partial^2\phi)], \tag{B11}$$

$$(\partial_0\psi)^\dagger\star(\partial_0\phi) = \partial_0^\dagger[\psi^\dagger\star(\kappa\Delta_+^{-1}\partial_0\phi)] + i\partial_i^\dagger[\psi^\dagger\star(\Delta_+^{-1}\partial_i\partial_0\phi)] - \frac{\kappa}{\Delta_+}[\psi\star(\partial_0^2\phi)]. \tag{B12}$$

Finally, we will also need the following identity:

$$m^2\phi^\dagger\star\psi = -\left(\frac{\Delta_+}{\kappa}-1\right)(m^2\phi^\dagger\star\psi) + \frac{\Delta_+}{\kappa}(m^2\phi^\dagger\star\psi) = -\frac{i\partial_0}{\kappa}(m^2\phi^\dagger\star\psi) - \frac{i\partial_4}{\kappa}(m^2\phi^\dagger\star\psi) + \frac{\Delta_+}{\kappa}(m^2\phi^\dagger\star\psi) \tag{B13}$$

and its Hermitian conjugate

$$m^2\psi^\dagger\star\phi = +\frac{i\partial_0^\dagger}{\kappa}(m^2\psi^\dagger\star\phi) + \frac{i\partial_4^\dagger}{\kappa}(m^2\psi^\dagger\star\phi) + \frac{\kappa}{\Delta_+}(m^2\psi^\dagger\star\phi). \tag{B14}$$

For the opposite ordering, we have instead

$$(\partial_i\psi)\star(\partial_i\phi)^\dagger = \kappa\partial_i(\psi\star[\Delta_+^{-1}(\partial_i\phi)^\dagger]) - \psi\star(\partial^2\phi)^\dagger, \tag{B15}$$

$$(\partial_0\psi)\star(\partial_0\phi)^\dagger = \partial_0(\psi\star[\kappa\Delta_+^{-1}(\partial_0\phi)^\dagger]) - i\partial_i(\psi\star[\Delta_+^{-1}\partial_i(\partial_0\phi)^\dagger]) - [\psi\star(\partial_0^2\phi)^\dagger] + \left(\frac{i}{\kappa}\partial_0 + i\frac{\partial_4}{\kappa}\right)[\psi\star(\partial_0^2\phi)^\dagger]. \tag{B16}$$

APPENDIX C: POINCARÉ SYMMETRY OF THE ACTION—SPACE-TIME APPROACH

We want to discuss the invariance of the action (45) under κ-Poincaré transformations. As a first step, let us notice that it is equivalent to

$$S = -\frac{1}{2}\int_{\mathbb{R}^4} d^4x[\phi^\dagger\star\partial_\mu\partial^\mu\phi + \phi\star(\partial_\mu\partial^\mu\phi)^\dagger + m^2(\phi^\dagger\star\phi + \phi\star\phi^\dagger)]. \tag{C1}$$

This is easy to see using Eqs. (B15) and (B16) for integrating by parts the second term, and Eqs. (B11) and (B12) for the first term, in the action (45), since the Lagrangians are the same up to a total divergence. Let us consider infinitesimal transformations. The basic assumption is that a scalar field transforms as

$$0 = \phi'(x') - \phi(x) = [\phi'(x') - \phi(x')] + [\phi(x') - \phi(x)] \simeq \delta\phi(x) + d\phi(x), \tag{C2}$$

where d is the differential operator corresponding to κ-Poincaré transformations. In order to show the invariance of the Lagrangian appearing in Eq. (C1), it is enough to prove that

$$\mathcal{L}[\phi'(x')] - \mathcal{L}[\phi(x)] \simeq \delta\mathcal{L}[\phi(x)] + d\mathcal{L}[\phi(x)] = 0, \tag{C3}$$

where $\delta\mathcal{L}$ is the functional variation $\mathcal{L}[\phi+\delta\phi] - \mathcal{L}[\phi]$.

The invariance of the Lagrangian is ensured if the differential satisfies the Leibniz rule with respect to the $\star$ product,

$$d(\phi(x)\star\psi(x)) = (d\phi(x))\star\psi(x) + \phi(x)\star d\psi(x), \tag{C4}$$

which is a standard requirement for the definition of a differential calculus. Two different prescriptions have been proposed in the literature [14,17,52,53]. We adopt here the one proposed in Ref. [14] that is based on a differential calculus that satisfies the "bicovariance" property [54]. In this case, the differential $\hat{d}$, generating infinitesimal κ-Poincaré transformations in κ-Minkowski space-time, takes the form

$$\hat{d} = i(\hat{\epsilon}^A P_A + \hat{\omega}^{\mu\nu}L_{\mu\nu})\triangleright, \tag{C5}$$

where P_A and $L_{\mu\nu}$ are, respectively, the κ-Poincaré translation and Lorentz generators (in classical basis). These are defined through their action on noncommutative plane waves as $P_A \equiv -i\hat{\partial}_A$ and $L_{\mu\nu} \equiv -\frac{i}{2}\hat{x}_{[\mu}\hat{\partial}_{\nu]}\frac{\kappa}{P_0+P_4}$, respectively. It can be proved, however (see [14]), that the action of the Lorentz generator on the field, through the Weyl map (18), reduces to the standard action

$$L_{\mu\nu}\triangleright\phi(x) = \mathcal{W}^{-1}(L_{\mu\nu}\triangleright\phi(\hat{x})) = -\frac{1}{2}x_{[\mu}\star\partial_{\nu]}\frac{\kappa}{\partial_0+\partial_4}\phi(x) = -\frac{i}{2}x_{[\mu}\partial_{\nu]}\phi(x). \tag{C6}$$

The parameters $\hat{\epsilon}^A$ and $\hat{\omega}^{\mu\nu}$ must obey commutation relations with $\hat{x}^\mu$ so that $\hat{d}$ satisfies the Leibniz rule in Minkowski space-time

$$\hat{d}(\phi(\hat{x})\psi(\hat{x})) = \hat{d}\phi(\hat{x})\psi(\hat{x}) + \phi(\hat{x})\hat{d}\psi(\hat{x}). \tag{C7}$$

The commutation properties of $\hat{\epsilon}^A$ and $\hat{\omega}^{\mu\nu}$ are reported in Appendix D, and the corresponding relations (D3) and (D4) between the images of the parameters under Weyl map and the associate $\star$ product lead to

$$\phi(x)\star\epsilon^A = \epsilon^B K_B^A(\partial)\star\phi(x) \tag{C8}$$

and

$$\phi(x)\star\omega^{\mu\nu} = \Omega^{\mu\nu}_{\rho\sigma}(\partial)\omega^{\rho\sigma}\star\phi(x), \tag{C9}$$

where the matrices K and Ω are also defined in Appendix D.

Two additional properties of the transformation parameters (see [14]) are that

$$\partial_A \epsilon^B = \partial_A \omega^{\mu\nu} = 0 \tag{C10}$$

and that

$$(d\phi)^\dagger = d\phi^\dagger. \tag{C11}$$

We can now write the image of $\hat{d}$ under the Weyl map as

$$d\phi(x) = \epsilon^A \star \partial_A \phi(x) + \frac{1}{2}\omega^{\mu\nu} \star x_{[\mu}\partial_{\nu]}\phi(x). \tag{C12}$$

Using relations (C8) and (C9), the Lorentz action (C6), and relations (B4)–(B6) and (B8), one can prove that the Leibniz rule (C4) is satisfied.

We can now prove the invariance of the Lagrangian for Eq. (C1). Considering that from Eq. (C2) $\delta\phi(x) = -d\phi(x)$, the functional variation of the Lagrangian gives

$$\begin{aligned}\delta\mathcal{L}[\phi(x)] = &-(d\phi(x))^\dagger \star \partial_\mu \partial^\mu \phi(x) - \phi^\dagger(x) \star \partial_\mu \partial^\mu d\phi(x)\\ &- (d\phi(x)) \star (\partial_\mu \partial^\mu \phi(x))^\dagger - \phi(x) \star (\partial_\mu \partial^\mu d\phi(x))^\dagger\\ &+ m^2[(d\phi(x))^\dagger \star \phi(x) + \phi^\dagger(x) \star d\phi(x)]\\ &+ m^2[(d\phi(x)) \star \phi^\dagger(x) + \phi(x) \star (d\phi(x))^\dagger].\end{aligned} \tag{C13}$$

Given that the action (C6) of $L_{\mu\nu}$ is the same as the standard one, and (C10), it is straightforward to prove that $[\partial^\mu \partial_\mu, d] = 0$. Indeed, the only part of d on which the derivatives act is the standard Lorentz term $\propto x_{[\rho}\partial_{\sigma]}$, so that the derivation is the same as in the standard case:

$$\begin{aligned}\partial_\mu \partial^\mu d\phi(x) &= \epsilon^A \star \partial_A \partial_\mu \partial^\mu \phi(x) + \frac{1}{2}\omega^{\rho\sigma} \star \partial_\mu \partial^\mu x_{[\rho}\partial_{\sigma]}\phi(x)\\ &= \epsilon^A \star \partial_A \partial_\mu \partial^\mu \phi(x) + \frac{1}{2}\omega^{\rho\sigma} \star x_{[\rho}\partial_{\sigma]}\partial_\mu \partial^\mu \phi(x)\\ &\quad + \omega^{\rho\sigma} \star \partial_{[\rho}\partial_{\sigma]}\phi(x)\\ &= d\partial_\mu \partial^\mu \phi(x).\end{aligned} \tag{C14}$$

Then, using the properties (C11) and (C4), it follows immediately that $\delta\mathcal{L}[\phi(x)] = -d\mathcal{L}[\phi(x)]$. We show as an example the derivation for the second row of Eq. (C13). Using the result (C14), we rewrite it first as

$$-(d\phi(x)) \star (\partial_\mu \partial^\mu \phi(x))^\dagger - \phi(x) \star (d\partial_\mu \partial^\mu \phi(x))^\dagger.$$

We now use properties (C11) and (C4) to rewrite it as

$$\begin{aligned}&-(d\phi(x)) \star (\partial_\mu \partial^\mu \phi(x))^\dagger - \phi(x) \star d(\partial_\mu \partial^\mu \phi(x))^\dagger\\ &\quad = -d[\phi(x) \star (\partial_\mu \partial^\mu \phi(x))^\dagger].\end{aligned}$$

We have thus shown under which hypotheses the action is κ-Poincaré invariant. To conclude, let us discuss briefly what the condition (C2) implies for the transformation of the field. If we consider the Fourier transform of the field generically as

$$\phi(x) = \int d\mu(p)\tilde{\phi}(p)e^{-ip\cdot x}, \tag{C15}$$

then

$$\begin{aligned}d\phi(x) &= \epsilon^A \star \int d\mu(p)\tilde{\phi}(p)\partial_A e^{-ip\cdot x} + \frac{1}{2}\omega^{\mu\nu} \star \int d\mu(p)\tilde{\phi}(p)x_{[\mu}\partial_{\nu]}e^{-ip\cdot x}\\ &= -i\epsilon^A \star \int d\mu(p)\tilde{\phi}(p)p_A e^{-ip\cdot x} - \frac{1}{2}\omega^{\mu\nu} \star \int d\mu(p)\tilde{\phi}(p)p_{[\mu}\frac{\partial}{\partial p^{\nu]}}e^{-ip\cdot x}\\ &= \int d\mu(p)\left(-ip_A\tilde{\phi}(p)\epsilon^A + \frac{1}{2}p_{[\mu}\frac{\partial}{\partial p^{\nu]}}\tilde{\phi}(p)\omega^{\mu\nu}\right) \star e^{-ip\cdot x}.\end{aligned}$$

The field variation $\delta\phi = -d\phi$ implies that formally we can state

$$\delta\tilde{\phi}(p) = \left(i\epsilon^A p_A \tilde{\phi}(p) - \frac{1}{2}\omega^{\mu\nu}p_{[\mu}\frac{\partial}{\partial p^{\nu]}}\tilde{\phi}(p)\right)\star. \tag{C16}$$

The last relation is very similar to its classical analog, which is given by[9]

$$U^{-1}(\Lambda, a)\tilde{\phi}(p)U(\Lambda, a) = e^{i(\Lambda^{-1}a)\cdot p}\tilde{\phi}(\Lambda^{-1}p) \simeq \tilde{\phi}(p) + i\epsilon^\mu p_\mu \tilde{\phi}(p) - \frac{1}{2}\omega^{\mu\nu}p_{[\mu}\frac{\partial}{\partial p^{\nu]}}\tilde{\phi}(p). \tag{C17}$$

[9]We are here using notations such that for a finite Poincaré transformation $\phi'(x') = U^{-1}(\Lambda, a)\phi(\Lambda x + a)U(\Lambda, a) \simeq \phi(x) + \delta\phi(x) + d\phi(x)$, so that $U^{-1}(\Lambda, a)\phi(x)U(\Lambda, a) \simeq \delta\phi(x) = -d\phi(x)$.

APPENDIX D: PROPERTIES OF THE NONCOMMUTATIVE PARAMETERS ϵ^A AND $\omega^{\mu\nu}$

The properties of the noncommutative parameters are derived in Ref. [14]. For the translation parameter, they amount to

$$[\hat{x}^\mu, \hat{\epsilon}^A] = (X^\mu)^A{}_B \hat{\epsilon}^B,$$

where

$$\hat{X}^0 = -\frac{i}{\kappa}\begin{pmatrix} 0 & \mathbf{0}^T & 1 \\ \mathbf{0} & \mathbf{0}_{3\times3} & \mathbf{0} \\ 1 & \mathbf{0}^T & 0 \end{pmatrix}, \qquad \hat{\mathbf{X}} = \frac{i}{\kappa}\begin{pmatrix} 0 & \mathbf{n}^T & 0 \\ \mathbf{n} & \mathbf{0}_{3\times3} & \mathbf{n} \\ 0 & -\mathbf{n}^T & 0 \end{pmatrix},$$

where $\mathbf{n}$ is a unit vector in standard basis. In terms of plane waves, they satisfy the relation

$$\hat{e}_k \hat{\epsilon}^A \hat{e}_k^{-1} = \hat{\epsilon}^B K^A_B(p(k)), \tag{D1}$$

with

$$K(p) = \frac{1}{\kappa}\begin{pmatrix} p_4 + \frac{\mathbf{p}^2}{p_0+p_4} & -\frac{\kappa}{p_0+p_4}\mathbf{p}^T & p_0 \\ -\mathbf{p} & \kappa \mathbf{1}_{3\times3} & -\mathbf{p} \\ p_0 - \frac{\mathbf{p}^2}{p_0+p_4} & \frac{\kappa}{p_0+p_4}\mathbf{p}^T & p_4 \end{pmatrix},$$

$$K^{-1}(p) = \frac{1}{\kappa}\begin{pmatrix} p_4 + \frac{\mathbf{p}^2}{p_0+p_4} & \mathbf{p}^T & -p_0 + \frac{\mathbf{p}^2}{p_0+p_4} \\ \frac{\kappa}{p_0+p_4}\mathbf{p} & \kappa \mathbf{1}_{3\times3} & \frac{\kappa}{p_0+p_4}\mathbf{p} \\ -p_0 & -\mathbf{p}^T & p_4 \end{pmatrix}.$$

Notice that $\hat{X}^\mu$ and $K(p(k))$ matrices coincide, respectively, with the 5D representations of $\hat{x}^\mu$ and $\hat{e}_k$ given in Eqs. (2) and (4), in agreement with the fact that $\hat{\epsilon}^A$ form a representation of κ-Minkowski algebra. For the Lorentz parameter, the commutation properties are given by

$$\hat{e}_k \hat{\omega}^{\mu\nu} \hat{e}_k^{-1} = \hat{\omega}^{\rho\sigma} \Omega^{\mu\nu}_{\rho\sigma}(p(k)), \tag{D2}$$

with

$$\Omega^{\mu\nu}_{\rho\sigma}(p) = \delta^\mu_{[\rho} \tau^\nu_{\sigma]}(p)$$

and

$$\tau(p) = \begin{pmatrix} 2\frac{\kappa}{p_0+p_4} - 1 & -2\frac{\mathbf{p}}{p_0+p_4} \\ 0 & \mathbf{1} \end{pmatrix}.$$

Using the (inverse) Weyl map (22) with Eqs. (D1) and (D2), we obtain the corresponding properties for the $\star$ product between the parameters and the plane waves:

$$e_p \star \epsilon^A = K^A_B(p)\epsilon^B \star e_p, \quad \epsilon^A \star e_p = (K^{-1})^A_B(p) e_p \star \epsilon^B \tag{D3}$$

and

$$e_p \star \omega^{\mu\nu} = \Omega^{\mu\nu}_{\rho\sigma}(p)\omega^{\rho\sigma} \star e_p, \quad \omega^{\mu\nu} \star e_p = (\Omega^{-1})^{\mu\nu}_{\rho\sigma}(p) e_p \star \omega^{\rho\sigma}. \tag{D4}$$

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