Conformal generation of an exotic rotationally invariant harmonic oscillator

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An exotic rotationally invariant harmonic oscillator (ERIHO) is constructed by applying a nonunitary isotropic conformal bridge transformation (CBT) to a free planar particle. It is described by the isotropic harmonic oscillator Hamiltonian supplemented by a Zeeman type term with a real coupling constant g. The model reveals the Euclidean ($|g|<1$) and Minkowskian ($|g|>1$) phases separated by the phases $g=+1$ and $g=-1$ of the Landau problem in the symmetric gauge with opposite orientation of the magnetic field. A hidden symmetry emerges in the system at rational values of g. Its generators, together with the Hamiltonian and angular momentum produce nonlinearly deformed $\mathfrak{u}(2)$ and $\mathfrak{gl}(2,\mathbb{R})$ algebras in the cases of $0<|g|<1$ and $\infty>|g|>1$, which transmute one into another under the inversion $g\to -1/g$. Similarly, the true, $\mathfrak{u}(2)$, and extended conformal, $\mathfrak{gl}(2,\mathbb{R})$, symmetries of the isotropic Euclidean oscillator ($g=0$) interchange their roles in the isotropic Minkowskian oscillator ($|g|=\infty$), while two copies of the $\mathfrak{gl}(2,\mathbb{R})$ algebra of analogous symmetries mutually transmute in Landau phases. We show that the ERIHO system is transformed by a peculiar unitary transformation into the anisotropic harmonic oscillator generated, in turn, by anisotropic CBT. The relationship between the ERIHO and the subcritical phases of the harmonically extended Landau problem, as well as with a plane isotropic harmonic oscillator in a uniformly rotating reference frame, is established.

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I. INTRODUCTION

The revival of interest in nonrelativistic conformal symmetry [1,2] was stimulated by nonrelativistic AdS/CFT correspondence [3–6], its relevance to black holes physics and cosmology [7–13], and its utility in the description of strongly coupled condensed matter systems [14–17] and QCD confinement problem [18,19]. In this context, the mechanism to improve the properties of the scale-free conformal mechanics proposed initially by de Alfaro, Fubini, and Furlan [1] amounts to an improved choice of the time coordinate in black holes physics since a usual time variable is not a good global evolution coordinate on AdS_2 [11]. Via the same basic mechanism, the mass and length scales are introduced in holographic QCD [19]. On the other hand, this mechanism corresponds to the Niederer's transformation [20], by which the relation between the free particle's and harmonic oscillator's dynamics was established at the classical and quantum levels. The latter relationship, it turn, corresponds to different forms of dynamics [21] with respect to the conformal symmetry.

In recent papers [22–24], the nonunitary conformal bridge transformation (CBT) was introduced, by which the noncompact and compact generators of the conformal symmetry can be related in the spirit of Dirac's different forms of dynamics. This allowed us to establish the relation between the quantum states and symmetries, including hidden symmetries, of different asymptotically free and associated harmonically trapped systems. The correspondence comprises not only energy eigenstates of the systems, but also coherent and squeezed states. The nonunitary CBT turns out to be closely related with a unitary transformation between the quantum coordinate (Schrödinger) and holomorphic (Fock-Bargmann) representations. Its classical analog yields a canonical transformation corresponding to the Hamiltonian vector flow produced by generators of the conformal symmetry taken with particular complex values of the parameters. Bearing in mind these two last properties, the CBT shows some not explored yet similarity with the $\mathcal{PT}$-symmetry [25–29].

The CBT employed in Refs. [22–24] in different geometric and dynamical backgrounds possesses the property of rotational invariance. In this work, we exploit the isotropy of the CBT to generate the exotic rotational invariant one-parametric family of harmonic oscillator systems which have a number of properties interesting from a physical point of view. As will be showed, in dependence on the value of the real parameter g, the family reveals two distinct phases with Euclidean ($|g|<1$) and Minkowskian ($|g|>1$) properties, which are separated by the phases $g=+1$ and $g=-1$ corresponding to the Landau problem in the symmetric gauge with opposite orientation of the magnetic field. At rational values of g,

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each system of the family possesses a hidden symmetry [30], and so, is maximally superintegrable. Symmetries of different phases mutually transmute under the inversion $g \to -1/g$, and in general case they generate nonlinear algebras of the W-type [31]. We also reveal a unitary relationship of the exotic rotationally invariant harmonic oscillator (ERIHO) with the anisotropic harmonic oscillator (AHO) systems, and establish the relation of our model with subcritical phases of the harmonically extended Landau problem.

The paper is organized as follows. In Sec. II, the basic aspects associated with the CBT are reviewed. In Sec. III, the nonunitary isotropic CBT is applied to a complex linear combination of the generators of dilatations and rotations of the two-dimensional free particle to produce the ERIHO. The dynamics and symmetries of the system are discussed in detail there at the classical and quantum levels. In Sec. IV, the anisotropic CBT is used to generate the AHO from the free particle. In Sec. V, the relationships of the ERIHO with the AHO, harmonically extended Landau problem, and a plane isotropic harmonic oscillator in a uniformly rotating reference frame are explored. The final Sec. VI is devoted to the discussion and outlook.

II. THE CONFORMAL BRIDGE TRANSFORMATION

The technique of CBT [22] allows us to relate asymptomatically free systems (such as a free particle, or conformal mechanics model), characterized by the $\mathfrak{so}(2,1)$ conformal symmetry, with harmonically confined models that are $\mathfrak{sl}(2,\mathbb{R})$ conformal invariant (such as the harmonic oscillator, or the conformal mechanics model of de Alfaro, Fubini and Furlan [1]). In this section we review the basic aspects associated with this transformation to apply it then for the construction of the ERIHO.

Consider the quantum conformal $\mathfrak{so}(2,1)$ algebra

$$[\hat{D},\hat{H}] = i\hbar\hat{H}, \qquad [\hat{D},\hat{K}] = -i\hbar\hat{K}, \qquad [\hat{K},\hat{H}] = 2i\hbar\hat{D}, \tag{2.1}$$

where $\hat{H}$, $\hat{D}$ and $\hat{K}$ are implied to be, respectively, the Hamiltonian, the dilatation generator, and the generator of special conformal transformations of a system, and we assume here that the operators $\hat{D}$ and $\hat{K}$ do not depend explicitly on time. By taking linear combinations

$$\hat{\mathcal{J}}_0 = \frac{1}{2\omega\hbar}(\hat{H}+\omega^2\hat{K}),$$
$$\hat{\mathcal{J}}_\pm = -\frac{1}{2\omega\hbar}(\hat{H}-\omega^2\hat{K}\pm 2i\omega\hat{D}), \tag{2.2}$$

we produce the $\mathfrak{sl}(2,\mathbb{R})$ algebra

$$[\hat{\mathcal{J}}_0,\hat{\mathcal{J}}_\pm] = \pm\hat{\mathcal{J}}_\pm, \qquad [\hat{\mathcal{J}}_-,\hat{\mathcal{J}}_+] = 2\hat{\mathcal{J}}_0. \tag{2.3}$$

Though both algebraic structures are isomorphic, from a physical point of view they describe the systems with essentially different properties if $\hat{\mathcal{J}}_0$ is identified as the Hamiltonian of another model. The generator $\hat{H}$ of the $\mathfrak{so}(2,1)$ symmetry is noncompact and has a continuous spectrum. It represents the Hamiltonian of an asymptomatically free (for $|x| \to \infty$) particle. On the other hand, the operator $\hat{\mathcal{J}}_0$ is a compact generator of the $\mathfrak{sl}(2,\mathbb{R})$ Newton-Hooke symmetry [20,32–35]. It is characterized by a discrete spectrum corresponding to the associated harmonically trapped system. The Planck constant $\hbar$ and parameter $\omega > 0$ of the dimension of frequency introduced in (2.2) guarantee the dimensionless character of the generators $\hat{\mathcal{J}}_0$ and $\hat{\mathcal{J}}_\pm$.

The *nonunitary* operators

$$\hat{\mathfrak{S}} = e^{-\frac{\omega}{\hbar}\hat{K}} e^{\frac{\hat{H}}{2\hbar\omega}} e^{\frac{i}{\hbar}\ln(2)\hat{D}} = e^{-\frac{\omega}{\hbar}\hat{K}} e^{\frac{i}{\hbar}\ln(2)\hat{D}} e^{\frac{\hat{H}}{\hbar\omega}},$$
$$\hat{\mathfrak{S}}^{-1} = e^{-\frac{i}{\hbar}\ln(2)\hat{D}} e^{-\frac{\hat{H}}{2\hbar\omega}} e^{\frac{\omega}{\hbar}\hat{K}}, \tag{2.4}$$

relate (intertwine) the sets of generators $(\hat{H},\hat{D},\hat{K})$ and $(\hat{\mathcal{J}}_-, \hat{\mathcal{J}}_0, \hat{\mathcal{J}}_+)$ by a similarity transformation,

$$\hat{\mathfrak{S}}(\hat{H})\hat{\mathfrak{S}}^{-1} = -\omega\hbar\hat{\mathcal{J}}_-, \qquad \hat{\mathfrak{S}}(i\hat{D})\hat{\mathfrak{S}}^{-1} = \hbar\hat{\mathcal{J}}_0,$$
$$\hat{\mathfrak{S}}(\hat{K})\hat{\mathfrak{S}}^{-1} = \frac{\hbar}{\omega}\hat{\mathcal{J}}_+. \tag{2.5}$$

The systems described by the Hamiltonians $\hat{H}$ and $\hat{\mathcal{J}}_0$ correspond, according to Dirac [21], to two different forms of dynamics associated here with conformal symmetry. This is the quantum version of the CBT, for some earlier applications of which see Refs. [22–24]. This transformation implies, in particular, that

$$\hat{D}|\lambda\rangle = i\hbar\lambda|\lambda\rangle \Rightarrow \hat{\mathcal{J}}_0(\hat{\mathfrak{S}}|\lambda\rangle) = \lambda\hat{\mathfrak{S}}|\lambda\rangle, \tag{2.6}$$

$$\hat{H}|E\rangle = E|E\rangle \Rightarrow \hat{\mathcal{J}}_-(\hat{\mathfrak{S}}|E\rangle) = -\frac{E}{\hbar\omega}\hat{\mathfrak{S}}|E\rangle. \tag{2.7}$$

One sees [22,24] then that to get normalizable eigenfunctions of the operator $\hat{\mathcal{J}}_0$, the formal eigenvector $|\lambda\rangle$ of the operator $\hat{D}$ has to satisfy the following properties:

(I) The series $\exp(\frac{\hat{H}}{2\hbar\omega})|\lambda\rangle = \sum_{n=0}^{\infty}\frac{1}{n!(2\hbar\omega)^n}(\hat{H})^n|\lambda\rangle$ has to reduce to a finite number of terms, i.e., $|\lambda\rangle$ should be a Jordan state of the operator $\hat{H}$ corresponding to zero energy.[1]

(II) The wave functions $\langle \boldsymbol{x}|\lambda\rangle$ must not have poles and have to be single-valued.

[1]The wave functions of generalized Jordan states corresponding to energy λ satisfy relations of the form $P(\hat{H})\Omega_\lambda = \psi_\lambda$, where $\hat{H}\psi_\lambda = \lambda\psi_\lambda$ and $P(\eta)$ is a polynomial [36–38]. Here we consider the Jordan states satisfying the relations $(\hat{H})^\ell\Omega_\lambda = \lambda\psi_\lambda$ with $\lambda = 0$ for a certain natural number ℓ.

On the other hand, the eigenvectors $|E\rangle$ (physical, or nonphysical, with complex eigenvalues in general case) of $\hat{H}$ are transformed into eigenvectors of the lowering operator $\hat{\mathcal{J}}_-$ of the $\mathfrak{sl}(2,\mathbb{R})$ algebra. Therefore, the resulting eigenstates in (2.7) are the coherent states of the system with the Hamiltonian $\hat{\mathcal{J}}_0$.

The classical analog of the CBT is given by the *complex* canonical transformation [22,24]

$$\mathcal{T}(\tau,\beta,\delta,\gamma,t) = T_{2\omega\mathcal{J}_0}(\tau)\circ T_{\beta\delta\gamma}\circ T_H(-t), \tag{2.8}$$

where

$$\begin{aligned}\exp(\gamma F)\star f(q,p) &:= f(q,p) + \sum_{n=1}^{\infty}\frac{\gamma^n}{n!}\{F,\{\ldots,\{F,f\underbrace{\}\ldots\}\}}_{n}\\ &=: T_F(\gamma)(f)\end{aligned} \tag{2.9}$$

is a Hamiltonian flux generated by a phase space function F, and

$$\begin{aligned}T_{\beta\delta\gamma} &:= T_{K_0}(\beta)\circ T_H(\delta)\circ T_{D_0}(\gamma)\\ &= T_{K_0}(\delta)\circ T_{D_0}(\gamma)\circ T_H(2\delta),\end{aligned} \tag{2.10}$$

$$\text{with}\quad \delta = \frac{i}{2\omega},\quad \beta = -i\omega,\quad \gamma = -\ln 2. \tag{2.11}$$

Here, $D_0 = D|_{t=0}$ and $K_0 = K|_{t=0}$, and we assume that the generators of dilatations, D, and special conformal transformations, K, are explicitly depending on time, *dynamical* integrals of motion satisfying a relation of the form $\dot{A} = \{A,H\} + \frac{\partial A}{\partial t} = 0$. In correspondence with this, in the composed Hamiltonian flux (2.8), the first transformation $T_H(-t)$ removes the t dependence in the dynamical integrals D and K. The second transformation relates these generators at $t=0$ with the generators of the $\mathfrak{sl}(2,\mathbb{R})$ algebra $\mathcal{J}_0$ and $\mathcal{J}_\pm$ taken at $\tau = 0$ (this is the classical analog of the quantum similarity transformation presented above). Finally, $T_{2\omega\mathcal{J}_0}(\tau)$ restores the τ dependence[2] of the generators $\mathcal{J}_\pm$.

In particular case of the d-dimensional quantum free particle, its conformal symmetry generators are given in the Schrödinger representation by

$$\hat{H} = \sum_{i=1}^{d}\hat{H}_i,\qquad \hat{D} = \sum_{i=1}^{d}\hat{D}_i,\qquad \hat{K} = \sum_{i=1}^{d}\hat{K}_i, \tag{2.12}$$

$$\hat{H}_i = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x_i^2},\quad \hat{D}_i = -i\frac{\hbar}{2}\left(x_i\frac{\partial}{\partial x_i}+\frac{1}{2}\right),\quad \hat{K}_i = \frac{m}{2}x_i^2. \tag{2.13}$$

Here, each set of generators $\hat{H}_i$, $\hat{D}_i$ and $\hat{K}_i$ satisfies the $\mathfrak{so}(2,1)$ algebraic relations (2.13), and so, the operators $\hat{H}$, $\hat{D}$ and $\hat{K}$ generate the same $\mathfrak{so}(2,1)$ Lie algebra.

By using generators (2.12) to construct the conformal bridge operators (2.4), one gets

$$\hat{\mathfrak{S}} = \Pi_{i=1}^{d}\hat{\mathfrak{S}}_i,\quad \hat{\mathfrak{S}}_i = e^{-\frac{\omega}{\hbar}\hat{K}_i}e^{\frac{\hat{H}_i}{2\hbar\omega}}e^{\frac{i}{\hbar}\ln(2)\hat{D}_i},\quad [\hat{\mathfrak{S}}_i,\hat{\mathfrak{S}}_j]=0. \tag{2.14}$$

The CBT produced by the operator $\hat{\mathfrak{S}}$ and its inverse is a composition of d independent transformations. Each of these transformations touches a particular spatial direction, leaving the rest invariant, and here we use the same parameter ω to guarantee the rotational invariance of the transformation. Later we will consider other possibilities. In correspondence with (2.5), the total transformation $\hat{\mathfrak{S}}$ produces the generators

$$\hat{\mathcal{J}}_0 = \sum_{i=0}^{d}\hat{\mathcal{J}}_0^i = \frac{1}{2\omega\hbar}\hat{H}_{\rm osc},\qquad \hat{\mathcal{J}}_\pm = \sum_{i=0}^{d}\hat{\mathcal{J}}_\pm^i, \tag{2.15}$$

$$\hat{\mathcal{J}}_0^i = \frac{1}{2}\left(\hat{a}_i^+\hat{a}_i^- + \frac{1}{2}\right) = \frac{1}{2\omega\hbar}\hat{H}_{\rm osc}^i,\qquad \hat{\mathcal{J}}_\pm^i = \frac{1}{2}(\hat{a}_i^\pm)^2, \tag{2.16}$$

where $\hat{H}_{\rm osc} = \sum_{i=1}^{d}\hat{H}_{\rm osc}^i$ is the quantum Hamiltonian of the d-dimensional isotropic harmonic oscillator, and

$$\begin{aligned}&\hat{a}_i^\pm = \sqrt{\frac{m\omega}{2\hbar}}\left(x_i \mp \frac{\hbar}{m\omega}\frac{\partial}{\partial x_i}\right),\qquad [\hat{a}_i^\pm,\hat{a}_j^\pm]=0,\\ &[\hat{a}_i^-,\hat{a}_j^+] = \delta_{ij},\end{aligned} \tag{2.17}$$

are the usual first-order ladder operators of the system.

In the free particle system, we also have the linear momenta operators and the Galilean boosts for each direction,

$$\begin{aligned}&\hat{p}_j = -i\hbar\frac{\partial}{\partial x_j},\quad \hat{\xi}_j = m x_j,\quad [\hat{\xi}_j,\hat{\xi}_k] = [\hat{p}_j,\hat{p}_k] = 0,\\ &[\hat{\xi}_j,\hat{p}_k] = i\hbar m\delta_{jk}.\end{aligned} \tag{2.18}$$

The application of the CBT to these operators produces (no summation over repeated index)

$$\hat{\mathfrak{S}}(\hat{p}_j)\hat{\mathfrak{S}}^{-1} = \hat{\mathfrak{S}}_j(\hat{p}_j)\hat{\mathfrak{S}}_j^{-1} = -i\sqrt{m\hbar\omega}\,\hat{a}_j^-, \tag{2.19}$$

[2]We are interested in time dependence of the classical dynamical integrals as we will investigate the classical dynamics of the corresponding systems. The time dependence in the operators in Heisenberg picture can be restored analogously by inclusion of the respective evolution operators, but we will not be interested in it at the quantum level.

$$\hat{\mathfrak{S}}(\hat{\xi}_j)\hat{\mathfrak{S}}^{-1} = \hat{\mathfrak{S}}_j(\hat{\xi}_j)\hat{\mathfrak{S}}_j^{-1} = \sqrt{\frac{m\hbar}{\omega}}\hat{a}_j^+. \quad (2.20)$$

The angular momentum tensor

$$\hat{M}_{ij} = \frac{1}{m}(\hat{\xi}_i\hat{p}_j - \hat{\xi}_j\hat{p}_i) = -i\hbar(\hat{a}_i^+\hat{a}_j^- - \hat{a}_j^+\hat{a}_i^-) \quad (2.21)$$

commutes with the operator $\hat{\mathfrak{S}}$ and its inverse.

As any symmetry generator of the d-dimensional free particle (harmonic oscillator) is a function of $\hat{\xi}_i$ and $\hat{p}_i$ ($\hat{a}_i^\pm$), the CBT allows us to map the integrals of one system to those of another system.

III. EXOTIC ROTATIONALLY INVARIANT HARMONIC OSCILLATOR

From now on we restrict ourselves to the case of $d = 2$ and extend the CBT of the previous section to generate and investigate the ERIHO. For this we exploit the invariance of the rotation generator $\hat{p}_\varphi = \hat{M}_{12}$ under the CBT described in the previous section, and consider a complex linear combination of the classical free particle symmetry generators $2iD_0 + gp_\varphi$, where g is a real parameter, to produce the associated system as a generalization of the isotropic harmonic oscillator that is obtained at $g = 0$. One has $2iD_0 + gp_\varphi = x_j\Delta_{jk}p_k$, where $\Delta_{jk} = i\delta_{jk} + g\epsilon_{jk}$ is the complex tensor satisfying the relations

$$\Delta_{jk}\Delta_{jl} = (g^2 - 1)\delta_{kl}, \qquad \det\Delta = g^2 - 1. \quad (3.1)$$

Based on (3.1), one can expect that the one-parametric family of the quantum planar rotationally invariant systems described by

$$\hat{H}_g = \hat{\mathfrak{S}}\omega(2i\hat{D} + g\hat{p}_\varphi)\hat{\mathfrak{S}}^{-1} = \hat{H}_{\rm osc} + g\omega\hat{p}_\varphi \quad (3.2)$$

should have essentially different physical properties and symmetries in the cases $g^2 < 1$ and $g^2 > 1$ separated by the special parameter values $g = \pm 1$. The symmetries and the states of the quantum (and corresponding classical) system $\hat{H}_g$ has to be related by the CBT to those of the operator $2i\hat{D} + g\hat{p}_\varphi$ of the free particle.

In terms of the "circular" ladder operators,

$$\hat{b}_1^- = \frac{1}{\sqrt{2}}(\hat{a}_1^- - i\hat{a}_2^-), \qquad \hat{b}_1^+ = (\hat{b}_1^-)^\dagger,$$
$$\hat{b}_2^- = \frac{1}{\sqrt{2}}(\hat{a}_1^- + i\hat{a}_2^-), \qquad \hat{b}_2^+ = (\hat{b}_2^-)^\dagger, \quad (3.3)$$

being unitary transformation of $\hat{a}_i^\pm$ and satisfying relations $[\hat{b}_i^\pm, \hat{b}_j^\pm] = 0$, $[\hat{b}_i^-, \hat{b}_j^+] = \delta_{ij}$, the Hamiltonian $\hat{H}_g$ takes the form

$$\hat{H}_g = \hbar\omega(\ell_1\hat{b}_1^+\hat{b}_1^- + \ell_2\hat{b}_2^+\hat{b}_2^- + 1),$$
$$\ell_1 = 1 + g, \qquad \ell_2 = 1 - g. \quad (3.4)$$

This expression for $\hat{H}_g$ reminds us the AHO Hamiltonian, but presented here in a rotationally invariant form. The angular momentum operator $\hat{p}_\varphi = \epsilon_{ij}\hat{x}_j\hat{p}_j = -i\hbar\epsilon_{ij}\hat{a}_i^+\hat{a}_j^-$ is represented as

$$\hat{p}_\varphi = \hbar(\hat{b}_1^+\hat{b}_1^- - \hat{b}_2^+\hat{b}_2^-), \quad (3.5)$$

and it commutes with $\hat{H}_g$, $[\hat{H}_g, \hat{p}_\varphi] = 0$.

The one-parameter family (3.4) of the ERIHOs is interesting as it interpolates between different types of mechanical systems depending on the value of the real parameter g:

(1) When $g = 0$, we have the planar isotropic harmonic oscillator.
(2) For $g = \pm 1$, the system corresponds to the Landau problem of a particle of charge q in magnetic field $B_3 = \epsilon_{ij}\partial_i A_j^\pm = \mp B$ given by the two-dimensional vector potential in symmetric gauge

$$A_i^\pm = \pm\frac{1}{2}B\epsilon_{ij}x_j \Rightarrow \omega = \omega_B \equiv \frac{qB}{2mc}, \quad (3.6)$$

where we assume $qB > 0$.
(3) The case $|g| < 1$ looks like the Euclidean AHO with different frequencies $\omega_1 \neq \omega_2$, $\omega_i = \ell_i\omega$. When $|g| > 1$ we have instead the form of the Minkowskian AHO with frequencies of two different signs. Notice that the family with $|g| > 1$ resembles the Pais-Uhlenbeck oscillator [39].[3]
(4) In the limit $g \to \infty$, one has $g^{-1}\hat{H}_g \to \omega\hat{p}_\varphi$. In terms of the operators $\hat{b}_i^\pm$ this corresponds to the isotropic Minkowskian oscillator, see Eq. (3.5).

The first two cases were analyzed in the light of the CBT in [22]. In this section we investigate the properties of the ERIHO (3.2) in the general case, at the classical and quantum levels. This will allow us to reveal rather nontrivial relations between dynamics and symmetries of the systems with different values of the parameter g. In particular, between those corresponding to the isotropic Minkowskian oscillator case, $g^2 = \infty$, on the one hand, and the cases of the isotropic harmonic oscillator, $g = 0$, and Landau problem, $g^2 = 1$.

A. Classical picture

Let us consider the classical system described by the Hamiltonian

[3]Pais-Uhlenbeck oscillator attracted recently considerable attention in relation to the $\mathcal{PT}$-symmetry, see Refs. [40–42].

$$H_g = H_{\rm osc} + g\omega p_\varphi, \qquad H_{\rm osc} = \frac{1}{2m} p_i p_i + \frac{1}{2} m\omega^2 x_i x_i, \tag{3.7}$$

being the classical analog of (3.2). As in the quantum case, this Hamiltonian arises by applying the classical CBT to the complex linear combination $\omega(2iD + gp_\varphi)$ of the symmetry generators of the free particle system.

In terms of the classical analogues of circular ladder operators

$$b_1^- = \frac{1}{\sqrt{2}}(a_1^- - ia_2^-), \qquad b_1^+ = (b_1^-)^*,$$
$$b_2^- = \frac{1}{\sqrt{2}}(a_1^- + ia_2^-), \qquad b_2^+ = (b_2^-)^*, \tag{3.8}$$

$$a_i^\pm = \sqrt{\frac{m\omega}{2}}\left(x_i \mp \frac{i}{m\omega} p_i\right), \tag{3.9}$$

Hamiltonian (3.7) takes the form

$$H_g = \omega(\ell_1 b_1^+ b_1^- + \ell_2 b_2^+ b_2^-), \quad \ell_1 = 1 + g, \quad \ell_2 = 1 - g. \tag{3.10}$$

The equations of motion and their solutions are

$$\dot{b}_i^\pm = \{b_i^\pm, H_g\} = \pm i\omega\ell_i b_i^\pm \Rightarrow b_i^\pm(t) = e^{\pm i\omega\ell_i t} b_i^\pm(0) \coloneqq b_i^\pm. \tag{3.11}$$

Using (3.8) and (3.9), we have $\sqrt{m\omega}(x_1 + ix_2) = b_1^+ + b_2^-$, and find the trajectory of the particle,

$$z(t) = x_1(t) + ix_2(t) = R_1 e^{i\gamma_1} e^{i\omega\ell_1 t} + R_2 e^{-i\gamma_2} e^{-i\omega\ell_2 t}, \tag{3.12}$$

where $R_i \geq 0$ and $\gamma_i \in \mathbb{R}$ are the integration constants. The exponents in (3.12) evolve in opposite, clockwise and counterclockwise, directions in the case $g^2 < 1$, while for $g^2 > 1$ they evolve in one of the two directions depending on the sign of g. At $g^2 = 1$ one of the frequencies $\omega_i = \ell_i\omega$ vanishes and (3.12) describes a closed circular trajectory. At $g = +1$ $(g = -1)$, $\omega_2 = 0$ $(\omega_1 = 0)$, and the orbit is a circumference of radius R_1 (R_2) centered at (X_1, X_2) with $Z = X_1 + iX_2 = R_2 e^{-i\gamma_2}$ $(Z = R_1 e^{i\gamma_1})$. In general, the trajectories will be closed for arbitrary choice of the initial data (integration constants) iff the condition $\ell_1/\ell_2 = q_2/q_1$ with $q_1, q_2 \in \mathbb{Z}$ is fulfilled, that implies rational values for the parameter $g = (q_2 - q_1)/(q_1 + q_2)$. Some trajectories for rational values of g are shown in Figs. 1 and 2.

Rescaling the frequency parameter, $\omega \to \omega/|g|$, and taking limit $|g| \to \infty$, Hamiltonian (3.10) reduces to the Hamiltonian of the isotropic Minkowskian oscillator

$$H_\infty = \epsilon_\infty \omega p_\varphi = \epsilon_\infty \omega (b_1^+ b_1^- - b_2^+ b_2^-), \tag{3.13}$$

where $\epsilon_\infty = \pm 1$ for $g \to \pm\infty$. Solution to equations of motion for system (3.13) are obtained from (3.12) by the same procedure $\omega \to \omega/|g|$, $|g| \to \infty$,

$$z(t) = x(t) + iy(t) = R_1 e^{i(\omega t + \gamma_1)} + R_2 e^{i(\omega t - \gamma_2)}. \tag{3.14}$$

For the sake of definiteness we assume here $\epsilon_\infty = +1$. Equation (3.14) describes a circular trajectory centered at the origin, for which the squared radius is given by

$$x^2(t) + y^2(t) = R_1^2 + R_2^2 + 2R_1 R_2 \cos(\gamma_1 + \gamma_2), \tag{3.15}$$

and so, $(R_1 - R_2)^2 \leq x^2(t) + y^2(t) \leq (R_1 + R_2)^2$. From the viewpoint of dynamics, the case of the isotropic Minkowskian oscillator is similar, on the one hand, to the case of Euclidean isotropic oscillator $(g = 0)$, whose trajectories also are centered at the origin, but which are ellipses that reduce to circular trajectories only for particular choice of the initial data. On the other hand, the isotropic Minkowskian oscillator is similar to the case of Landau problem $(g^2 = 1)$, where trajectories are circular, but which are centered at the origin only for a particular choice of the initial data.

The explicitly depending on time complex phase space functions

$$\beta_j^\pm = b_j^\pm e^{\mp i\omega\ell_j t}, \quad j = 1, 2, \tag{3.16}$$

that correspond to the integration constants $b_j^\pm(0) = R_j e^{\pm i\gamma_j}$, are the *dynamical* integrals of motion, which generate the two-dimensional Heisenberg algebra. In the case $g = +1$ $(g = -1)$, one pair of them transforms into the *true*, not depending explicitly on time mutually conjugate complex integrals of motion of the Landau problem. Being multiplied by $\frac{1}{\sqrt{m\omega}}$, they correspond to the coordinates (X_1, X_2) of the center of the circular orbit having nonzero Poisson brackets $\{X_1, X_2\} = \frac{g}{2m\omega}$, $g = \pm 1$. Using these, dynamical in the case $g^2 \neq 1$ integrals (3.16), one can construct ten quadratic integrals

$$\mathcal{J}_\pm = e^{\mp 2i\omega t} b_1^\pm b_2^\pm, \quad \mathcal{J}_0 = \frac{1}{2}(b_1^+ b_1^- + b_2^+ b_2^-) = \frac{1}{2\omega} H_{\rm osc}, \tag{3.17}$$

$$\mathcal{L}_2 = \frac{1}{2}(b_1^+ b_1^- - b_2^+ b_2^-) = \frac{1}{2} p_\varphi, \qquad \mathcal{L}_\pm = e^{\mp 2i\omega g t} b_1^\pm b_2^\mp, \tag{3.18}$$

$$\mathcal{B}_1^\pm = e^{\mp 2i\omega\ell_1 t}(b_1^\pm)^2, \qquad \mathcal{B}_2^\pm = e^{\mp 2i\omega\ell_2 t}(b_2^\pm)^2. \tag{3.19}$$

In this set, only $\mathcal{J}_0$ and $\mathcal{L}_2$ are the true integrals for general case of g since $H_g = 2\omega(\mathcal{J}_0 + g\mathcal{L}_2)$ and $\{\mathcal{J}_0, \mathcal{L}_2\} = 0$.

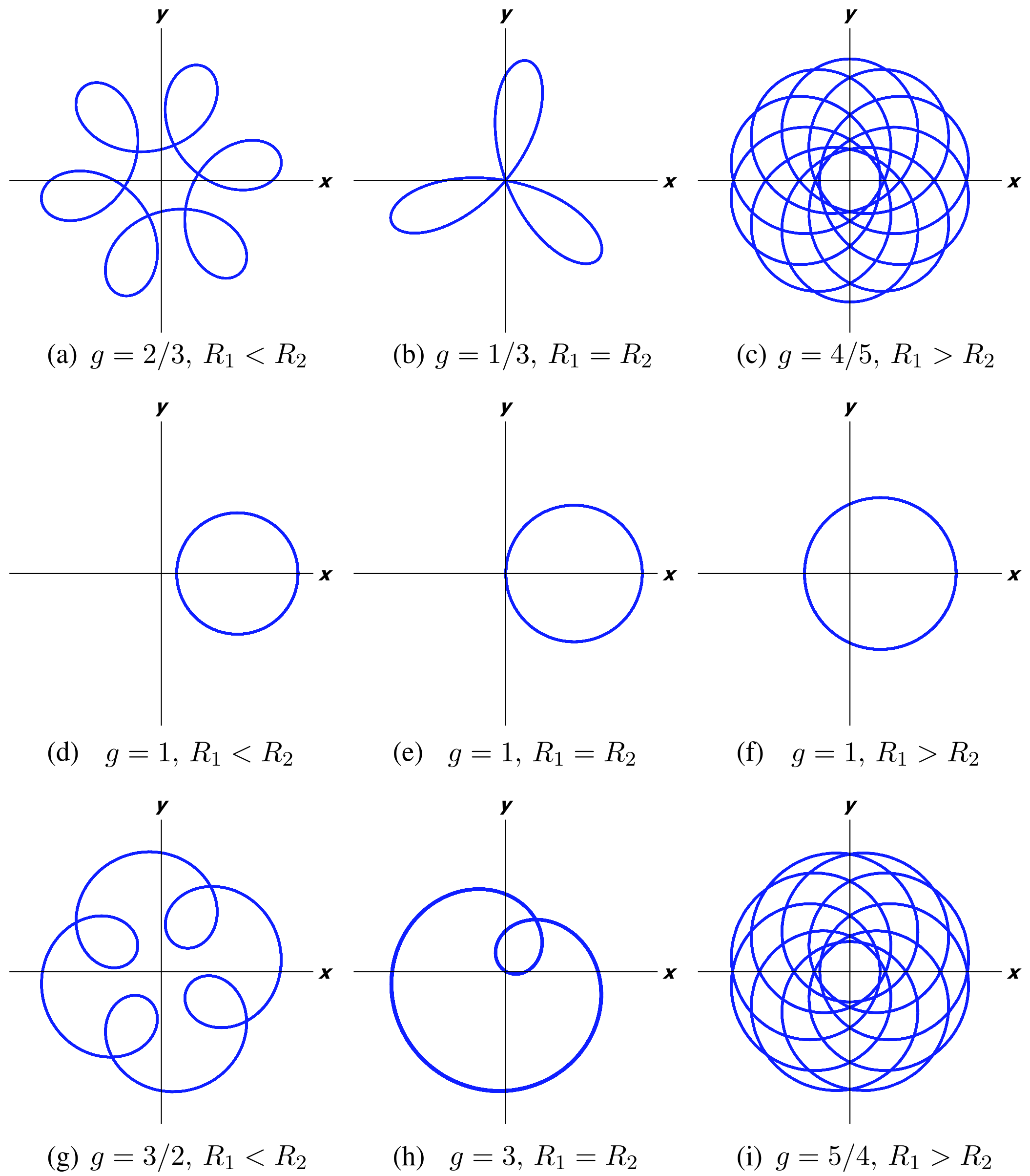


FIG. 1. Trajectories for some rational values of g. In cases (b), (e), and (h), $p_\varphi = 0$ and trajectories pass through the origin. In cases (a), (d) and (g), $p_\varphi < 0$, while cases (c), (f) and (i) correspond to $p_\varphi > 0$.

Here, the quantities (3.17) and (3.18) generate, respectively, the $\mathfrak{sl}(2,\mathbb{R})$ and the $\mathfrak{su}(2)$ algebras. All the quadratic integrals (3.17), (3.18), and (3.19) generate the $\mathfrak{sp}(4,\mathbb{R})$ algebra with the following nonzero Poisson brackets,

$$\{\mathcal{J}_0, \mathcal{J}_\pm\} = \mp\, i\mathcal{J}_\pm, \qquad \{\mathcal{J}_-, \mathcal{J}_+\} = -2i\mathcal{J}_0, \tag{3.20}$$

$$\{\mathcal{L}_2, \mathcal{L}_\pm\} = \mp\, i\mathcal{L}_\pm, \qquad \{\mathcal{L}_+, \mathcal{L}_-\} = -2i\mathcal{L}_2, \tag{3.21}$$

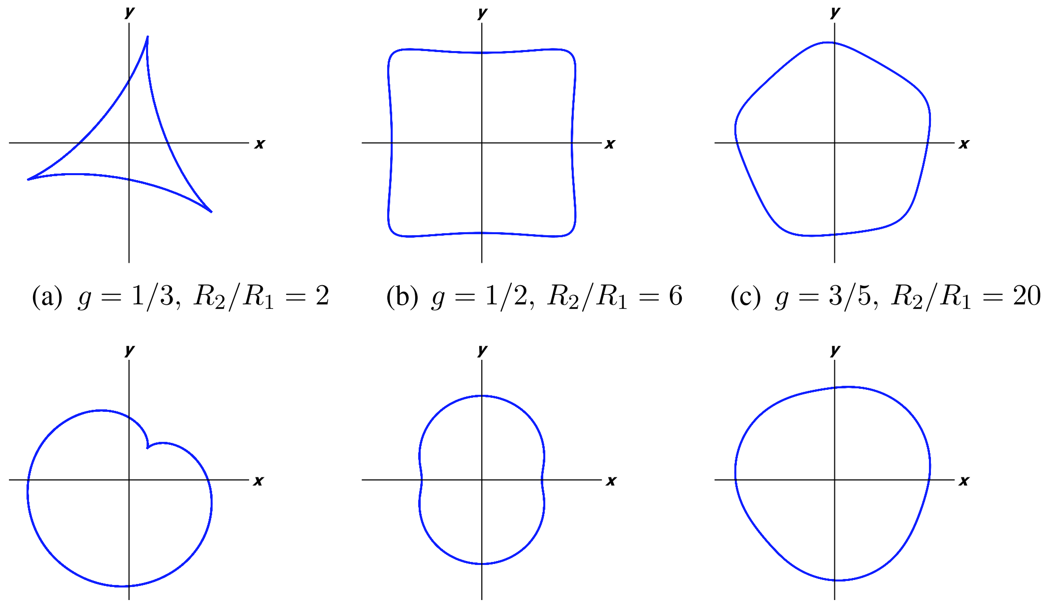
$$\{\mathcal{J}_\pm, \mathcal{L}_\mp\} = \pm i\mathcal{B}_2^\pm, \qquad \{\mathcal{J}_\pm, \mathcal{L}_\pm\} = \pm i\mathcal{B}_1^\pm, \tag{3.22}$$

$$\{\mathcal{J}_0, \mathcal{B}_j^\pm\} = \mp\, i\mathcal{B}_j^\pm, \qquad \{\mathcal{J}_\mp, \mathcal{B}_2^\pm\} = \mp\, 2i\mathcal{L}_\mp,$$
$$\{\mathcal{J}_\mp, \mathcal{B}_1^\pm\} = \mp\, 2i\mathcal{L}_\pm, \tag{3.23}$$

$$\{\mathcal{L}_2, \mathcal{B}_j^\pm\} = \pm(-1)^j i\mathcal{B}_j^\pm,$$
$$\{\mathcal{L}_\pm, \mathcal{B}_1^\mp\} = \pm 2i\mathcal{J}_\mp, \qquad \{\mathcal{L}_\pm, \mathcal{B}_2^\pm\} = \mp\, 2i\mathcal{J}_\pm, \tag{3.24}$$

$$\{\mathcal{B}_1^-, \mathcal{B}_1^+\} = -4i(\mathcal{J}_0 + \mathcal{L}_2),$$
$$\{\mathcal{B}_2^-, \mathcal{B}_2^+\} = -4i(\mathcal{J}_0 - \mathcal{L}_2). \tag{3.25}$$

By taking repeatedly the Poisson brackets of any of five sets of four integrals $(\mathcal{J}_\pm, \mathcal{L}_\pm)$, $(\mathcal{J}_\pm, \mathcal{B}_1^\pm)$, $(\mathcal{J}_\pm, \mathcal{B}_2^\pm)$, $(\mathcal{L}_\pm, \mathcal{B}_1^\pm)$ and $(\mathcal{L}_\pm, \mathcal{B}_2^\pm)$, all the $\mathfrak{sp}(4,\mathbb{R})$ algebra is produced. On the other hand, we notice that the algebra contains the $\mathfrak{su}(2) \oplus \mathfrak{u}(1) \cong \mathfrak{u}(2)$ subalgebra generated by the set of integrals



(a) $g = 1/3,\ R_2/R_1 = 2$ (b) $g = 1/2,\ R_2/R_1 = 6$ (c) $g = 3/5,\ R_2/R_1 = 20$

(d) $g = 3,\ R_2/R_1 = 2$ (e) $g = 2,\ R_2/R_1 = 6$ (f) $g = 5/3,\ R_2/R_1 = 20$

FIG. 2. Trajectories for some rational values of g and R_1/R_2. "Dual" figures (a) and (d), see below, correspond to a general case $R_1|\ell_1| = R_2|\ell_2|$ of the trajectories with cusps, in which velocity turns into zero. In dual cases (b)-(e) and (c)-(f) the indicated equality is violated, and corresponding trajectories are smooth.

$$(S1)\colon\ (\mathcal{L}_2, \mathcal{L}_\pm) \oplus \mathcal{J}_0, \qquad \mathcal{L}_2^2 + \mathcal{L}_+\mathcal{L}_- = \mathcal{J}_0^2, \tag{3.26}$$

where the second equality corresponds to relation between the Casimir element of the $\mathfrak{su}(2)$ subalgebra and central element $\mathcal{J}_0$ of the $\mathfrak{u}(2)$.

The $\mathfrak{sp}(4,\mathbb{R})$ algebra also contains three copies of the Lie algebra $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{u}(1) \cong \mathfrak{gl}(2,\mathbb{R})$ generated by any of the three sets of integrals

$$(S2)\colon\ (\mathcal{J}_0, \mathcal{J}_\pm) \oplus \mathcal{L}_2, \qquad -\mathcal{J}_0^2 + \mathcal{J}_+\mathcal{J}_- = -\mathcal{L}_2^2, \tag{3.27}$$

$$(S3)\colon\ \left(\frac{1}{2}(\mathcal{J}_0 - \mathcal{L}_2), \frac{1}{2}\mathcal{B}_2^\pm\right) \oplus (\mathcal{J}_0 + \mathcal{L}_2),$$
$$-\left(\frac{1}{2}(\mathcal{J}_0 - \mathcal{L}_2)\right)^2 + \frac{1}{2}\mathcal{B}_2^+ \frac{1}{2}\mathcal{B}_2^- = 0, \tag{3.28}$$

$$(S4)\colon\ \left(\frac{1}{2}(\mathcal{J}_0 + \mathcal{L}_2), \frac{1}{2}\mathcal{B}_1^\pm\right) \oplus (\mathcal{J}_0 - \mathcal{L}_2),$$
$$-\left(\frac{1}{2}(\mathcal{J}_0 + \mathcal{L}_2)\right)^2 + \frac{1}{2}\mathcal{B}_1^+ \frac{1}{2}\mathcal{B}_1^- = 0. \tag{3.29}$$

Here, analogously to (3.26), we indicated the values taken by the Casimir element of the $\mathfrak{sl}(2,\mathbb{R})$ subalgebra in each copy of $\mathfrak{gl}(2,\mathbb{R})$.

In the case $g = 0$, besides $\mathcal{J}_0$ and $\mathcal{L}_2$, the system has two additional true integrals of motion $\mathcal{L}_\pm$ not depending explicitly on time, and so, the $\mathfrak{u}(2)$ subalgebra is the true symmetry of the isotropic harmonic oscillator $H_{\rm osc}$. For it, the $\mathfrak{sl}(2,\mathbb{R})$ part of the (S2) subalgebra corresponds to the dynamical conformal symmetry.

For Landau problem with $g = 1$, the subalgebra (S3) corresponds to the true symmetry of the system, while its dynamical conformal symmetry is generated by the $\mathfrak{sl}(2,\mathbb{R})$ part of the (S4) subalgebra. In the case of Landau problem with $g = -1$, the true symmetry corresponds to subalgebra (S4), and its dynamical conformal symmetry is generated by the $\mathfrak{sl}(2,\mathbb{R})$ part of the subalgebra (S3).

In the case of the isotropic Minkowskian oscillator (obtained by $\omega \to \omega/|g|$, $|g| \to \infty$), the subalgebra (S2) with its $\mathfrak{sl}(2,\mathbb{R})$ part corresponds to the true symmetry, while the $\mathfrak{su}(2)$ part of the (S1) subalgebra is its dynamical conformal symmetry. We have here a kind of transmutation of the true symmetry into dynamical conformal symmetry and vise versa when we pass over from $g = 0$ to the $|g| = \infty$ case. We return to this point below in Sec. V B. Analogous phenomenon of transmutation of symmetries takes place for the Landau problem corresponding to the cases of $g = +1$ and $g = -1$. However, one notes that in the Landau problem in both cases the true and dynamical conformal symmetries correspond to different realizations of the same $\mathfrak{gl}(2,\mathbb{R})$ algebra.

In the case of the isotropic Euclidean oscillator ($g = 0$), the integrals $\mathcal{J}_0 = \frac{1}{2\omega}H_{\rm osc}$ and $\mathcal{L}_2 = \frac{1}{2}p_\varphi$ define the major and minor semiaxes of the elliptic orbit, while the pair of the integrals $\mathcal{L}_\pm$ defines its orientation in the plane via their phase $(\gamma_1 - \gamma_2)$, see Eq. (3.12). The modulus of $\mathcal{L}_\pm$ is fixed by the integrals $\mathcal{J}_0$ and $\mathcal{L}_2$ via the $\mathfrak{su}(2)$ Casimir value, see Eq. (3.26).

In the case of the isotropic Minkowskian oscillator, the radius of its circular orbit centered at the origin is defined by the three parameters R_1, R_2 and $(\gamma_1+\gamma_2)$, see Eq. (3.14), which are fixed, again, by the integrals $\mathcal{L}_2=\frac{1}{2\omega}H_\infty$, $\mathcal{J}_0$, and by the phase of the true integrals $\mathcal{J}_\pm$ of the system. The modulus of $\mathcal{J}_\pm$ is fixed, in turn, by the integrals $\mathcal{J}_0$ and $\mathcal{L}_2$ via the $\mathfrak{sl}(2,\mathbb{R})$ Casimir value, see Eq. (3.27).

In the Landau problem, the radius of the circular orbit is defined by the corresponding Hamiltonian $H_{g=1}=2\omega(\mathcal{J}_0+\mathcal{L}_2)$ ($H_{g=-1}=2\omega(\mathcal{J}_0-\mathcal{L}_2)$), while its center is given by the $\mathfrak{sl}(2,\mathbb{R})$ generators $\mathcal{B}_2^\pm$ ($\mathcal{B}_1^\pm$) (being the squares of the corresponding linear integrals $\beta_2^\pm$ ($\beta_1^\pm$)). Zero values of the $\mathfrak{sl}(2,\mathbb{R})$ classical Casimirs in Eqs. (3.28) and (3.29) reflect the fact that in the Landau problem dynamics is effectively governed by the corresponding one-dimensional harmonic oscillator Hamiltonians $H_{g=1}=\omega b_1^+b_1^-$ ($H_{g=-1}=b_2^+b_2^-$) [22].

The closed character of the trajectories for rational values of the parameter g different from the already discussed cases of $g=0,\pm1,\pm\infty$ indicates that some additional true integrals of motion also have to appear in the corresponding systems. Such integrals, however, are of higher order and, as we will see, produce nonlinear deformations of the $\mathfrak{u}(2)$ and $\mathfrak{gl}(2,\mathbb{R})$ symmetries in the cases of $g^2<1$ and $g^2>1$, respectively. They can be found by taking the products of dynamical integrals β_j^+ and β_j^- so that the time-dependent exponential factors in them will be canceled. For this, consider the dynamical integrals

$$\mathcal{L}^+_{j_1,j_2}=(\beta_1^+)^{j_1}(\beta_2^-)^{j_2}=e^{-i\omega(j_1\ell_1-j_2\ell_2)t}(b_1^+)^{j_1}(b_2^-)^{j_2}.\quad \mathcal{L}^-_{j_1,j_2}=(\mathcal{L}^+_{j_1,j_2})^*, \tag{3.30}$$

where, in principle, the indexes j_1, j_2 can take any nonnegative integer values. In order (3.30) would be true integrals for the system H_g, $\{H_g,\mathcal{L}^\pm_{j_1,j_2}\}=0$, there should exist the exponents $j_i=s_i$ that obey the relation

$$s_1\ell_1-s_2\ell_2=0. \tag{3.31}$$

The condition (3.31) is satisfied iff $g=(s_2-s_1)/(s_1+s_2)$. The positive integer numbers s_1 and s_2 can be chosen in such a way that the fraction is irreducible, that we will imply in what follows. So, two additional higher order true integrals $\mathcal{L}^\pm_{s_1,s_2}$ of the indicated form exist for rational values of g with $|g|<1$.

The time-independent integrals $\mathcal{L}^\pm_{s_1,s_2}$ are eigenstates of the true integrals $\mathcal{J}_0$, $\mathcal{L}_2$ in the sense of Poisson brackets, while dynamical integrals $\mathcal{J}_\pm$, change their indexes by transforming them into dynamical integrals:

$$\{\mathcal{L}_2,\mathcal{L}^\pm_{s_1,s_2}\}=\mp\frac{i}{2}(s_1+s_2)\mathcal{L}^\pm_{s_1,s_2},\quad \{\mathcal{J}_0,\mathcal{L}^\pm_{s_1,s_2}\}=\mp\frac{i}{2}(s_1-s_2)\mathcal{L}^\pm_{s_1,s_2}, \tag{3.32}$$

$$\{\mathcal{J}_\pm,\mathcal{L}^\pm_{s_1,s_2}\}=\pm is_2\mathcal{L}^\pm_{s_1+1,s_2-1},\quad \{\mathcal{J}_\mp,\mathcal{L}^\pm_{s_1,s_2}\}=\mp is_1\mathcal{L}^\pm_{s_1-1,s_2+1}. \tag{3.33}$$

Taking Poisson brackets of $\mathcal{J}_0$, $\mathcal{L}_2$, and $\mathcal{J}_\pm$ with the generated integrals $\mathcal{L}^\pm_{s_1+1,s_2-1}$ and $\mathcal{L}^\pm_{s_1-1,s_2+1}$, we continue the process, producing in this way the finite set of the integrals

$$\mathcal{L}^\pm_{s_1+s_2,0},\ldots,\mathcal{L}^\pm_{s_1,s_2},\ldots\mathcal{L}^\pm_{0,s_1+s_2}, \tag{3.34}$$

in which only $\mathcal{L}^\pm_{s_1,s_2}$ are the true, not depending explicitly on time, integrals, while the rest are dynamical, time-dependent higher order integrals. Using the Jacobi identity, one can show that the phase space functions $\{\mathcal{L}^-_{n_1,n_2},\mathcal{L}^+_{r_1,r_2}\}$ with $n_1+n_2=r_1+r_2=s_1+s_2$ Poisson commute with $\mathcal{L}_2$, and therefore, this bracket must be a function of the $\mathfrak{sl}(2,\mathbb{R})$ generators (3.17) and the angular momentum. Moreover, one has

$$\{\{\mathcal{L}^-_{s_1,s_2},\mathcal{L}^+_{s_1,s_2}\},\mathcal{J}_0\}=0\Rightarrow\{\mathcal{L}^-_{s_1,s_2},\mathcal{L}^+_{s_1,s_2}\}=F_g(H_g,\mathcal{L}_2), \tag{3.35}$$

where F_g is a polynomial function of H_g and $\mathcal{L}_2$. In conclusion, the set ($\mathcal{J}_0$, $\mathcal{J}_\pm$, $\mathcal{L}_2$, $\mathcal{L}^\pm_{k,s_1+s_2-k}$), where $k=0,\ldots,s_1+s_2$, produces a finite dimensional nonlinear algebra, in which the nonlinear subalgebra generated by true integrals H_g, $\mathcal{L}_2$, and $\mathcal{L}^\pm_{s_1,s_2}$ corresponds to a deformation of $\mathfrak{u}(2)$.

The already discussed in detail cases of the isotropic Euclidean oscillator and Landau problems with their $\mathfrak{sp}(4,\mathbb{R})$ Lie algebra can also be included in the described structure. The case $s_1=s_2=1$ reproduces here the case $g=0$, for which the integrals $\mathcal{L}_2$ and $\mathcal{L}^\pm_{1,1}=\mathcal{L}_\pm$, generate the $\mathfrak{su}(2)$ hidden symmetry of the system. On the other hand $s_1=0$ and $s_2=2$ ($s_1=2$ and $s_2=0$) yields $g=1$ ($g=-1$), for which $\mathcal{L}^\pm_{0,2}=2\mathcal{B}^\pm_2$ ($\mathcal{L}^\pm_{2,0}=2\mathcal{B}^\pm_1$). Taking $s_1=0$ and $s_2=1$ ($s_1=1$ and $s_2=0$), we reproduce the first order true integrals for the Landau problem with $g=1$ ($g=-1$), $\mathcal{L}^\pm_{0,1}=b_2^\mp$ ($\mathcal{L}^\pm_{1,0}=b_1^\mp$), that generate translations of particle's coordinates x_i.

If instead of (3.36) we consider the dynamical integrals

$$\mathcal{J}^+_{j_1,j_2}=(\beta_1^+)^{j_1}(\beta_2^+)^{j_2}=e^{-i\omega(j_1\ell_1+j_2\ell_2)t}(b_1^+)^{j_1}(b_2^+)^{j_2},\quad \mathcal{J}^-_{j_1,j_2}=(\mathcal{J}^+_{j_1,j_2})^*, \tag{3.36}$$

one notes that the $\mathcal{J}^\pm_{s_1,s_2}$ are the time-independent, true integrals of motion if an only if the condition

$$s_1\ell_1+s_2\ell_2=0, \tag{3.37}$$

is satisfied, i.e., when $g=(s_2+s_1)/(s_2-s_1)$, that implies $|g|>1$. In this case, we consider the Poisson bracket relations of $\mathcal{J}^\pm_{s_1,s_2}$ with the true integrals $\mathcal{J}_0$ and $\mathcal{L}_2$ and two other generators $\mathcal{L}_\pm$ of the dynamical $\mathfrak{su}(2)$ symmetry:

$$\{\mathcal{J}_0, \mathcal{J}^{\pm}_{s_1,s_2}\} = \mp \frac{i}{2}(s_1 + s_2)\mathcal{J}^{\pm}_{s_1,s_2},$$
$$\{\mathcal{L}_2, \mathcal{J}^{\pm}_{s_1,s_2}\} = \mp \frac{i}{2}(s_1 - s_2)\mathcal{J}^{\pm}_{s_1,s_2}, \tag{3.38}$$

$$\{\mathcal{L}_{\pm}, \mathcal{J}^{\pm}_{s_1,s_2}\} = \mp is_2\mathcal{J}^{+}_{s_1+1,s_2-1},$$
$$\{\mathcal{L}_{\mp}, \mathcal{J}^{\pm}_{s_1,s_2}\} = \pm is_1\mathcal{J}^{+}_{s_1-1,s_2+1}. \tag{3.39}$$

By the same reasoning as in the case of rational g with $|g| < 1$, we generate the set

$$\mathcal{J}^{\pm}_{s_1+s_2,0}, \ldots, \mathcal{J}^{\pm}_{s_1,s_2}, \ldots, \mathcal{J}^{\pm}_{0,s_1+s_2}, \tag{3.40}$$

in which only $\mathcal{J}^{\pm}_{s_1,s_2}$ are the true integrals, while the rest are the dynamical integrals of motion. They together with $\mathcal{J}_0$, $\mathcal{L}_2$, and $\mathcal{L}_{\pm}$ generate a finite nonlinear algebra, in which $\{\mathcal{J}^{-}_{j_1,j_2}, \mathcal{J}^{+}_{j_1,j_2}\}$ is a polynomial function of H_g and $\mathcal{L}_2$ only. Taking here $s_1 = 0$ and $s_2 = 2$ ($s_1 = 2$ and $s_2 = 0$), we reproduce the quadratic integrals $\mathcal{J}^{\pm}_{0,2} = 2\mathcal{B}^{\pm}_2$ ($\mathcal{J}^{\pm}_{2,0} = 2\mathcal{B}^{\pm}_1$) of the Landau problem with $g = 1$ ($g = -1$). The values $s_1 = 0$ and $s_2 = 1$ ($s_1 = 1$ and $s_2 = 0$) provide us with the corresponding linear integrals of the Landau problem with $g = 1$ ($g = -1$). On the other hand, setting formally $s_1 = s_2 = 1$, we reproduce the true integrals $\mathcal{J}^{\pm}_{1,1} = \mathcal{J}_{\pm}$ of the isotropic Minkowskian oscillator, which together with $\mathcal{J}_0$ and $H_\infty = 2\omega\mathcal{L}_2$ generate its Lie algebraic $\mathfrak{gl}(2,\mathbb{R})$ symmetry. In the case of finite rational values of g with $|g| > 1$, the subalgebra generated by $\mathcal{J}^{\pm}_{s_1,s_2}$, $\mathcal{J}_0$, and H_g is identified as a nonlinear deformation of the $\mathfrak{gl}(2,\mathbb{R})$.

We do not consider here a rather complicated complete nonlinear Poisson bracket algebraic structure generated by the true and dynamical integrals in the case of rational values of the parameter g different from the already discussed particular cases. We only note that the repeated Poisson brackets of higher order dynamical integral $\mathcal{L}^{+}_{s_1+s_2,0} = \mathcal{J}^{+}_{s_1+s_2,0}$ ($\mathcal{L}^{-}_{s_1+s_2,0} = \mathcal{J}^{-}_{s_1+s_2,0}$) of the case $g = (s_2 - s_1)/(s_2 + s_1) \coloneqq g^{s_1,s_2}_{<}$, $|g^{s_1,s_2}_{<}| < 1$, with quadratic dynamical integrals $\mathcal{L}_-$ ($\mathcal{L}_+$) generate all the set (3.40) of the integrals that we have had in the case of H_g with $g = (s_2 + s_1)/(s_2 - s_1) \coloneqq g^{s_1,s_2}_{>} = 1/g^{s_1,s_2}_{<}$, $|g^{s_1,s_2}_{>}| > 1$. All the integrals (3.40) are, however, dynamical for the system H_g with $g = g^{s_1,s_2}_{<}$. Analogously, the repeated Poisson brackets of higher order dynamical integral $\mathcal{J}^{+}_{s_1+s_2,0}$ ($\mathcal{J}^{-}_{s_1+s_2,0}$) of the case $g = g^{s_1,s_2}_{>}$ with quadratic dynamical integrals $\mathcal{L}_-$ ($\mathcal{L}_+$) generate all the set (3.34) of the integrals that we have had in the case of H_g with $g = g^{s_1,s_2}_{<}$. All integrals (3.34) are dynamical for the system H_g with $g = g^{s_1,s_2}_{>}$. From this point of view we also have a kind of transmutation of symmetries for the "dual" pairs of the systems with $g = g^{s_1,s_2}_{<}$ and $g = g^{s_1,s_2}_{>} = 1/g^{s_1,s_2}_{<}$, where the nonlinearly deformed $\mathfrak{u}(2)$ and $\mathfrak{gl}(2,\mathbb{R})$ subalgebras generated by the sets (H_g, $\mathcal{L}_2$, $\mathcal{L}^{\pm}_{s_1,s_2}$) and ($H_g$, $\mathcal{J}_0$, $\mathcal{J}^{\pm}_{s_1,s_2}$) change their role in the sense of the true and dynamical subsymmetries.

B. Quantum picture

Now, we return to the quantum system (3.2) to analyze, in the light of the CBT, its spectrum, integrals associated with degeneracy of the energy levels, and spectrum generating ladder operators.

To find normalizable eigenstates and the spectrum of the system (3.2) by means of the CBT, we exploit its rotational invariance reflected, particularly, in the form of classical solutions (3.12), and pass over from Cartesian coordinates x_i to the complex variable $z = x_1 + ix_2$, $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2})$. In terms of z and z^*, the equation for formal eigenfunctions $\phi_\lambda(x_1, x_2) = \langle x_1, x_2|\lambda\rangle$ of the non-Hermitian operator $2i\hat{D} + g\hat{p}_\varphi$ takes the form

$$(2i\hat{D} + g\hat{p}_\varphi)\phi_\lambda = \hbar\left((1+g)z\frac{\partial}{\partial z} + (1-g)z^*\frac{\partial}{\partial z^*}\right)\phi_\lambda = \lambda\phi_\lambda. \tag{3.41}$$

The well defined in $\mathbb{R}^2$ simultaneous eigenfunctions of the mutually commuting operators $2i\hat{D}$ and $\hat{p}_\varphi$ are

$$\phi_{n_1,n_2} = z^{n_1}(z^*)^{n_2}, \tag{3.42}$$

where n_1 and n_2 are non-negative integers. They satisfy relations

$$\hat{p}_-\phi_{n_1,n_2} = -2i\hbar n_1\phi_{n_1-1,n_2}, \quad \hat{p}_+\phi_{n_1,n_2} = -2i\hbar n_2\phi_{n_1,n_2-1}, \tag{3.43}$$

$$\hat{\xi}_+\phi_{n_1,n_2} = m\phi_{n_1+1,n_2}, \qquad \hat{\xi}_-\phi_{n_1,n_2} = m\phi_{n_1,n_2+1}, \tag{3.44}$$

where $\hat{p}_\pm = \hat{p}_1 \pm i\hat{p}_2$ and $\hat{\xi}_\pm = \hat{\xi}_1 \pm i\hat{\xi}_2$. Therefore, free particle quadratic operators $\hat{H}$, $\hat{D}$, $\hat{K}$ and $\hat{p}_\varphi$ act on states (3.42) as follows:

$$\hat{H}\phi_{n_1,n_2} = -\frac{2\hbar}{m}n_1n_2\phi_{n_1-1,n_2-1}, \quad \hat{K}\phi_{n_1,n_2} = \frac{m}{2}\phi_{n_1+1,n_2+1}, \tag{3.45}$$

$$2i\hat{D}\phi_{n_1,n_2} = \hbar(n_1 + n_2 + 1)\phi_{n_1,n_2},$$
$$\hat{p}_\varphi\phi_{n_1,n_2} = \hbar(n_1 - n_2)\phi_{n_1,n_2}. \tag{3.46}$$

From the first equation in (3.45), we see that the action of the free particle Hamiltonian on ϕ_{n_1,n_2} decreases both indexes n_1 and n_2, and annihilates $\phi_{n_1,0}$ and ϕ_{0,n_2}. It is clear then that functions (3.42) are the zero energy Jordan states of $\hat{H}$. Using Eq. (3.46), we find the normalized wave functions $\Psi_{n_1,n_2} = \mathcal{N}_{n_1,n_2}\hat{\mathfrak{S}}\phi_{n_1,n_2}$ (where $\mathcal{N}_{n_1,n_2}$ is a numerical factor, see below) of $\hat{H}_g$,

$$\hat{H}_g\Psi_{n_1,n_2} = E_{n_1,n_2}\Psi_{n_1,n_2}, \quad E_{n_1,n_2} = \hbar\omega(\ell_1 n_1 + \ell_2 n_2 + 1), \tag{3.47}$$

which simultaneously are eigenfunctions of the angular momentum operator, $\hat{p}_\varphi\Psi_{n_1,n_2} = \hbar(n_1 - n_2)\Psi_{n_1,n_2}$. Note that the spectrum of the system has degeneracies if an only if g is a rational number. All energy levels are positive if $|g| \leq 1$, while for $|g| > 1$ the energy levels can take negative values and the spectrum is not bounded from below. Moreover, for rational g with $|g| < 1$ each energy level is finitely degenerate and the ground state with $n_1 = n_2 = 0$ is nondegenerate, while for rational g with $|g| \geq 1$ each energy level has infinite degeneracy. In the limit case of the Landau phase $g = +1$ $(g = -1)$, one has $\ell_2 = 0$ $(\ell_1 = 0)$, and all the energy levels, including the lowest Landau level, become infinitely degenerate.

The explicit action of the CBT operator $\hat{\mathfrak{S}}$ on functions ϕ_{n_1,n_2} is computed by employing the inverse Weierstrass transformation [22,43]

$$e^{-\frac{1}{4}\frac{d^2}{d\eta^2}}\eta^n = 2^{-n}H_n(\eta), \tag{3.48}$$

where $H_n(\eta)$ are the Hermite polynomials. This gives us the normalized eigenfunctions

$$\Psi_{n_1,n_2} = \sqrt{\frac{m\omega}{\hbar\pi n_1!n_2!}}H_{n_1,n_2}\left(\sqrt{\frac{m\omega}{\hbar}}x_1, \sqrt{\frac{m\omega}{\hbar}}x_2\right)e^{-\frac{m\omega}{2\hbar}(x_1^2+x_2^2)}, \tag{3.49}$$

$$\mathcal{N}_{n_1,n_2} = \left(\frac{2\hbar}{m\omega}\right)^{\frac{n_1+n_2}{2}}\sqrt{n_1!n_2!\pi}. \tag{3.50}$$

Here, the functions

$$H_{n_1,n_2}(\eta_1,\eta_2) = 2^{n_1+n_2}\sum_{k=0}^{n_1}\sum_{l=0}^{n_2}(i)^{n_1-n_2+l-k}H_{l+k}(\eta_1) \times H_{n_1+n_2-l-k}(\eta_2), \tag{3.51}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H_{n_1,n_2}(\eta_1,\eta_2)H_{l_1,l_2}(\eta_1,\eta_2)e^{-(\eta_1^2+\eta_2^2)}d\eta_1 d\eta_2 = \pi n_1!n_2!\delta_{n_1,l_1}\delta_{n_2,l_2}, \tag{3.52}$$

correspond to the generalized Hermite polynomials of two indexes [44].

Using the CBT relations

$$\hat{\mathfrak{S}}(\hat{p}_-)\hat{\mathfrak{S}}^{-1} = -i\sqrt{2m\hbar\omega}\hat{b}_1^-, \quad \hat{\mathfrak{S}}(\hat{p}_+)\hat{\mathfrak{S}}^{-1} = -i\sqrt{2m\hbar\omega}\hat{b}_2^-, \tag{3.53}$$

$$\hat{\mathfrak{S}}(\hat{\xi}_+)\hat{\mathfrak{S}}^{-1} = \sqrt{\frac{2m\hbar}{\omega}}\hat{b}_1^+, \qquad \hat{\mathfrak{S}}(\hat{\xi}_-)\hat{\mathfrak{S}}^{-1} = \sqrt{\frac{2m\hbar}{\omega}}\hat{b}_2^+, \tag{3.54}$$

one finds the action of operators $\hat{b}_i^\pm$ on eigenstates (3.49),

$$\hat{b}_1^-\Psi_{n_1,n_2} = \sqrt{n_1}\Psi_{n_1-1,n_2}, \qquad \hat{b}_2^-\Psi_{n_1,n_2} = \sqrt{n_2}\Psi_{n_1,n_2-1} \tag{3.55}$$

$$\hat{b}_1^+\Psi_{n_1,n_2} = \sqrt{n_1+1}\Psi_{n_1+1,n_2}, \quad \hat{b}_2^+\Psi_{n_1,n_2} = \sqrt{n_2+1}\Psi_{n_1,n_2+1}, \tag{3.56}$$

wherefrom it is clear that they are spectrum generating operators.

Let us study the spectral characteristics for the cases in which g is a rational number. We do not consider the issue of the quantum algebra here which inherits the properties of the corresponding classical algebra, but only identify the integrals associated with the spectral degenerations of the system.

Case $|g| < 1$. Let us assume that $g = g_<^{s_1,s_2} = \frac{s_2-s_1}{s_2+s_1}$ is an irreducible fraction with some fixed non-negative integer values of s_1 and s_2. In this case the condition (3.31) is fulfilled, implying that $E_{n_1,n_2} = E_{n_1+js_1,n_2-js_2}$, where j is an integer number such that $n_1 + js_1 \geq 0$, $n_2 - js_2 \geq 0$. One can construct the quantum operators

$$\hat{\mathcal{L}}^+_{s_1,s_2} = (\hat{b}_1^+)^{s_1}(\hat{b}_2^-)^{s_2}, \qquad \hat{\mathcal{L}}^-_{s_1,s_2} = (\hat{\mathcal{L}}^-_{s_1,s_2})^\dagger, \tag{3.57}$$

which are the direct quantum analogs of (3.30) with $j_1 = s_1$, $j_2 = s_2$, $[\hat{H}_g, \hat{\mathcal{L}}^\pm_{s_1,s_2}] = 0$. These quantum integrals can be obtained by application of the conformal bridge transformation to the free particle higher order operators

$$\hat{S}^+_{s_1,s_2} = (\hat{\xi}_+)^{s_1}(\hat{p}_+)^{s_2}, \qquad \hat{S}^-_{s_1,s_2} = (\hat{p}_-)^{s_1}(\hat{\xi}_-)^{s_2}, \tag{3.58}$$

which commute with the operator $2i\hat{D} + g\hat{p}_\varphi$ with $g = g_<^{s_1,s_2}$.

Using relations (3.55) and (3.56) we get

$$\hat{\mathcal{L}}^+_{s_1,s_2}\Psi_{n_1,n_2} = \sqrt{\frac{n_2!\Gamma(n_1+s_1+1)}{n_1!\Gamma(n_2-s_2+1)}}\Psi_{n_1+s_1,n_2-s_2}, \tag{3.59}$$

$$\hat{\mathcal{L}}^-_{s_1,s_2}\Psi_{n_1,n_2} = \sqrt{\frac{n_1!\Gamma(n_2+s_2+1)}{n_2!\Gamma(n_1-s_1+1)}}\Psi_{n_1-s_1,n_2+s_2}. \tag{3.60}$$

These equalities imply that the operators $\hat{\mathcal{L}}^\pm_{s_1,s_2}$ allow us to obtain the complete set of physical eigenstates which have the same energy but different angular momentum eigenvalues starting from some fixed eigenstate Ψ_{n_1,n_2}, i.e., they correspond to integrals of motion associated with hidden symmetries of the system which are responsible for degeneracy of the spectrum. In the cases $g = 0$ and $g = \pm 1$ we recover the symmetry operators of the isotropic harmonic oscillator and the Landau system in the symmetric gauge, respectively.

Case $|g| > 1$. We suppose now that $g = g_{>}^{s_1,s_2} = \frac{s_2+s_1}{s_2-s_1}$ is an irreducible fraction with $s_1 \neq s_2$. In this case the condition $s_1\ell_1 + s_2\ell_2 = 0$ is fulfilled, implying that $E_{n_1,n_2} = E_{n_1+js_1,n_2+js_2}$, where now j is an integer number such that $n_1 + js_1 \geq 0$, $n_2 + js_2 \geq 0$. We can construct here the operators

$$\hat{\mathcal{J}}^{+}_{s_1,s_2} = (\hat{b}_1^{+})^{s_1}(\hat{b}_2^{+})^{s_2}, \qquad \hat{\mathcal{J}}^{-}_{s_1,s_2} = (\hat{\mathcal{J}}^{-}_{s_1,s_2})^{\dagger}, \quad (3.61)$$

which are the quantum analogs of the integrals (3.36) with $j_1 = s_1$ and $j_2 = s_2$, $[\hat{H}_g, \hat{\mathcal{J}}^{\pm}_{s_1,s_2}] = 0$.

These integrals are obtained by the application of the CBT to the free particle higher order operators

$$\hat{\Xi}^{+}_{s_1,s_2} = (\hat{\xi}_{+})^{s_1}(\hat{\xi}_{-})^{s_2}, \qquad \hat{\Xi}^{-}_{s_1,s_2} = (\hat{p}_{+})^{s_1}(\hat{p}_{-})^{s_2}, \quad (3.62)$$

which commute with the operator $2i\hat{D} + g\hat{p}_{\varphi}$ with $g = g_{>}^{s_1,s_2}$. Their action on the eigenstates is given by

$$\hat{\mathcal{J}}^{-}_{s_1,s_2}\Psi_{n_1,n_2} = \sqrt{\frac{n_1!n_2!}{\Gamma(n_1-s_1+1)\Gamma(n_2-s_2+1)}}\Psi_{n_1-s_1,n_2-s_2}, \quad (3.63)$$

$$\hat{\mathcal{J}}^{+}_{s_1,s_2}\Psi_{n_1,n_2} = \sqrt{\frac{\Gamma(n_1+s_1+1)\Gamma(n_2+s_2+1)}{n_1!n_2!}}\Psi_{n_1+s_1,n_2+s_2}. \quad (3.64)$$

All the normalizable eigenfunctions with the same energy can be obtained by repeated application of these operators to some fixed state Ψ_{n_1,n_2}. It is worth to note here the difference in the action of the integrals $\hat{\mathcal{L}}^{\pm}_{s_1,s_2}$ in the case $g = \frac{s_2-s_1}{s_2+s_1}$, and integrals $\hat{\mathcal{J}}^{\pm}_{s_1,s_2}$ for $g = \frac{s_2+s_1}{s_2-s_1}$. In the first case, after repeated application of the corresponding integral operator with index plus or minus, at some step we obtain zero due to appearance of a Gamma function pole in denominator of coefficients in (3.59) and (3.60). A similar situation we have only for the action of the operator $\hat{\mathcal{J}}^{-}_{s_1,s_2}$ in the case of $g = \frac{s_2+s_1}{s_2-s_1}$, but the repeated application of the operator $\hat{\mathcal{J}}^{+}_{s_1,s_2}$ will never produce zero, see the coefficient in Eq. (3.64). This difference is just another reflection of the finite and infinite degeneracy of energy levels in the two indicated cases.

In conclusion of this section, we show how the CBT can be used to construct coherent states for the system $\hat{H}_g$. For this we consider the exponential function $e^{\frac{1}{\sqrt{2}}(\alpha z+\beta z^*)}$, which, in dependence on the values of the parameters $\alpha, \beta \in \mathbb{C}$ is either the plane wave eigenfunction, or formal, nonphysical eigenfunction of the free particle Hamiltonian operator $\hat{H}$. The application of the CBT operator yields

$$\begin{aligned}\mathfrak{S}e^{\frac{1}{\sqrt{2}}(\alpha z+\beta z^*)} &= \sqrt{2}e^{-\frac{m\omega}{2}zz^*+\alpha z+\beta z^*-\frac{\hbar}{m\omega}\alpha\beta} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\left(\frac{\hbar}{m\omega}\right)^{\frac{n}{2}}\frac{\alpha^{n-k}\beta^{k}}{\sqrt{k!(n-k)!}}\alpha^{n-k}\beta^{k}\Psi_{n-k,k} \\ &= \mathcal{C}_{\alpha,\beta}\Phi(x_1,x_2,\alpha,\beta), \end{aligned} \quad (3.65)$$

where $\mathcal{C}_{\alpha,\beta} = \sqrt{\pi}e^{-\frac{\hbar}{2m\omega}(|\alpha|^2+|\beta|^2)}$, and $\Phi(x_1,x_2,\alpha,\beta)$ is a normalized function. The expansion of $\Phi(x_1,x_2,\alpha,\beta)$ over the orthonormal eigenstates of the system allows us to see how these states transform under time translations and rotations,

$$e^{-\frac{it}{\hbar}\hat{H}_g}\Phi(x,y,\alpha,\beta) = \Phi(x,y,\alpha e^{-i\omega\ell_1 t},\beta e^{-i\omega\ell_2 t}), \quad (3.66)$$

$$e^{\frac{i\gamma}{\hbar}\hat{p}_{\varphi}}\Phi(x,y,\alpha,\beta) = \Phi(x,y,\alpha e^{i\gamma},\beta e^{-i\gamma}). \quad (3.67)$$

On the other hand, the introduced exponential is an eigenstate of the operators $\hat{p}_{\pm}$, and then one has

$$\begin{aligned}\hat{b}_1^{-}\Phi(x,y,\alpha,\beta) &= \sqrt{\frac{\hbar}{m\omega}}\alpha\Phi(x,y,\alpha,\beta), \\ \hat{b}_2^{-}\Phi(x,y,\alpha,\beta) &= \sqrt{\frac{\hbar}{m\omega}}\beta\Phi(x,y,\alpha,\beta). \end{aligned} \quad (3.68)$$

As these functions are the eigenstates of the lowering operators $\hat{b}_i^{-}$, and they maintain their form without dispersion while time evolves, we conclude that $\Phi(x,y,\alpha,\beta)$ are coherent states for the system $\hat{H}_g$.

IV. GENERATION OF AHO BY ANISOTROPIC CBT

In the previous section we have discussed the ERIHO generated from the two-dimensional free particle by the rotationally invariant CBT. In this section we explore the possibility of connecting the free particle and the AHO which does not have rotational invariance. For this, we employ an anisotropic CBT.

Consider the following generator of the CBT (no summation over the repeated index)

$$\hat{\mathfrak{S}}_{\omega_1,\omega_2} = \hat{\mathfrak{S}}_{\omega_1}\hat{\mathfrak{S}}_{\omega_2}, \qquad \hat{\mathfrak{S}}_{\omega_i} = e^{-\frac{\omega_i}{\hbar}\hat{K}_i}e^{\frac{\hat{H}_i}{2\hbar\omega_i}}e^{\frac{i}{\hbar}\ln(2)\hat{D}_i}. \quad (4.1)$$

In the case $\omega_1 \neq \omega_2$, this generator is anisotropic (rotationally noninvariant) operator. Instead of the linear combination of the free particle symmetry generators $\omega(2i\hat{D} + g\hat{p}_{\varphi})$, we apply the anisotropic CBT to the two operators

$$2i\hat{\mathcal{D}}_{\pm} \coloneqq 2i(\omega_1\hat{D}_1 \pm \omega_2\hat{D}_2). \quad (4.2)$$

By using the decomposition (2.12) and the second relation in (2.5), we obtain[4]

$$\hat{\mathfrak{S}}_{\omega_1,\omega_2}(2i\hat{\mathcal{D}}_\pm)\hat{\mathfrak{S}}^{-1}_{\omega_1,\omega_2} = \sum_{i=1}^{2}(\pm 1)^{i-1}\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_i^2}+\frac{m\omega_i^2}{2}x_i^2\right) \coloneqq \hat{H}^{(\pm)}_{\omega_1,\omega_2}. \tag{4.3}$$

When we choose the positive sign, the operator $\hat{H}^{(+)}_{\omega_1,\omega_2}$ corresponds to the Hamiltonian operator of the usual, Euclidean AHO system. When the negative sign is selected, $\hat{H}^{(-)}_{\omega_1,\omega_2}$ corresponds to the Minkowskian AHO.

Acting on the momenta operators and the Galilean boost generators of the free particle, the anisotropic CBT produces

$$\hat{\mathfrak{S}}_{\omega_1,\omega_2}(\hat{p}_i)\hat{\mathfrak{S}}^{-1}_{\omega_1,\omega_2} = -i\sqrt{m\hbar\omega_i}\hat{a}^-_{\omega_i},\qquad \hat{\mathfrak{S}}_{\omega_1,\omega_2}(\hat{\xi}_i)\hat{\mathfrak{S}}^{-1}_{\omega_1,\omega_2} = \sqrt{\frac{m\hbar}{\omega_i}}\hat{a}^+_{\omega_i}, \tag{4.4}$$

where

$$\hat{a}^\pm_{\omega_i} = \sqrt{\frac{m\omega_i}{2\hbar}}\left(x_i \mp \frac{\hbar}{m\omega_i}\frac{\partial}{\partial x_i}\right) \tag{4.5}$$

are the first-order ladder operators for each direction. From here, we find that the angular momentum is not invariant under the anisotropic CBT,

$$\hat{\mathfrak{S}}_{\omega_1,\omega_2}(\hat{p}_\varphi)\hat{\mathfrak{S}}^{-1}_{\omega_1,\omega_2} = -i\hbar\left(\sqrt{\frac{\omega_2}{\omega_1}}\hat{a}^+_{\omega_1}\hat{a}^-_{\omega_2} - \sqrt{\frac{\omega_1}{\omega_2}}\hat{a}^+_{\omega_2}\hat{a}^-_{\omega_1}\right). \tag{4.6}$$

Instead of the angular momentum operator in the case of $\hat{H}_g$, the AHO systems

$$\hat{H}^{(\pm)}_{\omega_1,\omega_2} = \hbar\left(\omega_1\hat{a}^+_{\omega_1}\hat{a}^-_{\omega_1} \pm \omega_2\hat{a}^+_{\omega_2}\hat{a}^-_{\omega_2} + \frac{1}{2}(\omega_1\pm\omega_2)\right) \tag{4.7}$$

are characterized here by the obvious symmetry generators

$$\hat{L}^{(\pm)}_{\omega_1,\omega_2} = \hbar(\omega_1\hat{a}^+_{\omega_1}\hat{a}^-_{\omega_1} \mp \omega_2\hat{a}^+_{\omega_2}\hat{a}^-_{\omega_2}). \tag{4.8}$$

Therefore, in the case $\omega_1=\omega_2=\omega$ $(\hat{a}^\pm_{\omega_i}\to\hat{a}^\pm_i)$, the Hamiltonian of the isotropic, in the sense of $(1+1)$-dimensional Lorentzian metric, Minkowskian oscillator

$$\hat{H}^{(-)}_{\omega,\omega} = \hbar\omega(\hat{a}^+_1\hat{a}^-_1 - \hat{a}^+_2\hat{a}^-_2) \tag{4.9}$$

is invariant under the $\mathfrak{so}(1,1)$ transformations generated by the operator

$$\hat{L}_{1,1} = \hat{x}_1\hat{p}_2 + \hat{x}_2\hat{p}_1 = i\hbar(\hat{\mathcal{J}}_+ - \hat{\mathcal{J}}_-),\qquad \hat{\mathcal{J}}_\pm = \hat{a}^\pm_1\hat{a}^\pm_2. \tag{4.10}$$

The true integrals $\hat{\mathcal{J}}_\pm$, together with[5] $\hat{\mathcal{J}}_0 = \frac{1}{2\omega\hbar}L^{(-)}_{\omega,\omega} = \frac{1}{2\omega\hbar}(\hat{H}_{\rm osc}-\hbar\omega)$ produce the $\mathfrak{sl}(2,\mathbb{R})$ symmetry algebra of the system (4.9). As $\hat{\mathcal{J}}_0$ and the combination $\hbar(\hat{\mathcal{J}}_- + \hat{\mathcal{J}}_+) = \frac{\hat{p}_1\hat{p}_2}{2m} - m\omega\hat{x}_1\hat{x}_2$ are the second order in momenta integrals of motion, they correspond to the hidden symmetry operators [30].

The obvious choice for the well defined in $\mathbb{R}^2$ eigenfunctions that obey the eigenvalue equation $2i\hat{\mathcal{D}}_\pm\phi_\lambda = \lambda\phi_\lambda$ corresponds to

$$\phi_{n_1,n_2} = x_1^{n_1}x_2^{n_2} \tag{4.11}$$

with non-negative integer values of n_1 and n_2. These eigenfunctions satisfy the relations

$$\hat{p}_1\phi_{n_1,n_2} = -i\hbar n_1\phi_{n_1-1,n_2},\qquad \hat{p}_2\phi_{n_1,n_2} = -i\hbar n_2\phi_{n_1,n_2-1}, \tag{4.12}$$

$$\hat{\xi}_1\phi_{n_1,n_2} = m\phi_{n_1+1,n_2},\qquad \hat{\xi}_2\phi_{n_1,n_2} = m\phi_{n_1,n_2+1}. \tag{4.13}$$

They are the zero energy Jordan states (generally, of different orders in the case of $n_1\neq n_2$) of the one-dimensional Hamiltonian operators $\hat{H}_i$, and the action of $\hat{\mathfrak{S}}_{\omega_1,\omega_2}$ on them yields

$$\hat{\mathfrak{S}}_{\omega_1,\omega_2}\phi_{n_1,n_2}(x_1,x_2) = \left(\frac{\omega_1}{\omega_2}\right)^{-\frac{1}{4}}\left(\frac{\hbar}{m}\right)^{\frac{n_1+n_2+1}{2}}\sqrt{2n_1!n_2!\pi}\,\psi_{n_1,n_2}(x_1,x_2), \tag{4.14}$$

$$\psi_{n_1,n_2}(x_1,x_2) = \psi_{n_1}(x_1)\psi_{n_2}(x_2), \tag{4.15}$$

$$\psi_{n_i}(x_i) = \frac{1}{\sqrt{2^{n_i}n_i!}}\left(\frac{m\omega_i}{\pi\hbar}\right)^{\frac{1}{4}}H_{n_i}\left(\sqrt{\frac{m\omega_i}{\hbar}}x_i\right)e^{-\frac{m\omega_i}{2\hbar}x_i^2}, \tag{4.16}$$

where H_{n_i} are the Hermite polynomials. Meanwhile, the formal eigenvalue equation

[4] As before, one could introduce the notation $\ell_1 = 1+g$, $\ell_2 = 1-g$, $g\in\mathbb{R}$, $\omega_i = \ell_i\omega$, and instead of (4.2), apply the CBT (4.1) to the operator $2i\omega\hat{\mathcal{D}}_g \coloneqq 2i\omega(\ell_1 D_1 + \ell_2 D_2)$. This, however, will not change the final results, see below.

[5] In the context of the symmetry transmutation, the planar isotropic harmonic oscillator Hamiltonian appears here as the integral of the system we are dealing with. It is invariant under $\mathfrak{su}(2)$ transformations generated by (3.18) taken with $g=0$, which for the system (4.9) correspond to the explicitly depending on time, dynamical integrals.

$$2i\hat{\mathcal{D}}_{\pm}\phi_{n_1,n_2} = \hbar\left(\omega_1 n_1 \pm \omega_2 n_2 + \frac{\omega_1 \pm \omega_2}{2}\right)\phi_{n_1,n_2} \quad (4.17)$$

implies that

$$\hat{H}^{(\pm)}_{\omega_1,\omega_2}\psi_{n_1,n_2} = \hbar\left(\omega_1 n_1 \pm \omega_2 n_2 + \frac{\omega_1 \pm \omega_2}{2}\right)\psi_{n_1,n_2}. \quad (4.18)$$

From here it follows that the energy values of the Euclidean AHO $\hat{H}^{(+)}_{\omega_1,\omega_2}$ are positive, while in the case of Minkowskian AHO described by $\hat{H}^{(-)}_{\omega_1,\omega_2}$ the spectrum is not bounded from below.

From relations (4.4), (4.12), and (4.13), one concludes that (4.5) are the spectrum generating operators that satisfy relations

$$\hat{a}^{-}_{\omega_1}\psi_{n_1,n_2} = \sqrt{n_1}\psi_{n_1-1,n_2}, \qquad \hat{a}^{-}_{\omega_2}\psi_{n_1,n_2} = \sqrt{n_2}\psi_{n_1,n_2-1}, \quad (4.19)$$

$$\hat{a}^{+}_{\omega_1}\psi_{n_1,n_2} = \sqrt{n_1+1}\psi_{n_1+1,n_2}, \\ \hat{a}^{+}_{\omega_2}\psi_{n_1,n_2} = \sqrt{n_2+1}\psi_{n_1,n_2+1}. \quad (4.20)$$

On the other hand, it is well known that besides the integrals (4.8), both systems $\hat{H}^{(\pm)}_{\omega_1,\omega_2}$ have additional, higher order true integrals of motion when frequencies are commensurable, $\omega_1/\omega_2 = l_2/l_1$ [31]. In the case of $\hat{H}^{(+)}_{\omega_1,\omega_2}$, these integrals can be obtained by applying the anisotropic CBT to the higher order operators

$$\hat{\mathcal{S}}_{l_1,l_2} = (\hat{\xi}_1)^{l_1}(\hat{p}_2)^{l_2}, \qquad \hat{\mathcal{S}}_{l_2,l_1} = (\hat{p}_1)^{l_1}(\hat{\xi}_2)^{l_2}, \quad (4.21)$$

which commute with $\hat{\mathcal{D}}_+$ in this case. One has

$$\hat{\mathfrak{S}}_{\omega_1,\omega_2}(\hat{\mathcal{S}}_{l_1,l_2})\hat{\mathfrak{S}}^{-1}_{\omega_1,\omega_2} \propto \hat{\mathscr{L}}^{+}_{l_1,l_2}. \\ \hat{\mathfrak{S}}_{\omega_1,\omega_2}(\hat{\mathcal{S}}_{l_2,l_1})\hat{\mathfrak{S}}^{-1}_{\omega_1,\omega_2} \propto \hat{\mathscr{L}}^{-}_{l_1,l_2}, \quad (4.22)$$

$$\hat{\mathscr{L}}^{+}_{l_1,l_2} = (\hat{a}^{+}_{\omega_1})^{l_1}(\hat{a}^{-}_{\omega_2})^{l_2}, \qquad \hat{\mathscr{L}}^{-}_{l_1,l_2} = (\hat{\mathscr{L}}^{+}_{l_1,l_2})^{\dagger} = \hat{\mathscr{L}}^{+}_{l_2,l_1}. \quad (4.23)$$

The explicit action of these hidden symmetry operators is given by

$$\hat{\mathscr{L}}^{-}_{l_1,l_2}\psi_{n_1,n_2} = \sqrt{\frac{n_1!\Gamma(n_2+l_2+1)}{n_2!\Gamma(n_1-l_1+1)}}\psi_{n_1-l_1,n_2+l_2}, \quad (4.24)$$

$$\hat{\mathscr{L}}^{+}_{l_1,l_2}\psi_{n_1,n_2} = \sqrt{\frac{n_2!\Gamma(n_1+l_1+1)}{n_1!\Gamma(n_2-l_2+1)}}\psi_{n_1+l_1,n_2-l_2}. \quad (4.25)$$

In the special case in which $\omega_1 = \omega_2$, implying $l_1 = l_2 = 1$, integral operators $\hat{\mathscr{L}}^{\pm}_{1,1}$ and $\hat{\mathscr{L}}^{(+)}_{2} = \frac{1}{2}\hat{p}_{\varphi}$, to which the operator $\frac{1}{2\omega}\hat{L}^{(+)}_{\omega,\omega}$ from (4.8) is reduced, generate the $\mathfrak{su}(2)$ algebra. For $l_1 \neq l_2$, the integrals $\hat{L}^{(+)}_{\omega_1,\omega_2}$ and $\hat{\mathcal{L}}^{\pm}_{l_1,l_2}$ together with Hamiltonian $\hat{H}^{(+)}_{\omega_1,\omega_2}$ generate a nonlinear deformation of $\mathfrak{u}(2)$. Classical analogs of $\hat{\mathcal{L}}^{\pm}_{l_1,l_2}$ correspond to hidden symmetries because they generate the transformations that mix coordinates and momenta in phase space.

In the case of $\hat{H}^{(-)}_{\omega_1,\omega_2}$, additional true integrals are obtained from the operators

$$\hat{\mathcal{P}}_{l_1,l_2} = (\hat{p}_1)^{l_1}(\hat{p}_2)^{l_2}, \qquad \hat{\Xi}_{l_1,l_2} = (\hat{\xi}_1)^{l_1}(\hat{\xi}_2)^{l_2}, \quad (4.26)$$

which commute with $\hat{\mathcal{D}}_-$. They are transformed by the anisotropic CBT into

$$\hat{\mathfrak{S}}_{\omega_1,\omega_2}(\hat{\Xi}_{l_1,l_2})\hat{\mathfrak{S}}^{-1}_{\omega_1,\omega_2} \propto \hat{\mathcal{J}}^{+}_{l_1,l_2} \\ \hat{\mathfrak{S}}_{\omega_1,\omega_2}(\hat{\mathcal{P}}_{l_1,l_2})\hat{\mathfrak{S}}^{-1}_{\omega_1,\omega_2} \propto \hat{\mathcal{J}}^{-}_{l_1,l_2}, \quad (4.27)$$

$$\hat{\mathcal{J}}^{+}_{l_1,l_2} = (\hat{a}^{+}_{\omega_1})^{l_1}(\hat{a}^{+}_{\omega_2})^{l_2}, \qquad \hat{\mathcal{J}}^{-}_{l_1,l_2} = (\hat{\mathcal{J}}^{+}_{l_1,l_2})^{\dagger}. \quad (4.28)$$

They act on the eigenstates $\psi_{n_1,n_2}(x_1,x_2)$ as follows,

$$\hat{\mathcal{J}}^{-}_{l_1,l_2}\psi_{n_1,n_2} = \sqrt{\frac{n_1!n_2!}{\Gamma(n_1-l_1+1)\Gamma(n_2-l_2+1)}}\psi_{n_1-1,n_2-1}, \quad (4.29)$$

$$\hat{\mathcal{J}}^{+}_{l_1,l_2}\psi_{n_1,n_2} = \sqrt{\frac{\Gamma(n_1+l_1+1)\Gamma(n_2+l_2+1)}{n_1!n_2!}}\psi_{n_1+1,n_2+1}. \quad (4.30)$$

In the case $l_1 = l_2 = 1$ ($\omega_1 = \omega_2 = \omega$), the system $\hat{H}^{(-)}_{\omega,\omega}$, as we already noted, corresponds to the $\mathfrak{so}(1,1)$-invariant Minkowskian oscillator (4.9), for which the not depending explicitly on time integrals (4.10), $\hat{\mathcal{J}}^{\pm}_{1,1} = \hat{\mathcal{J}}_{\pm}$, $\hat{\mathcal{J}}_0 = \frac{1}{2\omega\hbar}(\hat{H}_{\rm osc} - \hbar\omega)$ and $\hat{H}^{(-)}_{\omega,\omega}$ generate the $\mathfrak{gl}(2,\mathbb{R})$ symmetry. Note that analogously to the ERIHO, here the bounded and unbounded from below character of the spectra in the cases of Euclidean and Minkowskian AHO systems is encoded in the structure of coefficients in Eqs. (4.24), (4.25) and (4.29), (4.30), respectively.

By applying the anisotropic CBT to the physical, or to nonphysical eigenstates $e^{\frac{1}{\sqrt{2}}(\alpha_1 x_1+\alpha_2 x_2)}$, $\alpha_1,\alpha_2 \in \mathbb{C}$, of the free particle Hamiltonian, we obtain

$$\Phi(x_1,x_2,\alpha_1,\alpha_2) \coloneqq \hat{\mathfrak{S}}_{\omega_1,\omega_2}e^{\frac{1}{\sqrt{2}}(\alpha_1 x_1+\alpha_2 x_2)} \\ = \sqrt{2}e^{-\frac{\hbar}{4m}\left(\frac{\alpha_1^2}{\omega_1}+\frac{\alpha_2^2}{\omega_2}\right)-\frac{m}{2\hbar}(\omega_1 x_1^2+\omega_2 x_2^2)+\alpha_1 x_1+\alpha_2 x_2}. \quad (4.31)$$

These functions correspond to coherent states for both systems $\hat{H}^{(\pm)}_{\omega_1,\omega_2}$ as they satisfy the relations

$$\hat{a}^{-}_{\omega_i}\Phi(x,y,\alpha_1,\alpha_2) = \sqrt{\frac{\hbar}{2m\omega_i}}\alpha_i\Phi(x,y,\alpha,\beta). \tag{4.32}$$

When comparing these results with the quantum analysis of the ERIHO systems, we observe that both models are really similar. Their spectra in Euclidean and Minkowskian cases have similar characteristics, while the corresponding spectrum-generating operators and the integral operators in both classes of models act in a similar way on their respective eigenstates. At the same time we note that the difference in their properties with respect to the planar rotations reveals itself in classical dynamics. This can be observed by comparing the form of trajectories in the ERIHO systems with those in the systems discussed in the present section, for which trajectories are described by equations

$$x_1(t) = A_1\cos(\omega_1 t) + B_1\sin(\omega_1 t),$$
$$x_2(t) = A_2\cos(\omega_2 t) + B_2\sin(\omega_2 t). \tag{4.33}$$

In the case of commensurable frequencies these trajectories also are closed but they are represented by Lissajous curves, some examples of which are shown in Fig. 3.

Essential difference also consists in the fact that Eqs. (4.33) describing the trajectories have exactly the same form for the Euclidean, $H^{(+)}_{\omega_1,\omega_2}$, and Minkowskian, $H^{(-)}_{\omega_1,\omega_2}$, AHOs with the same values of frequencies ω_1 and ω_2 (though the same trajectories correspond to different energies in the two indicated cases), while the form of dynamics and corresponding trajectories Euclidean ($|g| < 1$) and Minkowskian ($|g| > 1$) ERIHO systems is different.

Nevertheless, the similarities at the quantum level indicate that the two classes of the harmonic oscillator systems should be related somehow, and in the next section we describe their relationship.

V. RELATIONSHIPS OF THE ERIHO

With the help of a unitary transformation, which corresponds to a rotation in the three-dimensional "ambient space" of the $\mathfrak{su}(2)$ algebra, in this section we show that the ERIHO and AHO can be related to each other. We also relate the ERIHO with $g^2 < 1$ and $g^2 > 1$ with the Landau problem in the presence of the additional attractive and repulsive harmonic potentials.

A. Relationship of the ERIHO and AHO systems

Let us consider again the ERIHO Hamiltonian

$$\hat{H}_g = \hbar\omega(\ell_1\hat{b}^{+}_1\hat{b}^{-}_1 + \ell_2\hat{b}^{+}_2\hat{b}^{-}_2 + 1),$$
$$\ell_1 = 1+g, \qquad \ell_2 = 1-g, \tag{5.1}$$

and introduce the unitary operator [24]

$$\hat{U} = \exp\left(-i\frac{2\pi}{3}\frac{1}{\sqrt{3}}(\hat{\mathcal{L}}_1 + \hat{\mathcal{L}}_2 + \hat{\mathcal{L}}_3)\right), \tag{5.2}$$

$$\mathcal{L}_1 = \frac{1}{2}(\mathcal{L}_- + \mathcal{L}_+) = \frac{1}{2}(\hat{a}^{+}_1\hat{a}^{-}_2 + \hat{a}^{+}_2\hat{a}^{-}_1),$$
$$\mathcal{L}_3 = \frac{i}{2}(\mathcal{L}_- - \mathcal{L}_+) = \frac{1}{2}(\hat{a}^{+}_1\hat{a}^{-}_1 - \hat{a}^{+}_2\hat{a}^{-}_2), \tag{5.3}$$

see Eqs. (3.18) and (3.3). It produces the unitary transformation

$$\hat{U}\hat{a}^{\pm}_j\hat{U}^{\dagger} = e^{\pm i\frac{\pi}{4}}\hat{b}^{\pm}_j, \tag{5.4}$$

$$\hat{U}\hat{\mathcal{L}}_1\hat{U}^{\dagger} = \hat{\mathcal{L}}_3, \quad \hat{U}\hat{\mathcal{L}}_2\hat{U}^{\dagger} = \hat{\mathcal{L}}_1, \quad \hat{U}\hat{\mathcal{L}}_3\hat{U}^{\dagger} = \hat{\mathcal{L}}_2. \tag{5.5}$$

Using these relations, we find that

$$\hat{H}_g = \hat{U}(\hat{H}^{\text{n-i}}_g)\hat{U}^{\dagger}, \quad \text{where}$$
$$\hat{H}^{\text{n-i}}_g = \hbar\omega(\ell_1\hat{a}^{+}_1\hat{a}^{-}_1 + \ell_2\hat{a}^{+}_2\hat{a}^{-}_2 + 1). \tag{5.6}$$

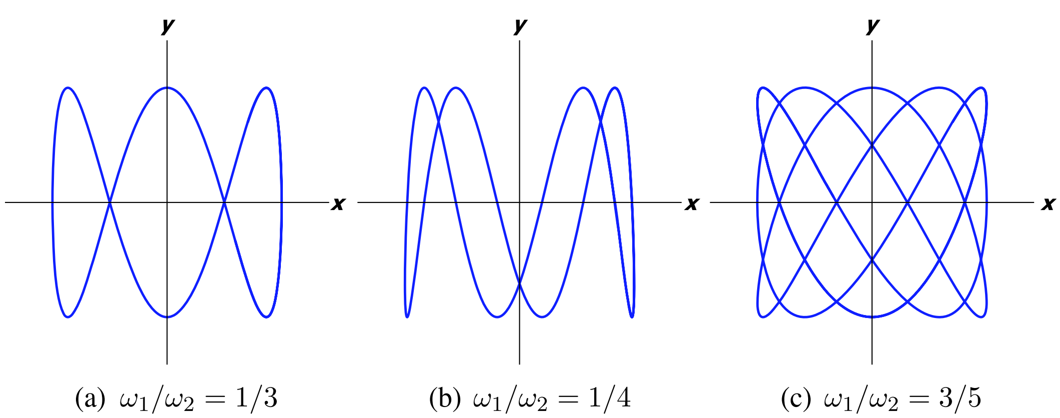


FIG. 3. Trajectories of the anisotropic harmonic oscillator with commensurable frequencies.

Unlike $\hat{H}_g$, Hamiltonian operator $\hat{H}_g^{\text{n-i}}$ is not invariant under the $\mathfrak{so}(2)$ rotations. In terms of the coordinates and momenta operators, $\hat{H}_g^{\text{n-i}}$ is presented as

$$\hat{H}_g^{\text{n-i}} = \frac{1}{2m}(\ell_1 \hat{p}_1^2 + \ell_2 \hat{p}_2^2) + \frac{m\omega^2}{2}(\ell_1 x_1^2 + \ell_2 x_2^2). \tag{5.7}$$

After the quantum canonical transformation of anisotropic rescaling,

$$x_i \to x_i' = \sqrt{|\ell_i|} x_i, \qquad \hat{p}_i \to \hat{p}_i{}' = \hat{p}_i / \sqrt{|\ell_i|}, \tag{5.8}$$

we recognize

$$\hat{H}_g^{\text{n-i}'} = \text{sign}(\ell_1)\left(\frac{\hat{p}_1'^2}{2m} + \frac{m\Omega_1^2}{2} x_1'^2\right) + \text{sign}(\ell_2)\left(\frac{\hat{p}_1'^2}{2m} + \frac{m\Omega_2^2}{2} x_2'^2\right), \qquad \Omega_i = |\ell_i|\omega. \tag{5.9}$$

When $|g| < 1$, the operator $\hat{H}_g^{\text{n-i}'}$ can be interpreted as the Hamiltonian of a one-parametric family of Euclidean AHO systems. On the other hand, when $|g| > 1$ we have a one-parametric family of Minkowskian AHO models.

By applying unitary transformation inverse to (5.8), and then the transformation inverse to (5.4), (5.5), to the true and dynamical integrals of the system $\hat{H}_g$, one gets, in particular,

$$\hat{b}_i^{\pm} \to \hat{a}_{\Omega_i}^{\pm}, \qquad \hat{\mathcal{L}}_{s_1,s_2}^{\pm} \to \hat{\mathscr{L}}_{s_1,s_2}^{\pm}, \qquad \hat{\mathcal{J}}_{s_1,s_2}^{\pm} \to \hat{\mathscr{J}}_{s_1,s_2}^{\pm}, \tag{5.10}$$

$$\hat{\mathcal{J}}_{\pm} \to \hat{\mathscr{J}}_{\pm}^{\Omega_1,\Omega_2} = \hat{a}_{\Omega_1}^{\pm}\hat{a}_{\Omega_2}^{\pm}, \qquad \hat{\mathcal{L}}_{\pm} \to \hat{\mathscr{L}}_{\pm}^{\Omega_1,\Omega_2} = \hat{a}_{\Omega_1}^{\pm}\hat{a}_{\Omega_2}^{\mp}, \tag{5.11}$$

$$\hat{\mathcal{J}}_0 - \frac{1}{2} \to \hat{\mathscr{J}}_0^{\Omega_1,\Omega_2} = \frac{1}{2}(\hat{a}_{\Omega_1}^{+}\hat{a}_{\Omega_1}^{-} + \hat{a}_{\Omega_2}^{+}\hat{a}_{\Omega_2}^{-}), \tag{5.12}$$

$$\hat{\mathcal{L}}_2 \to \hat{\mathscr{L}}_2^{\Omega_1,\Omega_2} = \frac{1}{2}(\hat{a}_{\Omega_1}^{+}\hat{a}_{\Omega_1}^{-} - \hat{a}_{\Omega_2}^{+}\hat{a}_{\Omega_2}^{-}). \tag{5.13}$$

So, we find that both, the ERIHO and AHO systems are unitary equivalent being related by the described composition of the two unitary transformations. The peculiarity of this relation is that (5.4) and (5.5) corresponds to a particular $\mathfrak{su}(2)$ rotation in a fictitious three-dimensional space corresponding to the index of the $\mathfrak{su}(2)$ generators, and the rotational invariance in the $\mathbb{R}^2$ configurational space is broken by the anisotropy of the rescaling transformation (5.8). Notice that unitary transformation (5.8) generated by the operator $\exp(i(\hat{D}_1 \ln|\ell_1| + \hat{D}_2 \ln|\ell_2|))$ is a Bogolyubov transformation [45] corresponding to a hyperbolic $\mathfrak{so}(1,1) \oplus \mathfrak{so}(1,1)$ rotation in terms of the operators $(\hat{a}_1^+, \hat{a}_1^-)$, $(\hat{a}_2^+, \hat{a}_2^-)$. In order to relate the isotropic Minkowskian case of the ERIHO with its $\mathfrak{so}(1,1)$ analog (4.9), it is necessary to make a change $\omega \to \omega/|g|$ in (5.9), and then take a limit $|g| \to \infty$.

According to Eq. (5.4), the classical analog of the unitary transformation generated by (5.2) mixes coordinates and momenta variables. Then, with taking into account the classical analog of the unitary anisotropic rescaling transformation (5.8), one can show that classical solutions (3.12) and (4.33) correspond to the same trajectories in four-dimensional phase space projected onto two different two-dimensional hyperplanes there which correspond to coordinate variables x_i and x_i' of the ERIHO and AHO systems.

B. Relationship with harmonically extended Landau problem

Let us consider the Landau problem in symmetric gauge assuming that the particle is subject to the action of the additional quadratic potential term. The Hamiltonian of the system is

$$H_{\pm} = \frac{1}{2m}(\Pi_i^{\pm})^2 + \frac{1}{2}m\Lambda x_i^2, \quad \Pi_i^{\pm} = p_i - \frac{q}{c}A_i^{\pm}, \tag{5.14}$$

where $\Lambda \in \mathbb{R}$ is a constant of dimension $[t^{-2}]$ and $A_i^{\pm}$ is given in (3.6). Note that when $q = 0$, Hamiltonian (5.14) with $\Lambda < 0$ corresponds to the inverted isotropic harmonic oscillator. If we choose the case of the positive upper index [that corresponds to $g = 1$ in (3.6)], and expand this Hamiltonian, we obtain

$$H_+ = \frac{1}{2m}(p_1^2 + p_2^2) + \frac{m}{2}(\omega_B^2 + \Lambda)(x_1^2 + x_2^2) + \omega_B p_\varphi, \qquad \omega_B = \frac{qB}{2mc}, \tag{5.15}$$

where now we do not restrict the sign of qB. Assuming that $\Lambda > -\omega_B^2$, i.e., that in the case of the inverted harmonic potential its coupling constant $|\Lambda|m$ is not too strong, we denote

$$\omega^2 = \omega_B^2 + \Lambda, \qquad \omega^2 > 0. \tag{5.16}$$

With this restriction, the system is confined: all its trajectories are bounded, while the quantum spectrum is discrete. Introducing the parameter g defined by relations $\sqrt{|\Lambda|} = \sqrt{|1-g^2|}\omega$, $\text{sign}(g) = \text{sign}(qB)$, Hamiltonian (5.15) takes then the form of the Hamiltonian of the ERIHO system,

$$H_+ = H_{\text{osc}} + g\omega p_\varphi = H_g. \tag{5.17}$$

Therefore, the case of Landau problem (in symmetric gauge) subjected to the additional action of the isotropic harmonic potential trap ($\Lambda > 0$) is equivalent to the Euclidean case of the ERIHO system with $g^2 < 1$, while the case of not too strong inverted isotropic harmonic potential ($0 > \Lambda > -\omega_B^2$) is equivalent to the Minkowskian case of the ERIHO system with $g^2 > 1$. Having this relationship of the ERIHO systems H_g with $g^2 < 1$ and $g^2 > 1$ with the Landau problem supplemented, respectively, with the attractive and repulsive harmonic potential terms, one can understand the phenomenon of transmutation of symmetries discussed in Sec. III A in the light of different realizations of conformal symmetries considered in [34].

In the critical case $\Lambda = -\omega_B^2$, frequency ω turns into zero, and Hamiltonian (5.15) takes the form $H_c = \frac{1}{2m} p_i p_i + \omega_B p_\varphi$. The system H_c is not confined anymore, its trajectories are infinite, and the corresponding quantum spectrum is continuous and not bounded from below. In supercritical case $\Lambda < -\omega_B^2$, ω^2 is negative, and Hamiltonian (5.15) takes the form of a two-dimensional inverted oscillator Hamiltonian plus a Zeeman type term, $H_{\rm sc} = \frac{1}{2m}(p_1^2 + p_2^2) - \frac{m}{2}|\omega^2|(x_1^2 + x_2^2) + \omega_B p_\varphi$. All the peculiar properties of the critical case, i.e., infinite classical trajectories and continuous spectrum not bound from below, are inherited by $H_{\rm sc}$.

One can notice that the critical case corresponds here to the dynamics of a free particle in a noninertial, uniformly rotating reference frame that is described by the Lagrangian

$$L_\Omega = \frac{1}{2} m(\dot{\vec{r}} + \vec{\Omega} \times \vec{r})^2. \quad (5.18)$$

Indeed, choosing $\vec{\Omega} = \frac{q}{2mc}\vec{B}$, and assuming that the uniform magnetic field $\vec{B}$ is oriented perpendicular to the plane with coordinates x_1, x_2, the dynamics in the direction orthogonal to this plane will be free. Neglecting this free part of the dynamics, Lagrangian (5.18) can be reduced to

$$L_\Omega^+ = L_L + \frac{1}{2} m\omega_B^2 (x_1^2 + x_2^2),$$
$$L_L = \frac{1}{2} m(\dot{x}_1^2 + \dot{x}_2^2) + \frac{q}{c} A_i^+ x_i, \quad (5.19)$$

where L_L is the Lagrangian of the Landau problem in the symmetric gauge. Hamiltonian (5.15) in the critical case of $\Lambda = -\omega_B^2$ corresponds exactly to the Lagrangian L_Ω^+. Then, our ERIHO system admits yet another interpretation as a plane isotropic harmonic oscillator described by a potential $U = \frac{1}{2} k(x_1^2 + x_2^2)$ in a noninertial, uniformly rotating reference frame. The cases $0 < k < m\omega_B^2$, $k = m\omega_B^2$ and $k > m\omega_B^2$ correspond to the phases $|g| > 1$, $|g| = 1$ and $0 < |g| < 1$, respectively, with $\mathrm{sing} g = \mathrm{sing}\Omega_3$. The inertial case $\Omega = 0$ with $k = m\omega^2$ corresponds, obviously, to the isotropic Euclidean oscillator ($g = 0$). From this point of view, the Minkowskian phase, $g^2 > 1$, corresponds to the case when the rotation frequency of the noninertial reference frame dominates the oscillator frequency, $\Omega^2 > k/m$, while their equality, $\Omega^2 = k/m$, corresponds to the Landau phases $g = \pm 1$.

VI. DISCUSSION AND OUTLOOK

We studied the ERIHO system that represents an isotropic Euclidean planar harmonic oscillator supplemented by a kind of Zeeman-like term with a dimensionless coupling constant g. The system was obtained by generalizing the conformal bridge transformation construction of Refs. [22–24] that allows to relate harmonically confined models with associated asymptotically free systems. To this aim, we applied a certain nonunitary rotationally invariant conformal intertwining operator to the complex linear combination of the free particle's dilatation and rotation integrals.

We showed that the Hamiltonian of the obtained ERIHO system H_g can be presented as a sum of the two circular oscillatory modes taken with the relative weights $(1 + g)$ and $(1 - g)$. As a consequence, the system reveals three different phases depending on the coupling constant value. In the case of $g^2 < 1$, the system represents the Euclidean phase of the ERIHO, that turns into isotropic planar oscillator at $g = 0$. The case $g^2 > 1$ corresponds to the Minkowskian phase of the ERIHO, which under frequency rescaling $\omega \to \omega/|g|$ and taking the infinite limit $|g| \to \infty$ transforms into the isotropic, $\mathfrak{so}(2)$-invariant Minkowskian oscillator. In the cases $g = \pm 1$ the system reduces to the Landau problem in the symmetric gauge with the opposite orientation of the magnetic field. The trajectories are closed for arbitrary choice of the initial data only for rational values of g. They have central symmetry except the cases $g = \pm 1$ with arbitrary-centered circular orbits. For $g = 0$ and $|g| = \infty$, the trajectories are, respectively, elliptic and circular.

The closed character of the trajectories at rational values of g is reflected in the presence of the hidden symmetry described by the pair of not depending explicitly on time, true integrals of motion that appear in addition to the Hamiltonian H_g and angular momentum p_φ. The additional integrals are quadratic in the circular oscillator variables in the cases $g = 0$, $g^2 = 1$ and $g^2 = \infty$. Moreover, in particular cases of $g = \pm 1$, they are quadratic in the true linear integrals corresponding to the translation symmetry generators which are the noncommuting coordinates of the circumference' center. In the indicated exceptional cases, the four true integrals together with other six explicitly depending on time, dynamical integrals of the second order generate the $\mathfrak{sp}(4, \mathbb{R})$ Lie algebra, and the pairs of exceptional cases ($g = 0$, $|g| = \infty$) and ($g = +1$, $g = -1$) are related by a transmutation of symmetries in the following sense. The $\mathfrak{sp}(4, \mathbb{R})$ algebra of the isotropic Euclidean case ($g = 0$) contains the $\mathfrak{u}(2) \cong \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ subalgebra, generated by the four true integrals, and the $\mathfrak{gl}(2,\mathbb{R}) \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{u}(1)$ algebra, in which $\mathfrak{sl}(2,\mathbb{R})$ corresponds to the conformal symmetry while the center $\mathfrak{u}(1)$ is generated by p_φ. In the isotropic Minkowskian oscillator case ($|g| = \infty$), these symmetries are interchanged: the subalgebra $\mathfrak{gl}(2,\mathbb{R})$ is generated by the true integrals, while $\mathfrak{u}(2)$ is associated with its conformal, dynamical symmetry. In the cases $g = \pm 1$ the true and extended conformal symmetries generate two copies of the same subalgebra $\mathfrak{gl}(2,\mathbb{R})$. The generators of these subalgebars are interchanged under the change $g = +1 \leftrightarrow g = -1$.

In general case of rational values of g the pair of additional true integrals is of higher order in circular oscillator variables. In the case of $g^2 < 1$ with $g = g_<^{s_1,s_2} = (s_2 - s_1)/(s_2 + s_1)$, where positive integer numbers s_1 and s_2 are chosen so that the fraction is irreducible, the pair of mutually complex conjugate additional integrals are of the order $s_1 + s_2$ in circular oscillator variables, and together with H_g and p_φ they generate a nonlinear deformation of the $\mathfrak{u}(2)$ algebra. When $g^2 > 1$ with $g = g_>^{s_1,s_2} = (s_2 + s_1)/(s_2 - s_1) = 1/g_<^{s_1,s_2}$, the corresponding additional true integrals have the same order $s_1 + s_2$ in circular oscillator variables, but together with H_g and p_φ they generate, instead, a nonlinear deformation of the $\mathfrak{gl}(2,\mathbb{R})$. So, in the case of rational g any system H_g is maximally superintegrable [46,47]. Notice that if to change the notation $s_1 \leftrightarrow s_2$ in the case $g^2 > 1$, we find that the corresponding generators of hidden symmetries of the same orders $s_1 + s_2$ in the phases with $g^2 < 1$ and $g^2 > 1$ will mutually transmute under the inversion $g \to -1/g$. Then the statement on transmutation (duality) under the inversion $g \to -1/g$ can also be extended for the sets of generators of the true and extended conformal symmetries in the Landau problem with the opposite orientation of the magnetic field ($g = +1$ and $g = -1$).

At the quantum level, the system with $g^2 < 1$ has a discrete positive spectrum with finite degeneracy of energy levels. The quantum analogs of the additional true integrals of motion control this degeneracy and allow to generate any state with a given energy value starting from any eigenstate with the same value of energy. In the case of rational g with $g^2 > 1$, the picture is similar, but there spectrum is not bounded from below and each energy level is infinitely degenerate.

On the other hand, we showed how the usual AHO systems can be generated from the free particle by using a certain rotationally noninvariant, anisotropic conformal bridge transformation. In such systems, as is well known, the trajectories are closed in the case of commensurable frequencies and represent the Lissajous curves. The peculiarity of the AHO systems in comparison with the ERIHO systems is that in them the trajectories are the same in the cases of Euclidean and Minkowskian planar oscillators with the same values of frequencies. We showed that the ERIHO and AHO systems with the corresponding parameter values can be related by a unitary canonical transformation that represents a composition of a certain $\mathfrak{su}(2)$ rotation in an "ambient" three-dimensional space and of anisotropic rescaling, which is an $\mathfrak{so}(1,1) \oplus \mathfrak{so}(1,1)$ Bogolyubov transformation.

We also showed that the ERIHO systems are equivalent to the Landau problem in symmetric gauge subjected to the action of the additional rotationally invariant harmonic potential term $\frac{1}{2} m\Lambda x_i^2$. In this case the systems with positive coupling constant $\Lambda > 0$ correspond to the ERIHO systems with $g^2 < 1$, while the negative values $0 > \Lambda > -(\frac{qB}{2mc})^2$, corresponding to the inverted oscillator in the subcritical phase, yield the ERIHO systems with $g^2 > 1$. In the case of critical, $\Lambda = -(\frac{qB}{2mc})^2$ and supercritical values $\Lambda < -(\frac{qB}{2mc})^2$ of the coupling constant of the inverted harmonic potential term, classical trajectories are infinite, and the quantum spectrum is continuous and not bounded from below. These phases with critical and super-critical values of the inverted potential term coupling have no analogs in the studied by us ERIHO systems. But we notice here that such phases appear in the systems with exotic Newton-Hooke symmetries and noncommutative geometry [32,48].

Finally, we showed that our ERIHO admits yet another interpretation as a plane isotropic harmonic oscillator in a uniformly rotating reference frame. From this point of view, the phases $g^2 < 1$, $g^2 = 1$ and $g^2 > 1$ correspond, respectively, to the cases $\Omega^2 < k/m$, $\Omega^2 = k/m$ and $\Omega^2 > k/m$, where Ω is the angular frequency of the reference frame, and k is the harmonic oscillator constant. Critical case of the harmonically extended Landau problem in this case corresponds to a free particle in a uniformly rotating reference frame.

A phase transition in rotating harmonically trapped Bose-Einstein condensates is expected at $\Omega^2 = k/m$, see [49–51] and references therein. It is interesting whether the described peculiar properties related to hidden symmetries emerging at rational values of g reveal themselves somehow in such systems. Taking into account the unitary equivalence of the ERIHO and AHO systems, in this context one also can expect that Bose-Einstein condensates in rotationally noninvarint harmonic anisotropic traps [52,53] should exhibit properties similar to those they have in rotating harmonic traps.

In conclusion, we note some problems that deserve attention for further research.

First of all, the considered CBTs of the rotationally invariant and rotationally noninvariant nature can be generalized for dimensions higher than two. Furthermore, the conformal potential term $\gamma/(x_i x_i)$ can be included into the initial free particle Hamiltonian, and analysis also can be extended for Calogero-type systems that would correspond to the case of anisotropic CBT. Having in mind the constructions based on exotic Galilean and Newton-Hooke symmetries [48,54], relevant to the physics of anyons

[55,56], it also would be interesting to apply our analysis to the case of noncommutative geometry.

In a recent paper [24], by applying the CBT to the study of dynamics in a cosmic string background, we revealed a kind of quantum anomaly there. It would be interesting to investigate what happens with the quantum anomaly in the presence of the Zeeman-like term. In fact, our interest in studying the ERIHO systems arose from an attempt to understand, in the light of symmetries, the Landau problem in a geometric background with topological defects such as a conical one [57,58] and a background of a rotating cosmic string [59,60].

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