Scattering amplitudes in affine gravity

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Affine gravity is a connection-based formulation of gravity that does not involve a metric. After a review of basic properties of affine gravity, we compute the tree-level scattering amplitude of scalar particles interacting gravitationally via the connection in a curved spacetime. We find that, while classically equivalent to general relativity, affine gravity differs from metric quantum gravity.

I. INTRODUCTION

Even more than 100 years after the formulation of general relativity (GR), and despite an enormous effort, we still lack a consistent and experimentally verified theory of quantum gravity. The standard approach that works excellently for the Standard Model of particle physics, perturbation theory, fails to provide an acceptable quantization of GR beyond the framework of effective field theory. The reason is that every new loop order introduces new interaction terms which are not part of the original action [1], and with it, new coupling constants appear that have to be fixed by some experiment. In pure gravity, this happens for the first time at the two-loop level with the famous Goroff-Sagnotti term [2–4], whereas, with matter, the first occurrence is at one-loop order. One thus finds oneself in the situation that in principle one would have to conduct infinitely many experiments to uniquely specify the theory, and thus any predictivity is lost. This behavior is also expected from naïve power counting: the coupling of GR, Newton's constant G_N, has a mass dimension of -2 in four spacetime dimensions.

Many different ideas have been put forward to circumvent this problem in one way or another. The most conservative possibility to solve this problem is that gravity can be renormalized in a nonperturbative way. Approaches in this direction include the asymptotic safety program [5–9], causal dynamical triangulations (CDT) [10,11], and loop quantum gravity [12–14]. Closely related are more geometry-based approaches like causal sets [15], with the

disadvantage that the connection to standard quantum field theory is less clear. An alternative is to give up on some of the symmetries, and for example break Lorentz symmetry at small scales [16–20]. More recently, Stelle's higher-derivative gravity [21,22] as a perturbative approach to quantum gravity has received renewed interest [23–27], but it is as of yet unclear whether the ghost modes can be completely avoided in this theory.

A central question in all of these approaches is what the fundamental degrees of freedom are. For example, whereas in general asymptotic safety is agnostic about this, an overwhelming majority of research conducted in this area is based on the metric as the fundamental field, with the occasional exception based, e.g., on the tetrad formalism [28]. CDT is based on purely geometric objects, on which the diffeomorphism group acts trivially. In loop quantum gravity, a central role is played by holonomies. Usually all these different choices have in common is that at the classical level they are equivalent to GR.

A particular choice of degrees of freedom has not received a lot of attention, even though it was first discussed quite early—the purely affine formulation, based only on the use of an affine connection. Originally discussed by Eddington and Schrödinger [29–31], most investigations involving the connection as a fundamental degree of freedom are carried out in a formulations where the relevant connection lives in a related gauge bundle, e.g., $SO(3)$ [32–37], or $SO(1,3)$ [38]. On paper, the purely affine theory has many interesting properties. First, it is much closer in spirit to the other fundamental forces in nature. In this picture, there is a unification of the description: forces are mediated by a connection (be it gravitational, strong or electroweak), whereas matter is described by fermions and the Higgs boson. Second, the classical theory is completely equivalent to GR with a cosmological constant; however, the latter appears as an integration constant similar to the case of unimodular gravity. A key difference to the metric formulation is that

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the equation of motion for the connection is *cubic* instead of nonpolynomial, once again emphasizing the similarity to classical Yang-Mills (YM) theory. Third, while having a metric is certainly useful, it is not necessary to describe, e.g., geodesic motion, which solely depends on the connection. Fourth, the implementation of torsion degrees of freedom is completely straightforward. While torsion is usually not considered (partially because in GR, torsion is not dynamical), it is a question of experiment whether our Universe has torsion degrees of freedom or not. Finally, and maybe most promising from the viewpoint of quantization, the coupling constant in the affine theory is dimensionless. This opens up the possibility of a perturbative quantization of gravity without making other sacrifices like assumptions on extra fields, dimensions or breaking of symmetries.

In this paper, we take all these reasons as a motivation to study affine gravity from a modern perspective. With the anticipation that not many readers will be familiar with the theory, we provide an overview of the basics in Sec. II, systematizing some of the earlier literature. In particular, we will discuss its action and equation of motion, the symmetry structure of the theory, as well as the coupling to matter. With this in place, in Sec. III we derive the scattering amplitude of a two-scalar-to-two-scalar process mediated by the gravitational connection and discuss the differences to GR. This will be done on a curved spacetime that fulfils the equations of motion. We then close with a summary of the results and an outlook in Sec. IV.

II. BASICS OF AFFINE GRAVITY

In this section we present the foundations of affine gravity. Since we do not have access to a metric, we will be very careful with the index positions of all tensors in any expression. To make matters simpler, in this work we restrict ourselves to a symmetric connection so that the torsion vanishes:

$$\Gamma^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}{}_{(\mu\nu)}. \tag{1}$$

The brackets indicate the symmetrization of the enclosed indices normalized to unit strength. The Riemann and symmetrized Ricci tensors in terms of this connection are then

$$R_{\mu\nu\rho}{}^{\alpha} = \Gamma^{\alpha}{}_{\nu\beta}\Gamma^{\beta}{}_{\mu\rho} - \Gamma^{\alpha}{}_{\mu\beta}\Gamma^{\beta}{}_{\nu\rho} - \partial_{\mu}\Gamma^{\alpha}{}_{\nu\rho} + \partial_{\nu}\Gamma^{\alpha}{}_{\mu\rho},$$
$$R_{\mu\nu} = R_{(\mu|\alpha|\nu)}{}^{\alpha}. \tag{2}$$

Vertical bars indicate that the enclosed indices are not symmetrized over. In affine gravity we will generally have to assume that the symmetrized Ricci tensor is nondegenerate so that it has a multiplicative inverse, which we denote by Я,

$$Я^{\mu\alpha}R_{\alpha\nu} = \delta_{\nu}{}^{\mu}. \tag{3}$$

Effectively, R and Я will be used to lower and raise indices, with the difference to the metric case that in general their covariant derivatives are not zero. In this sense, it is akin to not using the Levi-Civita connection in the metric case. From the invertibility requirement of the symmetrized Ricci tensor we conclude that the limit of having a flat spacetime will have to be implemented very carefully. Finally, we can define the equivalent of a d'Alembertian operator, which we will call Д, by

$$Д = -D_{\mu}Я^{\mu\nu}D_{\nu}. \tag{4}$$

In this, D is the covariant derivative associated to the connection Γ. Notice that, in contrast to the d'Alembertian, Д has a vanishing mass dimension, so its eigenvalues are pure numbers. We will see that this is a general property of the affine setup: the mass dimension of an expression is related to the number of its free indices.

A. Action and equation of motion

Let us first discuss the classical action of affine gravity. In a general spacetime dimension d, it is given by

$$S^{\text{Edd}} = \frac{1}{4\alpha_g} \int d^d x \sqrt{-\det R_{\mu\nu}}. \tag{5}$$

Several remarks are in order. First, the coupling α_g is dimensionless for all d, and the nomenclature is chosen to emphasize the closeness to YM theory. Second, it is nonpolynomial in the Ricci curvature tensor, so eventually we have to carefully assess whether the theory possesses any ghost or tachyonic modes. Finally, for the action to be real, we have to require that the Ricci tensor retains its signature. This is similar to the requirement in metric gravity that the metric has a fixed signature, and will constrain the domain of integration of the path integral.

To derive the equations of motion, we perform a standard variation of the action with respect to the connection Γ. After some algebra, and using that R is nondegenerate, one can show that the equation of motion has the very compact form

$$D_{\mu}R_{\alpha\beta} = 0. \tag{6}$$

Recalling the dependence of the Ricci tensor on the connection (2) this shows that the equation of motion is a cubic second-order partial differential equation, similar to that of YM theory. One can use the equation of motion to show that the connection is the Levi-Civita connection with respect to the Ricci tensor,

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2}Я^{\mu\nu}(\partial_{\alpha}R_{\nu\beta} + \partial_{\beta}R_{\nu\alpha} - \partial_{\nu}R_{\alpha\beta}). \tag{7}$$

The equation (6) is equivalent to GR with a cosmological constant. It states that the Ricci tensor is covariantly constant, so we can write suggestively

$$R_{\mu\nu} = \frac{2\Lambda}{d-2} g_{\mu\nu}, \qquad (8)$$

where Λ is a constant of mass dimension two, g is a covariantly constant tensor of mass dimension zero, and the normalization is chosen to allow a simple relation to GR. Clearly, (8) is just Einstein's equation with cosmological constant Λ and metric g. With this, we can relate the coupling α_g to Newton's constant G_N and the cosmological constant. Requiring that the classical Einstein-Hilbert action of GR,

$$S^{\mathrm{GR}} = \frac{1}{16\pi G_N} \int \mathrm{d}^d x \sqrt{-\det g_{\mu\nu}} (R - 2\Lambda), \qquad (9)$$

agrees with the Eddington action (5) on the solution of the equation of motion (8), we find

$$\alpha_g = 2\pi G_N \left(\frac{2\Lambda}{d-2} \right)^{\frac{d-2}{2}} \overset{d=4}{=} 2\pi G_N \Lambda. \qquad (10)$$

The observed value of α_g is extremely tiny (of the order of 10^{-120}), so that we can expect excellent convergence of any perturbative treatment in this coupling.

Let us stress the conceptual difference of the origin of Newton's constant and the cosmological constant in GR and affine gravity. In GR, both constants are part of the action, and thus are fundamental parameters of the theory. By contrast, in affine gravity, only the coupling α_g is part of the definition of the action. The cosmological constant arises as a constant of integration in the solution of the equation of motion, similar to unimodular gravity. In this way, the tiny observed value of the cosmological constant is just a statement about what initial condition is realized in the Universe. Trying to explain its small value from first principles is akin to asking for an explanation of why a harmonic oscillator starts at the exact position where it starts.

B. Coupling to matter

We will now discuss some aspects of the coupling to matter in affine gravity. Since we do not have access to a metric off shell, there are a few differences with regards to the metric case. The first observation is that the measure is already dimensionless in contradistinction to the metric case. This means that

$$\int \mathrm{d}^d x \sqrt{-\det R_{\mu\nu}}, \qquad (11)$$

is a pure number and carries the interpretation of a volume in units of the curvature. As a consequence, any Lagrangian density that is included will have to be dimensionless. This changes the mass dimension of some fields compared to the standard case. For example, writing down a kinetic term for a scalar field ϕ,

$$S^{\mathrm{scal}} = \frac{1}{2} \int \mathrm{d}^d x \sqrt{-\det R_{\mu\nu}} \mathcal{R}^{\alpha\beta} (D_\alpha \phi)(D_\beta \phi), \qquad (12)$$

it is evident that the scalar field has zero mass dimensions. By contrast, a gauge field continues to have a mass dimension of one, and a kinetic term for the photon reads

$$S^{\mathrm{phot}} = \frac{1}{4} \int \mathrm{d}^d x \sqrt{-\det R_{\mu\nu}} \mathcal{R}^{\alpha\gamma} \mathcal{R}^{\beta\delta} F_{\alpha\beta} F_{\gamma\delta}, \qquad (13)$$

where F is the field strength tensor.

For fermions, the situation is a bit more involved. A first step in the construction of the spinor bundle would be to define the Clifford algebra in terms of the Ricci tensor,

$$\{\gamma_\mu, \gamma_\nu\} = 2R_{\mu\nu} \mathbb{1}. \qquad (14)$$

Here, γ_μ are the mass dimension one gamma matrices, and the matrix $\mathbb{1}$ is the identity in Dirac space. However, the full construction of the spinor bundle in the absence of a Levi-Civita connection goes beyond the scope of this work.

We will now discuss how to obtain the mass dimension of any object. By definition, the partial derivative has a mass dimension of one, and comes with one lower spacetime index. To be consistent, any (gauge) connection must have the same mass dimension so that covariant derivatives have a uniform mass dimension. From this it follows that any curvature tensor has a mass dimension of two, and the inverse Ricci tensor then has a mass dimension of minus two. With these facts, it is straightforward to convince oneself that the mass dimension of any tensor is given by the difference of the number of lower spacetime indices minus the number of upper spacetime indices. In particular, any spacetime scalar has vanishing mass dimension. The interpretation of this compared to standard quantum field theory with a metric is that in affine gravity the fields are measured in units of the spacetime curvature (which is the on shell equivalent to measuring in units of the cosmological constant).

This has two very unconventional consequences: all conceivable coupling constants must have mass dimension zero, so naive power counting stops to be applicable. This might be a potential roadblock to the perturbative quantization of the theory, but more investigations are necessary. In particular, on solutions to the equation of motion, standard quantum field theory, including its power counting, must emerge under a suitable rescaling of the fields. Second, no *ad hoc* scale is introduced at any level. Every measurement of a dimensionful quantity is relational in the

sense that it will be expressed in units of some other dimensionful quantity of the same physical system.

C. Symmetry structure

We proceed our discussion of affine gravity by investigating its symmetry properties. Naturally, any action for affine gravity that we propose should be diffeomorphism invariant. Infinitesimally, this is made explicit by the condition

$$S^{\text{Edd}}(\Gamma + \mathcal{L}_\epsilon \Gamma) = S^{\text{Edd}}(\Gamma), \qquad (15)$$

where $\mathcal{L}_\epsilon$ denotes the Lie derivative with respect to the vector field ϵ that generates a diffeomorphism. The Lie derivative acting on the connection Γ is given by

$$(\mathcal{L}_\epsilon \Gamma)^\alpha{}_{\mu\nu} = D_\mu D_\nu \epsilon^\alpha + R_{\mu\beta\nu}{}^\alpha \epsilon^\beta. \qquad (16)$$

In order to compute the connection propagator in affine gravity, it is necessary to gauge-fix the diffeomorphism symmetry. For this, we employ the background field method. Here, the connection is split into a background connection $\bar{\Gamma}$ and a fluctuation $\delta\Gamma$,

$$\Gamma = \bar{\Gamma} + \delta\Gamma. \qquad (17)$$

The first step in the gauge-fixing procedure is to define a DeWitt inner product on the space of fluctuations. This is constructed in complete analogy to its metric counterpart, introduced in [39]. For the affine case, we employ the symmetrized Ricci tensor with respect to the background connection. The most general ultralocal inner product reads

$$\langle \delta\Gamma, \delta\Gamma \rangle = \int \mathrm{d}^d x \sqrt{-\det \bar{R}_{\kappa\lambda}} \, \delta\Gamma^\alpha{}_{\mu\nu} \mathcal{G}_\alpha{}^{\mu\nu}{}_\beta{}^{\rho\sigma} \delta\Gamma^\beta{}_{\rho\sigma}, \quad (18)$$

where the tensor $\mathcal{G}$ is given by

$$\begin{aligned}
\mathcal{G}_\alpha{}^{\mu\nu}{}_\beta{}^{\rho\sigma} &= \bar{R}_{\alpha\beta} \bar{\mathcal{R}}^{\mu(\rho} \bar{\mathcal{R}}^{\sigma)\nu} \\
&+ \frac{\beta_1}{2} (\delta_\alpha{}^\mu \delta_\beta{}^{(\rho} \bar{\mathcal{R}}^{\sigma)\nu} + \delta_\alpha{}^\nu \delta_\beta{}^{(\rho} \bar{\mathcal{R}}^{\sigma)\mu}) \\
&+ \beta_2 \bar{R}_{\alpha\beta} \bar{\mathcal{R}}^{\mu\nu} \bar{\mathcal{R}}^{\rho\sigma} \\
&+ \frac{\beta_3}{2} (\bar{\mathcal{R}}^{\mu\nu} \delta_\alpha{}^{(\rho} \delta_\beta{}^{\sigma)} + \bar{\mathcal{R}}^{\rho\sigma} \delta_\alpha{}^{(\mu} \delta_\beta{}^{\nu)}) \\
&+ \beta_4 \delta_\alpha{}^{(\rho} \bar{\mathcal{R}}^{\sigma)(\mu} \delta_\beta{}^{\nu)}. \qquad (19)
\end{aligned}$$

The parameters $\{\beta_i\}$ are gauge parameters. The DeWitt metric is invertible for most choices of $\{\beta_i\}$, except for a few isolated cases.

The gauge fixing action is now constructed via

$$S^{\text{gf}} = \frac{1}{2\alpha} \int \mathrm{d}^d x \sqrt{-\det \bar{R}_{\mu\nu}} \, \bar{\mathcal{R}}^{\gamma\delta} F_\gamma[\delta\Gamma, \bar{\Gamma}] F_\delta[\delta\Gamma, \bar{\Gamma}], \qquad (20)$$

where the gauge fixing condition is implemented by the functional F using the deWitt metric,

$$F_\gamma[\delta\Gamma, \bar{\Gamma}] = \mathcal{G}_\alpha{}^{\mu\nu}{}_\beta{}^{\rho\sigma} (\delta_\gamma{}^\beta \bar{D}_\sigma \bar{D}_\rho + \bar{R}_{\rho\gamma\sigma}{}^\beta) \delta\Gamma^\alpha{}_{\mu\nu}. \qquad (21)$$

D. Connection to metric theory

In this section we explain how to recover standard metric theory from affine gravity in cases where we can define a metric, e.g., via the equation of motion (8). As we have observed earlier, mass dimensions of fields and couplings in the theory are different from the standard case, so all fields and couplings have to be rescaled by appropriate powers of the only scale available, the cosmological constant, introduced by the equation of motion. For example, a scalar field in the affine theory is dimensionless,

$$[\phi] = 0, \qquad (22)$$

whereas in a standard metric theory, it has a mass dimension

$$[\phi_g] = \frac{d-2}{2}. \qquad (23)$$

Here the subscript g indicates that the quantity belongs to a metric theory. Accordingly, up to constant rescalings by numbers, the two fields are related by

$$\phi = \Lambda^{\frac{2-d}{2}} \phi_g. \qquad (24)$$

From this, we can derive the scaling of any coupling constant. For example, consider an term in the action of the structural form

$$\kappa \int \mathrm{d}^d x \sqrt{-\det R_{\mu\nu}} \; \Delta^m \phi^n, \qquad (25)$$

where it is understood that the derivatives are distributed in some way among the fields, and κ is a coupling constant. Replacing the curvatures via (8) and the scalar fields by (24) while dropping overall constants, this term becomes

$$\kappa \Lambda^{d-m+\frac{2-d}{2}n} \int \mathrm{d}^d x \sqrt{-\det g_{\mu\nu}} \Delta^m \phi_g^n. \qquad (26)$$

Once again, the standard Laplacian $\Delta = -g^{\mu\nu} D_\mu D_\nu$ is understood to be distributed among the scalar fields in the same way as above. From this, we can read off the necessary rescaling of the dimensionless coupling κ to define its dimensionful metric counterpart κ_g,

$$\kappa = \Lambda^{-d+m-\frac{2-d}{2}n} \kappa_g. \qquad (27)$$

The same algorithm applies to relate other matter fields to their metric counterparts. This emphasizes again the

interpretation that all quantities in affine gravity are measured in units relative to the only scale spacetime itself gives us in the theory, the cosmological constant.

E. Synopsis of previous results

In this section, we collect results on affine gravity that have been previously obtained. We will first consider classical aspects of the theory. Next, we will briefly review known quantum aspects.

We start our discussion by noting that the affine formulation, the metric formulation, and metric-affine formulations such as the Palatini formalism can all be shown to be classically equivalent [40,41]. For a discussion on quantum equivalence see [42]. The proof of the classical equivalence shows that the metric degree of freedom arises in the affine formulation as the canonical conjugate of the symmetric part of the Ricci tensor. The canonical structure of the affine theory was discussed in [43]. The Newtonian limit of affine gravity was studied in [44].

Cosmological implications of affine gravity, for example in the context of inflation, as well as the impact of nonminimal couplings, were studied in [45–51].

The identification of the metric as the symmetric part of the Ricci tensor can be generalized to unify gravity and $U(1)$ gauge fields [40,52–54]. Here, the electromagnetic field strength tensor $F_{\mu\nu}$ is identified with the antisymmetric part of the Ricci tensor. Variation of the action (13) with respect to the connection then gives both the Einstein equations with cosmological constant, and the Maxwell equations.

An initial study of the quantum properties of affine gravity was performed in [55]. In this work, the one-loop renormalization group (RG) running of the coupling α_g was computed. This beta function exhibits an infrared-attractive fixed point at $\alpha_g = 0$, similar to the electromagnetic fine-structure constant in QED. Since α_g is related to the cosmological constant, such a fixed point is particularly favored by the small observed value of Λ. We note that [55] first recasts the Eddington action (5) into an on shell equivalent, polynomial action. While the quantum properties of this theory are certainly appealing, it is presently unknown whether the beta function of the original theory (5) agrees with these findings.

III. SCATTERING AMPLITUDE

While affine gravity agrees with GR when it comes to the equation of motion, the off shell degrees of freedom are expected to differ, so that their respective quantized versions potentially give rise to different results. A straightforward way to probe this is the calculation of a tree-level scattering amplitude. Since pure gravitational scattering is technically challenging, we will discuss a gravity-mediated scalar scattering of a single minimally coupled massless scalar field.

A. Setup

Our ansatz for the action is

$$S = \int d^d x \sqrt{-\det R_{\mu\nu}} \left[\frac{1}{4\alpha_g} + \frac{1}{2} Я^{\alpha\beta} (D_\alpha \phi)(D_\beta \phi) \right]. \quad (28)$$

For the calculation of the amplitude of the scattering process $\phi\phi \to \phi\phi$, we need the connection propagator. Due to the underlying diffeomorphism symmetry, we have to employ a gauge fixing as discussed in Sec. II C. To further simplify the calculation, we will use a covariantly constant background,

$$\bar{D}_\mu \bar{R}_{\alpha\beta} = 0, \quad (29)$$

and specify the Riemann tensor to be maximally symmetric,

$$\bar{R}_{\mu\nu\rho}{}^\sigma = \frac{1}{d-1} (\bar{R}_{\mu\rho} \delta_\nu{}^\sigma - \bar{R}_{\nu\rho} \delta_\mu{}^\sigma). \quad (30)$$

This is similar to, but clearly more general than, standard calculations of gravitationally mediated scattering amplitudes in a flat spacetime, and consistent since we do not discuss external gravitational legs. In order to ease the notation, we omit the bar to denote background quantities.

As a consequence of this choice of background, the on shell condition for the scalar field ϕ simply reads

$$Д\, \phi = 0. \quad (31)$$

Additionally, we assume that we can actually define asymptotic states on the given background. We expect that the technical progress of calculating scattering amplitudes in curved spacetimes that we put forward in the following will also be useful for metric-based calculations.

B. Calculation

We will now outline the calculation of the curved space scattering amplitude. Since we do not have access to momentum space, we will work in position space throughout the calculation.

The gravity-mediated amplitude consists of the contraction of two three-point vertices with a single connection propagator. In position space, a two-to-two amplitude is then an operator which acts on four different arguments. With this in mind, it is in practice easier to not take the derivatives with respect to the external fields; that is, we formulate the amplitude in terms of its action on test fields. In that way, what we actually have to calculate is the contraction of the connection propagator 𝕲 sandwiched between two connection-energy-momentum tensors:

$$\mathcal{A}(\{\phi_i\}) = \int d^d x \sqrt{-\det R_{\mu\nu}} \, T_{\phi_1\phi_2} \, 𝕲 \, T_{\phi_3\phi_4}. \quad (32)$$

We will call this object the amplitude functional, and the actual amplitude is the fourth functional derivative of it. The amplitude (32) corresponds to the s channel, and the labels 1,2 indicate the incoming scalars whereas the labels 3,4 indicate the outgoing scalars. The t and u channels follow by crossing symmetry.

With this general approach settled, the central object that we have to compute is the propagator, which is the inverse of the two-point function. Structurally, the propagator is a linear combination of functions of Д and tensor structures in such a way that the complete object is self-adjoint. We can choose a specific ordering of propagator functions and tensor structures to simplify the calculation. The choice that we implemented is to sort all propagator functions to the right of the respective tensor structure, so that the propagator reads

$$\mathfrak{G}^{\mu}{}_{\alpha\beta}{}^{\nu}{}_{\kappa\lambda} = \sum_{\ell=1}^{32} \mathcal{T}_{\ell}{}^{\mu}{}_{\alpha\beta}{}^{\nu}{}_{\kappa\lambda} G_{\ell}(Д). \tag{33}$$

Here, the $\mathcal{T}_{\ell}$ are 32 independent tensor structures that span a basis, and the G_{ℓ} are the propagator functions that have to be computed.

To compute the propagator functions $\{G_{\ell}\}$, we demand that

$$\frac{\delta^2 S^{\text{Edd}}}{\delta\Gamma^{\rho}{}_{\tau\omega}\delta\Gamma^{\nu}{}_{\kappa\lambda}}\mathfrak{G}^{\nu}{}_{\kappa\lambda}{}^{\mu}{}_{\alpha\beta} = \mathbf{i}\delta_{\rho}{}^{\mu}\delta_{\alpha}{}^{(\tau}\delta_{\beta}{}^{\omega)}. \tag{34}$$

To do this in practice, we act with this equation onto a test tensor $\Gamma_{\mu}{}^{\alpha\beta}$ with an index structure opposite to the connection but with the same symmetry. Next, all occurrences of Д are sorted to the right, so they act first on Γ. Finally, we symmetrize all remaining covariant derivatives to arrive at an expression in canonical form, where all tensor structures are independent. The propagator functions can then be read off.

With the propagator at hand, it remains to contract it with two first connection derivatives of the matter action to derive the scattering amplitude functional. Partial integration can be used to simplify the expression—these are nothing else than the curved space equivalent of momentum conservation. Furthermore, the external test fields are assumed to be on shell, so that any Д acting on an individual field vanishes. The full calculation is rather cumbersome, and we carried it out with the tensor algebra package *xAct* [56–59]. For the convenience of the reader, the complete *Mathematica* notebook detailing the calculation is provided as an ancillary file [60].

C. Result

To obtain a manifestly gauge-invariant result, the scattering amplitude has to be brought into a canonical form. Due to the structure of the action, we know that the amplitude functional can be written as

$$\begin{aligned}
\frac{\mathcal{A}}{4\alpha_g} = \int \mathrm{d}^d x \sqrt{-\det R_{\alpha\beta}} \Big[&\phi_1\phi_2 a_0(Д)\phi_3\phi_4 \\
&+ \mathfrak{R}^{\mu\nu}(D_{\mu}\phi_1)\phi_2 a_1(Д)(D_{\nu}\phi_3)\phi_4 \\
&+ \mathfrak{R}^{\mu\nu}\mathfrak{R}^{\rho\sigma}(D_{(\mu}D_{\rho)}\phi_1)\phi_2 a_2(Д)(D_{(\nu}D_{\sigma)}\phi_3)\phi_4 \\
&+ \mathfrak{R}^{\mu\nu}\mathfrak{R}^{\rho\sigma}\mathfrak{R}^{\kappa\lambda}(D_{(\mu}D_{\rho}D_{\kappa)}\phi_1)\phi_2 a_3(Д)(D_{(\nu}D_{\sigma}D_{\lambda)}\phi_3)\phi_4 \Big].
\end{aligned} \tag{35}$$

The reason for this form of the functional is that each of the vertices carries at most three uncontracted derivatives. All other uncontracted derivatives that come from the propagator must be contracted in some way to form a Д acting either on a single ϕ which is eliminated using the equation of motion, or is absorbed in the one of the functions a_i. To arrive at the form (35), one has to use the commutator rules derived in the Appendix together with partial integrations. With some effort, one arrives at the result

$$\begin{aligned}
a_0(Д) &= 0, \\
a_1(Д) &= -(1+Д), \\
a_2(Д) &= \frac{2(2d+(d-1)Д)}{2+(d-1)Д}, \\
a_3(Д) &= 0.
\end{aligned} \tag{36}$$

The calculation of the functions a_i is the main result of this work. We have checked that the result does not depend on the gauge fixing parameters $\alpha, \{\beta_i\}$.

To make contact to standard quantum field theory notions of scattering in momentum space, it is useful to introduce curved-spacetime generalizations of the Mandelstam variables,

$$\begin{aligned}
\mathfrak{s}(\phi_1\phi_2\phi_3\phi_4) &\equiv \phi_1\phi_2(Д\phi_3\phi_4), \\
\mathfrak{t}(\phi_1\phi_2\phi_3\phi_4) &\equiv \phi_2\phi_4(Д\phi_1\phi_3).
\end{aligned} \tag{37}$$

It should be noted that these operators do not commute in curved spacetime. Our convention is that all $\mathfrak{t}$ operators are sorted to the right. All other contractions of derivatives can be eliminated by partial integration, the equation of

motion and by using the commutator rules derived in the Appendix.

With these operators, we define the amplitude operator $\mathfrak{A}$, which is the operator equivalent of an amplitude in momentum space. It is related to the amplitude functional via

$$\mathcal{A} = \int d^d x \sqrt{-\det R_{\alpha\beta}} \, \mathfrak{A} \phi_1 \phi_2 \phi_3 \phi_4. \qquad (38)$$

It reads

$$\mathfrak{A} = 4\alpha_g \left(\mathfrak{s} + \frac{2}{d-1} \right)^{-1}$$
$$\times (a_{01}\mathfrak{t} + a_{11}\mathfrak{s}\mathfrak{t} + a_{21}\mathfrak{s}^2\mathfrak{t} + a_{02}\mathfrak{t}^2 + a_{12}\mathfrak{s}\mathfrak{t}^2), \qquad (39)$$

where the coefficients a_{ij} are given by

$$a_{01} = \frac{2d+1}{1-d}, \qquad a_{11} = \frac{1}{1-d} - \frac{3}{2},$$
$$a_{21} = -\frac{1}{2}, \qquad a_{02} = \frac{d}{1-d},$$
$$a_{12} = -\frac{1}{2}. \qquad (40)$$

D. Discussion

The amplitude computed in Sec. III C possesses several striking features. First, the amplitude is gauge-invariant, as it must be since it is directly related to an observable, the cross section.

Second, we note that the interpretation of (35) as a scattering amplitude is to be carefully considered. Due to the difficulty of defining asymptotic states in a constantly curved background, we focus on the S matrix in the flat limit. Using the equation of motion (8), we find that the Mandelstam operators are related to the usual Mandelstam variables by $\mathfrak{s} \propto s/\Lambda$, $\mathfrak{t} \propto t/\Lambda$. The amplitude A, in the limit $\Lambda \to 0$, has the leading order behavior

$$A \propto \alpha_g \left[\frac{(s+t)t}{\Lambda^2} + \mathcal{O}(\Lambda^{-1}) \right]$$
$$\propto G_N [(s+t)t\Lambda^{\frac{d-6}{2}} + \mathcal{O}(\Lambda^{\frac{d-4}{2}})]. \qquad (41)$$

Subleading contributions of $\mathcal{O}(\Lambda^{\frac{d-4}{2}})$ depend on the ordering of the operators, and are therefore nonuniversal.

The momentum scaling, and correspondingly the scaling with Λ, in the flat limit can also be derived by power counting of derivatives in the action. We observe that the propagator scales like $\propto E^{-2}$, where E denotes the center-of-mass energy. The vertices each contribute a factor of E^3, yielding an amplitude $\propto E^4$. Since the amplitude is dimensionless, this has to be compensated by the only dimensionful quantity that we have at our disposal, namely Λ. This is in contrast to metric gravity, where the vertex scales as E^2.

Specializing to $d = 4$, we find that the amplitude (41) resembles the amplitude obtained in GR, $A^{\mathrm{GR}} \propto \frac{(s+t)t}{s}$, except that the factor $1/s$ is replaced by $1/\Lambda$. The similarity is affirmed by performing a partial-wave analysis of (41). The resulting spectrum contains contributions with spin 0 and spin 2, in agreement with GR.

Since the amplitude diverges as Λ^{-1}, the flat space limit in $d = 4$ is formally ill defined. Inserting the observed values of Λ and G_N, we observe that the amplitude becomes of order unity around the meV scale. This implies that quantum gravitational effects would become relevant at this scale. Taking the scaling with energy at face value, this would rule out the theory experimentally.

Nevertheless, there are at least two reasons why this result might still be modified. First, using dimensional analysis, it is straightforward to deduce that loop diagrams of any order contribute to the flat limit. It is therefore conceivable that a resummation of loop diagrams will ameliorate the divergent behavior. Second, the amplitude can be modified by introducing nonminimal interactions [61,62]. While this may not cancel the divergence in general, this could be achieved by a fine-tuning of the couplings, or be imposed automatically by a fixed point of the theory's RG.

IV. SUMMARY AND OUTLOOK

In this paper, we have given a modern perspective on affine gravity. This theory is classically equivalent to GR, but its dynamical degrees of freedom are based on the connection rather than the metric. In affine gravity, the cosmological constant Λ has the role of an integration constant, providing a possible explanation for its small observed value.

The main result of this work comprises the tree-level amplitude functional (35) of a two-scalar-to-two-scalar process in a curved background, giving insight into the quantum degrees of freedom of the theory. Taking the flat-space limit, we find that, in four dimensions, the amplitude scales like Λ^{-1}. While this is worrying at first sight, we propose that the introduction of nonminimal interactions and loop corrections may ameliorate this behavior. The computation of these modifications is left for future work. The calculation also shows that in the flat space limit, the off shell gauge-invariant degrees of freedom of the theory have a spin two and a spin zero contribution, in agreement with metric gravity. At finite curvature, this structure appears to be more intricate.

A systematic search of nonminimal interactions can be done in terms of form factors, which can be successfully used to parametrize metric actions [61–63]. In affine gravity, the occurrence of the inverse symmetric Ricci tensor $\Re^{\mu\nu}$ prohibits an expansion in powers of the curvature tensor. A first step in investigating interactions can be done in $d = 3$ dimensions, where the Riemann tensor is fully determined by $R_{\mu\nu}$ only.

Due to the treatment of the affine connection as a standard gauge field, the affine formalism lends itself to making a relation to other approaches. First, affine gravity is straightforwardly generalized to include torsion. Second, treating the affine connection and gauge fields on equal footing, one could speculate about grand unification scenarios of gravity. Finally, affine gravity may allow for a lattice formulation, opening up new computational toolkits to investigate the quantum properties of this theory.

ACKNOWLEDGMENTS

APPENDIX: COMMUTATORS

To sort the scattering amplitude into a canonical form, we need to know how we can commute functions of the operator Д with covariant derivatives. In alignment with the assumptions of the main text, we will consider a covariantly constant Ricci tensor,

$$D_\mu R_{\alpha\beta} = 0, \qquad (A1)$$

together with a maximally symmetric Riemann tensor of the form

$$R_{\mu\nu\rho}{}^\sigma = \frac{1}{d-1}(R_{\mu\rho}\delta_\nu{}^\sigma - R_{\nu\rho}\delta_\mu{}^\sigma). \qquad (A2)$$

As a direct consequence, we find that

$$Я^{\mu\rho}R_{\mu\nu\rho}{}^\sigma = \delta_\nu{}^\sigma. \qquad (A3)$$

This equality indeed follows directly from a covariantly constant Ricci tensor without invoking (A2), since

$$0 = Я^{\mu\kappa}Я^{\lambda\tau}[D_\mu, D_\nu]R_{\kappa\lambda} = \delta_\nu{}^\tau - Я^{\kappa\lambda}R_{\kappa\nu\lambda}{}^\tau. \qquad (A4)$$

The basic commutators are

$$[D_\mu, D_\nu]X_\rho = R_{\mu\nu\rho}{}^\alpha X_\alpha = \frac{1}{d-1}(R_{\mu\rho}X_\nu - R_{\nu\rho}X_\mu), \qquad (A5)$$

and

$$[D_\mu, D_\nu]Y^\rho = -R_{\mu\nu\alpha}{}^\rho Y^\alpha,$$
$$= -\frac{1}{d-1}(R_{\mu\alpha}\delta_\nu{}^\rho Y^\alpha - R_{\nu\alpha}\delta_\mu{}^\rho Y^\alpha). \qquad (A6)$$

By the assumption that the Ricci tensor is covariantly constant, we have

$$[D_\mu, D_\nu]Y^\rho = [D_\mu, D_\nu]Я^{\rho\alpha}R_{\alpha\beta}Y^\beta,$$
$$= Я^{\rho\alpha}[D_\mu, D_\nu](RY)_\alpha, \qquad (A7)$$

thus it is enough to derive formulas for tensors with only lower indices.

We want to derive a formula of the form

$$f(Д)D_\alpha \mathbf{X} = D_\alpha f(Д)\mathbf{X} + ..., \qquad (A8)$$

where f is some arbitrary function and $\mathbf{X}$ is a tensor of rank $(0, n)$,

$$\mathbf{X} = X_{\mu_1\cdots\mu_n}. \qquad (A9)$$

To derive this equation, we employ a standard trick and write the function f as an inverse Laplace transform, so that we can use the Baker-Campbell-Hausdorff formula,

$$f(Д)D_\alpha\mathbf{X} = \int_0^\infty ds\, \tilde{f}(s)e^{-sД}D_\alpha\mathbf{X},$$
$$= \int_0^\infty ds\, \tilde{f}(s)\sum_{\ell\geq 0}\frac{(-s)^\ell}{\ell!}[Д, D_\alpha]_\ell e^{-sД}\mathbf{X}. \qquad (A10)$$

Here, we used the multicommutator, which is defined recursively by

$$[A, B]_n = [A, [A, B]_{n-1}], \qquad [A, B]_0 = B. \qquad (A11)$$

To make progress, let us first calculate the standard commutator. Using the standard rules for commutators and our assumptions of covariant constancy and maximal symmetry, we find

$$[Д, D_\alpha]X_{\mu_1\cdots\mu_n} = -D_\alpha X_{\mu_1\cdots\mu_n} - \frac{2}{d-1}\sum_{k=1}^n [D_{\mu_k}X_{\mu_1\cdots\mu_{k-1}\alpha\mu_{k+1}\cdots\mu_n} - Я^{\kappa\lambda}R_{\alpha\mu_k}D_\kappa X_{\mu_1\cdots\mu_{k-1}\lambda\mu_{k+1}\cdots\mu_n}]$$
$$\equiv C_{\alpha\mu_1\ldots\mu_n}{}^{\beta\nu_1\ldots\nu_n}D_\beta X_{\nu_1\ldots\nu_n}. \qquad (A12)$$

Here, we made the observation that the commutator is in fact a multiplication with a covariantly constant tensor **C**, so that structurally we find,

$$[Д, D]_k \mathbf{X} = \mathbf{C}^k D \mathbf{X}. \qquad (A13)$$

Plugging this into the original equation (A10), we find

$$f(Д)D\mathbf{X} = \int_0^\infty ds \, \tilde{f}(s) \, \mathrm{Texp}\,[-s\mathbf{C}] \, De^{-s\,Д}\mathbf{X}. \quad (A14)$$

Here, Texp is the tensor exponential, defined in terms of its power series.

While the calculation of this exponential for an arbitrary rank tensor seems difficult, we can easily calculate it for tensors of low rank, and, in practice, this is all that is needed. We will illustrate this for scalars and vectors. For a scalar,

$$[Д, D_\alpha]X = -D_\alpha X, \qquad (A15)$$

so that

$$C_\alpha{}^\beta = -\delta_\alpha{}^\beta. \qquad (A16)$$

Correspondingly, we have

$$f(Д)D_\alpha X = D_\alpha f(Д - 1)X. \qquad (A17)$$

This emphasizes again a structural difference to standard metric-based calculations—since Д is dimensionless, such commutator formulas involve shifts by constants instead of shifts by curvatures. In complete analogy, for a vector, we have

$$C_{\mu\nu}{}^{\alpha\beta} = -\delta_\mu{}^\alpha \delta_\nu{}^\beta - \frac{2}{d-1}\delta_\mu{}^\beta \delta_\nu{}^\alpha + \frac{2}{d-1}R_{\mu\nu}Я^{\alpha\beta}, \qquad (A18)$$

which gives

$$
\begin{aligned}
f(Д)D_\mu X_\nu = {}& D_{(\mu} f\left(Д - \frac{d+1}{d-1}\right) X_{\nu)} \\
& + \frac{1}{d} R_{\mu\nu} Я^{\alpha\beta} D_\alpha \left[f(Д+1) - f\left(Д - \frac{d+1}{d-1}\right) \right] X_\beta \\
& + D_{[\mu} f\left(Д - \frac{d-3}{d-1}\right) X_{\nu]}.
\end{aligned}
\qquad (A19)
$$

We shall not give the formula for a rank two tensor, since it is rather lengthy. The formula can be found in the *Mathematica* notebook attached to this work.

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