

Possible connection between the reflection symmetry and existence of equatorial circular orbit

Sayak Datta^{*} and Sajal Mukherjee[†]

Inter-University Centre for Astronomy and Astrophysics, Post Bag 4, Pune-411007, India

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We study a viable connection between the circular-equatorial orbits and reflection symmetry across the equatorial plane of a vacuum stationary axis-symmetric spacetime in general relativity. The behavior of the circular equatorial orbits in the direction perpendicular to the equatorial plane is studied, and different outcomes in the presence and in the absence of the reflection symmetry are discussed. We conclude that in the absence of the equatorial reflection symmetry neither stable nor unstable circular orbit can exist on the equatorial plane. Moreover, to address the observational aspects, we provide two possible examples relating gravitational wave astronomy and the thin accretion disk which can put constraints on the symmetry breaking parameters.

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I. INTRODUCTION

The Kerr metric uniquely describes a stationary, axis-symmetric, and asymptotically flat black hole (BH) solution of vacuum Einstein's field equations in four dimensions (assuming the regularity on and outside of the horizon) [1–5]. Besides the stationary and axis-symmetry properties, Kerr spacetime is also endowed with an additional feature of reflection symmetry across the equatorial plane. Even if the former characteristics are likely to be associated with astrophysical objects with rotation, both asymptotic flatness and equatorial reflection symmetry can be relaxed in order to probe a larger domain of compact objects other than BH. Various possible distinctions between BHs and other exotic compact objects based on tidal deformability [6–9], tidal heating [8,10–13], multipole moments [10,14], *echoes* in postmerger [15–20], and electromagnetic observations [20–24] have been proposed in the literature. Similarly, distinguishing them on the basis of equatorial reflection symmetry can be useful to detect them or rule them out as viable astrophysical bodies. Not only these studies provide a fresh outlook to model astrophysical objects, but they also assign an observational impact to it. In the present article, we aim to elaborate on the equatorial reflection symmetry in a generic spacetime and outline its possible theoretical and observational implications in depth.

To study any particular effect appearing from spacetime geometry, the ideal approach is, to begin with, the orbital dynamics. Based on how orbits behave in a given spacetime, more involved astrophysical searches are constructed.

In Kerr, the orbital properties are well studied and extensively explored in literature [25–27]. Similar exploration is carried out for Kerr-NUT spacetime as well [28–30]. NUT charge introduces a unique extension of the Kerr spacetime, and violates the equatorial reflection symmetry and describes an asymptotically nonflat geometry [31,32]. While in Kerr we know stable/unstable equatorial circular orbits exist, the same is not true in the presence of NUT charge. In particular, neither stable nor unstable equatorial circular orbits can exist for massive or massless particles in Kerr-NUT spacetime [28,29]. In fact, this stark contrast between Kerr and Kerr-NUT is the primary source of our motivation to study further and investigate whether equatorial reflection symmetry and the existence of the equatorial circular orbits can be generically connected. We introduce a perturbative approach for confronting equatorial circular geodesics in geometries where equatorial reflection symmetry is absent. By assuming that the orbits reside on the equatorial plane initially, we study the growth of perturbation in time and aim to realize which parameters engineer any possible deviation from the equatorial plane. Assuming the perpendicular to $\theta = \pi/2$ is along the z axis, we consider the z perturbation in our work. Because of the absence of the equatorial reflection symmetry, it is likely that the potential will not be an even function of z . As a result, there will be an intrinsic force in the z direction that will lift the orbits from the equatorial plane. To study this in the context of a general stationary axis-symmetric metric, we will consider Ernst's potential and write metric components in terms of it.

The existence of the planner circular orbits is crucial for discussing physical effects relevant for various astrophysical models, such as binary, spectra of accreting black holes, etc. The orbits in extreme mass ratio inspirals, which will

^{*}sayak@iucaa.in
[†]sajal@iucaa.in

be observed with LISA [33], are most likely to be generic [34–36]. Binaries with stellar masses, as already detected by gravitational wave detectors LIGO and VIRGO, can have precession [37]. This means that the understandings found in the current paper will not only have theoretical grounds but also will have an observational impact, which will be discussed later.

The rest of the manuscript is organized as follows. In Sec. II, we start with the Ernst potential, and in Sec. III, we will briefly discuss the geodesic equations in terms of the metric components. The primary findings of the paper are given in Secs. IV and V, respectively. In Sec. VI, we will study the observational impacts of current findings, and finally, we will conclude in Sec. VII.

II. METRIC COMPONENTS AND THE ERNST POTENTIAL

We start with a stationary, axis-symmetric, and vacuum spacetime written in cylindrical coordinates (t, ρ, z, ϕ) within general relativity as [38,39]

$$ds^2 = -F(dt - \omega d\phi)^2 + \frac{1}{F}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2], \quad (1)$$

where F , ω , and γ are functions of ρ and z . Substituting the metric in the Einstein equation, it is possible to find governing equations for these entities. In passing, we should note that the above metric may or may not be asymptotically flat and remains general otherwise.

The above metric components can be written in a more compact form by using the complex Ernst potential, which is a combination of both a *norm* (λ) and *twist* (ω) timelike Killing vector. In particular, $\lambda = -g_{tt} = F$, and $\omega_\mu = \sqrt{-g}\epsilon_{\mu\nu\gamma\delta}\xi^\nu\nabla^\gamma\xi^\delta$, where ξ^μ is the timelike killing vector [39]. Given that the spacetime is stationary and axis symmetric, both norm and twist are expected to be nonzero. Finally, the Ernst potential takes the form

$$\mathcal{E} = F + i\psi = \frac{(\rho^2 + z^2)^{1/2} - \tilde{\xi}}{(\rho^2 + z^2)^{1/2} + \tilde{\xi}}, \quad (2)$$

where $\tilde{\xi}$ can be written as [40]

$$\tilde{\xi} = \sum_{j,k=0}^{\infty} a_{jk} \frac{\rho^j z^k}{(\rho^2 + z^2)^{j+k}}. \quad (3)$$

The reasons to choose Ernst's potential formalism as a tool to express metric components are twofold. First, the metric components can be written directly in terms of \mathcal{E} . As a result, from the behavior of \mathcal{E} under the absence of the symmetries, the nature of the orbits can easily be extracted. Second, reflection symmetry manifests itself through the values of a_{jk} , making it easier to impose the presence or the absence of equatorial reflection symmetry. The a_{jk} is

nonzero only for non-negative, even j and non-negative k . If there is reflection symmetry across the equatorial plane, then a_{jk} is real for even k and imaginary for odd k [38,40–42]. However, as the present study remains general as far as the equatorial reflection symmetry is concerned, we restrain ourselves from making such assumptions. We assume that a_{jk} has both real and imaginary components for both even and odd values of k .

In terms of F and ψ , the metric components g_{tt} , $g_{t\phi}$, and $g_{\phi\phi}$, which will be of particular use, can be written as [38]

$$g_{tt} = -F, \quad g_{\phi\phi} = (g_{t\phi}^2 - \rho^2)/g_{tt},$$

$$g_{t\phi} = -F \int_{z=\text{const}} \frac{\rho}{F^2} \frac{\partial\psi}{\partial z} d\rho + F \int_{\rho=\text{const}} \frac{\rho}{F^2} \frac{\partial\psi}{\partial\rho} dz, \quad (4)$$

where the detailed calculations to arrive at the above expressions are shown in Appendix A.

III. OFF-EQUATORIAL PERTURBATION OF THE EQUATORIAL GEODESICS

To study the existence of equatorial circular orbits in a generic spacetime with metric given in Eq. (1), we start with the geodesic equations

$$\ddot{\rho} = \frac{1}{2} \frac{\partial\mathcal{V}_\rho}{\partial\rho}, \quad \ddot{z} = \frac{1}{2} \frac{\partial\mathcal{V}_z}{\partial z}, \quad (5)$$

where the dot defines a derivative with respect to the affine parameter which we may call τ and \mathcal{V}_ρ and \mathcal{V}_z are radial and angular potentials, respectively. The above equations will determine the locations of an orbit, while the conserved energy and momentum are dictated by t and ϕ components. Given that we are interested in orbits confined on a plane and circular in nature, the above two equations would give $\dot{\rho} = \dot{z} = 0$ in principle. However, as the equatorial reflection symmetry is not respected, the potential \mathcal{V}_z is likely to contain terms with the odd power of z such that $\mathcal{V}_z(z) \neq \mathcal{V}_z(-z)$. One should be mindful that the condition of circularity is not expected to be affected by the equatorial reflection symmetry, and we may safely impose $\dot{\rho} = \dot{z} = 0$. From the timelike constraint, $\mathcal{U}^\alpha\mathcal{U}_\alpha = -1$, we arrive at the expression

$$\mathcal{U}^t\mathcal{U}_t + \mathcal{U}^\rho\mathcal{U}_\rho + \mathcal{U}^z\mathcal{U}_z + \mathcal{U}^\phi\mathcal{U}_\phi = -1, \quad (6)$$

and finally [43]

$$\mathcal{V}_z = g_{zz}(\mathcal{U}^z)^2 = -1 - g^{tt}E^2 - g^{\phi\phi}L_z^2 - 2g^{t\phi}EL_z, \quad (7)$$

where E and L_z are given as conserved energy and momentum, respectively, appearing from the spacetime symmetries. We will expand \mathcal{V}_z about the equatorial plane, i.e., $z = 0$. Neglecting terms approximately $\mathcal{O}(\delta z)^3$ and beyond, i.e., $\mathcal{V}_z(\delta z) = \mathcal{V}_0 + (\delta z)\mathcal{V}_1 + (\delta z)^2\mathcal{V}_2$, we may rewrite z equation in Eq. (5) as follows:

$$2\ddot{\delta z} = \mathcal{V}_1 + 2\mathcal{V}_2\delta z. \quad (8)$$

By solving the above equation, we arrive at

$$\delta z(\tau) = \frac{\mathcal{V}_1}{2\omega^2} + A \exp[-i\omega\tau] + B \exp[i\omega\tau], \quad (9)$$

where we set $\mathcal{V}_2 = -\omega^2$, and ω can be both real and imaginary. If we assume that the initial conditions are $\delta z(0) = \dot{\delta z}(0) = 0$, the above equation may be rewritten as

$$\delta z(\tau) = -\frac{\mathcal{V}_1}{2\omega^2} + \frac{\mathcal{V}_1}{4\omega^2} \{\exp(-i\omega\tau) + \exp(i\omega\tau)\}. \quad (10)$$

Depending on the value of ω , the above equation either represent a hyperbola ($\text{Im}[\omega] \neq 0$), and a oscillatory ($\text{Im}[\omega] = 0$) motion. For the Kerr-NUT spacetime, as we have shown in Appendix B, $\text{Im}[\omega] = 0$, and the solution is always oscillatory. For future purposes, we may note that in case of a oscillatory solution we have

$$\delta z(\tau) = -\frac{\mathcal{V}_1}{\omega^2} \sin^2(\omega\tau/2), \quad (11)$$

which hints at an interesting property of this motion written as follows. For a particular radius, this perturbation is either positive or negative depending on the sign of \mathcal{V}_1 , but never switches sign. It indicates that it would never cross the equatorial plane but approach it in each cycle. In short, the hobbling from the equatorial plane would be one sided.

It should be mentioned that $\delta z(\tau)$ in Eq. (11) vanishes at $\tau = 0$ which reflects that no external perturbation is applied. Any additional external perturbation may result in some changes in the final expression, which we have not studied in this article. Besides, one should also note how \mathcal{V}_1 and \mathcal{V}_2 are affecting the perturbation. While \mathcal{V}_1 is directly proportional to its value, \mathcal{V}_2 is responsible for engineering its nature. By setting $\mathcal{V}_2 = 0$, we obtain a diverging nature of the perturbation, as $\delta z(\tau) \sim \tau^2$, and may not appropriately model $\delta z(\tau)$. Keeping these points in mind, we will keep \mathcal{V}_1 and \mathcal{V}_2 in the expression and ignore higher-order corrections. Eventually, we will show that nonzero \mathcal{V}_1 is connected to the absence of equatorial reflection symmetry, which, as a result, does not allow a circular orbit to exist in the equatorial plane, not even perturbatively.

IV. DECOMPOSITION OF THE ERNST POTENTIAL

Referring to Eq. (3), we may state that equatorial reflection symmetry in the potential comes through a_{jk} . For a clear exposition of our results, we separate out the real and imaginary parts of a_{jk} , i.e.,

$$a_{jk} = \hat{a}_{jk} + i\check{a}_{jk}, \quad (12)$$

where \hat{a}_{jk} and \check{a}_{jk} are the real and imaginary parts of the a_{jk} , respectively. If the equatorial reflection symmetry exists, then a_{jk} is real for even k and imaginary for odd k [38,40–42],

$$\check{a}_{j(2m)} = 0, \quad \hat{a}_{j(2m+1)} = 0, \quad (13)$$

for all non-negative values of m . However, in the present context, we should note again that the above equations are not valid and are expected to be nonzero. With the above expressions in hand, we now attempt to connect the potential with a_{jk} and start with decomposing $\tilde{\xi}$ in real and imaginary parts as follows:

$$\tilde{\xi} = \hat{\xi} + i\check{\xi}. \quad (14)$$

Therefore, we gather

$$\hat{\xi} = \sum_{j,k=0}^{\infty} \hat{a}_{jk} \frac{\rho^j z^k}{(\rho^2 + z^2)^{j+k}}, \quad \check{\xi} = \sum_{j,k=0}^{\infty} \check{a}_{jk} \frac{\rho^j z^k}{(\rho^2 + z^2)^{j+k}}. \quad (15)$$

By using Eqs. (3), (14), and (15), we may arrive at the expression

$$F = \frac{R - \hat{\xi}^2 - \check{\xi}^2}{(R^{1/2} + \hat{\xi})^2 + \check{\xi}^2}, \quad \psi = \frac{-2R^{1/2}\check{\xi}}{(R^{1/2} + \hat{\xi})^2 + \check{\xi}^2}, \quad (16)$$

where $R = \rho^2 + z^2$. For our purpose, we need to understand the properties of the $\partial\mathcal{V}_z/\partial z$ up to order of z , and considering \mathcal{V}_z up to order of z^2 . This would need the knowledge of g^{tt} , $g^{\phi\phi}$, $g^{\phi\phi}$, F , and $g_{t\phi}$ up to the order of z^2 and ψ up to the order of z^3 . Therefore, we need contribution from \hat{a}_{j0} , \hat{a}_{j1} and \check{a}_{j0} , \check{a}_{j1} for our analysis. For the expansion, we take $z/\rho \ll 1$, and as a result, it is not necessary for z to be very small as long as $z \ll \rho$ is satisfied. Keeping up to the order of z^3/ρ^3 , we rewrite $\hat{\xi}$ and $\check{\xi}$ as follows:

$$\hat{\xi} = \sum_j \left[\frac{\hat{a}_{j0}}{\rho^j} \left\{ 1 - j \frac{z^2}{\rho^2} \right\} + \frac{\hat{a}_{j1}}{\rho^{j+1}} \left\{ \frac{z}{\rho} - (j+1) \frac{z^3}{\rho^3} \right\} \right],$$

$$\check{\xi} = \sum_j \left[\frac{\check{a}_{j0}}{\rho^j} \left\{ 1 - j \frac{z^2}{\rho^2} \right\} + \frac{\check{a}_{j1}}{\rho^{j+1}} \left\{ \frac{z}{\rho} - (j+1) \frac{z^3}{\rho^3} \right\} \right]. \quad (17)$$

Using the results found in this section, we will find the $\partial\mathcal{V}_z/\partial z$ in the next section.

V. RESULTS

In this section, we express the relevant quantities as a series expansion in z and only keep terms up to z^2 terms in the potential. Based on our earlier discussions, here, we will

derive \mathcal{V}_1 for a general stationary, axis-symmetric metric. By substituting Eq. (17) into Eq. (16), we arrive at

$$F = \frac{1 + \sum_0^3 g_i z^i / \rho^i}{1 + \sum_0^3 f_i z^i / \rho^i} \equiv \sum_0^3 F_i z^i, \\ \psi = \frac{\sum_0^3 \Psi_i z^i / \rho^i}{\rho^2 (1 + \sum_0^3 f_i z^i / \rho^i)} \equiv \sum_{i=0}^3 \Psi_i \frac{z^i}{\rho^i}, \quad (18)$$

where the coefficients can be expressed as follows:

$$f_0 = \sum_j \hat{a}_{j0} \frac{2}{\rho^{j+1}} + \left[\sum_{j,j'} \frac{1}{\rho^{j+j'+2}} \{ \hat{a}_{j0} \hat{a}_{j'0} + \hat{a} \rightarrow \check{a} \} \right], \\ f_1 = \sum_j \hat{a}_{j1} \frac{2}{\rho^{j+2}} + \left[\sum_{j,j'} \frac{1}{\rho^{j+j'+3}} 2 \{ \hat{a}_{j0} \hat{a}_{j'1} + \hat{a} \rightarrow \check{a} \} \right] \\ f_2 = 1 + \sum_j -\hat{a}_{j0} \frac{2}{\rho^{j+1}} \left(j - \frac{1}{2} \right) + \left[\sum_{j,j'} \frac{1}{\rho^{j+j'+2}} \right. \\ \left. \times \left\{ -\hat{a}_{j0} \hat{a}_{j'0} (j + j') + \frac{\hat{a}_{j1} \hat{a}_{j'1}}{\rho^2} + \hat{a} \rightarrow \check{a} \right\} \right] \\ g_0 = \sum_j \hat{a}_{j0} \frac{2}{\rho^{j+1}} - f_0, \quad g_1 = \sum_j \hat{a}_{j1} \frac{2}{\rho^{j+2}} - f_1 \\ g_2 = \sum_j -\hat{a}_{j0} \frac{2}{\rho^{j+1}} \left(j - \frac{1}{2} \right) + 2 - f_2 \\ \psi_0 = \sum_j \check{a}_{j0} \frac{2}{\rho^{j-1}}, \quad \psi_2 = \sum_j -\check{a}_{j0} \frac{2}{\rho^{j-1}} \left(j - \frac{1}{2} \right), \\ \psi_1 = \sum_j \check{a}_{j1} \frac{2}{\rho^j}, \quad \psi_3 = \sum_j -\check{a}_{j1} \frac{2}{\rho^j} \left(j + \frac{1}{2} \right) \quad (19)$$

$$F_0 = \frac{1+g_0}{1+f_0}, \quad F_1 = \frac{g_1(1+f_0) - f_1(1-g_0)}{(f_0+1)^2}, \\ F_2 = \frac{g_2(1+f_0)^2 + (f_2 g_0 + g_1 f_1 - f_2)(1+f_0) + f_1^2(1+g_0)}{(f_0+1)^3}, \\ \Psi_0 = \frac{\psi_0}{\rho^2(1+f_0)}, \quad \Psi_1 = \frac{\psi_1(1+f_0) - f_1 \psi_0}{\rho^2(1+f_0)^2} \\ \Psi_2 = \frac{(1+f_0)^2 \psi_2 - (f_1 \psi_1 + f_2 \psi_0)(1+f_0) + f_1^2 \psi_0}{\rho(1+f_0)^3} \quad (20) \\ \Psi_3 = \frac{(1+f_0)^3 \psi_3 - (1+f_0)^2 (f_1 \psi_2 + f_2 \psi_1 + f_3 \psi_0)}{\rho^2(1+f_0)^4} \\ + \frac{(1+f_0)(f_1^2 \psi_1 + 2f_1 f_2 \psi_0) - \psi_0 f_1^3}{\rho^2(1+f_0)^4}, \quad (21)$$

which are functions of ρ only and independent of z . By employing these expressions, we can obtain the derivatives of the metric components, g_{tt} , $g_{t\phi}$, and $g_{\phi\phi}$ [given in Eq. (4)], which are essential for our study. We start with

the following expression for $g_{t\phi}$ and obtain the derivative of $g_{t\phi}$ and $g_{\phi\phi}$,

$$g_{t\phi} = -F \sum_i z^i \int I_i d\rho' + N = -F \sum_i z^i \mathcal{I}_i + \sum_i N_i z^i, \quad (22)$$

$$\rho^2 \frac{\partial g^{t\phi}}{\partial z} = -(F_1 \mathcal{I}_0 - N_1 + F_0 \mathcal{I}_1) \\ - 2z(F_1 \mathcal{I}_1 + F_2 \mathcal{I}_0 - N_2 + F_0 \mathcal{I}_2), \\ \frac{\partial g^{\phi\phi}}{\partial z} = \frac{1}{\rho^2} [F_1 + 2F_2 z], \quad (23)$$

where we have used Eq. (18), and the expressions for N_1 , N_2 , I_0 , I_1 , and I_2 are given as

$$N_1 = \frac{\rho \Psi_0'}{F_0}, \\ N_2 = \frac{\Psi_1'}{2F_0} - \frac{\Psi_1}{2\rho F_0}, \\ I_0 = \frac{\Psi_1}{F_0^2}, \quad I_1 = \frac{2\Psi_2 F_0 - 2F_1 \Psi_1 \rho}{\rho F_0^3}, \\ I_2 = \frac{3\Psi_3 F_0^2 - 4F_1 \Psi_2 F_0 \rho - 2F_2 F_0 \Psi_1 \rho^2 + 3F_1^2 \Psi_1 \rho^2}{F_0^4 \rho^2}, \quad (24)$$

where $\Psi_i' = \frac{\partial \Psi_i}{\partial \rho}$. It is now easy to evaluate the derivative of g_{tt} by using Eqs. (23) and (25). Finally, we can obtain the expression for $\partial \mathcal{V}_z(z)/\partial z$ by using Eq. (7) and the derivatives of the metric components. The result is as follows:

$$\mathcal{V}_1 = -E^2 \left(-\frac{\mathcal{I}_0^2 F_1}{\rho^2} + \frac{2\mathcal{I}_0 \mathcal{I}_1 F_0}{\rho^2} + \frac{F_1}{F_0^2} - \frac{2\mathcal{I}_0 N_1}{\rho^2} + 2\frac{\mathcal{I}_0^2 F_1}{\rho^2} \right) \\ - \frac{L_z^2}{\rho^2} F_1 + 2\frac{EL_z}{\rho^2} (F_0 \mathcal{I}_1 + F_1 \mathcal{I}_0 - N_1). \quad (25)$$

It is easy to notice that in general $\mathcal{V}_1 \neq 0$ from Eq. (25). The consequence of nonzero \mathcal{V}_1 has already been discussed in Sec. III. It may be possible that, even though the z -independent term is nonzero for each of the derivatives, in some special cases, these terms may add up to give a vanishing z -independent term. In particular, one may ask whether it is possible to choose conserved energy and momentum in such a way that it would result in $\mathcal{V}_1 = 0$. Indeed, we find this can be a possibility; however, both the energy and momentum need to be in consonance with $\dot{\rho} = \dot{\phi} = 0$, too, which can put the further restriction in its motion. For example, in the Kerr-NUT spacetime, it is not possible to choose energy and momentum in such a way that it would describe a circular geodesic confined on the

equatorial plane [29]. Therefore, while this is a valid possibility, it does not describe a general outcome of our findings.

For a quick follow-up of the above analysis where the equatorial reflection symmetry is respected, we focus on the z -independent term. Here, the z -independent part depends only on F_0, F_1, I_0, I_1 , and N_1 . When equatorial reflection symmetry is present, $\check{a}_{j(2m)} = 0$ and $\hat{a}_{j(2m+1)} = 0$; hence, $f_1 = g_1 = \psi_0 = \psi_2 = 0$. This implies $F_1 = \Psi_0 = \Psi_2 = I_1 = \mathcal{I}_1 = N_1 = 0$. Therefore, the \mathcal{V}_1 in Eq. (25) vanishes. It is remarkable that in the case of equatorial reflection symmetry the z^0 terms in every term that constitutes $\frac{\partial \mathcal{V}_1}{\partial z}$ vanish and, as a result, \mathcal{V}_1 vanishes. This indicates that the presence of circular equatorial orbit implies the presence of the reflection symmetry.

VI. OBSERVATIONAL PROSPECTS AND POSSIBLE CONSTRAINTS

We explore a possible connection between the equatorial reflection symmetry and equatorial circular orbits and arrive at the conclusion that in the absence of the equatorial reflection symmetry circular orbit cannot exist on the equatorial plane. The parameters which break the reflection symmetry will engineer to elevate the orbit from the equatorial plane and boost it with a force. However, if we assume these parameters are small enough, we can still naively assume the orbits to be equatorial and still perform some astrophysical calculations [44]. Nonetheless, this would introduce a test bed to execute several observational expeditions to confirm the existence of stable orbits and equatorial reflection symmetry. In fact, the observation of stable circular equatorial orbit will be a telltale signature of the presence of reflection symmetry or a very mild violation of it. For the present purpose, we outline a few such examples where the violation of equatorial reflection symmetry may be detectable by observation. For other theoretical and observable impacts, check Refs. [45–47].

A. Gravitational wave astronomy

Consider a binary system prepared in such a way that the spins of the components are either aligned or antialigned with the orbital angular momentum. With time due to the push from \mathcal{V}_1 , the spins will not stay (anti)aligned even if they were prepared in an (anti)aligned manner. This will result in a nonzero in-plane spin component (χ_p) (check Ref. [48] for the definition). Therefore, for equatorial reflection symmetry violating bodies in a binary, χ_p measurement should be nonzero. Hence, nonzero χ_p can arise in several different ways. One is due to the formation mechanism of binaries, which has components that respect equatorial reflection symmetry yet introduce a nonzero χ_p possibly due to spin effects. Another reason for nonzero χ_p would be due to the absence of equatorial reflection

symmetry. This means that there will be a degeneracy between the formation channel and equatorial reflection symmetry violation, and it becomes uncertain to arrive at a unique conclusion.

This possible degeneracy can be broken by measuring the multipole moments of a compact object. It is well known that for axis-symmetric bodies there can be two sets of multipole moments: one is the mass moment (M_l), and another is the current moment (S_l). For a metric solution with the equatorial reflection symmetry like Kerr, both the odd mass and even current moments would identically vanish. However, for a simple illustration of these moments in a Kerr-NUT spacetime which is known to break the equatorial reflection symmetry, one immediately notices that all the orders for mass and current multipole moments would survive [44]. From an observation perspective, there now exists contemporary tools to measure the quadrupole moment in a binary [10,14]. If reflection symmetry is violated, then it is likely that the metric will have nonzero M_{2l+1} and/or S_{2l} , i.e., classes of fuzzball solutions [49–56]. Hence, measuring a nonzero S_2 will be a signature of breaking of reflection symmetry, along with the nonzero χ_p observation.

B. Constraints from the accretion disk

If the metric of these objects does not respect equatorial reflection symmetry, then there should be some imprint of such violation on the matter distribution around it. In such cases, depending on the value of \mathcal{V}_1 , we may expect the matter to be distributed in off-equatorial planes. This, as a result, can give a possible opportunity to constraint \mathcal{V}_1 from observation. For example, if we assume that there is a deviation from the reflection symmetry, we gather from Eq. (11) that farthest a particle can go from the equatorial plane is $\mathcal{D} \sim |\frac{\mathcal{V}_1}{\omega^2}|$. Depending on the model, the radius of an accretion disk around a BH can have radius $r \sim R(\frac{M}{M_\odot})$, where $R \sim (10 - 10^6)$ km. Traditional thin accretion disks can have scale height $h \sim .01r$ [57,58]. Therefore, to ensure a disk structure consistent with most of the observation, we have $\mathcal{D} < h$, which translates to

$$\left| \frac{\mathcal{V}_1}{\omega^2} \right| < 10^4 \text{ km} \left(\frac{\zeta}{.01} \right) \left(\frac{R}{10^6 \text{ km}} \right) \left(\frac{M}{M_\odot} \right), \quad (26)$$

where $\zeta = h/r$. For a given accretion disk, we may be able to constrain the reflection breaking parameters from observation, which may provide useful information about the central object. Since the accretion phenomenon is observed with x-ray observations, it requires investigating if it is possible to probe equatorial reflection symmetry from the x-ray observations. The observed time variability in the x-ray flux emitted by accreting compact objects (i.e., quasiperiodic oscillations [59–62]) can possibly shed some light in this regard. Currently, the underlying mechanism is

not very well understood, except the belief that these fluxes originate from the innermost region of the accretion flow [63]. Since the physics of accretion disks is very complex, it is challenging to extract accurate information. We will leave such studies for the future.

VII. CONCLUSION

We have studied the perturbation of the circular-equatorial orbits of a general stationary axis-symmetric metric in the z direction. In the process, we have identified a set of parameters that are the potential source of the equatorial reflection symmetry violation, namely, $\check{a}_{j(2m)}$ and $\hat{a}_{j(2m+1)}$. We have shown that, in general, when these parameters are zero (nonzero) equatorial reflection symmetry is present (absent), and the small oscillation solution across the equatorial plane is present (absent). This leads us to conclude that the very existence of equatorial circular orbit is an indication that the geometry respects equatorial reflection symmetry, while it may not be true the way around. To be precise, equatorial circular geodesics may not exist even if the spacetime respects the equatorial reflection symmetry. For example, this can be simply stemmed from the fact that the conserved momentum and energy are not favorable to host any bound equatorial circular geodesic.

In this paper, we argue that in the case of equatorial reflection symmetry violation it is unlikely to have a measurement with $\chi_p = 0$. This can be addressed by properly identifying the orbital parameters that will be representative of symmetry violation, and possibly depend on $\check{a}_{j(2m)}$ and $\hat{a}_{j(2m+1)}$. Therefore, to break this stalemate, we need to confront the multipolar structure of the object which would consist of both odd mass multipole moments and even current multipole moments in case the symmetry is violated. By measuring the S_2 and M_3 components, it would be sufficient to confirm this claim. We also constrained equatorial reflection symmetry violating parameters for the stellar and supermassive objects. To our knowledge, this is the first time such a kind of constraint has been found.

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APPENDIX A: METRIC COMPONENTS IN TERMS OF THE ERNST POTENTIAL

In this section, we will derive $g_{t\phi}$ in terms of the Ernst potential. To our knowledge, this has not been computed explicitly in the literature. If the metric is stationary and axis symmetric, then there will exist a timelike Killing vector field ξ^α and an axial Killing vector field \mathfrak{A}^α . Then, it is possible to define a vector field ω^α , defined as

$$\omega_\alpha = \epsilon_{\alpha\beta\gamma\delta} \xi^\alpha \nabla^\beta \xi^\gamma, \quad (\text{A1})$$

which satisfies $\nabla_{[\alpha} \omega_{\beta]} = 0$. Therefore, we can define a twist potential ψ as $\omega_\alpha = \partial_\alpha \psi$.

In the cylindrical coordinate system (t, ρ, z, ϕ) , the nonzero components can be found as

$$\begin{aligned} \omega_\rho &= -(g_{\rho\rho} g_{zz})^{1/2} g^{zz} [g^{t\phi} \partial_z g_{tt} + g^{\phi\phi} \partial_z g_{t\phi}] \\ &= \frac{1}{\rho} g_{tt}^2 \partial_z \left(\frac{g_{t\phi}}{g_{tt}} \right), \\ \omega_z &= (g_{\rho\rho} g_{zz})^{1/2} g^{\rho\rho} [g^{\phi\phi} \partial_\rho g_{t\phi} + g^{t\phi} \partial_\rho g_{tt}] \\ &= -\frac{1}{\rho} g_{tt}^2 \partial_\rho \left(\frac{g_{t\phi}}{g_{tt}} \right). \end{aligned} \quad (\text{A2})$$

From Eq. (A2), it is simple to find $g_{t\phi}$ as

$$\frac{g_{t\phi}}{g_{tt}} = \int \frac{\rho}{g_{tt}^2} \partial_\rho \psi dz - \int \frac{\rho}{g_{tt}^2} \partial_z \psi d\rho. \quad (\text{A3})$$

APPENDIX B: EXAMPLE OF A EQUATORIAL REFLECTION SYMMETRY BREAKING SPACETIME—KERR-NUT GEOMETRY

To display the connection between equatorial symmetry and circular orbits explicitly, we consider an example, namely, Kerr-NUT spacetime, where the reflection symmetry is known to be violated [29]. The nonexistence of equatorial circular orbits in Kerr-NUT geometry was first claimed in Ref. [28] and recently explored further in Ref. [29]. In the present context, though, we will be more involved in studying the perturbation equation in the theta direction and discuss the near equatorial plane behavior.

The Kerr-NUT spacetime is a solution to vacuum Einstein field equations and describes a stationary, axis-symmetric, and asymptotically nonflat spacetime. To discuss the nature of the angular perturbation, we need to consider the angular geodesic equation in Boyer-Lindquist coordinates (t, r, θ, ϕ) ,

$$\begin{aligned} \mathcal{V}_\theta &= (\dot{\theta})^2 = \left(\frac{d\theta}{d\tau_m} \right)^2 = (\lambda \sin^2 \theta + (L_z - aE)^2 \sin^2 \theta \\ &\quad - (EP - L_z)^2 - \sin^2 \theta (l + a \cos \theta)^2), \end{aligned} \quad (\text{B1})$$

where τ_m is the Mino time, a dot defines a derivative with respect to it, and \mathcal{V}_θ can be defined as angular potential [29]. The quantities λ , E , L_z , l , and a are defined as Carter constant, energy, momentum, NUT charge, and angular momentum, respectively, and ρ and P are given by $\rho^2 = r^2 + (l + a \cos \theta)^2$ and $P = a \sin^2 \theta - 2l \cos \theta$. Let us now assume that the particle is initially confined on the equatorial plane and we expect to study the perturbation originated from the term $\ddot{\theta}$. By setting $\dot{\theta} = 0$ and $\theta = \pi/2$, we gather $\lambda = l^2$, which identically vanishes for zero NUT charge, i.e., in the case of Kerr spacetime. In addition to $\dot{\theta} = 0$, to have a planner orbit, we need to ensure $\ddot{\theta} = 0$, too, which warrants that $\dot{\theta} = 0$ remains satisfied along the trajectory. This is where the NUT charge comes into play and engineers to disobey $\ddot{\theta} = 0$. Even then, the $\ddot{\theta}$ equation can be useful to extract information about the variation of θ near to the equatorial plane, which we will do next. Let us start by introducing the equation

$$\ddot{\theta} = \frac{1}{2} \frac{d\mathcal{V}(\theta)}{d\theta}, \quad (\text{B2})$$

which we need to write in terms of the coordinate time, such that it can be useful for an asymptotic observer. To execute this task, we may unfold $\ddot{\theta}$ as

$$\frac{d^2\theta}{d\tau^2} = \left(\frac{d^2\theta}{dt^2} \right) \left(\frac{dt}{d\tau} \right)^2 + \left(\frac{d\theta}{d\tau} \right) \left(\frac{dt}{d\tau} \right)^{-1} \left[\frac{d}{dt} \left(\frac{dt}{d\tau} \right) \frac{dt}{d\tau} \right], \quad (\text{B3})$$

and one easily notices that the second term goes to zero, as we assume $\dot{\theta} = 0$ in the first place. Bringing together Eqs. (B2) and (B3) and writing $\theta(t)$ as $\theta(t) = \pi/2 + \delta\theta(t)$, we arrive at the following expression:

$$\left(\frac{d^2\delta\theta(t)}{dt^2} \right) \left(\frac{dt}{d\tau} \right)^2 = \frac{1}{2} \frac{d\mathcal{V}(\theta)}{d\theta}. \quad (\text{B4})$$

The expression of $U^t = dt/d\tau$ can be derived from $U^t = -g^{tt}E + g^{t\phi}L_z$, where the metric components can be found in Ref. [29]. Finally, assuming $\delta\theta(t) \ll 1$ and terms with $\delta\theta(t)^2$, $\delta\theta(t)\delta\theta'(t)$, $\delta\theta(t)\delta\theta''(t)$, and beyond are neglected, we gather

$$C_3\delta\theta''(t) + C_2\delta\theta(t) + C_1 = 0, \quad (\text{B5})$$

where a prime denotes a differentiation with respect to t . The expressions for C_1 , C_2 , and C_3 are given as

$$\begin{aligned} C_1 &= l\{a(2E^2 - 1) - 2EL_z\}, \\ C_2 &= L_z^2 + 4E^2l^2 - a^2(E^2 - 1), \\ C_3 &= \frac{1}{\Delta^2} \{E(r^2 + l^2)^2 - 2aL_z(l^2 + Mr) \\ &\quad + a^2E(r(r + 2M) + 3l^2)\}, \end{aligned} \quad (\text{B6})$$

where $\Delta = r^2 - 2Mr + a^2 - l^2$ becomes zero on the event horizon. The above equation has a generic solution of the form

$$\delta\theta(t) = -\frac{C_1}{C_2} + A \cos \omega t + B \sin \omega t, \quad (\text{B7})$$

where A and B are integration constants to be evaluated from the initial conditions and $\omega = \sqrt{C_2/C_3}$. It is interesting to point out that C_3 diverges on the horizon, and ω becomes zero, which no longer describes equatorial timelike circular orbits. This is in consonance with the fact that on the null surface of event horizon no timelike circular orbit can exist. With the initial condition $\delta\theta(t=0) = \delta\theta'(t=0) = 0$, the above equation turns out to be

$$\delta\theta(t) = -\frac{C_1}{C_2}(1 - \cos \omega t) = -\frac{2C_1}{C_2} \sin^2(\omega t/2). \quad (\text{B8})$$

which may oscillate on either side of $\theta = \pi/2$, depending on the signs of C_1 and C_2 . Finally, we should say some words regarding the nature of the oscillations and how it is different from the Kerr case. It is easy to realize that the oscillation solely depends on C_1 and C_2 , and among them, C_2 is always positive as far as we are concerned with bound circular geodesics, i.e., $E \leq 1$. Coming to C_1 , it can only vanish if we have $l = 0$ or find a radius which satisfies $(2E^2 - 1) = 2EL_z$. It turns out that the later option is ruled out as far as one is interested with equatorial circular orbits in NUT spacetime [29], and one is left with no choice but to set $l = 0$ to stop the oscillation. Therefore, the NUT charge, which is entirely responsible for breaking the equatorial

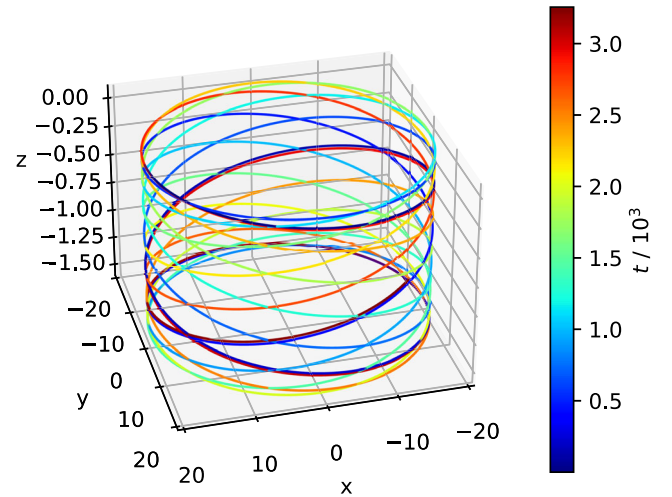


FIG. 1. In this figure, we demonstrate how a circular orbit in Kerr-NUT spacetime is slowly evolving from the equatorial plane. The radius of the circular orbit is considered to be $r = 20M$, where M is the mass of the black hole.

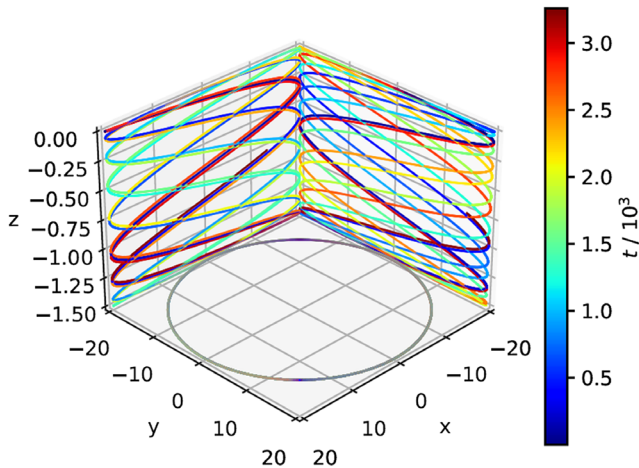


FIG. 2. In this figure, we demonstrate the evolution by projecting it at constant x , y , and z plane. The radius of the circular orbit is considered to be $r = 20M$, and the NUT charge $l = 0.1M$. x , y , z , and t are in the units of M .

symmetry, also engineers to angular perturbation. In passing, we should also mention the similar scenario in connection to massless particles. In this case, too, it is possible to arrive at an equation equivalent to Eq. (B8),

only with the expressions of C_1 , C_2 , and C_3 changed as follows:

$$\begin{aligned} C_1 &= 2El\{aE - L_z\}, \\ C_2 &= L_z^2 + E^2(4l^2 - a^2) + l^2, \\ C_3 &= \frac{1}{\Delta^2} \{E(r^2 + l^2)^2 - 2aL_z(l^2 + Mr) \\ &\quad + a^2E(r(r + 2M) + 3l^2)\}, \end{aligned} \quad (\text{B9})$$

The expression for C_1 can be set to zero by two possible ways, namely, $l = 0$ and $aE = L_z$. Like the earlier case, the later option can never give rise to a circular geodesic, and we need to choose $l = 0$. Therefore, the massless case is also in agreement with our claim that the NUT charge is solely responsible for having no equatorial circular orbits. For a typical set of parameters, we have shown in Figs. 1 and Fig. 2 how the NUT charge is influencing a circular orbit which starts from the equatorial plane. Along with time, the NUT charge pushes the orbit from the equatorial plane, and the orbit slowly deviates. Note that the general study representing the same for an arbitrary spacetime with no equatorial symmetry is already presented in Sec. V. For other specific examples, check Refs. [64,65].

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