

Deflection of light rays by a spherically symmetric black hole in a dispersive medium

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The influence of the medium on the gravitational deflection of light rays is widely discussed in literature for the simplest nontrivial case: cold nonmagnetized plasma. In this article, we generalize these studies to the case of an arbitrary transparent dispersive medium with a given refractive index. We calculate the deflection angle of light ray moving in a general spherically symmetric metric in the presence of medium with the spherically symmetric refractive index. The equation for the radius of circular light orbits is also derived. We discuss in detail the properties of these results and various special cases. In particular, we show that multiplying the refractive index by a constant does not affect the deflection angle and radius of circular orbits. At the same time, the presence of dispersion makes the trajectories different from the case of a vacuum even in a spatially homogeneous medium. As one of the applications of our results, we calculate the correction to the angle of vacuum gravitational deflection for the case when a massive object is surrounded by homogeneous but dispersive medium. As another application, we present the calculation of the shadow of a black hole surrounded by medium with arbitrary refractive index. Our results can serve as a basis for studies of various plasma models beyond the cold plasma case.

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I. INTRODUCTION

Light rays moving in an inhomogeneous medium propagate along curved paths. Similarly, if a gravitating body is surrounded by some kind of medium, the trajectories of the rays will differ from the case of motion near the gravitating body in a vacuum. In the presence of a medium around the gravitating body, the deflection of light rays will be determined by a complex combination of various effects: gravity, refraction, dispersion.

The theory of the propagation of light rays in curved spacetime in the presence of a medium was developed in the classic monograph by Synge [1], see also papers [2–4]. The light deflection in the linearized regime, when the total deflection can be calculated as the sum of the vacuum gravitational deflection and refraction in an inhomogeneous medium, was discussed for the case of plasma, e.g., in [5,6]. In his monograph on ray optics in general relativity, Perlick [7] derived an integral formula for the exact deflection angle of a light ray moving in the equatorial plane of a Kerr black hole surrounded by a spherically symmetric plasma distribution.

Recent studies of the influence of medium on the propagation of rays in a gravitational field are mainly

devoted to the various effects of gravitational lensing in the presence of plasma, which fills cosmic space, and is also concentrated around compact objects. In the works of Bisnovaty-Kogan and Tsupko [8,9], we investigated the case of the weak deflection of photons in the Schwarzschild metric in the presence of plasma. These results were further generalized to the case of weak deflection in the Kerr metric [10]. In the work of Tsupko and Bisnovaty-Kogan [11], gravitational lensing in plasma in case of a strong deflection was investigated. In particular, the deflection angles were calculated in the strong deflection limit, and the properties of high-order images [12,13] were calculated for the case of a homogeneous plasma. Er and Mao [14] numerically examined strong lens systems in the presence of plasma. Various astrophysical situations associated with strong deflection of light rays near compact objects in the presence of plasma were studied in a series of works by Rogers [15–17]. Perlick, Tsupko and Bisnovaty-Kogan [18] have fully analytically investigated the shadow of spherically symmetric black holes in the presence of plasma. In another article, we analytically investigated the effect of plasma on light ray propagation and on formation of the shadow for Kerr black hole [19]. Later these issues were further discussed in articles of Yan [20], Huang, Dong and Liu [21] and Kimpson, Wu and Zane [22,23]. Different problems of light deflection in presence of plasma have been recently studied in series of papers of Crisnejo and

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Gallo with coauthors [24–27]. In particular, Crisnejo, Gallo and Jusufi [27] have calculated the higher order terms of the deflection angles. Wave effects in presence of solar gravity and solar corona were studied in works of Turyshev and Toth [28,29]. In a recent paper [30], we have investigated gravitational microlensing in the presence of plasma (“hill-hole” effect). For some other recent studies see [31–39]. Recent review of plasma effects in gravitational lensing can be found in [40].

In most of the mentioned works, the simplest plasma model is considered—cold nonmagnetized plasma. In this case, the plasma has a simple expression for refractive index, and many formulas take on a relatively simple form. In this paper, we go beyond this approximation and consider a medium with an arbitrary refractive index. The formulas we obtained make it possible to easily calculate the deflection angles, circular orbits and the shadow of a black hole for any known spherically symmetric refractive index. For example, these can be more complex plasma models.

The paper is organized as follows. In Sec. II, we briefly describe the relativistic geometrical optics in medium on basis of Sygne’s approach. In Sec. III, we calculate the deflection angle for light ray moving in spherically symmetric spacetime in presence of spherically symmetric medium. In Sec. IV, we find the equation for the circular light orbits. In Sec. V, we present the discussion of our results and consider various particular cases. As an application of our results, in Sec. VI we calculate the correction to vacuum gravitational deflection due to presence of homogeneous dispersive medium. As another application, in Sec. VII the angular radius of shadow of black hole surrounded by transparent spherically symmetric medium is analytically calculated. Concluding remarks are presented in Sec. VIII.

We use units such that $G = c = 1$, and signature is $\{-, +, +, +\}$. Latin indices take the values $i, k = 0, 1, 2, 3$, whereas Greek indices take the values $\alpha, \beta = 1, 2, 3$. We denote the differentiation with respect to affine parameter by dot and the differentiation with respect to radial coordinate with a prime.

II. GENERAL RELATIVISTIC GEOMETRICAL OPTICS IN DISPERSIVE MEDIUM

General relativistic geometrical optics in curved spacetime filled with isotropic transparent medium was developed by Sygne [1]. Let us consider spacetime with metric coefficients g_{ik} which are known functions of coordinates. In this spacetime, we consider medium specified by its refractive index n (which is reciprocal of the phase speed) and its four-velocity V^i . Refractive index n is considered as a given function of the coordinates x^i and photon frequency ω ; the four-velocity V^i of medium is a given function of coordinates. Phase-speed and frequency

are measured in the instantaneous rest frame of the medium.

In Sygne’s approach, relativistic geometric optics is based on so-called medium equation [1]:

$$n^2 = 1 + \frac{p_i p^i}{(p_k V^k)^2}. \quad (1)$$

The photon frequency $\omega(x^i)$ measured by observer at position x^i is related with p_i and V^i by the following formula:

$$p_i V^i = -\omega(x^i). \quad (2)$$

Note that, in comparison with Eq. (6) and p. 373 of [1], we also choose the Planck constant as unity. This action is also equivalent to rescaling the affine parameter, which does not affect the final results.

To apply the Hamiltonian method, the medium equation is rewritten in the form

$$H(x^i, p_i) = 0, \quad (3)$$

where the Hamiltonian is

$$H(x^i, p_i) = \frac{1}{2} \{g^{ik} p_i p_k - (n^2 - 1)(p_k V^k)^2\}. \quad (4)$$

Then, propagation of the light rays is described by the following Hamiltonian equations:

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x^i}, \quad (5)$$

where a dot means differentiation with respect to an affine parameter λ changing along the light trajectory.

Let us now consider static spacetime,

$$g_{ik} dx^i dx^k = g_{\alpha\beta} dx^\alpha dx^\beta + g_{00} (dx^0)^2, \quad (6)$$

where the coefficients g_{ik} are independent of t . Let us also consider the static medium, so the medium refractive index n is a function of the frequency ω and the space coordinates x^α but is independent of t . Therefore, the whole Hamiltonian is independent of t . In such a case, from the equation for p_0 from (5) we read that $p_0 = \text{const}$.

Additionally, for a static medium we have [1]

$$V^\alpha = 0, \quad V^0 = \sqrt{-g^{00}}, \quad (7)$$

and therefore the relation (2) is simplified to

$$\omega(x^\alpha) = -p_0 \sqrt{-g^{00}}. \quad (8)$$

Using (8), we write the Hamiltonian (4) as

$$\begin{aligned} H(x^i, p_i) &= \frac{1}{2} \left\{ g^{ik} p_i p_k - (n^2 - 1) \left(p_0 \sqrt{-g^{00}} \right)^2 \right\} \\ &= \frac{1}{2} \{ g^{00} p_0 p_0 + g^{\alpha\beta} p_\alpha p_\beta + (n^2 - 1) p_0^2 g^{00} \} \\ &= \frac{1}{2} \{ g^{\alpha\beta} p_\alpha p_\beta + n^2 g^{00} p_0^2 \}. \end{aligned} \quad (9)$$

III. DEFLECTION ANGLE IN SPHERICALLY SYMMETRIC STATIC SPACETIME FILLED BY SPHERICALLY SYMMETRIC MEDIUM

In this section we derive the exact integral expression for the deflection angle of light ray moving in general spherically symmetric static spacetime filled by transparent static medium with spherically symmetric refractive index n .

We consider spherically symmetric and static spacetime:

$$\begin{aligned} g_{ik} dx^i dx^k &= -A(r) dt^2 + B(r) dr^2 \\ &\quad + D(r) (d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \end{aligned} \quad (10)$$

where coefficients $A(r)$, $B(r)$ and $D(r)$ are positive and independent of t . In this spacetime we consider a spherically symmetric static medium where the refractive index is a given function of the radial coordinate r and photon frequency ω : $n(\omega, r)$.

Due to spherical symmetry, we can restrict ourselves by motion in an equatorial plane: $\vartheta = \pi/2$, $p_\vartheta = 0$. Then the Hamiltonian (9) is simplified to

$$H(x^i, p_i) = \frac{1}{2} \left\{ \frac{p_r^2}{B(r)} + \frac{p_\varphi^2}{D(r)} - \frac{n^2 p_0^2}{A(r)} \right\}. \quad (11)$$

For the frequency $\omega(r)$ measured by static observer at position r we obtain from Eq. (8):

$$\omega(r) = -\frac{p_0}{\sqrt{A(r)}}, \quad (12)$$

where p_0 is the constant of motion. If we further consider only asymptotically flat spacetimes, then $A(r) \rightarrow 1$ for $r \rightarrow \infty$. Then we have:

$$p_0 = -\omega_0, \quad (13)$$

where $\omega_0 \equiv \omega(\infty)$. Therefore for the frequency $\omega(r)$ we have gravitational redshift formula

$$\omega(r) = \frac{\omega_0}{\sqrt{A(r)}}. \quad (14)$$

Equations of motion:

$$\dot{p}_\varphi = -\frac{\partial H}{\partial \varphi} = 0, \quad (15)$$

$$\begin{aligned} \dot{p}_r &= -\frac{\partial H}{\partial r} \\ &= \frac{1}{2} \left\{ \frac{p_r^2 B'(r)}{B^2(r)} + \frac{p_\varphi^2 D'(r)}{D^2(r)} + \frac{(n^2)'\omega_0^2}{A(r)} - \frac{n^2 \omega_0^2 A'(r)}{A^2(r)} \right\}, \end{aligned} \quad (16)$$

$$\dot{\varphi} = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{D(r)}, \quad (17)$$

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{B(r)}, \quad (18)$$

with $H = 0$, i.e.,

$$\frac{p_r^2}{B(r)} + \frac{p_\varphi^2}{D(r)} - \frac{n^2 \omega_0^2}{A(r)} = 0. \quad (19)$$

From Eq. (15) we find that p_φ is the constant of motion.

To derive the orbit equation, we write

$$\frac{d\varphi}{dr} = \frac{\dot{\varphi}}{\dot{r}} = \frac{p_\varphi B(r)}{p_r D(r)}. \quad (20)$$

Expressing p_r from (19), we find:

$$\frac{d\varphi}{dr} = \pm \frac{\sqrt{B(r)}}{\sqrt{D(r)}} \left(\frac{\omega_0^2}{p_\varphi^2} h^2(r) - 1 \right)^{-1/2}. \quad (21)$$

Here we have introduced the function

$$h^2(r) = \frac{D(r)}{A(r)} n^2(\omega(r), r). \quad (22)$$

Function (22) is the generalization of function $h(r)$ used in [7,11,18] for cold plasma. In the particular case of the photon motion near Schwarzschild black hole in vacuum, function (22) coincides with the ‘‘effective potential of photon’’ introduced in book of Misner, Thorne and Wheeler [41].

Let us consider the situation when the trajectory of a photon passes through the point of closest approach. Then it is convenient to express everything in terms of the distance of the closest approach R (minimal value of the coordinate r for this trajectory). As R corresponds to the turning point of the trajectory, we should have $dr/d\varphi = 0$ if $r = R$. From (21) we find:

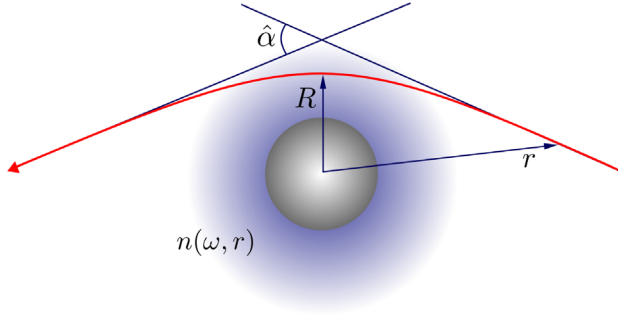


FIG. 1. Deflection angle $\hat{\alpha}$ of light ray moving near the black hole surrounded by transparent dispersive medium with the refractive index $n(\omega, r)$. The distance of the closest approach R is the minimum value of the radial coordinate r for this trajectory.

$$h^2(R) = \frac{p_\varphi^2}{\omega_0^2}. \quad (23)$$

Then the orbit equation (21) takes the following form:

$$\frac{d\varphi}{dr} = \pm \frac{\sqrt{B(r)}}{\sqrt{D(r)}} \left(\frac{h^2(r)}{h^2(R)} - 1 \right)^{-1/2}. \quad (24)$$

Now we calculate the deflection angle $\hat{\alpha}$ of light ray moving from infinity towards the center, reaching the distance of the closest approach R and again flying away to infinity (Fig. 1). Without loss of generality, we will assume that the ray moves in such a way that its angular coordinate φ increases. Then, when moving towards the center, the coordinate r is decreasing, and one need to use a “minus” sign in the equation (24), and when moving from the center to infinity, a “plus” sign should be used. Change to the angular coordinate can be written as follows:

$$\Delta\varphi = 2 \int_R^\infty \frac{\sqrt{B(r)}}{\sqrt{D(r)}} \left(\frac{h^2(r)}{h^2(R)} - 1 \right)^{-1/2} dr. \quad (25)$$

Since motion along a straight line corresponds to a change $\Delta\varphi = \pi$, then π must be subtracted to obtain the deflection angle $\hat{\alpha}$:

$$\hat{\alpha} = 2 \int_R^\infty \frac{\sqrt{B(r)}}{\sqrt{D(r)}} \left(\frac{h^2(r)}{h^2(R)} - 1 \right)^{-1/2} dr - \pi. \quad (26)$$

Formula (26) with (22) allows to calculate the deflection angle in any spherically symmetric spacetime in presence of spherically symmetric medium defined by known refractive index. This formula is generalization of plasma formula in [18]. We have managed to write the formula for a more general case in the same form as in [18], reducing all the differences to a more general definition of a function $h(r)$ only.

To apply the formula (26) for some medium with given refractive index $n(\omega, r)$, one should use ω in the form of $\omega(r)$ from (14). Then $n(\omega(r), r)$ should be substituted to Eq. (22) to obtain $h(r)$. The function $h(r)$ will also contain the frequency at infinity ω_0 . To get $h(R)$, one should take the refractive index as $n(\omega(R), R)$.

For example, cold nonmagnetized plasma has the index of refraction:

$$n^2 = 1 - \frac{\omega_p^2(r)}{\omega^2}, \quad (27)$$

where ω_p is electron plasma frequency and ω is photon frequency. In the presence of gravity we have:

$$n^2 = 1 - \frac{\omega_p^2(r)}{\omega^2(r)} = 1 - A(r) \frac{\omega_p^2(r)}{\omega_0^2}. \quad (28)$$

Therefore

$$h^2(r) = \frac{D(r)}{A(r)} \left(1 - A(r) \frac{\omega_p^2(r)}{\omega_0^2} \right), \quad (29)$$

and

$$h^2(R) = \frac{D(R)}{A(R)} \left(1 - A(R) \frac{\omega_p^2(R)}{\omega_0^2} \right). \quad (30)$$

We emphasize that in our approach we neglect the gravitational field created by the particles of the medium, compared to the gravitational field created by the black hole itself. In another words, we assume that the medium does not backreact to the geometry. In particular, in the formulas for plasma, the plasma number density is included in the refractive index through the plasma frequency, but we neglect the mass of plasma particles in comparison with the mass of the black hole when calculating the gravitational field. To characterize this approximation, a dimensionless parameter of the following form can be introduced: $\eta = \rho_m R_S^3 / m$, where ρ_m is the density of medium, m is the black hole mass and $R_S = 2Gm/c^2$ is the Schwarzschild radius (in dimensional units). The method works if $\eta \ll 1$. Otherwise, it is necessary to take into account the mass of the particles of the medium when calculating gravity. Taking into account the mass of distributed matter around a black hole was also discussed in Sec. 6 of Ref. [9].

IV. CIRCULAR LIGHT ORBITS

Along a circular light orbit, it should be $\dot{r} = 0$ and $\ddot{r} = 0$. Due to relation (18), $\dot{r} = 0$ leads to $p_r = 0$. From (19) we obtain:

$$\frac{p_\varphi^2}{D(r)} = \frac{n^2 \omega_0^2}{A(r)}. \quad (31)$$

Additionally, from (18), we read:

$$\dot{p}_r = \frac{d}{d\lambda}(B(r)\dot{r}) = B'(r)\dot{r}^2 + B(r)\ddot{r}. \quad (32)$$

Therefore, two conditions, $\dot{r} = 0$ and $\ddot{r} = 0$, together lead to $\dot{p}_r = 0$. Using Eq. (16), we find the second equation for circular light orbits:

$$\frac{p_\varphi^2 D'(r)}{D^2(r)} + \frac{(n^2)'\omega_0^2}{A(r)} - \frac{n^2 \omega_0^2 A'(r)}{A^2(r)} = 0. \quad (33)$$

Substituting p_φ^2 from (31) into (33), we find that the equation for the radius of a circular light orbit can be compactified to the following simple form:

$$\frac{d}{dr} h^2(r) = 0. \quad (34)$$

Analogous equation for cold plasma case was derived in our previous paper [18].

V. DISCUSSION AND PARTICULAR CASES

Two properties of the results obtained can be distinguished:

- (i) First, since the refractive index is represented in the formulas (24), (25) and (26) as a ratio, it is clear that multiplying the refractive index by a constant (i.e., replacing n with $C_0 n$) does not change the deflection angle. It means that, for a light ray with a given R and ω_0 , the deflection angle (26) will remain the same after such a change of the refractive index. Note that the speed of ray propagation in this case, of course, will change by a factor of C_0 . In the same way, it is seen that such a change will not affect the solutions of Eq. (34), i.e., on the radii of circular orbits. In the absence of gravity, this property is well known: the equations of ray trajectories describing the refraction in an optically inhomogeneous medium include quantities of the type $\nabla n/n$.
- (ii) At the same time, in the presence of a gravitational field, a new property appears. If the medium is dispersive, but at the same time spatially homogeneous, then in the absence of a gravitational field the ray will not bend. The situation changes with the addition of a gravitational field. Since the photon frequency in the gravitational field changes according (14), then, even in the case of a spatially homogeneous medium, a dependence on spatial coordinates, i.e., effective nonhomogeneity, appears: $n(\omega(r))$. As a result, the ray trajectories in the dispersive medium will differ from the vacuum

ones, even if the medium is homogeneous. This property was already mentioned in our previous papers with particular attention to homogeneous plasma case. Here we find the deflection angle (26) for a general case of dispersive medium, and below we will also calculate the correction to vacuum case.

We emphasize the difference between the items (i) and (ii). In the first case, we are talking about multiplying by a constant that does not depend on either frequency or position in space, so it is just a number. In the second case, we consider a homogeneous medium with dispersion, which means that the refractive index is not a constant but a function of frequency $n(\omega)$. This, in turn, leads to the fact that, in the presence of gravity, the refractive index of a homogeneous medium acquires a dependence on the radial coordinate $n(\omega(r))$ due to gravitational redshift relation (14).

In particular, the noticeable difference between homogeneous medium with and without dispersion follows from this for deflection angle in presence of gravitational field. If we consider a homogeneous medium *without dispersion* and compare the deflection angle with the case of a vacuum, we will get that the deflection angle will not change, because such a medium will have the refractive index as a constant, and this constant will be simply canceled in equations. But if we consider a homogeneous medium *with dispersion*, then we will have the refractive index as $n(\omega(r))$, and the angle of deflection will change in comparison with vacuum.

Let us further consider the following particular cases:

A. Vacuum or homogeneous medium in absence of gravity

This is trivial case: the deflection is absent in vacuum [Fig. 2(a)] and in homogeneous medium, both in non-dispersive [Fig. 2(b)] or dispersive [Fig. 2(g)]. Taking $A(r) = B(r) = 1$, $D(r) = r^2$ and $n = \text{const}$ in (26), we find

$$\hat{\alpha} = 2 \int_R^\infty \left(\frac{r^2}{R^2} - 1 \right)^{-1/2} \frac{dr}{r} - \pi = \pi - \pi = 0. \quad (35)$$

The use $n(\omega)$ does not change the result.

B. Nonhomogeneous nondispersive medium, in absence of gravity

To begin with, see Fig. 2(c). Taking $A(r) = B(r) = 1$, $D(r) = r^2$ and $n = n(r)$ in (26), we find

$$\hat{\alpha} = 2 \int_R^\infty \left(\frac{r^2 n^2(r)}{R^2 n^2(R)} - 1 \right)^{-1/2} \frac{dr}{r} - \pi. \quad (36)$$

For propagation in the absence of gravity, this formula is known, compare, for example, with Born and Wolf [42], Zheleznyakov [43].

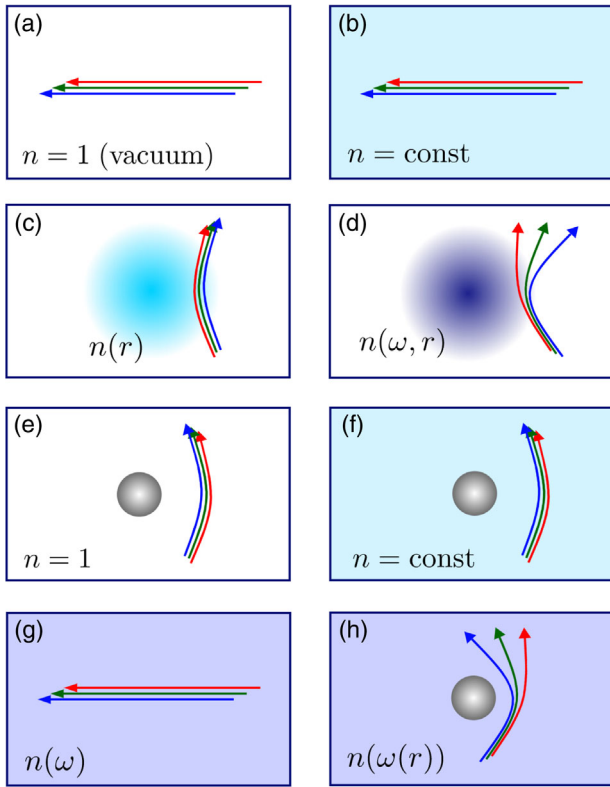


FIG. 2. Comparison of light deflection in different media, see discussion in Sec. V. (a) Light rays of three different frequencies in vacuum, without gravity. Rays are not deflected. (b) Homogeneous nondispersive medium in absence of gravity. There is no deflection. (c) Nonhomogeneous nondispersive medium in absence of gravity. There is achromatic deflection (refraction), because the refractive index depends on space coordinate. (d) Nonhomogeneous dispersive medium in absence of gravity. There is chromatic refraction, because the refractive index depends on both ω and r . (e) Achromatic gravitational deflection in vacuum. (f) Achromatic deflection near a gravitating body surrounded by homogeneous nondispersive medium. Trajectories are the same as in previous case, although light speed is smaller. (g) Homogeneous dispersive medium in absence of gravity. Rays of any frequency are not deflected due to the uniformity of the medium (no refraction). (h) Homogeneous dispersive medium in presence of gravity. In contrast to the case of homogeneous nondispersive medium in presence of gravity, the deflection differs from the vacuum gravitational deflection. This is because the photon frequency in gravitational field changes according (14), and the medium becomes effectively nonhomogeneous. This deflection is chromatic.

C. Nonhomogeneous dispersive medium, in absence of gravity

To begin with, see Fig. 2(d). It means that refractive index depends on the space coordinates and the photon frequency: $n = n(\omega, r)$. In this case we have refractive deflection, which is calculated with (36) with the only difference being that the refractive index now depends on ω also, and the trajectory and deflection angle will be different for different ω .

D. Gravitational deflection in vacuum

To begin with, see Fig. 2(e). Taking $n = 1$ in (26), we obtain the deflection angle of light ray moving in a spherically symmetric metric (10), compare, e.g., with (8.5.6) of Weinberg [44]:

$$\hat{\alpha} = 2 \int_R^\infty \frac{\sqrt{B(r)}}{\sqrt{D(r)}} \left(\frac{D(r)A(R)}{A(r)D(R)} - 1 \right)^{-1/2} dr - \pi. \quad (37)$$

E. Homogeneous nondispersive medium, in presence of gravity

To begin with, see Fig. 2(f). It means that the refractive index does not depend on the photon frequency and space coordinates: $n = \text{const} > 1$. In this case n is eliminated and we get the same trajectory and deflection angle (37) as in vacuum. The radius of photon circular orbits also will not change in comparison with vacuum.

F. Homogeneous dispersive medium in presence of gravity

This case is already discussed in the beginning of this section. While there is no deflection in a homogeneous dispersive medium in the absence of gravity [Fig. 2(g)], it occurs if gravity is added, and this deflection is chromatic [Fig. 2(h)].

G. Nonhomogeneous cold plasma

The propagation of light rays in the presence of gravity and cold plasma is well studied in the literature mentioned, so here we will focus only on certain properties that distinguish plasma from other media.

We consider cold, nonmagnetized plasma with electron plasma frequency

$$\omega_p^2(r) = \frac{4\pi e^2}{m_e} N(r), \quad (38)$$

where e and m_e are the electron charge and mass correspondingly, and $N(r)$ is the electron number density.

The refractive index of such a plasma has the form (28). The Hamiltonian of such medium is reduced to the following form [7–9,11,18]:

$$H(x^i, p_i) = \frac{1}{2} \{g^{ik} p_i p_k + \omega_p^2(r)\}. \quad (39)$$

We find that for plasma the photon frequency $\omega(r)$ is eliminated from the Hamiltonian. In particular, this leads to the fact that the deflection angles do not depend on the velocity of the medium [7], because it can be included into the Hamiltonian by means of the formula (2) only. This conclusion agrees, for example, with the consideration of light deflection by cold plasma stream in absence of

gravity, see Ko and Chuang [45]. They found that the normally incident ray passes through the moving cold plasma without any bending of the ray direction.

For other media, the motion of the medium will affect the propagation of light (we remind the reader that in this article we restrict ourselves to the case of static media). Note also that the absence of the photon frequency $\omega(r)$ inside the Hamiltonian (39) does not mean that the deflection angle become achromatic: the Hamiltonian contains the constant $p_0 = -\omega_0$.

H. Homogeneous cold plasma

An important special case is a homogeneous plasma. The number density, and hence the plasma frequency of such a plasma, are constants, $\omega_p = \text{const}$, and the refraction index is

$$n^2 = 1 - \frac{\omega_p^2}{\omega^2(r)}. \quad (40)$$

The Hamiltonian takes the following form:

$$H(x^i, p_i) = \frac{1}{2} \{g^{ik} p_i p_k + \omega_p^2\}, \quad (41)$$

where ω_p is constant. This Hamiltonian can be compared with Hamiltonian describing the motion of massive particle in vacuum:

$$H(x^i, p_i) = \frac{1}{2} \{g^{ik} p_i p_k + \mu^2\}, \quad (42)$$

where μ is the mass of test particle. As shown by Kulsrud and Loeb [4], in a homogeneous plasma, photons move along exactly the same trajectories as massive particles in a vacuum, with an effective mass equal to the plasma frequency, with an energy equal to the photon energy, and with speed equal to the group velocity of light ray. This conclusion is valid only in homogeneous plasma, i.e., in the absence of refraction.

As already discussed, the deflection of light in homogeneous plasma differs from vacuum case, because plasma is an example of dispersive medium. As shown in [8,9], the deflection angle by Schwarzschild black hole surrounded by homogeneous plasma can be written in approximation of weak deflection ($\hat{\alpha} \ll 1$) as

$$\hat{\alpha} = \frac{2m}{R} \left(1 + \frac{1}{1 - \omega_p^2/\omega_0^2} \right). \quad (43)$$

It is worthwhile to emphasize that the formula (43) contains the ratio ω_p^2/ω_0^2 , whereas the refractive index of plasma (28) used in derivation of this formula contains $\omega_p^2/\omega^2(r)$.

If we additionally assume that $\omega_p^2/\omega_0^2 \ll 1$, then we find a correction to the vacuum gravitational deflection due to a homogeneous plasma:

$$\hat{\alpha} = \frac{4m}{R} \left(1 + \frac{\omega_p^2}{2\omega_0^2} \right). \quad (44)$$

In the next section, we will calculate a similar correction for an arbitrary dispersive medium.

VI. APPLICATION I: CORRECTION TO EINSTEIN VACUUM GRAVITATIONAL DEFLECTION DUE TO PRESENCE OF HOMOGENEOUS DISPERSIVE MEDIUM

As we already discussed in the beginning of previous section, in the presence of gravitational field, the ray trajectories in the dispersive medium will differ from the vacuum ones, even if the medium is homogeneous. Let us now calculate the correction related to homogeneous dispersive medium for the vacuum Einstein deflection.

We start with our general expression for deflection angle (26) and apply it to the Schwarzschild spacetime,

$$A(r) = 1 - \frac{2m}{r}, \quad B(r) = \left(1 - \frac{2m}{r} \right)^{-1}, \quad D(r) = r^2, \quad (45)$$

and refractive index $n(\omega)$. We deal with spatially homogeneous medium $n(\omega)$, so the refractive index does not depend explicitly on space coordinate r . In presence of gravity, we have the gravitational redshift (8) for the photon frequency, so the refractive index reads as $n(\omega(r))$.

Let us expand the refractive index as

$$n(\omega(r)) \simeq n_0 + n_1(\omega(r) - \omega_0), \quad (46)$$

where we denote the values

$$n_0 = n(\omega_0), \quad n_1 = \left. \frac{\partial n}{\partial \omega} \right|_{\omega=\omega_0}, \quad (47)$$

which do not depend on r . In absence of gravity, the second term in (46) is absent, because $\omega = \omega_0$.

We further use the approximation that

$$\frac{n_1(\omega(r) - \omega_0)}{n_0} \ll 1 \quad (48)$$

in Eq. (46), and expand the deflection angle (26). After that, the expressions are still big, and we consider the approximation of weak deflection of the light ray: $R \gg m$. Expanding with small variables $m/r \ll 1$ and $m/R \ll 1$, we can then perform integration analytically.

Finally, we obtain in approximation $\hat{\alpha} \ll 1$ that

$$\hat{\alpha} = \frac{4m}{R} \left(1 + \frac{n_1 \omega_0}{2n_0} \right). \quad (49)$$

We remind the reader that the correction is present despite the fact that we deal with a homogeneous medium, that is, in the absence of refraction. Depending on the sign of the derivative n_1 , the presence of a homogeneous medium can both increase and decrease the deflection angle. For example, in a homogeneous plasma (40), the deflection angle increases [8,9].

In the particular case of homogeneous plasma (40) we can approximately change the condition (48) by the condition $\omega_p \ll \omega_0$. With this approximation in n_0 and n_1 , we recover (44).

VII. APPLICATION II: ANGULAR RADIUS OF BLACK HOLE SHADOW

As another important example, we will calculate the angular size of the shadow of a black hole immersed in a medium with a given refractive index. Black hole shadow [46–60] has now become the active research topic due to the recent observational discovery [61], see, e.g., Refs. [62–82]. Using the same approach as in our previous paper [18], we write down the angle α between the direction of the emission of the light ray and the radial direction (Fig. 3):

$$\cot \alpha = \frac{\sqrt{g_{rr}}}{\sqrt{g_{\varphi\varphi}}} \frac{dr}{d\varphi} \Big|_{r=r_0} = \frac{\sqrt{B(r)}}{\sqrt{D(r)}} \frac{dr}{d\varphi} \Big|_{r=r_0}. \quad (50)$$

Here r_0 is the radial coordinate of observer. Using (24) in (50), we obtain:

$$\cot^2 \alpha = \frac{h^2(r_0)}{h^2(R)} - 1. \quad (51)$$

This leads to

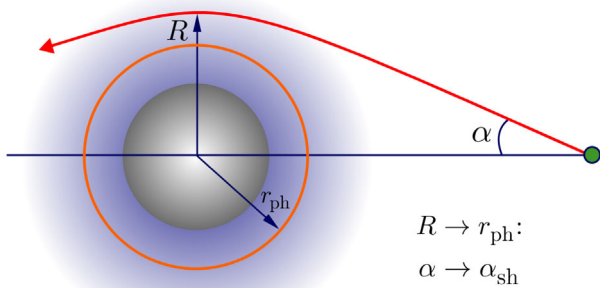


FIG. 3. Observer emits light ray into the past at angle α with respect to the radial direction, and this ray passes by the black hole at the distance of the closest approach R . If we substitute R by the photon sphere radius r_{ph} , then the angle α becomes the angular radius α_{sh} of the black hole shadow.

$$\sin^2 \alpha = \frac{h^2(R)}{h^2(r_0)}. \quad (52)$$

The size of the shadow is determined by the rays that asymptotically spiral towards the photon sphere. Therefore angular radius of the shadow α_{sh} is obtained if we put $R \rightarrow r_{\text{ph}}$:

$$\sin^2 \alpha_{\text{sh}} = \frac{h^2(r_{\text{ph}})}{h^2(r_0)}. \quad (53)$$

Here the radius r_{ph} should be found from Eq. (34).

The final formula (53) agrees with the formula derived for plasma case in [18], with only the difference being in the definition of function $h(r)$, see (22). As with the deflection angle, we can see that the size of the shadow does not change when the refractive index is multiplied by a constant.

VIII. CONCLUDING REMARKS

- (i) We calculate the deflection angle of a light ray moving in an general spherically symmetric metric in the presence of a transparent static dispersive medium with a spherically symmetric refractive index, see Eq. (26) with (22). The equation for the radius of the circular light orbits is also derived, see Eq. (34). The properties of these results and various special cases are discussed, see Sec. V.
- (ii) We find that multiplying the refractive index by a constant does not affect the trajectories: the deflection angle and the radius of circular orbits do not change. In particular, this means that the trajectories of light rays in a spatially homogeneous and nondispersive medium are the same as in vacuum [see Figs. 2(e) and 2(f)], although light velocity is smaller.
- (iii) At the same time, the presence of dispersion leads to a difference of trajectories from the vacuum case, even in a spatially homogeneous medium. This is due to the fact that in the presence of a gravitational field, the frequency of the photon changes depending on the position in space according to the gravitational redshift (14), and, accordingly, an effective inhomogeneity appears. As an application of our results, we calculate the correction to the vacuum gravitational deflection angle for the case when a massive object is surrounded by a homogeneous but dispersive medium, see Sec. VI.
- (iv) As another application, we present the calculation of the shadow of black hole surrounded by a medium with an arbitrary refractive index, see Sec. VII. This can be used for analytical calculation of shadow size beyond the widely used cold plasma case, e.g., for different warm, hot or collisional plasma models.

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- [1] J. L. Synge, *Relativity: The General Theory* (North-Holland, Amsterdam, 1960).
- [2] J. Bičák and P. Hadrava, General-relativistic radiative transfer theory in refractive and dispersive media, *Astron. Astrophys.* **44**, 389 (1975).
- [3] R. Breuer and J. Ehlers, Propagation of high-frequency electromagnetic waves through a magnetized plasma in curved space-time. I, *Proc. R. Soc. Ser. A* **370**, 389 (1980).
- [4] R. Kulsrud and A. Loeb, Dynamics and gravitational interaction of waves in nonuniform media, *Phys. Rev. D* **45**, 525 (1992).
- [5] D. O. Muhleman and I. D. Johnston, Radio Propagation in the Solar Gravitational Field, *Phys. Rev. Lett.* **17**, 455 (1966).
- [6] P. V. Bliokh and A. A. Minakov, *Gravitational Lenses* (Naukova Dumka, Kiev, 1989) (in Russian).
- [7] V. Perlick, *Ray Optics, Fermat's Principle, and Applications to General Relativity* (Springer, Berlin, 2000).
- [8] G. S. Bisnovaty-Kogan and O. Yu. Tsupko, Gravitational radiospectrometer, *Gravitation Cosmol.* **15**, 20 (2009).
- [9] G. S. Bisnovaty-Kogan and O. Yu. Tsupko, Gravitational lensing in a non-uniform plasma, *Mon. Not. R. Astron. Soc.* **404**, 1790 (2010).
- [10] V. S. Morozova, B. J. Ahmedov, and A. A. Tursunov, Gravitational lensing by a rotating massive object in a plasma, *Astrophys. Space Sci.* **346**, 513 (2013).
- [11] O. Yu. Tsupko and G. S. Bisnovaty-Kogan, Gravitational lensing in plasma: Relativistic images at homogeneous plasma, *Phys. Rev. D* **87**, 124009 (2013).
- [12] K. S. Virbhadra and G. F. R. Ellis, Schwarzschild black hole lensing, *Phys. Rev. D* **62**, 084003 (2000).
- [13] V. Bozza, S. Capozziello, G. Iovane, and G. Scarpetta, Strong field limit of black hole gravitational lensing, *Gen. Relativ. Gravit.* **33**, 1535 (2001).
- [14] X. Er and S. Mao, Effects of plasma on gravitational lensing, *Mon. Not. R. Astron. Soc.* **437**, 2180 (2014).
- [15] A. Rogers, Frequency-dependent effects of gravitational lensing within plasma, *Mon. Not. R. Astron. Soc.* **451**, 17 (2015).
- [16] A. Rogers, Escape and trapping of low-frequency gravitationally lensed rays by compact objects within plasma, *Mon. Not. R. Astron. Soc.* **465**, 2151 (2017).
- [17] A. Rogers, Gravitational lensing of rays through the levitating atmospheres of compact objects, *Universe* **3**, 3 (2017).
- [18] V. Perlick, O. Yu. Tsupko, and G. S. Bisnovaty-Kogan, Influence of a plasma on the shadow of a spherically symmetric black hole, *Phys. Rev. D* **92**, 104031 (2015).
- [19] V. Perlick and O. Yu. Tsupko, Light propagation in a plasma on Kerr spacetime: Separation of the Hamilton-Jacobi equation and calculation of the shadow, *Phys. Rev. D* **95**, 104003 (2017).
- [20] H. Yan, Influence of a plasma on the observational signature of a high-spin Kerr black hole, *Phys. Rev. D* **99**, 084050 (2019).
- [21] Y. Huang, Y.-P. Dong, and D.-J. Liu, Revisiting the shadow of a black hole in the presence of a plasma, *Int. J. Mod. Phys. D* **27**, 1850114 (2018).
- [22] T. Kimpson, K. Wu, and S. Zane, Spatial dispersion of light rays propagating through a plasma in Kerr spacetime, *Mon. Not. R. Astron. Soc.* **484**, 2411 (2019).
- [23] T. Kimpson, K. Wu, and S. Zane, Pulsar timing in extreme mass ratio binaries: A general relativistic approach, *Mon. Not. R. Astron. Soc.* **486**, 360 (2019).
- [24] G. Crisnejo and E. Gallo, Weak lensing in a plasma medium and gravitational deflection of massive particles using the Gauss-Bonnet theorem. A unified treatment, *Phys. Rev. D* **97**, 124016 (2018).
- [25] G. Crisnejo, E. Gallo, and A. Rogers, Finite distance corrections to the light deflection in a gravitational field with a plasma medium, *Phys. Rev. D* **99**, 124001 (2019).
- [26] G. Crisnejo, E. Gallo, and J. R. Villanueva, Gravitational lensing in dispersive media and deflection angle of charged massive particles in terms of curvature scalars and energy-momentum tensor, *Phys. Rev. D* **100**, 044006 (2019).
- [27] G. Crisnejo, E. Gallo, and K. Jusufi, Higher order corrections to deflection angle of massive particles and light rays in plasma media for stationary spacetimes using the Gauss-Bonnet theorem, *Phys. Rev. D* **100**, 104045 (2019).
- [28] S. G. Turyshev and V. T. Toth, Diffraction of light by plasma in the solar system, *J. Opt.* **21**, 045601 (2019).
- [29] S. G. Turyshev and V. T. Toth, Diffraction of light by the gravitational field of the Sun and the solar corona, *Phys. Rev. D* **99**, 024044 (2019).
- [30] O. Yu. Tsupko and G. S. Bisnovaty-Kogan, Hills and holes in the microlensing light curve due to plasma environment around gravitational lens, *Mon. Not. R. Astron. Soc.* **491**, 5636 (2020).

- [31] K. Schulze-Koops, V. Perlick, and D. J. Schwarz, Sachs equations for light bundles in a cold plasma, *Classical Quantum Gravity* **34**, 215006 (2017).
- [32] X. Er and A. Rogers, Two families of astrophysical diverging lens models, *Mon. Not. R. Astron. Soc.* **475**, 867 (2018).
- [33] X. Er and A. Rogers, Two families of elliptical plasma lenses, *Mon. Not. R. Astron. Soc.* **488**, 5651 (2019).
- [34] M. Sáreny and V. Balek, Effect of black hole-plasma system on light beams, *Gen. Relativ. Gravit.* **51**, 141 (2019).
- [35] X.-H. Jin, Y.-X. Gao, and D.-J. Liu, Strong gravitational lensing of a 4-dimensional Einstein-Gauss-Bonnet black hole in homogeneous plasma, *Int. J. Mod. Phys. D* **29**, 2050065 (2020).
- [36] J. Wagner and X. Er, Plasma lensing in comparison to gravitational lensing—Formalism and degeneracies, [arXiv:2006.16263](https://arxiv.org/abs/2006.16263).
- [37] M. Fathi and J. R. Villanueva, The role of elliptic integrals in calculating the gravitational lensing of a charged Weyl black hole surrounded by plasma, [arXiv:2009.03402v2](https://arxiv.org/abs/2009.03402v2).
- [38] K. Matsuno, Light deflection by squashed Kaluza-Klein black holes in a plasma medium, *Phys. Rev. D* **103**, 044008 (2021).
- [39] A. Chowdhuri and A. Bhattacharyya, Shadow analysis for rotating black holes in the presence of plasma for an expanding universe, [arXiv:2012.12914](https://arxiv.org/abs/2012.12914).
- [40] G. S. Bisnovatyi-Kogan and O. Yu. Tsupko, Gravitational lensing in presence of plasma: Strong lens systems, black hole lensing and shadow, *Universe* **3**, 57 (2017).
- [41] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman and Company, San Francisco, 1973).
- [42] M. Born and E. Wolf, *Principles of Optics Electromagnetic Theory of Propagation, Interference and Diffraction of Light* (Pergamon Press, New York, NY, 1959).
- [43] V. V. Zheleznyakov, *Radiation in Astrophysical Plasmas* (Kluwer Academic Publishers, Springer Netherlands, 1996), <https://www.springer.com/gp/book/9780792339076>.
- [44] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (John Wiley & Sons, Inc., New York, 1972).
- [45] H. C. Ko and C. W. Chuang, On the passage of radiation through moving astrophysical plasma, *Astrophys. J.* **222**, 1012 (1978).
- [46] J. L. Synge, The escape of photons from gravitationally intense stars, *Mon. Not. R. Astron. Soc.* **131**, 463 (1966).
- [47] Ya. B. Zeldovich and I. D. Novikov, Relativistic astrophysics. II, *Usp. Fiz. Nauk* **86**, 447 (1965) [*Sov. Phys. Usp.* **8**, 522 (1966)].
- [48] J. M. Bardeen, Timelike and null geodesics in the Kerr metric, in *Black Holes*, edited by C. DeWitt and B. DeWitt (Gordon and Breach, New York, 1973), p. 215.
- [49] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, Oxford, 1983).
- [50] I. G. Dymnikova, Motion of particles and photons in the gravitational field of a rotating body (In memory of Vladimir Afanas'evich Ruban), *Sov. Phys. Usp.* **29**, 215 (1986).
- [51] H. Falcke, F. Melia, and E. Agol, Viewing the shadow of the black hole at the galactic center, *Astrophys. J. Lett.* **528**, L13 (2000).
- [52] R. Takahashi, Shapes and positions of black hole shadows in accretion disks and spin parameters of black holes, *Astrophys. J.* **611**, 996 (2004).
- [53] A. F. Zakharov, A. A. Nucita, F. DePaolis, and G. Ingrosso, Measuring the black hole parameters in the galactic center with RADIOASTRON, *New Astron.* **10**, 479 (2005).
- [54] T. Johannsen and D. Psaltis, Testing the no-hair theorem with observations in the electromagnetic spectrum. II. Black hole images, *Astrophys. J.* **718**, 446 (2010).
- [55] T. Johannsen, Photon rings around Kerr and Kerr-like black holes, *Astrophys. J.* **777**, 170 (2013).
- [56] A. Grenzebach, V. Perlick, and C. Lämmerzahl, Photon regions and shadows of Kerr-Newman-NUT black holes with a cosmological constant, *Phys. Rev. D* **89**, 124004 (2014).
- [57] A. Grenzebach, V. Perlick, and C. Lämmerzahl, Photon regions and shadows of accelerated black holes, *Int. J. Mod. Phys. D* **24**, 1542024 (2015).
- [58] P. V. P. Cunha, C. A. R. Herdeiro, E. Radu, and H. F. Rúnarsson, Shadows of Kerr Black Holes with Scalar Hair, *Phys. Rev. Lett.* **115**, 211102 (2015).
- [59] J. O. Shipley and S. R. Dolan, Binary black hole shadows, chaotic scattering and the Cantor set, *Classical Quantum Gravity* **33**, 175001 (2016).
- [60] O. Yu. Tsupko, Analytical calculation of black hole spin using deformation of the shadow, *Phys. Rev. D* **95**, 104058 (2017).
- [61] The Event Horizon Telescope Collaboration *et al.*, First M87 Event Horizon Telescope Results. I. The shadow of the supermassive black hole, *Astrophys. J. Lett.* **875**, L1 (2019).
- [62] E. F. Eiroa and C. M. Sendra, Shadow cast by rotating braneworld black holes with a cosmological constant, *Eur. Phys. J. C* **78**, 91 (2018).
- [63] N. Tsukamoto, Black hole shadow in an asymptotically flat, stationary, and axisymmetric spacetime: The Kerr-Newman and rotating regular black holes, *Phys. Rev. D* **97**, 064021 (2018).
- [64] P. V. P. Cunha and C. A. R. Herdeiro, Shadows and strong gravitational lensing: A brief review, *Gen. Relativ. Gravit.* **50**, 42 (2018).
- [65] M. Mars, C. F. Paganini, and M. A. Oancea, The fingerprints of black holes—Shadows and their degeneracies, *Classical Quantum Gravity* **35**, 025005 (2018).
- [66] D. Ayzenberg and N. Yunes, Black hole shadow as a test of general relativity: Quadratic gravity, *Classical Quantum Gravity* **35**, 235002 (2018).
- [67] S.-W. Wei, Y.-X. Liu, and R. B. Mann, Intrinsic curvature and topology of shadows in Kerr spacetime, *Phys. Rev. D* **99**, 041303(R) (2019).
- [68] S. O. Alexeyev, B. N. Latosh, V. A. Prokopov, and E. D. Emtsova, Phenomenological extension for tidal charge black hole, *J. Exp. Theor. Phys.* **128**, 720 (2019).
- [69] D. Psaltis, Testing general relativity with the Event Horizon Telescope, *Gen. Relativ. Gravit.* **51**, 137 (2019).
- [70] R. A. Konoplya, Shadow of a black hole surrounded by dark matter, *Phys. Lett. B* **795**, 1 (2019).
- [71] T. M. Eubanks, Anchored in shadows: Tying the celestial reference frame directly to black hole event horizons, [arXiv:2005.09122](https://arxiv.org/abs/2005.09122).
- [72] V. I. Dokuchaev and N. O. Nazarova, Silhouettes of invisible black holes, *Phys. Usp.* **63**, 583 (2020).

- [73] V. Perlick, O. Y. Tsupko, and G. S. Bisnovaty-Kogan, Black hole shadow in an expanding universe with a cosmological constant, *Phys. Rev. D* **97**, 104062 (2018).
- [74] G. S. Bisnovaty-Kogan and O. Y. Tsupko, Shadow of a black hole at cosmological distances, *Phys. Rev. D* **98**, 084020 (2018).
- [75] S. E. Gralla, D. E. Holz, and R. M. Wald, Black hole shadows, photon rings, and lensing rings, *Phys. Rev. D* **100**, 024018 (2019).
- [76] O. Yu. Tsupko, Z. Fan, and G. S. Bisnovaty-Kogan, Black hole shadow as a standard ruler in cosmology, *Classical Quantum Gravity* **37**, 065016 (2020).
- [77] J. C. S. Neves, Constraining the tidal charge of brane black holes using their shadows, *Eur. Phys. J. C* **80**, 717 (2020).
- [78] J. R. Farah, D. W. Pesce, M. D. Johnson, and L. Blackburn, On the approximation of the black hole shadow with a simple polar curve, *Astrophys. J.* **900**, 77 (2020).
- [79] P.-C. Li, M. Guo, and B. Chen, Shadow of a spinning black hole in an expanding universe, *Phys. Rev. D* **101**, 084041 (2020).
- [80] Z. Chang and Q.-H. Zhu, Black hole shadow in the view of freely falling observers, *J. Cosmol. Astropart. Phys.* **06** (2020) 055.
- [81] O. Yu. Tsupko and G. S. Bisnovaty-Kogan, First analytical calculation of black hole shadow in McVittie metric, *Int. J. Mod. Phys. D* **29**, 2050062 (2020).
- [82] R. Kumar and S. G. Ghosh, Black hole parameter estimation from its shadow, *Astrophys. J.* **892**, 78 (2020).