Second-order perturbations of Kerr black holes: Formalism and reconstruction of the first-order metric

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Motivated by gravitational wave observations of binary black hole mergers, we present a procedure to compute the leading-order nonlinear gravitational wave interactions around a Kerr black hole. We describe the formalism used to derive the equations for second-order perturbations. We develop a procedure that allows us to reconstruct the first-order metric perturbation solely from knowledge of the solution to the first-order Teukolsky equation, without the need of Hertz potentials. Finally, we illustrate this metric reconstruction procedure in the asymptotic limit for the first-order quasinormal modes of Kerr. In a companion paper [J. L. Ripley *et al.*, Phys. Rev. D **103**, 104018 (2021)] we present a numerical implementation of these ideas.

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I. INTRODUCTION

The coalescence of binary black holes generally proceeds through three phases: the inspiral, merger, and ringdown. In the inspiral phase, the orbital velocity is typically small compared to the speed of light, and one can solve the field equations of general relativity (GR) using the perturbative post-Newtonian approximation [1]. In the merger phase, where the gravitational waves from the binary achieve their maximum amplitude, the nonlinearities of GR cannot be neglected, and one usually has to solve the field equations numerically [2]. Finally, the ringdown phase constitutes the response of the final black hole and is believed to be well described by the quasinormal modes computed using black hole perturbation theory [3].

The ringdown phase of the coalescence not only provides us with useful information regarding the remnant of binary mergers, it also gives us a means of testing the conjectured uniqueness of black holes in GR. Several properties of black holes are related to uniqueness: the no-hair theorems, stating that the only *stationary* black hole solutions in asymptotically flat four-dimensional spacetime with known matter fields are the three-parameter (mass, spin angular momentum, and electric charge) Kerr-Newman family [4–8]; Penrose's weak cosmic censorship conjecture that when gravitational collapse occurs the spacetime exterior to the black hole horizon is complete; and the final state conjecture [9], a special case of which is the conjectured nonlinear stability of the Kerr-Newman solutions, whereby all dynamical perturbations (however large) are absorbed by the black hole or radiated away, leaving behind another member of the Kerr-Newman family.

The uniqueness properties of black holes offer many avenues for testing the dynamical, strong-field regime of GR. Regarding the ringdown, the black hole spectroscopy proposal [10–13] exploits that the three parameters of the remnant (or two in an astrophysical setting where charge is expected to be insignificant) uniquely determine the frequencies and decay constants of the infinitely many quasinormal modes (QNMs) of the black hole; hence, measurements of multiple modes do not provide novel information about the black hole but instead are constraints to test uniqueness. This just scratches the surface of what is theoretically possible: for a ringdown produced by a binary black hole merger, the small set of parameters of the progenitor binary not only uniquely determines the remnant parameters (and hence the QNM complex frequencies), but also all the "initial" amplitudes and phases of all the QNM modes (this forms the basis of the proposal to coherently stack multiple detected events to enhance the ability to search for subdominant modes [14]). Moreover, all nonlinear effects, such as mode coupling at second-order, are also uniquely governed by the progenitor parameters. If the nonlinear phase of ringdown can be understood quantitatively, this regime of a merger will also be accessible to uniqueness tests.

We should note however that if our only goal were to confirm GR using black hole mergers, the residual test [15] is adequate and does not require us to understand or interpret phases of a merger; all one needs are full waveforms computed with enough accuracy that subtraction of a "best-fit" waveform from the data leaves a residual signal consistent with noise in the detectors. Though if such a test were to fail, it would be crucial to have a detailed knowledge of which part of the waveform led to the residual, and what novel physics or astrophysics that might point to (whether exotic alternatives to black holes, black holes with "hair," or the usual GR black holes embedded in a circumbinary environment sufficiently massive to measurably alter the uniqueness constraints an isolated binary is subject to).

Each quasinormal mode of the ringdown is identified by three integers, two (l, m) describing the angular dependence of the modes and one (n) describing the overtone [3]. Generally, the late time behavior of the ringdown phase is dominated by the leading-order $(l, m, n) = (2, 2, 0)$ quadrupole mode, but higher-order modes become relevant under particular circumstances. Higher angular modes have comparable decay time to the $(l, m, n) = (2, 2, 0)$ dominant mode but are more efficiently activated in systems with inherent asymmetries, such as an unequal mass binary (i.e., mass ratio $q \neq 1$) [16]. The first evidence for a nonquadrupole mode in the inspiral phase came from the recent merger event GW190412 [17]; however, this was not loud enough for a corresponding QNM to be detected.

Overtones generally decay faster than the $n = 0$ fundamental modes and thus can only be detected at higher signal-to-noise ratios (SNRs) or possibly, as with nonlinear effects, if the analysis can be extended closer to the merger phase. Intriguingly, [18,19] showed that for a merger of comparable mass nonspinning black holes, as consistent with GW150914, the waveform from peak amplitude onward can be well fit with linear modes if a sufficient number of overtones are included in the ringdown model. There are caveats with this analysis, but if it turns out to be sound, then there is already some evidence for observation of the first overtone of the quadrupole mode with GW150914 [20]. One of these caveats is, because of the rapid decay of the overtones, with low SNR (or low accuracy in the model) rapidly decaying nonlinear features could be fit by overtones and be erroneously ascribed to them. The study in [19] gave some evidence that this was not occurring in their fits; however, back of the envelope estimates suggest second-order mode coupling should be visible at comparable levels to the higher overtones they included. Without a detailed model of how the remnant black hole is "excited" during a merger to offer predictions for the various components of the ringdown, rather than fitting, it would be difficult to disentangle nonlinearity from overtones.

Most analyses of the ringdown of black holes stop at first-order in perturbation theory. In generic perturbative problems, second-order perturbations are sourced by the square of first-order perturbations, constituting the leading-order nonlinear effects. This holds true for black hole perturbation theory. Historically, second-order black hole perturbation theory was originally considered [21,22] to extend the close-limit approximation to black hole mergers [23]. These second-order calculations were later applied in the context of quasinormal modes of Schwarzschild black holes, where it was found that the second-order amplitudes could be as much as 10% of the first-order amplitudes [24–26]. A rigorous proof of the stability of fully nonlinear perturbations of a Schwarzschild black hole is only known restricted to a symmetry class [27]. More recently, second-order perturbation theory has been employed in the self-force formalism as a necessity for computing accurate waveforms for extreme mass ratio inspirals (EMRIs) (see e.g., [28–32]).

This being said, much about second-order perturbations of *spinning* black holes in the contexts of black hole ringdown and EMRI remain open problems. A promising approach to study such perturbations was initiated by Campanelli and Lousto [33], who employed the Newman-Penrose (NP) formalism [34,35] to derive an equation for second-order gravitational wave perturbations of Kerr black holes.

In the NP formalism, linear gravitational waves are described by the linear part of the Weyl scalar $\Psi_4^{(1)}$. (Here and below we use the notation $f^{(n)}$ to denote the nth-order perturbation of f about its background value $f^{(0)}$.) Campanelli and Lousto's equation takes the form of a Teukolsky equation for the second-order $\Psi_4^{(2)}$ with a source term quadratic in first-order perturbations. The chief challenge to computing this source term in a practical manner is that it depends on many more first-order geometric quantities than simply $\Psi_4^{(1)}$, and finding the set consistent with the given $\Psi_4^{(1)}$ is what we refer to as *reconstruction*. (All the above can equivalently be performed in terms of the NP scalar Ψ_0 instead of Ψ_4.)

An early method developed for reconstruction was given by Chrzanowski [36] (see also [37,38] for a more recent review), who showed that there exist "Hertz" potentials for gravitational (and electromagnetic) perturbations in the Kerr background. The gravitational Hertz potential solves the spin-weight −2 Teukolsky equation (which we simply call the "Teukolsky equation" for brevity). Effectively then from a solution Ψ to the Teukolsky equation one can generate a perturbed metric that solves the linearized Einstein equations about a Kerr background. The complication with this approach is that while the Hertz potential Ψ solves the Teukolsky equation, it does not relate in a simple way to the linearly perturbed Weyl scalar $\Psi_4^{(1)}$ (or $\Psi_0^{(1)}$). Therefore, it is not possible to *directly* apply Chrzanowski's method if one wants to find the perturbed metric associated with a particular $\Psi_4^{(1)}$.

A further drawback of Chrzanowski's method is that one is required to work in one of two radiation gauges, first described by Chrzanowski [36] and later expanded on in [39]. These gauge conditions can only be applied in type II or more special spacetimes and force particular conditions on the matter stress energy tensor. This limits the Hertz potential method from directly dealing with matter sources that do not satisfy those conditions, such as with EMRIs for example. Further, this technique cannot be applied at second-order in perturbation theory to recover the second-order metric perturbation, since the source terms coming from the first-order perturbation act as effective matter sources that are not consistent with the conditions required for the radiation gauges.

Recently, a new approach was proposed in [40] to extend the Hertz potential approach to allow for arbitrary matter sources. The approach starts by giving an ansatz for the metric perturbation of the form $h_{ab} \sim \mathrm{Re}[S^{\dagger}\Phi]_{ab} + x_{ab}$, where $S^{\dagger}$ is a second-order differential operator, Φ is the Hertz potential, and x_{ab} is a "correction" tensor. The first term on the right-hand side is essentially Chrzanowski's method that will give a linearized solution to the Einstein equations if the radiation gauge conditions can be met; if not, x_{ab} provides a correction proportional to the matter terms so that the net h_{ab} does solve the linearized Einstein equations. Thus an additional benefit of this procedure is that it allows for a path to calculating metric perturbations of the Kerr spacetime beyond linear order.

There are other workarounds to the above mentioned problems (see e.g., [41–43]), though there are also procedures [35,44] to directly reconstruct the metric from $\Psi_4^{(1)}$, which avoid the use of intermediate Hertz potentials. In this work we describe a formalism building on the latter methods, to compute the second-order gravitational wave perturbation of an arbitrary type D spacetime that satisfies the vacuum Einstein equations. The initial step is to write all first-order NP quantities (spin coefficients and Weyl scalars) in terms of the background metric and null tetrad projections of the first-order metric perturbation and its gradients. We use outgoing radiation gauge, though note that in principle our method does not require such a gauge; rather, it reduces the number of equations we need to solve in the end.

We then show how in this gauge, all first-order NP quantities can be derived from the solution of the Teukolsky equation for $\Psi_4^{(1)}$, several additional null transport equations, and some algebraic relations between spin coefficients and the first-order metric perturbation. This then allows us to compute the source term necessary to solve the Teukolsky equation for the second-order gravitational wave perturbation represented by $\Psi_4^{(2)}$.

At future null infinity in outgoing radiation gauge $\Psi_4^{(2)}$ relates to the two polarizations of the second-order metric perturbation ($h_\times^{(2)}$ and $h_+^{(2)}$) in exactly the same way $\Psi_4^{(1)}$ relates to the linear metric [33]:

$$\Psi_4^{(1,2)} = -\frac{1}{2}(\partial_t^2 h_+^{(1,2)} - i\partial_t^2 h_\times^{(1,2)}). \tag{1}$$

Thus by reading off $\Psi_4^{(1)}$ and $\Psi_4^{(2)}$ at future null infinity in outgoing radiation gauge we have a direct measure of the relative magnitude of second-order effects for a given choice of initial data.

To preview the detailed derivation later in the paper, in Fig. 1 we show a schematic of our metric reconstruction procedure. In the outgoing radiation gauge, the only nonzero metric perturbations $h_{\mu\nu}$ are the tetrad projections $h_{mm} = h_{\mu\nu}m^\mu m^\nu$, $h_{lm} = h_{\mu\nu}l^\mu m^\nu$ and $h_{ll} = h_{\mu\nu}l^\mu l^\nu$, with the tetrad consisting of a complex angular null vector m^μ and the real radially outgoing (ingoing) null vectors l^μ (n^μ). The starting point is to solve the Teukolsky equation for the first-order Weyl scalar $\Psi_4^{(1)}$. One can then solve for the spin coefficient $\lambda^{(1)}$ through Eq. (23), which can then be use to obtain h_{mm} through Eq. (24). Separately to this, one can obtain $\Psi_3^{(1)}$ from $\Psi_4^{(1)}$ using Eq. (25). The spin coefficient $\pi^{(1)}$ can then be obtained from Eq. (28), which then allows us to solve for h_{lm} through Eq. (29). Finally, from $\Psi_3^{(1)}$ we can obtain $\Psi_2^{(1)}$ from Eq. (30), which in turn allows us to solve for h_{ll} using Eq. (33). The remaining first-order spin coefficients can then be obtained from Eqs. (C1a)–(C1l) and the first-order Weyl scalars from Eqs. (D4) and (D5).

This kind of approach to metric reconstruction has a few advantages over the typical Hertz potential approach. First, using Hertz potentials requires one to work within one of the two radiation gauges, which place additional constraints on the matter sources or need to be corrected via the method in [40]. Here, though we have also chosen to work within the outgoing radiation gauge, this is simply because it is one of the easiest gauges to identify the necessary transport equations to fully reconstruct the metric. The basic strategy can be applied in essentially an arbitrary gauge, the only difference being the eventual number and complication of the transport equations to solve to obtain the first-order metric. Second, the Hertz potentials are spin-weight ± 2 quantities and thus only have support for modes with $l \geq 2$. However, there are nonradiative modes with $l < 2$ associated with shifts in the mass and spin of the black hole and thus cannot be obtained from the Hertz potential. Our approach is able to reconstruct these effects from homogeneous solutions to some of the transport equations, which

$$\overbrace{\mathcal{T}_{-2}\left[\Psi_4^{(1)}\right] = 0}^{\text{Teukolsky equation}} \quad \Longrightarrow \Psi_4^{(1)} \overset{\text{Eq. (23)}}{\Longrightarrow} \lambda^{(1)} \overset{\text{Eq. (24)}}{\Longrightarrow} h_{mm}$$
$$\Big\Downarrow \text{Eq. (25)}$$
$$\Psi_3^{(1)} \overset{\text{Eq. (28)}}{\Longrightarrow} \pi^{(1)} \overset{\text{Eq. (29)}}{\Longrightarrow} h_{lm}$$
$$\Big\Downarrow \text{Eq. (30)}$$
$$\Psi_2^{(1)} \overset{\text{Eq. (33)}}{\Longrightarrow} h_{ll}$$

FIG. 1. Schematic of our procedure for metric reconstruction. From the Teukolsky equation, one can solve for the Weyl scalar $\Psi_4^{(1)}$. In the outgoing radiation gauge detailed in Sec. III B, one can then directly reconstruct the three nonzero metric perturbations h_{mm}, h_{lm}, and h_{ll} using the Bianchi and Ricci identities of the Newman-Penrose formalism.

we will detail in an upcoming paper. A third issue with the use of a Hertz potential is additional steps must be taken beyond simply applying Chrzanowski's operator if one needs the resultant metric to be consistent with a desired $\Psi_4^{(1)}$. In particular, a fourth-order null transport equation needs to be solved; see e.g., Eq. (11) of [42] and the discussion of its solution therein.

The remainder of the paper is organized as follows. In Sec. II we list the equations that govern perturbations of type D spacetimes to first- and second-order in perturbation theory, a derivation of which is given in the Appendix B. In Sec. III we derive relations between first-order NP quantities and the linearized metric (with the full list of expressions for the spin coefficients given in Appendix C) and then describe the outgoing radiation gauge condition we use to fix the form of the first-order metric perturbation. In Sec. IV we describe our reconstruction procedure. The path to go from Ψ_4 to (h_{mm}, h_{lm}, h_{ll}) described there and illustrated in Fig. 1 is not unique, and in Appendix D we mention some alternative steps. As an illustration, in Sec. V we apply this method to the case of quasinormal modes of the Kerr spacetime in the limit of spatial infinity; i.e., we expand about $r \to \infty$. As explained in that section, there is a complication to finding the nonradiative metric perturbation associated with changes in the mass and spin of the black hole due to the gravitational wave perturbation; we leave it to future work to address that issue. In a companion paper [45] we detail the numerical code that implements the full method. We conclude with a discussion of future work in Sec. VI. Throughout this work, we use units with $G = c = 1$. For the NP formalism, a brief review of which is given in Appendix A, we use the conventions of [35], except that we use Greek letters to denote spacetime indices (e.g., our metric sign convention is $+---$, and we use $\bar{f}$ to denote the complex conjugate of f).

II. PERTURBATIONS OF TYPE D SPACETIMES

In the nonspinning limit, perturbation theory can be performed at the level of the metric; i.e., the metric can be written as $g_{\mu\nu} = g_{\mu\nu}^{\rm Schw} + \zeta h_{\mu\nu} + \mathcal{O}(\zeta^2)$, where $g_{\mu\nu}^{\rm Schw}$ is the background Schwarzschild metric, $h_{\mu\nu}$ is the first-order metric perturbation, and ζ is an order-keeping parameter. One can then write out the field equations for $h_{\mu\nu}$, which can be separated using spin-weighted spherical harmonics [46]. The gravitational waves are then described by the Regge-Wheeler (even-parity) [47] and Zerilli (odd-parity) [48,49] equations. For Kerr black holes, and any generic type D spacetime, the equations for the metric perturbation are not known to be separable.

The problem of finding separable equations for perturbations of Kerr spacetimes was solved by Teukolsky using the NP formalism [50], and Campanelli and Lousto [33] extended this beyond linear order. Here we list the equations, leaving a review of the derivations to Appendix B. In the NP formalism, a gravitational wave perturbation is characterized by the NP scalar Ψ_4 (or equivalently Ψ_0). The equation for the linear vacuum perturbation $\Psi_4^{(1)}$ is

$$\mathcal{T}[\Psi_4^{(1)}] = 0, \tag{2}$$

where $\mathcal{T}$ is the Teukolsky operator for a spin $= -2$ field (B13). The equation for the second-order vacuum perturbation $\Psi_4^{(2)}$ is

$$\mathcal{T}[\Psi_4^{(2)}] = \mathcal{S}_4^{(2)}, \tag{3}$$

where $\mathcal{T}$ is the same operator as in (2) and $\mathcal{S}_4^{(2)}$ is a second-order "source" term:

$$\begin{aligned}\mathcal{S}_4^{(2)} \equiv &-[d_4^{(0)}(D+4\epsilon-\rho)^{(1)} - d_3^{(0)}(\delta+4\beta-\tau)^{(1)}]\Psi_4^{(1)} \\ &+ [d_4^{(0)}(\bar{\delta}+2\alpha+4\pi)^{(1)} - d_3^{(0)}(\Delta+2\gamma+4\mu)^{(1)}]\Psi_3^{(1)} \\ &- 3[d_4^{(0)}\lambda^{(1)} - d_3^{(0)}\nu^{(1)}]\Psi_2^{(1)} \\ &+ 3\Psi_2^{(0)}[(d_4^{(1)} - 3\mu^{(1)})\lambda^{(1)} - (d_3^{(1)} - 3\pi^{(1)})\nu^{(1)}]. \end{aligned} \tag{4}$$

The source term is a function of first-order perturbed NP spin coefficients $\epsilon^{(1)}$, $\rho^{(1)}$, $\beta^{(1)}$, $\tau^{(1)}$, $\alpha^{(1)}$, $\pi^{(1)}$, $\gamma^{(1)}$, $\mu^{(1)}$, $\lambda^{(1)}$, and $\nu^{(1)}$, Weyl scalars $\Psi_2^{(1)}$, $\Psi_3^{(1)}$, and $\Psi_4^{(1)}$, and their derivatives through the background $d_3^{(0)}$ and $d_4^{(0)}$ and first-order $D^{(1)}$, $\Delta^{(1)}$, and $\delta^{(1)}$ gradient operators (see Appendixes A and B for the relevant definitions). This equation does not require imposing any particular coordinate system on the background, although it does require using a background tetrad that aligns with the two principal null directions of Kerr (such as the Kinnersley tetrad).

We see that in this approach, computing the leading nonlinear gravitational effects around a Kerr black hole is reduced to computing the source term and then solving the Teukolsky equation with that source term. If one has the first-order metric perturbation, it is trivial to compute all the NP quantities needed for the source term simply from their definitions. However, what is more typical is to only have $\Psi_4^{(1)}$ from a solution to the first-order Teukolsky equation. As mentioned above then, the main technical challenge for the second-order problem is reconstructing the remaining NP quantities required for the source from only one's knowledge of $\Psi_4^{(1)}$. In the remainder of this paper we describe a method for doing so for vacuum perturbations (see [40] for a different reconstruction procedure claimed to also work with gravity coupled to matter that is smooth and of compact support).

III. LINEARIZED METRIC AND GAUGE CONDITIONS

Before describing our reconstruction procedure in the following section, here we show the relation between linearized metric and tetrad components and linearized NP scalars (Sec. III A) and then discuss the radiation gauge conditions we employ to fix the form of the first-order metric perturbation (Sec. III B).

A. Linearized NP scalars in terms of the linearized metric

We write out the metric to first-order in perturbation theory as $g_{\mu\nu} = g^B_{\mu\nu} + \zeta h_{\mu\nu} + \mathcal{O}(\zeta^2)$, where $g^B_{\mu\nu}$ is a Petrov type D background spacetime and $h_{\mu\nu}$ is the first-order metric perturbation. For notational convenience, we write the components of $h_{\mu\nu}$ in the tetrad frame as $h_{ab} = h_{\mu\nu} e^\mu_a e^\nu_b$, reserving Latin (Greek) indices for tetrad (coordinate) components; for example $h_{nn} = h_{\mu\nu} n^\mu n^\nu$. We assume that the background tetrad $(l^\mu_{(0)}, n^\mu_{(0)}, m^\mu_{(0)}, \bar{m}^\mu_{(0)})$ is chosen such that $\Psi_0^{(0)} = \Psi_1^{(0)} = \Psi_3^{(0)} = \Psi_4^{(0)} = \kappa^{(0)} = \sigma^{(0)} = \nu^{(0)} = \lambda^{(0)} = 0$. Note that the results in this subsection do not rely on the choice of gauge for the metric but do depend on the choice of the linearized tetrad.

Our starting point is to calculate the first-order tetrad in terms of the metric perturbation. The background tetrad forms a complete basis, so it is natural to decompose the first-order tetrad in terms of these vectors, specifically

$$\begin{pmatrix} l^{(1)}_\mu \\ n^{(1)}_\mu \\ m^{(1)}_\mu \\ \bar{m}^{(1)}_\mu \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & c_{13} & \bar{c}_{13} \\ b_{21} & b_{22} & c_{23} & \bar{c}_{23} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ \bar{c}_{31} & \bar{c}_{32} & \bar{c}_{34} & \bar{c}_{33} \end{pmatrix} \begin{pmatrix} l^{(0)}_\mu \\ n^{(0)}_\mu \\ m^{(0)}_\mu \\ \bar{m}^{(0)}_\mu \end{pmatrix}, \tag{5}$$

where the b_{ij} are real coefficients and the c_{ij} are complex coefficients. Following [33,51], we can use our six degrees of freedom for the linearized tetrad vectors to choose $b_{11} = c_{13} = c_{23} = \mathrm{Im} c_{33} = 0$. We now solve for the coefficients of the matrix in Eq. (5) using the completeness relation $g_{\mu\nu} = 2l_{(\mu}n_{\nu)} - 2m_{(\mu}\bar{m}_{\nu)}$. Expanding to first-order, we have

$$h_{\mu\nu} = 2l^{(1)}_{(\mu}n^{(0)}_{\nu)} + 2l^{(0)}_{(\mu}n^{(1)}_{\nu)} - 2m^{(1)}_{(\mu}\bar{m}^{(0)}_{\nu)} - 2m^{(0)}_{(\mu}\bar{m}^{(1)}_{\nu)}. \tag{6}$$

Inserting the representation of the first-order tetrad in Eq. (5) and projecting into the tetrad frame gives us a set of linear equations that can be solved to obtain the b and c coefficients in terms of h_{ab}, specifically

$$l^{(1)}_\mu = \frac{1}{2} h_{ll} n^{(0)}_\mu, \tag{7a}$$

$$n^{(1)}_\mu = \frac{1}{2} h_{nn} l^{(0)}_\mu + h_{ln} n^{(0)}_\mu, \tag{7b}$$

$$m^{(1)}_\mu = h_{nm} l^{(0)}_\mu + h_{lm} n^{(0)}_\mu - \frac{1}{2} h_{m\bar{m}} m^{(0)}_\mu - \frac{1}{2} h_{mm} \bar{m}^{(0)}_\mu. \tag{7c}$$

Raising the coordinate indices on these expressions involves flipping the signs of the h_{ij} terms (since the relative signs of the covariant versus contravariant components of the first-order metric tensor perturbation are opposite). For convenience, we also write out the first-order directional derivatives $(D, \Delta, \delta, \bar{\delta})$ using these relations:

$$D^{(1)} = -\frac{1}{2} h_{ll} \Delta^{(0)}, \tag{8a}$$

$$\Delta^{(1)} = -\frac{1}{2} h_{nn} D^{(0)} - h_{ln} \Delta^{(0)}, \tag{8b}$$

$$\delta^{(1)} = -h_{nm} D^{(0)} - h_{lm} \Delta^{(0)} + \frac{1}{2} h_{m\bar{m}} \delta^{(0)} + \frac{1}{2} h_{mm} \bar{\delta}^{(0)}. \tag{8c}$$

The next step is to write out the spin coefficients in terms of the metric perturbations h_{ab}. To achieve this, we make use of the commutation relations in Eqs. (A5a)–(A5d) and the first-order tetrad in Eqs. (7a)–(7c). We expand out both sides of the commutation relations and match the coefficients of the directional derivatives to obtain linear equations for the first-order spin coefficients. As an example of this, consider Eq. (A5a). Expanding out the left-hand side, we have

$$\begin{aligned}
[\delta, D]^{(1)} = {} & \frac{1}{2}[2D^{(0)} h_{nm} + (\bar{\alpha}^{(0)} + \beta^{(0)} - \bar{\pi}^{(0)}) h_{m\bar{m}} + (\alpha^{(0)} + \bar{\beta}^{(0)} - \pi^{(0)}) h_{mm} \\
& - 2(\gamma^{(0)} + \bar{\gamma}^{(0)}) h_{lm} + \bar{\nu}^{(0)} h_{ll}] D^{(0)} + \frac{1}{2}[2D^{(0)} h_{lm} - \delta^{(0)} h_{ll} + (\bar{\alpha}^{(0)} + \beta^{(0)} - \tau^{(0)}) h_{ll} \\
& - 2(\epsilon^{(0)} + \bar{\epsilon}^{(0)}) h_{lm} + \kappa^{(0)} h_{m\bar{m}} + \bar{\kappa}^{(0)} h_{mm}] \Delta^{(0)} + \frac{1}{2}[-D^{(0)} h_{m\bar{m}} + (-\epsilon^{(0)} + \bar{\epsilon}^{(0)} - \bar{\rho}^{(0)}) h_{m\bar{m}} \\
& - (-\gamma^{(0)} + \bar{\gamma}^{(0)} + \mu^{(0)}) h_{ll} - \bar{\sigma}^{(0)} h_{mm} + 2(\pi^{(0)} + \bar{\tau}^{(0)}) h_{lm}] \delta^{(0)} \\
& + \frac{1}{2}[-D^{(0)} h_{mm} + (\epsilon^{(0)} - \bar{\epsilon}^{(0)} - \rho^{(0)}) h_{mm} - \bar{\lambda}^{(0)} h_{ll} - \sigma^{(0)} h_{m\bar{m}} + 2(\bar{\pi}^{(0)} + \tau^{(0)}) h_{lm}] \bar{\delta}^{(0)}.
\end{aligned} \tag{9}$$

Next, expanding out the right-hand side, we obtain

$$\begin{aligned}&\left[\bar{\alpha}^{(1)}+\beta^{(1)}-\bar{\pi}^{(1)}+(\epsilon^{(0)}-\bar{\epsilon}^{(0)})h_{nm}-\frac{1}{2}\kappa^{(0)}h_{nn}+\bar{\rho}^{(0)}h_{nm}+\sigma^{(0)}h_{n\bar{m}}\right]D^{(0)}\\&+\left[\kappa^{(1)}-\frac{1}{2}(\bar{\alpha}^{(0)}+\beta^{(0)}-\bar{\pi}^{(0)})h_{ll}+(\epsilon^{(0)}-\bar{\epsilon}^{(0)}+\bar{\rho}^{(0)})h_{lm}-\kappa^{(0)}h_{ln}+\sigma^{(0)}h_{l\bar{m}}\right]\Delta^{(0)}\\&+\left[-\epsilon^{(1)}+\bar{\epsilon}^{(1)}-\bar{\rho}^{(1)}-\frac{1}{2}h_{m\bar{m}}(\epsilon^{(0)}-\bar{\epsilon}^{(0)}+\bar{\rho}^{(0)})-\frac{1}{2}\sigma^{(0)}h_{\bar{m}\bar{m}}\right]\delta^{(0)}\\&+\left[-\sigma^{(1)}-\frac{1}{2}(\epsilon^{(0)}-\bar{\epsilon}^{(0)}+\bar{\rho}^{(0)})h_{mm}-\frac{1}{2}\sigma^{(0)}h_{m\bar{m}}\right]\bar{\delta}^{(0)}.\end{aligned}\tag{10}$$

Matching the coefficients of $\Delta^{(0)}$ allows us to solve for $\kappa^{(1)}$, i.e.,

$$\kappa^{(1)}=(D-2\epsilon-\bar{\rho})^{(0)}h_{lm}-\frac{1}{2}(\delta-2\bar{\alpha}-2\beta+\bar{\pi}+\tau)^{(0)}h_{ll}.\tag{11}$$

Repeating this method for the remaining commutation relations, we obtain the rest of the linearized Newman-Penrose scalars written in terms of the linearized metric components. We provide the complete listing of these quantities in Appendix C. The first-order spin coefficients are now completely determined in terms of the metric perturbation.

The final step to complete the description in terms of the metric perturbation is to obtain the Weyl scalars. This can be done readily from the transport equations in Eqs. (A9a)–(A9r). As an example, we may obtain $\Psi_0^{(1)}$ directly from Eq. (A9b), due to the fact that $\sigma^{(0)}=0=\kappa^{(0)}$, specifically

$$\begin{aligned}\Psi_0^{(1)}&=(D-\rho-\bar{\rho}-3\epsilon+\bar{\epsilon})^{(0)}\sigma^{(1)}\\&\quad-(\delta+\tau-\bar{\pi}+\bar{\alpha}+3\beta)^{(0)}\kappa^{(1)}.\end{aligned}\tag{12}$$

Likewise, from Eq. (A9j), we have

$$\begin{aligned}\Psi_4^{(1)}&=(\bar{\delta}+3\alpha+\bar{\beta}+\pi-\bar{\tau})^{(0)}\nu^{(1)}\\&\quad-(\Delta+\mu+\bar{\mu}+3\gamma-\bar{\gamma})^{(0)}\lambda^{(1)}.\end{aligned}\tag{13}$$

The remaining Weyl scalars must be found by taking linear combinations of Eqs. (A9a)–(A9r). We here provide the exact representation of these without linearizing:

$$\Psi_1=(D-\bar{\rho}+\bar{\epsilon})\beta-(\delta+\bar{\alpha}-\bar{\pi})\epsilon-(\alpha+\pi)\sigma+(\mu+\gamma)\kappa,\tag{14a}$$

$$\begin{aligned}\Psi_2=\frac{1}{3}[&(\bar{\delta}-2\alpha+\bar{\beta}-\pi-\bar{\tau})\beta-(\delta-\bar{\alpha}+\bar{\pi}+\tau)\alpha\\&+(D+\epsilon+\bar{\epsilon}+\rho-\bar{\rho})\gamma-(\Delta-\bar{\gamma}-\gamma+\bar{\mu}-\mu)\epsilon\\&+(\bar{\delta}-\alpha+\bar{\beta}-\bar{\tau}-\pi)\tau-(\Delta-\bar{\gamma}-\gamma+\bar{\mu}-\mu)\rho\\&+2(\nu\kappa-\lambda\sigma)],\end{aligned}\tag{14b}$$

$$\begin{aligned}\Psi_3&=(\bar{\delta}+\bar{\beta}-\bar{\tau})\gamma-(\Delta-\bar{\gamma}+\bar{\mu})\alpha+(\rho+\epsilon)\nu\\&\quad-(\tau+\beta)\lambda.\end{aligned}\tag{14c}$$

This completes the description of NP quantities in terms of the metric perturbation.

B. Radiation gauges

As mentioned, the form of the Teukolsky equation given in the previous section is independent of the coordinate system and only requires the radial null tetrad vectors to be aligned with the principle null directions of Kerr. Solving these equations in practice requires choosing coordinates for the background metric and first-order perturbations. Here, we describe our gauge to fix the form of the first-order metric and tetrad perturbations.

Under an infinitesimal gauge transformation $x^\mu\to x^\mu+\xi^\mu$ of the background metric, $h_{\mu\nu}$ transforms as

$$h_{\mu\nu}\to h_{\mu\nu}-\xi_{(\mu;\nu)}.\tag{15}$$

We make use of the radiation gauges developed by Chrzanowski [36], in which the metric perturbation is required to be transverse to one of the principal null directions. This condition can only be imposed in type II spacetimes or more symmetric spacetimes, like type D [39]. For the outgoing radiation gauge, we begin by imposing

$$n^\mu(h_{\mu\nu}-\xi_{(\mu;\nu)})=0.\tag{16}$$

This set of four equations for the vector ξ^μ implies we have freedom to choose ξ^μ such that four of the components of $h_{\mu\nu}$ are zero, specifically $h_{ln}=h_{nn}=h_{nm}=h_{n\bar{m}}=0$ in this gauge. However, in Petrov type D (or more generally Petrov type II) spacetimes it turns out that we still have some residual gauge freedom [related to the homogeneous solutions of Eq. (16)] that we can use to enforce a traceless condition [39]

$$h_\mu{}^\mu=g^{\mu\nu}h_{\mu\nu}=0.\tag{17}$$

Taken together with the previous conditions, this sets $h_{m\bar{m}}=0$, leaving the only nonzero components of the metric to be the real-valued h_{ll} and the complex-valued h_{lm} and h_{mm}. It then follows from Eqs. (C1b), (C1d), and (C1f) that

$$\nu^{(1)} = \mu^{(1)} = \gamma^{(1)} = 0. \tag{18}$$

If coupling to matter, the traceless condition also imposes a constraint on the stress energy tensor from Eq. (A9n), namely

$$\Phi_{22} = 0 \Rightarrow T_{\mu\nu} n^\mu n^\nu = 0. \tag{19}$$

Equations (16)–(19) specify the necessary and sufficient conditions for the outgoing radiation gauge. This gauge has the properties of being transverse and traceless on future null infinity and the past horizon for the Kerr spacetime.

Complementary to the outgoing radiation gauge, one can also specify the ingoing radiation gauge through the condition

$$l^\mu (h_{\mu\nu} - \xi_{(\mu;\nu)}) = 0. \tag{20}$$

Combining with the traceless condition in Eq. (17), we have the necessary conditions of the ingoing radiation gauge:

$$\epsilon^{(1)} = \kappa^{(1)} = \rho^{(1)} = 0, \tag{21}$$

$$\Phi_{00} = 0 \Rightarrow T_{\mu\nu} l^\mu l^\nu = 0. \tag{22}$$

This gauge has the property of being transverse and traceless on past null infinity and the future null horizon of the Kerr spacetime. Either one of these gauges allow for metric reconstruction as outlined in this paper, so long as the matter stress energy tensor satisfies either Eq. (19) or (22). Since we are most interested in the problem of quasinormal modes of Kerr black holes as the end state of a binary coalescence, we can restrict to the case of vacuum and both of these conditions are satisfied. For the remainder of this paper, we work within the outgoing radiation gauge.

IV. RECONSTRUCTING THE METRIC FROM $\Psi_4^{(1)}$

In this section, we describe a procedure to reconstruct the metric coefficients h_{ll}, $h_{l\bar{m}}$, and $h_{\bar{m}\bar{m}}$ in the outgoing radiation gauge from the Weyl curvature scalar $\Psi_4^{(1)}$.

In the NP formalism, there are eight complex equations from the Bianchi identities Eqs. (A10a)–(A10h), 36 complex equations (20 independent) from the Riemann identities Eqs. (A9a)–(A9r), and 12 complex equations for the spin coefficients Eqs. (C1a)–(C1l). However, in our chosen gauge, we only need to solve for five real-valued (one real and two complex) quantities. Thus, the problem of solving for the metric perturbation is overdetermined. The procedure that we detail below is, as a result, not unique, but it is sufficient to reconstruct the metric. Some alternative choices are outlined in Appendix D.

To begin, we focus on solving for h_{mm}. Consider the Riemann identity in Eq. (A9j). This is one of the equations used to derive the Teukolsky equation and, as explained there, is already of first-order smallness. Further, due to the choice of gauge, $\nu^{(1)} = 0$, and so we obtain the following transport equation for $\lambda^{(1)}$:

$$(\Delta + \mu + \bar{\mu} + 3\gamma - \bar{\gamma})^{(0)} \lambda^{(1)} = -\Psi_4^{(1)}. \tag{23}$$

Thus, once one has solved the Teukolsky equation for $\Psi_4^{(1)}$, one can naturally obtain $\lambda^{(1)}$. Now, consider the relationship between $\lambda^{(1)}$ and the metric perturbation in Eq. (C1a). Once again, our choice of gauge eliminates all of the metric coefficients in this expression, except for $h_{\bar{m}\bar{m}}$. Thus, we obtain a transport equation for $h_{\bar{m}\bar{m}}$, namely

$$[\Delta + 2(\bar{\gamma} - \gamma) + \bar{\mu} - \mu]^{(0)} h_{\bar{m}\bar{m}} = -2\lambda^{(1)}. \tag{24}$$

Of course, this also yields h_{mm} since $h_{mm} = [h_{\bar{m}\bar{m}}]^\dagger$. The real and imaginary parts of h_{mm} encode the gravitational waves at null infinity, and the above two equations are effectively equivalent to the statement $\Psi_4 = (1/2)\partial_t^2 (h_+ - i h_\times)$ in a far field expansion, where $h_{+,\times}$ are the polarization states of gravitational waves. This will become more explicit when we present our case study in Sec. V.

Having solved for h_{mm}, we now turn our attention to h_{lm}. Consider the Bianchi identity in Eq. (A10h). Just like our starting point for $\lambda^{(1)}$, this equation was used to derive the Teukolsky equation and is already of first-order smallness. Also, by virtue of $\nu^{(1)} = 0$, this gives us a transport equation that we may solve to obtain $\Psi_3^{(1)}$, namely

$$(\Delta + 2\gamma + 4\mu)^{(0)} \Psi_3^{(1)} = (\delta - \tau + 4\beta)^{(0)} \Psi_4^{(1)} + \mathcal{R}_h^{(1)}. \tag{25}$$

For generality, we have kept the terms dependent on the Ricci scalars in the above equation. We will do so throughout the metric reconstruction procedure. However, these terms must satisfy the gauge condition in (19). Having solved for $\Psi_3^{(1)}$, we now consider the Riemann identity in Eq. (A9i). After linearizing, we have

$$(\Delta + \gamma - \bar{\gamma})^{(0)} \pi^{(1)} = -\mu^{(0)} (\pi + \bar{\tau})^{(1)} - \lambda^{(1)} (\bar{\pi} + \tau)^{(0)} - \Psi_3^{(1)} - \Phi_{21}^{(1)}. \tag{26}$$

By combining Eqs. (C1l) and the complex conjugate of Eq. (C1k), we find

$$\pi^{(1)} + \bar{\tau}^{(1)} = -\frac{1}{2} h_{\bar{m}\bar{m}} (\bar{\pi} + \tau)^{(0)}. \tag{27}$$

Combining this with Eq. (26), we obtain a transport equation for $\pi^{(1)}$:

$$(\Delta+\gamma-\bar{\gamma})^{(0)}\pi^{(1)}=\left(\frac{1}{2}\mu^{(0)}h_{\bar{m}\bar{m}}-\lambda^{(1)}\right)(\bar{\pi}+\tau)^{(0)} - \Psi_3^{(1)}-\Phi_{21}^{(1)}. \tag{28}$$

Finally, by our choice of gauge, Eq. (C1l) gives us the transport equation for $h_{l\bar{m}}$, namely

$$(\Delta+\bar{\mu}-2\bar{\gamma})^{(0)}h_{l\bar{m}}=-2\pi^{(1)}-h_{\bar{m}\bar{m}}\tau^{(0)}. \tag{29}$$

Once again, we can obtain h_{lm} by taking the complex conjugate of $h_{l\bar{m}}$. Also, since we now have h_{lm} and h_{mm}, we can directly calculate $\alpha^{(1)}$, $\beta^{(1)}$, and $\tau^{(1)}$ from Eqs. (C1i), (C1j), and (C1k), respectively.

We now proceed with the final step and turn our attention to h_{ll}. Consider the Bianchi identity in Eq. (A10g). Linearizing, and applying our gauge conditions, we obtain a transport equation for the Weyl scalar $\Psi_2^{(1)}$:

$$(\Delta+3\mu)^{(0)}\Psi_2^{(1)}=(\delta+2\beta-2\tau)^{(0)}\Psi_3^{(1)}+\mathcal{R}_g^{(1)}. \tag{30}$$

Now consider the Riemann identity in Eq. (A9f), which after linearizing and applying gauge conditions becomes

$$\begin{aligned}D^{(1)}\gamma^{(0)}+(-\Delta+\gamma+\bar{\gamma})^{(0)}\epsilon^{(1)}-\gamma^{(0)}(\epsilon+\bar{\epsilon})^{(1)} &= \left(\alpha^{(1)}-\frac{1}{2}h_{\bar{m}\bar{m}}\beta^{(0)}\right)(\tau+\bar{\pi})^{(0)} \\ &\quad+\left(\beta^{(0)}-\frac{1}{2}h_{mm}\alpha^{(0)}\right)(\pi+\bar{\tau})^{(0)} \\ &\quad+\tau^{(1)}\pi^{(0)}+\tau^{(0)}\pi^{(1)}+\Psi_2^{(1)},\end{aligned} \tag{31}$$

where we have used Eq. (27). The left-hand side of this equation depends on h_{ll} and its derivatives, while the right-hand side is known from quantities already computed in the previous steps of metric reconstruction. Using Eq. (C1g) and its complex conjugate, we have

$$\epsilon^{(1)}+\bar{\epsilon}^{(1)}=\frac{1}{2}(-\Delta+\gamma+\bar{\gamma})^{(0)}h_{ll}-(\bar{\pi}+\tau)^{(0)}h_{l\bar{m}} - (\pi+\bar{\tau})^{(0)}h_{lm}. \tag{32}$$

Meanwhile, $D^{(1)}$ is given algebraically in terms of h_{ll} through Eq. (8a). Combining these expressions with Eq. (31), we obtain the following second-order transport equation for h_{ll}:

$$\begin{aligned}&\left[\frac{1}{4}(-\Delta+\gamma+\bar{\gamma})^{(0)}(-\Delta+2\bar{\gamma}+\mu-\bar{\mu})^{(0)}+\frac{1}{2}\gamma^{(0)}(-\Delta+\gamma+\bar{\gamma})^{(0)}-\frac{1}{2}\Delta^{(0)}\gamma^{(0)}\right]h_{ll} \\ &\quad=\left[-\frac{1}{4}(-\Delta+\gamma+\bar{\gamma})^{(0)}(-\delta+2\bar{\alpha}-\bar{\pi}-2\tau)^{(0)}+\gamma^{(0)}(\bar{\pi}+\tau)^{(0)}\right]h_{l\bar{m}} \\ &\qquad+\left[-\frac{1}{4}(-\Delta+\gamma+\bar{\gamma})^{(0)}(\bar{\delta}-2\alpha-3\pi-2\bar{\tau})^{(0)}+\gamma^{(0)}(\pi+\bar{\tau})^{(0)}\right]h_{lm} \\ &\qquad+\left(\alpha^{(1)}-\frac{1}{2}\beta^{(0)}h_{\bar{m}\bar{m}}\right)(\bar{\pi}+\tau)^{(0)}+\left(\beta^{(1)}-\frac{1}{2}\alpha^{(0)}h_{mm}\right)(\pi+\bar{\tau})^{(0)} \\ &\qquad+\pi^{(0)}\tau^{(1)}+\pi^{(1)}\tau^{(0)}+\Psi_2^{(1)}.\end{aligned} \tag{33}$$

Thus, we now have all of the necessary equations to solve for the components of the first-order metric perturbation. The remaining spin coefficients and Weyl scalars not computed from the transport equations in this reconstruction procedure may be derived from these metric components through Eqs. (11)–(C1l) and Eqs. (12)–(14c), respectively. In the next section, we give a practical example of this procedure.

V. CASE STUDY: QUASINORMAL MODES OF KERR BLACK HOLES

Having developed a procedure to reconstruct the metric in the outgoing radiation gauge, we illustrate the method with a concrete example, namely the first-order metric perturbation in the limit $r\to\infty$ corresponding to a single quasinormal mode of a Kerr black hole. To address issues of mode coupling at second-order will require reconstruction near the black hole; however, this is sufficiently complicated that we will do so numerically, as described in the companion paper [45].

We work in Boyer-Lindquist coordinates

$$\begin{aligned}ds^2&=\left(1-\frac{2Mr}{\Sigma}\right)dt^2+\frac{4Mra\sin^2\theta}{\Sigma}dtd\phi \\ &\quad-\frac{\Sigma}{\Delta}dr^2-\Sigma d\theta^2 \\ &\quad-\left(r^2+a^2-\frac{2Mra^2}{\Sigma}\sin^2\theta\right)d\phi^2,\end{aligned} \tag{34}$$

where $\Delta=r^2-2Mr+a^2$, and $\Sigma=r^2+a^2\cos^2\theta$, and choose the Kinnersley tetrad [52] (which sets l^μ and n^μ to be parallel to the principal null directions of the Kerr spacetime):

$$l^{\mu} = \frac{1}{\Delta}(r^2 + a^2, \Delta, 0, a), \tag{35a}$$

$$n^{\mu} = \frac{1}{2\Sigma}(r^2 + a^2, -\Delta, 0, a), \tag{35b}$$

$$m^{\mu} = \frac{1}{\sqrt{2}\Gamma}(ia\sin\theta, 0, 1, i\csc\theta), \tag{35c}$$

where $\Gamma = r + ia\cos\theta$. The spin coefficients and Weyl scalars are

$$\begin{aligned} &\kappa = \sigma = \lambda = \nu = \epsilon = \Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0, \\ &\rho = -\frac{1}{\bar{\Gamma}}, \qquad \beta = \frac{\cot\theta}{2^{3/2}\Gamma}, \qquad \pi = \frac{ia\sin\theta}{2^{1/2}\bar{\Gamma}^2}, \\ &\tau = -\frac{ia\sin\theta}{2^{1/2}\Gamma\bar{\Gamma}}, \qquad \mu = -\frac{\Delta}{2\Gamma\bar{\Gamma}^2}, \qquad \gamma = \mu + \frac{r-M}{2\Gamma\bar{\Gamma}}, \\ &\alpha = \pi - \bar{\beta}, \qquad \Psi_2 = -\frac{M}{\bar{\Gamma}^3}. \end{aligned} \tag{36}$$

A. Solving the Teukolsky equation

Before we can reconstruct the metric, we need a solution for $\Psi_4^{(1)}$. Teukolsky showed that by defining $\psi = \rho_{(0)}^{-4}\Psi_4^{(1)}$, Eq. (B12) can be solved by separation of variables [50]; we review that here. In Boyer-Lindquist coordinates, and in vacuum (i.e., all of the Ricci scalars are zero), the Teukolsky equation is

$$\begin{aligned} &\left\{\left[\frac{(r^2+a^2)}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2}{\partial t\partial\phi} - 4\left[r + ia\cos\theta - \frac{M(r^2+a^2)}{\Delta}\right]\frac{\partial}{\partial t}\right. \\ &\quad - \Delta^2\frac{\partial}{\partial r}\left(\Delta^{-1}\frac{\partial}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \left(\frac{1}{\sin^2\theta} - \frac{a^2}{\Delta}\right)\frac{\partial^2}{\partial\phi^2} \\ &\quad \left. + 4\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial}{\partial\phi} + (4\cot^2\theta + 2)\right\}\psi = 0. \end{aligned} \tag{37}$$

By writing

$$\psi = e^{-i\omega t}e^{im\phi}R(r)S(\theta), \tag{38}$$

we can separate the above equation into

$$\Delta^2\frac{d}{dr}\left(\Delta^{-1}\frac{dR}{dr}\right) + \left(\frac{K^2 + 4i(r-M)K}{\Delta} - 8i\omega r - B\right)R = 0, \tag{39a}$$

$$\begin{aligned} &\frac{1}{s_\theta}\frac{d}{d\theta}\left(s_\theta\frac{dS}{d\theta}\right) + \left(a^2\omega^2 c_\theta^2 - \frac{m^2}{s_\theta^2} + 4a\omega c_\theta\right. \\ &\quad \left. + \frac{4mc_\theta}{s_\theta^2} - \frac{4c_\theta^2}{s_\theta^2} - 2 + A\right)S = 0, \end{aligned} \tag{39b}$$

where $K = (r^2 + a^2)\omega - am$, $B = A + a^2\omega^2 - 2am\omega$, $A = A_{lm}(a\omega)$ is a separation constant with eigenvalue l, and $(c_\theta, s_\theta) = (\cos\theta, \sin\theta)$. Equation (39) provides the definition of spin-weighted spheroidal harmonics [53], which reduce to the well-known spin-weighted spherical harmonics in the limit $a \to 0$. We will write the solution to Eq. (39) as $S(\theta) = {}_{-2}S_{lm}(\theta)$.

To solve Eq. (39a), it is natural to make the transformation

$$Y = \frac{(r^2+a^2)^{1/2}}{\Delta}R, \qquad \frac{dr_\star}{dr} = \frac{r^2+a^2}{\Delta}. \tag{40}$$

Equation (39a) then reduces to

$$\begin{aligned} Y'' + &\left[\frac{K^2 + 4i(r-M)K - \Delta(8ir\omega + B)}{(r^2+a^2)^2}\right. \\ &\left. - G^2 - G'\right]Y = 0, \end{aligned} \tag{41}$$

where the prime corresponds to differentiation with respect to $r_\star$ and

$$G = \frac{r\Delta}{(r^2+a^2)^2} - \frac{2(r-M)}{r^2+a^2}. \tag{42}$$

We are interested in a solution near spatial infinity $(r \to \infty, r_\star \to \infty)$; expanding in this limit, Eq. (41) becomes

$$Y'' + \left(\omega^2 - \frac{4i\omega}{r}\right)Y = 0 \tag{43}$$

with solution $Y = (a_0/r^2)e^{-i\omega r_\star} + b_0 r^2 e^{i\omega r_\star}$. Since $(r^2+a^2)^{1/2}/\Delta \sim 1/r$, this implies $R = (a_0/r)e^{-i\omega r_\star} + b_0 r^3 e^{i\omega r_\star}$. Transforming back to the original variable $\Psi_4^{(1)}$, we have

$$\Psi_4^{(1)} = \left(\frac{a_0}{r^5}e^{-i\omega r_\star} + \frac{b_0}{r}e^{i\omega r_\star}\right)e^{-i\omega t+im\phi}{}_{-2}S_{lm}(\theta). \quad (44)$$

This solution corresponds to a superposition of ingoing ($e^{-i\omega r_\star}$) and outgoing ($e^{i\omega r_\star}$) radiation. To model the situation describing the ringdown of a black hole following a binary merger, we enforce the boundary condition that there is no ingoing radiation from infinity, i.e., $a_0 = 0$. Writing $b_0 = {}_{-2}\mathcal{A}_{lm}$, we arrive at the desired asymptotic solution

$$\Psi_4^{(1)} = \frac{{}_{-2}\mathcal{A}_{lm}}{r}e^{i[m\phi-\omega(t-r_\star)]}{}_{-2}S_{lm}(\theta). \quad (45)$$

The complex constant ${}_{-2}\mathcal{A}_{lm}$ is determined by initial conditions, which in the case of a binary coalescence is determined by the inspiral and merger phases.

B. First-order metric

Having a solution for $\Psi_4^{(1)}$, we may now proceed to reconstruct the first-order metric perturbation. Before we begin, there are a couple of important points to mention, one related to modes for $l < 2$, the other about initial data. In general, perturbations of black holes can have $l = 0$ and $l = 1$ angular modes, which physically correspond to shifts in the mass M and spin a of the black hole. Such modes cannot be captured by the spin $s = -2$ field Ψ_4 (or the $s = +2$ field Ψ_0), since spin-weighted fields of spin s can only have support over $l \geq \max(|s|, |m|)$. As demonstrated in the companion paper [45], lack of knowledge of the $l = 0$, 1 modes does not affect the source term or second-order mode coupling from first-order modes with $|m| \geq 2$. For radiative modes with $|m| < 2$ the influence of the nonradiative pieces needs to be incorporated through a combination of nontrivial initial conditions for the transport equations and their homogeneous ($\Psi_4^{(1)} = 0$) solutions, which we leave to future work to investigate (for more discussion of these issues see e.g., Appendix B of [44,54] in the context of the self-force problem). The example of metric reconstruction we provide here therefore does not include these nonradiative terms.

In regards to the specification of initial data, this is a nontrivial problem if posed on a spacelike (Cauchy) slice Σ, and, as with the issues related to $l < 2$ solutions of the transport equations, we leave to future work to investigate. Though in brief, the difficulty stems from the fact that initial data for the Einstein equations (linearized or not) when posed on a spacelike hypersurface are subject to the Hamiltonian and momentum constraints, most easily expressed in terms of geometric objects and their gradients intrinsic to Σ. In the NP formalism, only the two angular null tetrad vectors m and $\bar{m}$ can straightforwardly be rotated to be tangent to Σ (see e.g., [27]); the other two null vectors, and more importantly the corresponding gradient operators D and Δ they define, contain pieces orthogonal to Σ. Hence it is not easy to disentangle what data are freely specifiable (beyond $\Psi_4^{(1)}$) versus constrained if reconstruction is to begin on Σ. Here, the imposition of the QNM ansatz for $\Psi_4^{(1)}$ for all time t, together with only solving the equations in the large r limit, skirts the initial data issue.[1]

Our starting point will be to solve for h_{mm}. An intermediate step is to determine $\lambda^{(1)}$ from $\Psi_4^{(1)}$, with the relevant transport equation given in Eq. (23). We assume that $\lambda^{(1)}$ can be separated in a similar fashion to $\Psi_4^{(1)}$; specifically we write $\lambda^{(1)} = e^{-i\omega t}e^{im\phi}R_\lambda(r)S_\lambda(\theta)$, and our goal will be to determine $R_\lambda(r)$ and $S_\lambda(\theta)$. Inserting this ansatz into Eq. (23), applying the NP operators in Boyer-Lindquist coordinates, and expanding in $r \to \infty (r_\star \to \infty)$, we obtain

$$-\frac{1}{2}e^{-i\omega t+im\phi}S_\lambda(\theta)\left(\frac{dR_\lambda}{dr} + i\omega R_\lambda(r)\right) = -\frac{{}_{-2}\mathcal{A}_{lm}}{r}e^{-i\omega(t-r_\star)+im\phi}{}_{-2}S_{lm}(\theta). \quad (46)$$

A necessary condition to separate this equation is $S_\lambda(\theta) = {}_{-2}S_{lm}(\theta)$. Applying this, we obtain the following equation for $R_\lambda(r)$:

$$\frac{dR_\lambda}{dr} + i\omega R_\lambda = -\frac{2}{r}{}_{-2}\mathcal{A}_{lm}e^{i\omega r_\star}. \quad (47)$$

The homogeneous solution to this equation scales as $e^{-i\omega r_\star}$, and thus the $(t, r_\star)$ dependence of the full homogeneous solution goes as $\lambda^{(1)} \sim e^{-i\omega(t+r_\star)}$. This corresponds to an ingoing mode, which we set to zero, and so we only need to worry about the particular solution to the above equation. Due to the behavior of the right-hand side of Eq. (47), the particular solution will scale as $e^{i\omega r_\star}$. Writing $R_\lambda = a_0 e^{i\omega r_\star}/r^n$, we can insert this into Eq. (47) and solve for a_0 and n in an asymptotic expansion about spatial infinity. Doing so, we obtain $n = 1$ and $a_0 = -i\,{}_{-2}\mathcal{A}_{lm}/\omega$, and thus

$$\lambda^{(1)} = -\frac{i}{\omega r}{}_{-2}\mathcal{A}_{lm}e^{-i\omega(t-r_\star)+im\phi}{}_{-2}S_{lm}(\theta). \quad (48)$$

Now that we have $\lambda^{(1)}$, we turn our attention to the transport equation for $h_{\bar{m}\bar{m}}$ given by Eq. (24). The procedure for determining $h_{\bar{m}\bar{m}}$ follows the same steps as finding $\lambda^{(1)}$.

[1]For the numerical solution discussed in the companion paper we cannot make a QNM ansatz and do not limit the domain to large r. We still do not solve the initial data problem on Σ there but instead circumvent the problem by a particular restriction of the class of initial data and only performing self-consistent reconstruction within a related null wedge interior to the domain of the Cauchy evolution. Also, not all the NP equations are used to reconstruct the metric, and a subset are redundant (essentially stemming from the Bianchi identities). These are used in the code to check that the reconstruction is in fact self-consistent within the null wedge. For details see [45].

Writing $h_{\bar{m}\bar{m}} = e^{-i\omega t + im\phi} R_{\bar{m}\bar{m}}(r) S_{\bar{m}\bar{m}}(\theta)$, the necessary condition for separability is $S_{\bar{m}\bar{m}}(\theta) = {}_{-2}S_{lm}(\theta)$. We then obtain the equation

$$\frac{dR_{\bar{m}\bar{m}}}{dr} + i\omega R_{\bar{m}\bar{m}} = -\frac{4i}{\omega r}\,{}_{-2}\mathcal{A}_{lm} e^{i\omega r_\star}. \tag{49}$$

Using our boundary condition to set the homogeneous solution to zero, we solve for the particular solution to obtain

$$h_{\bar{m}\,\bar{m}} = -\frac{2}{\omega^2 r}\,{}_{-2}\mathcal{A}_{lm} e^{-i\omega(t-r_\star)+im\phi}\,{}_{-2}S_{lm}(\theta). \tag{50}$$

Finally, taking the complex conjugate, we have

$$h_{mm} = -\frac{2}{\bar{\omega}^2 r}\,{}_{-2}\bar{\mathcal{A}}_{lm} e^{i\bar{\omega}(t-r_\star)-im\phi}\,{}_{-2}\bar{S}_{lm}(\theta). \tag{51}$$

We have made it explicit here that one has to take the complex conjugate of ω and ${}_{-2}S_{lm}$, as well as ${}_{-2}\mathcal{A}_{lm}$. In general, the frequency of the quasinormal modes is complex, and since ${}_{-2}S_{lm}$ depends on ω, then it is also complex.

We now turn our attention to solving for h_{lm}. The starting point is to solve for the Weyl scalar $\Psi_3^{(1)}$ from Eq. (25). Expanding the right-hand side of this equation, we obtain

$$(\delta - \tau + 4\beta)^{(0)}\Psi_4^{(1)} = \frac{{}_{-2}\bar{\mathcal{A}}_{lm}}{\sqrt{2}r^2} e^{im\phi - i\omega(t-r_\star)} \times \mathcal{L}_{-2}[{}_{-2}S_{lm}(\theta)], \tag{52}$$

where $\mathcal{L}_s = \partial_\theta - m\csc\theta - s\cot\theta + a\omega\sin\theta$. These are the same operators that appear in the well-known Teukolsky-Starobinsky identities [35]. It is worth pointing out, however, that the operation $\mathcal{L}_{-2}[{}_{-2}S_{lm}(\theta)]$ does not generate the spin-weight -1 spheroidal harmonic ${}_{-1}S_{lm}(\theta)$, which can be verified by direct application of the angular Teukolsky equation (39) for spin weight -1. In fact, this is the reason why one cannot decouple the equations governing electromagnetic and gravitational perturbations of the Kerr-Newman spacetime [35,55]. Note that, in the nonspinning limit (i.e., $a = 0$), the operator $\mathcal{L}_s$ does reduce to the raising operator for spin-weighted spherical harmonics ${}_sY_{lm}(\theta, \phi)$ and in the Geroch-Held-Penrose formalism [56] is the asymptotically expanded ð operator which raises the spin weight of quantities. Thus, we may expect that the operation $\mathcal{L}_{-2}[{}_{-2}S_{lm}(\theta)]$ does produce an angular function of spin weight -1 but that it does not satisfy the corresponding angular Teukolsky equation.

To solve for $\Psi_3^{(1)}$, we propose the ansatz $\Psi_3^{(1)} = e^{im\phi - i\omega t}R_3(r)S_3(\theta)$. In order to perform separation of variables, we must have $S_3(\theta) = \mathcal{L}_{-2}[{}_{-2}S_{lm}(\theta)]$. This gives us the equation for the radial function $R_3(r)$ in the limit $r \to \infty$:

$$\frac{dR_3}{dr} + i\omega R_3(r) = \frac{{}_{-2}\mathcal{A}_{lm}}{\sqrt{2}r^2} e^{i\omega r_\star}. \tag{53}$$

Solving this equation, we obtain

$$\Psi_3^{(1)} = \frac{i}{\sqrt{2}} \frac{{}_{-2}\mathcal{A}_{lm}}{\omega r^2} e^{im\phi - i\omega(t-r_\star)} \mathcal{L}_{-2}[{}_{-2}S_{lm}(\theta)]. \tag{54}$$

The remainder of the procedure to obtain $\pi^{(1)}$ and h_{lm} follows these exact same steps. The angular dependence of these functions is $\mathcal{L}_{-2}[{}_{-2}S_{lm}(\theta)]$ in order to perform separation of variables. The end result of this computation is

$$\pi^{(1)} = \frac{1}{\sqrt{2}} \frac{{}_{-2}\mathcal{A}_{lm}}{\omega^2 r^2} e^{im\phi - i\omega(t-r_\star)} \mathcal{L}_{-2}[{}_{-2}S_{lm}(\theta)], \tag{55a}$$

$$h_{l\bar{m}} = -i\sqrt{2} \frac{{}_{-2}\mathcal{A}_{lm}}{\omega^3 r^2} e^{im\phi - i\omega(t-r_\star)} \mathcal{L}_{-2}[{}_{-2}S_{lm}(\theta)]. \tag{55b}$$

By virtue of having solved for $\pi^{(1)}$ and h_{lm}, we may also compute $\alpha^{(1)}$, $\beta^{(1)}$, and $\tau^{(1)}$, with the end result being

$$\alpha^{(1)} = \frac{1}{2^{3/2}} \frac{{}_{-2}\mathcal{A}_{lm}}{\omega^2 r^2} e^{im\phi - i\omega(t-r_\star)} \times \{2\mathcal{L}_{-2}[{}_{-2}S_{lm}(\theta)] - \cot\theta\,{}_{-2}S_{lm}(\theta)\}, \tag{56a}$$

$$\beta^{(1)} = \frac{1}{2^{3/2}} \frac{{}_{-2}\bar{\mathcal{A}}_{lm}}{\bar{\omega}^2 r^2} e^{-im\phi + i\bar{\omega}(t-r_\star)} \cot\theta\,{}_{-2}\bar{S}_{lm}(\theta), \tag{56b}$$

$$\tau^{(1)} = -\frac{1}{\sqrt{2}} \frac{{}_{-2}\bar{\mathcal{A}}_{lm}}{\bar{\omega}^2 r^2} \{\mathcal{L}_{-2}[{}_{-2}S_{lm}(\theta)]\}^\dagger, \tag{56c}$$

where † corresponds to complex conjugation of the angular function.

Finally, we consider the solution for h_{ll}. The first step is to solve for $\Psi_2^{(1)}$ using Eq. (30). Expanding the right-hand side, we have

$$(\delta + 2\beta - 2\tau)^{(0)}\Psi_3^{(1)} = \frac{i}{2} \frac{{}_{-2}\mathcal{A}_{lm}}{\omega r^3} \mathcal{L}_{-1}\mathcal{L}_{-2}[{}_{-2}S_{lm}(\theta)]. \tag{57}$$

Writing the ansatz $\Psi_2^{(1)} = e^{im\phi - i\omega t}R_2(r)S_2(\theta)$ and expanding the left-hand side of Eq. (30), we have that $S_2(\theta) = \mathcal{L}_{-1}\mathcal{L}_{-2}[{}_{-2}S_{lm}(\theta)]$ in order to achieve separation of variables. We are then left with

$$\frac{dR_2}{dr} + i\omega R_2(r) = \frac{i}{2} \frac{{}_{-2}\mathcal{A}_{lm}}{\omega r^3} e^{i\omega r_\star}, \tag{58}$$

which can be solved in a $1/r$ expansion to obtain

$$\Psi_2^{(1)} = -\frac{1}{2} \frac{{}_{-2}\mathcal{A}_{lm}}{\omega^2 r^3} e^{im\phi - i\omega(t-r_\star)} \mathcal{L}_{-1}\mathcal{L}_{-2}[{}_{-2}S_{lm}(\theta)]. \tag{59}$$

With $\Psi_2^{(1)}$ in hand, we now turn to Eq. (33). Consider the source terms on the right-hand side of this equation. In an

$r \to \infty$ expansion, the terms containing $h_{l\bar{m}}$, h_{lm}, and $\Psi_2^{(1)}$ dominate and scale as $1/r^3$. This expanded source term is real valued, since h_{ll} must be real valued. Writing $h_{ll} = e^{im\phi - i\omega t} R_+(r) S_+(\theta) + e^{-im\phi + i\omega t} R_-(r) S_-(\theta)$, the necessary conditions to perform separation of variables are $S_+(\theta) = \mathcal{L}_{-1}\mathcal{L}_{-2}[{}_{-2}S_{lm}(\theta)]$ and $S_-(\theta) = \{\mathcal{L}_{-1}\mathcal{L}_{-2}[{}_{-2}S_{lm}(\theta)]\}^{\dagger}$. Expanding about $r \to \infty$, we obtain

$$\frac{d^2 R_+}{dr^2} + 2i\omega \frac{dR_+}{dr} - \omega^2 R_+(r) = -4 \frac{{}_{-2}\mathcal{A}_{lm}}{\omega^2 r^3} e^{i\omega r_\star}, \tag{60a}$$

$$\frac{d^2 R_-}{dr^2} - 2i\bar{\omega} \frac{dR_-}{dr} - \bar{\omega}^2 R_-(r) = -4 \frac{{}_{-2}\bar{\mathcal{A}}_{lm}}{\bar{\omega}^2 r^3} e^{-i\bar{\omega} r_\star}. \tag{60b}$$

These equations can be solved with the methods we have previously employed to obtain

$$h_{ll} = 4 \frac{{}_{-2}\mathcal{A}_{lm}}{\omega^4 r^3} e^{im\phi - i\omega(t - r_\star)} \mathcal{L}_{-1}\mathcal{L}_{-2}[{}_{-2}S_{lm}(\theta)] + \text{c.c}, \tag{61}$$

where c.c. is shorthand for the complex conjugate of the preceding term.

Now that we have all of the components of the metric in our chosen gauge, we may complete the first-order description of the NP quantities. Applying Eqs. (11)–(C11), the remaining spin coefficients are

$$\kappa^{(1)} = -\sqrt{2} i \frac{{}_{-2}\bar{\mathcal{A}}_{lm}}{\bar{\omega}^3 r^3} e^{-im\phi + i\bar{\omega}(t - r_\star)} \{\mathcal{L}_{-2}[{}_{-2}S_{lm}(\theta)]\}^{\dagger}, \tag{62a}$$

$$\sigma^{(1)} = \frac{{}_{-2}\bar{\mathcal{A}}_{lm}}{\bar{\omega}^2 r^2} e^{-im\phi + i\bar{\omega}(t - r_\star)} {}_{-2}\bar{S}_{lm}(\theta), \tag{62b}$$

$$\begin{aligned}\epsilon^{(1)} = {} & \frac{5i}{4} \frac{{}_{-2}\mathcal{A}_{lm}}{\omega^3 r^3} e^{im\phi - i\omega(t - r_\star)} \mathcal{L}_{-1}\mathcal{L}_{-2}[{}_{-2}S_{lm}(\theta)] \\ & - \frac{3i}{4} \frac{{}_{-2}\bar{\mathcal{A}}_{lm}}{\bar{\omega}^3 r^3} e^{-im\phi + i\bar{\omega}(t - r_\star)} \{\mathcal{L}_{-1}\mathcal{L}_{-2}[{}_{-2}S_{lm}(\theta)]\}^{\dagger},\end{aligned} \tag{62c}$$

$$\rho^{(1)} = \frac{i}{2} \frac{{}_{-2}\mathcal{A}_{lm}}{\omega^3 r^3} e^{im\phi - i\omega(t - r_\star)} \mathcal{L}_{-1}\mathcal{L}_{-2}[{}_{-2}S_{lm}(\theta)] + \text{c.c.} \tag{62d}$$

To obtain the remaining Weyl scalar $\Psi_1^{(1)}$ and $\Psi_0^{(1)}$, we use the linearize Bianchi identities in Eqs. (63a) and (63b). The methods for solving these are the exact same methods we detailed for the metric coefficients. The end result is

$$\Psi_1^{(1)} = \frac{i}{\sqrt{2}} \frac{{}_{-2}\mathcal{A}_{lm}}{\omega^3 r^4} e^{im\phi - i\omega(t - r_\star)} \mathcal{L}_0\mathcal{L}_{-1}\mathcal{L}_{-2}[{}_{-2}S_{lm}(\theta)], \tag{63a}$$

$$\begin{aligned}\Psi_0^{(1)} = {} & \frac{{}_{-2}\mathcal{A}_{lm}}{\omega^4 r^5} e^{im\phi - i\omega(t - r_\star)} \mathcal{L}_1\mathcal{L}_0\mathcal{L}_{-1}\mathcal{L}_{-2}[{}_{-2}S_{lm}(\theta)] \\ & - 6iM \frac{{}_{-2}\bar{\mathcal{A}}_{lm}}{\bar{\omega}^3 r^5} e^{-im\phi + i\bar{\omega}(t - r_\star)} {}_{-2}\bar{S}_{lm}(\theta).\end{aligned} \tag{63b}$$

This completes the derivation of all NP quantities at first-order.

VI. DISCUSSION

Here we have laid some of the ground work necessary for the study of second-order perturbations of Kerr black holes. Working in outgoing radiation gauge, we showed that the first-order metric perturbations of a Kerr black hole can be reconstructed starting from a single NP quantity, namely $\Psi_4^{(1)}$. As an example we have applied this to obtain the first-order metric perturbations associated with the quasinormal modes of Kerr black holes in the asymptotic limit.

There are several directions for future work. As mentioned, reconstructing the metric over the entire spacetime is complicated and might not be analytically tractable. We have developed a numerical code to implement the solution of the Teukolsky equation, and reconstruction procedure, over the full spacetime exterior to the horizon [45]. This is particularly relevant regarding questions of mode coupling after binary black hole mergers, as this phenomena will be governed by sources strongest in the near-horizon region. Another direction of future study would thus be to investigate whether, in addition to our numerical analysis, analytic solutions may be obtained there. Also as discussed in Sec. V B, additional work is needed to solve for corrections to the metric corresponding to changes in the spin and mass of the black hole.

As mentioned in the introduction, crucial to understanding the nonlinear regime of ringdown is the question of what the "initial conditions" of the perturbed black hole following a merger are. If this is not known, it would be difficult to distinguish the higher overtones of linear modes from second-order effects, which could have similar amplitudes, frequencies and decay rates.[2] The close limit approximation to black hole mergers [23] seems like a natural avenue to address the question of initial conditions. Insight could also be gained from recent studies investigating this in the EMRI limit [57,58]. Also, numerical simulations of mergers can be used to at least constrain the initial conditions via measurement of "final conditions," i.e., the amplitudes and phases of modes in the ringdown once all the nonlinear effects have sorted themselves out, as well as measure-driven second-order modes that will persist and look like QNMs with amplitudes and complex frequencies that are squares of their parent modes (see e.g., [59]).

A further interesting application is investigating the energy cascade between modes due to nonlinear effects in ringdown. In asymptotically anti–de Sitter (AdS) spacetime, several studies of black holes and black branes have shown that horizon perturbations, modulo the natural

[2]If—as argued in [19]—linear theory can very accurately describe postmerger ringdown dynamics from peak amplitude onward, second-order analysis presumably then should be able to extend this to some time *before* peak amplitude.

decay, become turbulent [60–62]. This may be a peculiarity of AdS spacetime, though a study in [63] suggested similar phenomenology might be present for very rapidly rotating Kerr black holes in asymptotically flat spacetime. Those researchers used a scalar field on a Kerr background as a model for gravitational wave perturbations; with the tools presented here and in [45] it should be possible to repeat this for tensor perturbations. Their work suggests that turbulent dynamics might only be apparent for very rapidly spinning black holes; whether these exist in nature is unknown, nevertheless this is still an interesting open theoretical problem.

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APPENDIX A: NEWMAN-PENROSE FORMALISM

For completeness in this Appendix we review the NP formalism. We use the conventions of [35] (e.g., our metric sign convention is $+---$, and we use $\bar{f}$ to denote the complex conjugate of f), except that we use Greek letters to denote spacetime indices.

The NP formalism is a reformulation of the Einstein field equations in a null tetrad frame, defined by four null vectors $e_a^\mu = (l^\mu, n^\mu, m^\mu, \bar{m}^\mu)$ satisfying

$$l^\mu n_\mu = 1, \qquad m^\mu \bar{m}_\mu = -1, \tag{A1}$$

where the overbar corresponds to complex conjugation and the remaining dot products are zero. The metric $g_{\mu\nu}$ is related to the null vectors via $g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b$, where

$$\eta_{ab} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}. \tag{A2}$$

This leads to the completeness relation

$$g_{\mu\nu} = 2l_{(\mu} n_{\nu)} - 2m_{(\mu} \bar{m}_{\nu)}. \tag{A3}$$

We further define the derivatives along the null directions as

$$D = l^\mu \partial_\mu, \qquad \Delta = n^\mu \partial_\mu, \qquad \delta = m^\mu \partial_\mu, \qquad \bar{\delta} = \bar{m}^\mu \partial_\mu. \tag{A4}$$

These differential operators satisfy the following commutation relations:

$$[\Delta, D] = (\gamma + \bar{\gamma})D + (\epsilon + \bar{\epsilon})\Delta - (\bar{\tau} + \pi)\delta - (\tau + \bar{\pi})\bar{\delta}, \tag{A5a}$$

$$[\delta, D] = (\bar{\alpha} + \beta - \bar{\pi})D + \kappa\Delta - (\bar{\rho} + \epsilon - \bar{\epsilon})\delta - \sigma\bar{\delta}, \tag{A5b}$$

$$[\delta, \Delta] = -\bar{\nu}D + (\tau - \bar{\alpha} - \beta)\Delta + (\mu - \gamma + \bar{\gamma})\delta + \bar{\lambda}\bar{\delta}, \tag{A5c}$$

$$[\bar{\delta}, \delta] = (\bar{\mu} - \mu)D + (\bar{\rho} - \rho)\Delta + (\alpha - \bar{\beta})\delta + (\beta - \bar{\alpha})\bar{\delta}, \tag{A5d}$$

where $\{\alpha, \beta, \gamma, \epsilon, \rho, \lambda, \pi, \mu, \nu, \tau, \sigma, \kappa\}$ are the complex spin coefficients. The components of curvature in the NP formalism are characterized by contractions of the null tetrad with the Weyl tensor and Ricci tensor; specifically, the Weyl tensor contractions are

$$\Psi_0 = -C_{\mu\nu\rho\sigma} l^\mu m^\nu l^\rho m^\sigma, \tag{A6a}$$

$$\Psi_1 = -C_{\mu\nu\rho\sigma} l^\mu n^\nu l^\rho m^\sigma, \tag{A6b}$$

$$\Psi_2 = -C_{\mu\nu\rho\sigma} l^\mu m^\nu \bar{m}^\rho n^\sigma, \tag{A6c}$$

$$\Psi_3 = -C_{\mu\nu\rho\sigma} l^\mu n^\nu \bar{m}^\rho n^\sigma, \tag{A6d}$$

$$\Psi_4 = -C_{\mu\nu\rho\sigma} n^\mu \bar{m}^\nu n^\rho \bar{m}^\sigma, \tag{A6e}$$

and the contractions with the Ricci tensor are

$$\Phi_{00} = -\frac{1}{2} R_{\mu\nu} l^\mu l^\nu, \qquad \Phi_{22} = -\frac{1}{2} R_{\mu\nu} n^\mu n^\nu, \tag{A7a}$$

$$\Phi_{02} = -\frac{1}{2} R_{\mu\nu} m^\mu m^\nu, \qquad \Phi_{20} = -\frac{1}{2} R_{\mu\nu} \bar{m}^\mu \bar{m}^\nu \tag{A7b}$$

$$\Phi_{11} = -\frac{1}{4} R_{\mu\nu} (l^\mu n^\nu + m^\mu \bar{m}^\nu), \tag{A7c}$$

$$\Phi_{01} = -\frac{1}{2} R_{\mu\nu} l^\mu m^\nu, \qquad \Phi_{10} = -\frac{1}{2} R_{\mu\nu} l^\mu \bar{m}^\nu, \tag{A7d}$$

$$\Lambda = \frac{1}{12} R_{\mu\nu} (l^\mu n^\nu - m^\mu \bar{m}^\nu), \tag{A7e}$$

$$\Phi_{12} = -\frac{1}{2} R_{\mu\nu} n^\mu m^\nu, \qquad \Phi_{21} = -\frac{1}{2} R_{\mu\nu} n^\mu \bar{m}^\nu. \tag{A7f}$$

When the Einstein equations are imposed, this latter set of curvature scalars can be related to the stress energy tensor $T_{\mu\nu}$ of matter through the trace-reversed field equations:

$$R_{\mu\nu} = 8\pi\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right), \tag{A8}$$

where $T = T_\mu{}^\mu$.

The decomposition of the Riemann tensor in terms of the Weyl and Ricci tensors provides the necessary transport equations describing the evolution of the spin coefficients in terms of the above quantities; specifically

$$D\rho - \bar{\delta}\kappa = (\rho^2 + \sigma\bar{\sigma}) + \rho(\epsilon + \bar{\epsilon}) - \bar{\kappa}\tau - \kappa(3\alpha + \bar{\beta} - \pi) + \Phi_{00}, \tag{A9a}$$

$$D\sigma - \delta\kappa = \sigma(\rho + \bar{\rho} + 3\epsilon - \bar{\epsilon}) - \kappa(\tau - \bar{\pi} + \bar{\alpha} + 3\beta) + \Psi_0, \tag{A9b}$$

$$D\tau - \Delta\kappa = \rho(\tau + \bar{\pi}) + \sigma(\bar{\tau} + \pi) + \tau(\epsilon - \bar{\epsilon}) - \kappa(3\gamma + \bar{\gamma}) + \Psi_1 + \Phi_{01}, \tag{A9c}$$

$$D\alpha - \bar{\delta}\epsilon = \alpha(\rho + \bar{\epsilon} - 2\epsilon) + \beta\bar{\sigma} - \bar{\beta}\epsilon - \kappa\lambda - \bar{\kappa}\gamma + \pi(\epsilon + \rho) + \Phi_{10}, \tag{A9d}$$

$$D\beta - \delta\epsilon = \sigma(\alpha + \pi) + \beta(\bar{\rho} - \bar{\epsilon}) - \kappa(\mu + \gamma) - \epsilon(\bar{\alpha} - \bar{\pi}) + \Psi_1, \tag{A9e}$$

$$D\gamma - \Delta\epsilon = \alpha(\tau + \bar{\pi}) + \beta(\bar{\tau} + \pi) - \gamma(\epsilon + \bar{\epsilon}) - \epsilon(\gamma + \bar{\gamma}) + \tau\pi - \nu\kappa + \Psi_2 + \Phi_{11} - \Lambda, \tag{A9f}$$

$$D\lambda - \bar{\delta}\pi = (\rho\lambda + \bar{\sigma}\mu) + \pi(\pi + \alpha - \bar{\beta}) - \nu\bar{\kappa} - \lambda(3\epsilon - \bar{\epsilon}) + \Phi_{20}, \tag{A9g}$$

$$D\mu - \delta\pi = (\bar{\rho}\mu + \sigma\lambda) + \pi(\bar{\pi} - \bar{\alpha} + \beta) - \mu(\epsilon + \bar{\epsilon}) - \nu\kappa + \Psi_2 + 2\Lambda, \tag{A9h}$$

$$D\nu - \Delta\pi = \mu(\pi + \bar{\tau}) + \lambda(\bar{\pi} + \tau) + \pi(\gamma - \bar{\gamma}) - \nu(3\epsilon + \bar{\epsilon}) + \Psi_3 + \Phi_{21}, \tag{A9i}$$

$$\Delta\lambda - \bar{\delta}\nu = -\lambda(\mu + \bar{\mu} + 3\gamma - \bar{\gamma}) + \nu(3\alpha + \bar{\beta} + \pi - \bar{\tau}) - \Psi_4, \tag{A9j}$$

$$\delta\rho - \bar{\delta}\sigma = \rho(\bar{\alpha} + \beta) - \sigma(3\alpha - \bar{\beta}) + \tau(\rho - \bar{\rho}) + \kappa(\mu - \bar{\mu}) - \Psi_1 + \Phi_{01}, \tag{A9k}$$

$$\delta\alpha - \bar{\delta}\beta = \mu\rho - \lambda\sigma + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta + \gamma(\rho - \bar{\rho}) + \epsilon(\mu - \bar{\mu}) - \Psi_2 + \Phi_{11} + \Lambda, \tag{A9l}$$

$$\delta\lambda - \bar{\delta}\mu = \nu(\rho - \bar{\rho}) + \pi(\mu - \bar{\mu}) + \mu(\alpha + \bar{\beta}) + \lambda(\bar{\alpha} - 3\beta) - \Psi_3 + \Phi_{21}, \tag{A9m}$$

$$\delta\nu - \Delta\mu = (\mu^2 + \lambda\bar{\lambda}) + \mu(\gamma + \bar{\gamma}) - \bar{\nu}\pi + \nu(\tau - 3\beta - \bar{\alpha}) + \Phi_{22}, \tag{A9n}$$

$$\delta\gamma - \Delta\beta = \gamma(\tau - \bar{\alpha} - \beta) + \mu\tau - \sigma\nu - \epsilon\bar{\nu} - \beta(\gamma - \bar{\gamma} - \mu) + \alpha\bar{\lambda} + \Phi_{12}, \tag{A9o}$$

$$\delta\tau - \Delta\sigma = (\mu\sigma + \bar{\lambda}\rho) + \tau(\tau + \beta - \bar{\alpha}) - \sigma(3\gamma - \bar{\gamma}) - \kappa\bar{\nu} + \Phi_{02}, \tag{A9p}$$

$$\Delta\rho - \bar{\delta}\tau = -\rho\bar{\mu} + \sigma\lambda + \tau(\bar{\beta} - \alpha - \bar{\tau}) + \rho(\gamma + \bar{\gamma}) + \nu\kappa - \Psi_2 - 2\Lambda, \tag{A9q}$$

$$\Delta\alpha - \bar{\delta}\gamma = \nu(\rho + \epsilon) - \lambda(\tau + \beta) + \alpha(\bar{\gamma} - \bar{\mu}) + \gamma(\bar{\beta} - \bar{\tau}) - \Psi_3. \tag{A9r}$$

Meanwhile, the Bianchi identities provide the following transport equations for the Weyl scalar:

$$-\bar{\delta}\Psi_0 + D\Psi_1 + (4\alpha - \pi)\Psi_0 - 2(2\rho + \epsilon)\Psi_1 + 3\kappa\Psi_2 + \mathcal{R}_a = 0, \tag{A10a}$$

$$\bar{\delta}\Psi_1 - D\Psi_2 - \lambda\Psi_0 + 2(\pi - \alpha)\Psi_1 + 3\rho\Psi_2 - 2\kappa\Psi_3 + \mathcal{R}_b = 0, \tag{A10b}$$

$$-\bar{\delta}\Psi_2 + D\Psi_3 + 2\lambda\Psi_1 - 3\pi\Psi_2 + 2(\epsilon - \rho)\Psi_3 + \kappa\Psi_4 + \mathcal{R}_c = 0, \tag{A10c}$$

$$\bar{\delta}\Psi_3 - D\Psi_4 - 3\lambda\Psi_2 + 2(2\pi + \alpha)\Psi_3 - (4\epsilon - \rho)\Psi_4 + \mathcal{R}_d = 0, \tag{A10d}$$

$$-\Delta\Psi_0 + \delta\Psi_1 + (4\gamma - \mu)\Psi_0 - 2(2\tau + \beta)\Psi_1 + 3\sigma\Psi_2 + \mathcal{R}_e = 0, \tag{A10e}$$

$$-\Delta\Psi_1 + \delta\Psi_2 + \nu\Psi_0 + 2(\gamma - \mu)\Psi_1 - 3\tau\Psi_2 + 2\sigma\Psi_3 + \mathcal{R}_f = 0, \tag{A10f}$$

$$-\Delta\Psi_2 + \delta\Psi_3 + 2\nu\Psi_1 - 3\mu\Psi_2 + 2(\beta - \tau)\Psi_3 + \sigma\Psi_4 + \mathcal{R}_g = 0, \tag{A10g}$$

$$-\Delta\Psi_3 + \delta\Psi_4 + 3\nu\Psi_2 - 2(\gamma + 2\mu)\Psi_3 - (\tau - 4\beta)\Psi_4 + \mathcal{R}_h = 0, \tag{A10h}$$

where the $\mathcal{R}$ terms only depend on the Ricci scalars

$$\mathcal{R}_a = -D\Phi_{01} + \delta\Phi_{00} + 2(\epsilon + \bar{\rho})\Phi_{01} + 2\sigma\Phi_{10} - 2\kappa\Phi_{11} - \bar{\kappa}\Phi_{02} + (\bar{\pi} - 2\bar{\alpha} - 2\beta)\Phi_{00}, \tag{A11a}$$

$$\mathcal{R}_b = \bar{\delta}\Phi_{01} - \Delta\Phi_{00} - 2(\alpha+\bar{\tau})\Phi_{01} + 2\rho\Phi_{11} + \bar{\sigma}\Phi_{02} - (\mu - 2\gamma - 2\bar{\gamma})\Phi_{00} - 2\tau\Phi_{10} - 2D\Lambda, \tag{A11b}$$

$$\mathcal{R}_c = -D\Phi_{21} + \delta\Phi_{20} + 2(\bar{\rho}-\epsilon)\Phi_{21} - 2\mu\Phi_{10} + 2\pi\Phi_{11} - \bar{\kappa}\Phi_{22} - (2\bar{\alpha} - 2\beta - \bar{\pi})\Phi_{20} - 2\bar{\delta}\Lambda, \tag{A11c}$$

$$\mathcal{R}_d = -\Delta\Phi_{20} + \bar{\delta}\Phi_{21} + (2\alpha - \bar{\tau})\Phi_{21} + 2\nu\Phi_{10} + \bar{\sigma}\Phi_{22} - 2\lambda\Phi_{11} - (\bar{\mu} + 2\gamma - 2\bar{\gamma})\Phi_{20}, \tag{A11d}$$

$$\mathcal{R}_e = -D\Phi_{02} + \delta\Phi_{01} + 2(\bar{\pi} - \beta)\Phi_{01} - 2\kappa\Phi_{12} - \bar{\lambda}\Phi_{00} + 2\sigma\Phi_{11} + (\bar{\rho} + 2\epsilon - 2\bar{\epsilon})\Phi_{02}, \tag{A11e}$$

$$\mathcal{R}_f = \Delta\Phi_{01} - \bar{\delta}\Phi_{02} + 2(\bar{\mu} - \gamma)\Phi_{01} - 2\rho\Phi_{12} - \bar{\nu}\Phi_{00} + 2\tau\Phi_{11} + (\bar{\tau} - 2\bar{\beta} + 2\alpha)\Phi_{02} + 2\delta\Lambda, \tag{A11f}$$

$$\mathcal{R}_g = -D\Phi_{22} + \delta\Phi_{21} + 2(\bar{\pi} + \beta)\Phi_{21} - 2\mu\Phi_{11} - \bar{\lambda}\Phi_{20} + 2\pi\Phi_{12} + (\bar{\rho} - 2\epsilon - 2\bar{\epsilon})\Phi_{22} - 2\Delta\Lambda, \tag{A11g}$$

$$\mathcal{R}_h = \Delta\Phi_{21} - \bar{\delta}\Phi_{22} + 2(\bar{\mu} + \gamma)\Phi_{21} - 2\nu\Phi_{11} - \bar{\nu}\Phi_{20} + 2\lambda\Phi_{12} + (\bar{\tau} - 2\alpha - 2\bar{\beta})\Phi_{22}. \tag{A11h}$$

Finally, the evolution equations for the Ricci scalars are obtained through the divergence-free property of the Einstein tensor $\nabla_\mu G^{\mu\nu} = 0$:

$$\begin{aligned}&\bar{\delta}\Phi_{01} + \delta\Phi_{10} - D(\Phi_{11} + 3\Lambda) - \Delta\Phi_{00}\\ &\quad = \bar{\kappa}\Phi_{12} + \kappa\Phi_{21} + (2\alpha + 2\bar{\tau} - \pi)\Phi_{01}\\ &\qquad + (2\bar{\alpha} + 2\tau - \bar{\pi})\Phi_{10} - 2(\rho + \bar{\rho})\Phi_{11} - \bar{\sigma}\Phi_{02} - \sigma\Phi_{20}\\ &\qquad + [\mu + \bar{\mu} - 2(\gamma + \bar{\gamma})]\Phi_{00},\end{aligned} \tag{A12a}$$

$$\begin{aligned}&\bar{\delta}\Phi_{12} + \delta\Phi_{21} - \Delta(\Phi_{11} + 3\Lambda) - D\Phi_{22}\\ &\quad = -\nu\Phi_{01} - \bar{\nu}\Phi_{10} + (\bar{\tau} - 2\bar{\beta} - 2\pi)\Phi_{12}\\ &\qquad + (\tau - 2\beta - 2\bar{\pi})\Phi_{21} + 2(\mu + \bar{\mu})\Phi_{11}\\ &\qquad - (\rho + \bar{\rho} - 2\epsilon - 2\bar{\epsilon})\Phi_{22} + \lambda\Phi_{02} + \bar{\lambda}\Phi_{20},\end{aligned} \tag{A12b}$$

$$\begin{aligned}&\delta(\Phi_{11} - 3\Lambda) - D\Phi_{12} - \Delta\Phi_{01} + \bar{\delta}\Phi_{02}\\ &\quad = \kappa\Phi_{22} - \bar{\nu}\Phi_{00} + (\bar{\tau} - \pi + 2\alpha - 2\bar{\beta})\Phi_{02} - \sigma\Phi_{21}\\ &\qquad + \bar{\lambda}\Phi_{10} + 2(\tau - \bar{\pi})\Phi_{11} - (2\rho + \bar{\rho} - 2\bar{\epsilon})\Phi_{12}\\ &\qquad + (2\bar{\mu} + \mu - 2\gamma)\Phi_{01}.\end{aligned} \tag{A12c}$$

APPENDIX B: MASTER EQUATIONS FOR PERTURBATIONS OF A PETROV TYPE D SPACETIME

Here we review the derivation of the equations governing the first- and second-order perturbations of a Petrov type D spacetime satisfying the vacuum Einstein equations. The equation for first-order perturbations was originally derived by Teukolsky [50] and was later generalized to nth-order perturbations by Campanelli and Lousto [33]. We recall that a spacetime is a Petrov type D spacetime if it admits two double principal null directions, with respect to which

$$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0. \tag{B1}$$

By the Goldberg-Sachs theorem [64], we also have

$$\kappa = \sigma = \nu = \lambda = 0. \tag{B2}$$

Finally, if the outgoing null vector l^μ is chosen to be affinely parameterized, then we have, additionally, $\epsilon = 0$. We distinguish between the background quantities and perturbations with superscripts. For example, for the Weyl curvature component Ψ_0, we consider perturbations of the form

$$\Psi_0 = \Psi_0^{(0)} + \zeta\Psi_0^{(1)} + \zeta^2\Psi_0^{(2)} + \mathcal{O}(\zeta^3), \tag{B3}$$

where ζ is an order-keeping parameter, $\Psi_0^{(0)}$ denotes the background value, $\Psi_0^{(1)}$ denotes the first-order perturbations and $\Psi_0^{(2)}$ denotes the second-order perturbation. We similarly have second-order perturbations of all the Weyl curvature, Ricci coefficients and differential derivatives in the NP formalism. Since the background spacetime is of Petrov type D we have

$$\begin{aligned}\Psi_0^{(0)} &= \Psi_1^{(0)} = \Psi_3^{(0)} = \Psi_4^{(0)}\\ &= \kappa^{(0)} = \sigma^{(0)} = \nu^{(0)} = \lambda^{(0)} = 0.\end{aligned} \tag{B4}$$

By virtue of the fact that the spacetime satisfies the vacuum field equations, the Ricci scalars in Eqs. (A7a)–(A7f) all vanish on the background. For generality, we do allow these scalars to be nonzero at first and second-order in perturbation theory.

1. First-order perturbations

Consider the Bianchi identities Eqs. (A10d) and (A10h) and the Riemann identity Eq. (A9j) which can be written as

$$(D + 4\epsilon - \rho)\Psi_4 - (\bar{\delta} + 2\alpha + 4\pi)\Psi_3 + 3\lambda\Psi_2 = -\mathcal{R}_d, \tag{B5a}$$

$$-(\delta + 4\beta - \tau)\Psi_4 + (\Delta + 2\gamma + 4\mu)\Psi_3 - 3\nu\Psi_2 = -\mathcal{R}_h, \tag{B5b}$$

$$(\Delta + \mu + \bar{\mu} + 3\gamma - \bar{\gamma})\lambda - (\bar{\delta} + 3\alpha + \bar{\beta} + \pi - \bar{\tau})\nu = -\Psi_4. \tag{B5c}$$

The quantities $\{\Psi_4, \Psi_3, \lambda, \nu\}$ and the Ricci terms $\{\mathcal{R}_d, \mathcal{R}_h\}$ all vanish on the background, and thus these equations are "of first-order smallness," meaning that they describe the evolution of first-order quantities. Following [33], we define the derivatives

$$d_3 \equiv \bar{\delta} + 3\alpha + \bar{\beta} + 4\pi - \bar{\tau}, \tag{B6}$$

$$d_4 \equiv \Delta + 4\mu + \bar{\mu} + 3\gamma - \bar{\gamma}. \tag{B7}$$

We act on (B5a) with $d_4^{(0)}$ and on (B5b) with $d_3^{(0)}$ and sum the two equations to obtain

$$\begin{aligned}
&[d_4^{(0)}(D + 4\epsilon - \rho) - d_3^{(0)}(\delta + 4\beta - \tau)]\Psi_4 \\
&\quad + [-d_4^{(0)}(\bar{\delta} + 2\alpha + 4\pi) + d_3^{(0)}(\Delta + 2\gamma + 4\mu)]\Psi_3 \\
&\quad + 3[d_4^{(0)}\lambda - d_3^{(0)}\nu]\Psi_2 = -d_4^{(0)}\mathcal{R}_d - d_3^{(0)}\mathcal{R}_h.
\end{aligned} \tag{B8}$$

So far, we have not performed any perturbative expansions, and the above equation applies at all orders in perturbation theory.

We now show how the first-order term[3] of the above equation corresponds to the Teukolsky equation for Petrov type D spacetimes. By expanding Eq. (B8) to first-order we obtain

$$\begin{aligned}
&[d_4^{(0)}(D+4\epsilon-\rho)^{(0)} - d_3^{(0)}(\delta+4\beta-\tau)^{(0)}]\Psi_4^{(1)} \\
&\quad + [-d_4^{(0)}(\bar{\delta}+2\alpha+4\pi)^{(0)} + d_3^{(0)}(\Delta+2\gamma+4\mu)^{(0)}]\Psi_3^{(1)} \\
&\quad + 3[d_4^{(0)}\lambda^{(1)} - d_3^{(0)}\nu^{(1)}]\Psi_2^{(0)} = -d_4^{(0)}\mathcal{R}_d^{(1)} - d_3^{(0)}\mathcal{R}_h^{(1)},
\end{aligned} \tag{B9}$$

where we used that $\Psi_4^{(0)} = \Psi_3^{(0)} = \lambda^{(0)} = \nu^{(0)} = \mathcal{R}_d^{(0)} = \mathcal{R}_h^{(0)} = 0$. Now observe that using (A5d), (A9i), (A9m) and (A9r), one can prove that in a vacuum Petrov type D spacetime

$$[-d_4(\bar{\delta} + 4\pi + 2\alpha) + d_3(\Delta + 4\mu + 2\gamma)]f = 0 \tag{B10}$$

for any scalar f. As a result of this, the second line of Eq. (B9) now vanishes. Also, observe that using (A9j), (A10g) and (A10c) for a type D background, we can derive that

$$[d_4^{(0)}\lambda^{(1)} - d_3^{(0)}\nu^{(1)}]\Psi_2^{(0)} = -\Psi_2^{(0)}\Psi_4^{(1)}. \tag{B11}$$

Putting the above together, we obtain the Teukolsky equation

$$\mathcal{T}\Psi_4^{(1)} = \mathcal{R}_4^{(1)}, \tag{B12}$$

where $\mathcal{T}$ is the Teukolsky operator [see [50], Eq. (2.14)]

$$\begin{aligned}
\mathcal{T} \equiv{}& [d_4^{(0)}(D + 4\epsilon - \rho)^{(0)} - d_3^{(0)}(\delta + 4\beta - \tau)^{(0)}] \\
&- 3\Psi_2^{(0)}
\end{aligned} \tag{B13}$$

and $\mathcal{R}_4^{(1)} = -d_4^{(0)}\mathcal{R}_d^{(1)} - d_3^{(0)}\mathcal{R}_h^{(1)}$. Equation (B12) governs the gravitational wave perturbations in any type D spacetime satisfying the vacuum field equations. A solution $\Psi_4^{(1)}$ to (B12) can represent both ingoing and outgoing radiation, though is better adapted to describing outgoing waves far from a source. A similar procedure can be used to obtain a decoupled equation for $\Psi_0^{(1)}$, which likewise can represent both ingoing and outgoing waves, though is better adapted to describing the former [50].

2. Second-order perturbations

We now turn our attention to second-order perturbations of type D spacetimes. Returning to Eq. (B8), we expand to second-order to obtain

$$\begin{aligned}
&[d_4^{(0)}(D + 4\epsilon - \rho)^{(0)} - d_3^{(0)}(\delta + 4\beta - \tau)^{(0)}]\Psi_4^{(2)} \\
&\quad + [d_4^{(0)}(D + 4\epsilon - \rho)^{(1)} - d_3^{(0)}(\delta + 4\beta - \tau)^{(1)}]\Psi_4^{(1)} \\
&\quad + [-d_4^{(0)}(\bar{\delta} + 2\alpha + 4\pi)^{(1)} + d_3^{(0)}(\Delta + 2\gamma + 4\mu)^{(1)}]\Psi_3^{(1)} \\
&\quad + 3[d_4^{(0)}\lambda^{(1)} - d_3^{(0)}\nu^{(1)}]\Psi_2^{(1)} + 3[d_4^{(0)}\lambda^{(2)} - d_3^{(0)}\nu^{(2)}]\Psi_2^{(0)} \\
&= -d_4^{(0)}\mathcal{R}_d^{(2)} - d_3^{(0)}\mathcal{R}_h^{(2)},
\end{aligned} \tag{B14}$$

where we used Eq. (B10). We once again make use of Eqs. (A9j), (A10g), and (A10c) to derive

$$\begin{aligned}
&[d_4^{(0)}\lambda^{(2)} - d_3^{(0)}\nu^{(2)}]\Psi_2^{(0)} \\
&= -\Psi_2^{(0)}\Psi_4^{(2)} \\
&\quad + \Psi_2^{(0)}[-(d_4^{(1)} - 3\mu^{(1)})\lambda^{(1)} + (d_3^{(1)} - 3\pi^{(1)})\nu^{(1)}].
\end{aligned} \tag{B15}$$

We can thus write the second-order vacuum Teukolsky equation as

$$\mathcal{T}\Psi_4^{(2)} = \mathcal{S}_4^{(2)} + \mathcal{R}_4^{(2)}, \tag{B16}$$

where $\mathcal{R}_4^{(2)} = -d_4^{(0)}\mathcal{R}_d^{(2)} - d_3^{(0)}\mathcal{R}_h^{(2)}$ and the source term $\mathcal{S}_4^{(2)}$ is

$$\begin{aligned}
\mathcal{S}_4^{(2)} \equiv{}& -[d_4^{(0)}(D+4\epsilon-\rho)^{(1)} - d_3^{(0)}(\delta+4\beta-\tau)^{(1)}]\Psi_4^{(1)} \\
&+ [d_4^{(0)}(\bar{\delta}+2\alpha+4\pi)^{(1)} - d_3^{(0)}(\Delta+2\gamma+4\mu)^{(1)}]\Psi_3^{(1)} \\
&- 3[d_4^{(0)}\lambda^{(1)} - d_3^{(0)}\nu^{(1)}]\Psi_2^{(1)} \\
&+ 3\Psi_2^{(0)}[(d_4^{(1)} - 3\mu^{(1)})\lambda^{(1)} - (d_3^{(1)} - 3\pi^{(1)})\nu^{(1)}],
\end{aligned} \tag{B17}$$

as was derived in [33] [Eq. (9)]. In particular, the source term $\mathcal{S}_4^{(2)}$ only involves derivatives of the Ricci and

[3]Observe that the zeroth-order term of Eq. (B8) is trivially satisfied since $\Psi_4^{(0)} = \Psi_3^{(0)} = \lambda^{(0)} = \nu^{(0)} = 0$ in a type D spacetime.

curvature components of the background or of the first-order perturbation. Further, recall that we have not yet imposed any gauge conditions on the background or the first-order terms.

APPENDIX C: LINEARIZED NP SPIN COEFFICIENTS IN TERMS OF THE LINEARIZED METRIC

Using a choice of tetrad first described by Chrzanowski [51] and the commutation relations for the NP derivative operators, one can rewrite the linearized NP scalars in terms of the linearized metric components (see Sec. III A). Here we provide a complete listing of these relations [compare also to Eq. (A4) of [33]]:

$$\lambda^{(1)} = \frac{1}{2}[-\Delta + 2(\bar{\gamma} - \gamma) + \mu - \bar{\mu}]^{(0)} h_{\bar{m}\bar{m}} - (\pi + \bar{\tau})^{(0)} h_{n\bar{m}}, \tag{C1a}$$

$$\nu^{(1)} = \frac{1}{2}(\bar{\delta} + 2\alpha - \pi + 2\bar{\beta} - \bar{\tau})^{(0)} h_{nn} - (\Delta + 2\gamma + \bar{\mu})^{(0)} h_{n\bar{m}}, \tag{C1b}$$

$$\sigma^{(1)} = \frac{1}{2}[D + 2(\bar{\epsilon} - \epsilon) + \rho - \bar{\rho}]^{(0)} h_{mm} - (\tau + \bar{\pi})^{(0)} h_{lm}, \tag{C1c}$$

$$\begin{aligned}\gamma^{(1)} = &\frac{1}{4}(\bar{\delta} + 2\bar{\beta} - 2\pi - \bar{\tau})^{(0)} h_{nm} \\ &- \frac{1}{4}(\delta + 2\beta + 2\bar{\pi} + 3\tau)^{(0)} h_{n\bar{m}} + \frac{1}{4}(D + 2\bar{\epsilon} + \rho - \bar{\rho})^{(0)} h_{nn} \\ &+ \frac{1}{4}(\mu - \bar{\mu} - 4\gamma)^{(0)} h_{ln} + \frac{1}{4}(\mu - \bar{\mu})^{(0)} h_{m\bar{m}},\end{aligned} \tag{C1d}$$

$$\kappa^{(1)} = (D - 2\epsilon - \bar{\rho})^{(0)} h_{lm} - \frac{1}{2}(\delta - 2\bar{\alpha} - 2\beta + \bar{\pi} + \tau)^{(0)} h_{ll}, \tag{C1e}$$

$$\begin{aligned}\mu^{(1)} = &\frac{1}{2}(\bar{\delta} + 2\bar{\beta} - 2\pi - \bar{\tau})^{(0)} h_{nm} - \frac{1}{2}(\delta + 2\beta + \tau)^{(0)} h_{n\bar{m}} \\ &- \frac{1}{2}(\Delta - \mu + \bar{\mu})^{(0)} h_{m\bar{m}} \\ &+ \frac{1}{2}\rho^{(0)} h_{nn} - \frac{1}{2}(\mu + \bar{\mu})^{(0)} h_{ln},\end{aligned} \tag{C1f}$$

$$\begin{aligned}\epsilon^{(1)} = &\frac{1}{4}(-\Delta + 2\bar{\gamma} + \mu - \bar{\mu})^{(0)} h_{ll} + \frac{1}{4}(2D + \rho - \bar{\rho})^{(0)} h_{ln} \\ &+ \frac{1}{4}(-\delta + 2\bar{\alpha} - \bar{\pi} - 2\tau)^{(0)} h_{l\bar{m}} \\ &+ \frac{1}{4}(\bar{\delta} - 2\alpha - 3\pi - 2\bar{\tau})^{(0)} h_{lm} + \frac{1}{4}(\rho - \bar{\rho})^{(0)} h_{m\bar{m}},\end{aligned} \tag{C1g}$$

$$\begin{aligned}\rho^{(1)} = &\frac{1}{2}(D + \rho - \bar{\rho})^{(0)} h_{m\bar{m}} - \frac{1}{2}(\delta + \bar{\pi} + 2\tau - 2\bar{\alpha})^{(0)} h_{l\bar{m}} \\ &+ \frac{1}{2}(\bar{\delta} - \pi - 2\alpha)^{(0)} h_{lm} \\ &+ \frac{1}{2}\mu^{(0)} h_{ll} + \frac{1}{2}(\rho - \bar{\rho}) h_{ln},\end{aligned} \tag{C1h}$$

$$\begin{aligned}\alpha^{(1)} = &\frac{1}{4}(\bar{\delta} + 2\alpha - \pi - \bar{\tau})^{(0)} h_{m\bar{m}} - \frac{1}{4}(\delta - 2\bar{\alpha} + \bar{\pi} + \tau)^{(0)} h_{\bar{m}\bar{m}} \\ &- \frac{1}{4}(\Delta + 4\gamma - 2\bar{\gamma} + \bar{\mu} - 2\mu)^{(0)} h_{l\bar{m}} + \frac{1}{4}(\bar{\delta} - \pi - \bar{\tau})^{(0)} h_{ln} \\ &+ \frac{1}{4}(D - 2\epsilon - \rho - 2\bar{\rho})^{(0)} h_{n\bar{m}},\end{aligned} \tag{C1i}$$

$$\begin{aligned}\beta^{(1)} = &\frac{1}{4}(D - 4\epsilon + 2\bar{\epsilon} + 2\rho - \bar{\rho})^{(0)} h_{nm} + \frac{1}{4}(\delta - \bar{\pi} - \tau)^{(0)} h_{ln} \\ &- \frac{1}{4}(\delta - 2\beta + \bar{\pi} + \tau)^{(0)} h_{m\bar{m}} - \frac{1}{4}(\Delta + 2\gamma + \mu + 2\bar{\mu})^{(0)} h_{lm} \\ &+ \frac{1}{4}(\bar{\delta} + 2\bar{\beta} - \pi - \bar{\tau})^{(0)} h_{mm},\end{aligned} \tag{C1j}$$

$$\begin{aligned}\tau^{(1)} = &\frac{1}{2}(D + 2\bar{\epsilon} - \bar{\rho})^{(0)} h_{nm} + \frac{1}{2}(\Delta - 2\gamma + \mu)^{(0)} h_{lm} \\ &- \frac{1}{2}(\delta + \bar{\pi} + \tau)^{(0)} h_{ln} - \frac{1}{2}\pi^{(0)} h_{mm} - \frac{1}{2}\bar{\pi}^{(0)} h_{m\bar{m}},\end{aligned} \tag{C1k}$$

$$\begin{aligned}\pi^{(1)} = &-\frac{1}{2}(D + 2\epsilon - \rho)^{(0)} h_{n\bar{m}} - \frac{1}{2}(\Delta - 2\bar{\gamma} + \bar{\mu})^{(0)} h_{l\bar{m}} \\ &+ \frac{1}{2}(\bar{\delta} - \pi - \bar{\tau})^{(0)} h_{ln} - \frac{1}{2}\bar{\tau}^{(0)} h_{m\bar{m}} - \frac{1}{2}\tau^{(0)} h_{\bar{m}\bar{m}}.\end{aligned} \tag{C1l}$$

APPENDIX D: ALTERNATIVE METRIC RECONSTRUCTION EQUATIONS

The metric reconstruction procedure detailed in Sec. IV is not unique in the sense that one could derive alternative equations for the metric components h_{ll}, h_{lm}, and h_{mm}. The reason for this is that we have more equations than are necessary to solve for these components. We here provide an alternative equation for one of these components, namely h_{ll}. Consider the Riemann identity in Eq. (A9h). Linearizing this equation, we have

$$\begin{aligned}(D - \bar{\rho} + \epsilon + \bar{\epsilon})^{(1)}\mu^{(0)} = &(\delta + \bar{\pi} - \bar{\alpha} + \beta)^{(0)}\pi^{(1)} \\ &+ (\delta + \bar{\pi} - \bar{\alpha} + \beta)^{(1)}\pi^{(0)} \\ &+ \Psi_2^{(1)} + 2\Lambda^{(1)}.\end{aligned} \tag{D1}$$

The left-hand side of this equation contains all of the dependence on h_{ll}. Expanding out the left-hand side, we have

$$(D-\bar{\rho}+\epsilon+\bar{\epsilon})^{(1)}\mu^{(0)} = -\frac{\mu^{(0)}}{2}[\Delta-\gamma-\bar{\gamma}+\bar{\mu}]^{(0)}h_{ll}-\frac{1}{2}h_{ll}\Delta^{(0)}\mu^{(0)} -\frac{\mu^{(0)}}{2}(\delta-2\bar{\alpha}+\bar{\pi}+2\tau)h_{l\bar{m}} +\frac{\mu^{(0)}}{2}(\bar{\delta}-2\alpha-\pi)h_{lm}. \tag{D2}$$

This can be simplified by making use of the Riemann identity in Eq. (A9n) evaluated on the background, specifically $-\Delta^{(0)}\mu^{(0)}=(\mu^{(0)})^2+\mu^{(0)}(\gamma+\bar{\gamma})^{(0)}$. Applying this, we obtain a first-order transport equation for h_{ll}, specifically

$$[\Delta-2(\gamma+\bar{\gamma})-\mu+\bar{\mu}]^{(0)}h_{ll} = -(\delta-2\bar{\alpha}+\bar{\pi}+2\tau)h_{l\bar{m}}+(\bar{\delta}-2\alpha-\pi)h_{lm} -\frac{2}{\mu^{(0)}}[(\delta+\bar{\pi}-\bar{\alpha}+\beta)^{(0)}\pi^{(1)} +(\delta+\bar{\pi}-\bar{\alpha}+\beta)^{(1)}\pi^{(0)}+\Psi_2^{(1)}+2\Lambda^{(1)}]. \tag{D3}$$

Why did we not make use of this equation in our case study in Sec. V. The issue with this equation is the behavior of the source term in a $1/r$ expansion. To leading order, the terms on the right-hand side of Eq. (D3) are those containing $\pi^{(1)}$ and $\Psi_2^{(1)}$ and which scale as $1/r^2$. However, these terms exactly cancel one another, and we are left with an undetermined remainder of $\mathcal{O}(1/r^3)$. This happens to be the same order as the h_{lm} and $h_{l\bar{m}}$ terms. Thus, in order to get the correct behavior of the source term in Eq. (D3) one would have to obtain $\pi^{(1)}$ and $\Psi_2^{(1)}$ to higher order in $1/r$, which in turn means that we would have to start by calculating the higher order in $1/r$ corrections to $\Psi_4^{(1)}$. The second-order transport equation in Eq. (33) does not have this issue. We make use of Eq. (D3) in our numerical computations in [45], where this problem does not occur as we do not make any $1/r$ approximations.

This same issue arises if one tries to compute the Weyl scalar $\Psi_0^{(1)}$ from the expanded Riemann identity in Eq. (12). The terms containing $\sigma^{(1)}$ are the leading-order terms, which scale as $1/r^3$, and all cancel one another with a remainder of $\mathcal{O}(1/r^4)$, which is the same order as those terms containing $\kappa^{(1)}$. By the peeling theorem, $\Psi_0^{(1)}=\mathcal{O}(1/r^5)$, and thus all $\mathcal{O}(1/r^4)$ terms in this equation must also cancel one another. Alternatively, one can solve for the remaining Weyl scalars $\Psi_0^{(1)}$ and $\Psi_1^{(1)}$ using the Bianchi identities in Eqs. (A10e) and (A10f), respectively. Expanding these equations to first-order, we have

$$[-\Delta+2(\gamma-\mu)]^{(0)}\Psi_1^{(1)}+(\delta-3\tau)^{(0)}\Psi_2^{(1)} +(\delta-3\tau)^{(1)}\Psi_2^{(0)}=-\mathcal{R}_f^{(1)}, \tag{D4}$$

$$(-\Delta+4\gamma-\mu)^{(0)}\Psi_0^{(1)}+[\delta-2(2\tau+\beta)]\Psi_1^{(1)} +3\sigma^{(1)}\Psi_2^{(0)}=-\mathcal{R}_e^{(1)}. \tag{D5}$$

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