Flavor-dependent U(3) Nambu–Jona-Lasinio coupling constant

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A nonperturbative one-gluon exchange quark-antiquark interaction is considered to compute flavor-dependent U(3) Nambu–Jona-Lasinio-type (NJL-type) interactions of the form $G_{ij,\Gamma}(\bar{\psi}\lambda_i\Gamma\psi)(\bar{\psi}\lambda_j\Gamma\psi)$ for $i, j = 0 \ldots 8$ and $\Gamma = I, i\gamma_5$ from the one-loop polarization process with nondegenerate u-d-s quark effective masses. The resulting NJL-type coupling constants in all channels are resolved in the long-wavelength limit and the numerical results are presented for different choices of an effective gluon propagator. Leading deviations with respect to a flavor symmetric coupling constant are found to be of the order $(M^*_{f_2} - M^*_{f_1})^n/(M^*_{f_2} + M^*_{f_1})^n$, for $n = 1$, 2, where $M^*_{f_i}$ are the effective masses of quarks $f_1, f_2 = u$, d and s. The scalar channel coupling constants $G_{ij,s}$ can be considerably smaller than the pseudoscalar ones. The effect of the flavor-dependence of coupling constants for the masses of pions and kaons may be nearly of the same order of magnitude as the effect of the u, d, and s quark mass nondegeneracy. The effect of these coupling constants is also verified for some of the light scalar meson masses, usually described by quark-antiquark states, and for some observables of the pseudoscalar mesons.

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I. INTRODUCTION

Theoretical investigations and predictions for low energy strong interacting systems have important support from QCD effective models among which are the Nambu–Jona-Lasinio-type (NJL-type) models [1–3]. They are suitable for describing phenomena related to dynamical chiral symmetry breaking (DChSB), according to which massive constituent quarks can be defined and are responsible for most part of the hadron masses. The NJL model can be derived in terms of QCD degrees of freedom in different ways [4–9]. Lately, an effective gluon mass was found to be appropriate for parametrizing the deep infrared behavior of a gluon propagator, and the quark-NJL coupling constant has been identified roughly as $G_{NJL} \propto 1/M_G^2$. Nonperturbative or effective gluon propagators take into account part of the non-Abelian gluon dynamics and they might be suitable enough to provide numerical estimates for hadron properties. Eventually they make possible a clear relation of fundamental processes and fundamental degrees of freedom with the NJL model framework and description since they are expected to at least produce DChSB. In addition to the explicit chiral and flavor symmetry breakings due to the nondegenerate current quark mass, the couplings to electromagnetic fields also break these symmetries contributing to masses [2,10–14] and coupling constants [15]. It might be interesting to verify if, and to what extent, NJL coupling constants receive flavor-dependent contributions. This should be of relevance for a fine-tuned description of hadron masses and dynamics.

Current quark masses, at the energy scale $\mu = 2$ GeV, are approximately $m_u \simeq 2.1$ MeV, $m_d \simeq 4.7$ MeV, and $m_s \simeq 93$ MeV [16] and they are amplified due to the DChSB with the formation of quark-antiquark scalar condensates. In spite of the need for the electromagnetic corrections to fully describe hadron masses, there are well known strong-interaction contributions; for example, for masses of pions and kaons, the quasi-Goldstone bosons that are, respectively, of the order of $m_{\pi^\pm} - m_{\pi^0} \simeq 0.1$ MeV and $m_{K^\pm} - m_{K^0,\bar{K}^0} \simeq -5.3$ MeV [17]. This contribution to the pion mass difference is very small due to the small difference between up and down quark masses. The electromagnetic neutral and charged pion mass difference is somewhat larger, and of the order of 4 MeV [11–14]. The charged and neutral kaon mass difference has the opposite sign of the pion mass difference and is larger due to the larger strange quark mass. Flavor symmetry breaking corrections are small for light hadrons but important for a good description of hadrons. In QCD, the quark current masses are the only parameters that control flavor symmetry breaking. This issue has far consequences in some effective approaches as in chiral perturbation theory (ChPT), as an effective field theory (EFT) for the low energy regime [18]. By starting from QCD to understand effective models, one might expect that the flavor symmetry breaking encoded in different quark current masses might spread and have consequences for a variety of parameters and coupling constants in effective models by means of quantum effects. Therefore NJL coupling constants might also be expected to receive flavor-dependent contributions from quantum effects. The Schwinger-Dyson equation approach for light and heavy hadrons indicates coupling constants might be flavor-dependent [19]. The lightest scalar mesons that could be expected to be chiral partners of the pseudoscalar ones do not seem to be compatible with the usual quark model due to the apparent impossibility of fitting all the experimental data with quark-antiquark structures as prescribed by the simplest quark model [20,21], although it might be partially appropriated [16,20,22,23]. One of the specific problems with the attempt to describe some of the lightest scalar mesons in a U(3) nonet from the standard NJL model scheme is the inverted mass behavior of $a_0(I(J^C) = 1(0^+))$ and $K^*(\frac{1}{2}(0^+))$ [24]. In the present work this issue appears again and although no complete solution for this problem is obtained or proposed, we expect to show further insights. In fact, there are several different theoretical calculations with different proposals for their structures, such as whether they are composed by mixed states with tetraquarks, glueballs, meson molecules, or coupled channels resonances [21,25–32]. The full problem of the light scalar meson structure will not really be addressed in the present work. Nevertheless, it becomes interesting to introduce as many different effects as possible to test their individual contributions and the predictive power of the model.

In this work, explicit chiral and flavor symmetry breaking contributions to the NJL coupling constant are derived by considering vacuum polarization in a flavor U(3) model in which quark-antiquark interaction is mediated by a (nonperturbative) gluon exchange. These coupling constants are resolved in the local long-wavelength limit in terms of quark and gluon propagators and they will be used to calculate light pseudoscalar and scalar meson masses. An effective nonperturbative gluon propagator will be considered to incorporate non-Abelian dynamics to some extent, with the crucial requirement being to produce dynamical chiral symmetry breaking and the large constituent quark masses due to the gluon cloud. This method extends previous works for u-d-s or u-d degenerate quarks [5,33]. Because the logic and steps of the calculation has been shown with details in previous works, in the next section the main steps are only briefly outlined. In Sec. III, numerical estimations for the long-wavelength local limit for the resulting four point quark interaction are presented as NJL-type flavor-dependent interactions. This is done for two types of effective gluon propagators and different values of Lagrangian and effective quark masses. Since the flavor-dependent coupling constants G_{ij} were found to describe the light pseudoscalar mesons masses, further observables will be presented in Sec. III C to assess the change in their values if G_{ij} are used to redefine the gap equations. In Sec. IV, the effects of such flavor-dependent NJL coupling constants are verified on some light meson masses. Because not all of the scalars are seemingly described by quark-antiquark states, and the pseudoscalar $\eta - \eta'$ mesons require further interactions [34], the masses of pseudoscalar pions and kaons, and of the light scalar a_0 and K^* (or κ), will be investigated. In the last section there is a summary.

II. QUARK DETERMINANT AND LEADING CURRENT-CURRENT INTERACTIONS

The following low energy quark effective action [37–39] will be considered:

$$Z = N \int \mathcal{D}[\bar{\psi}, \psi] \exp i \int_x \Big[\bar{\psi}(i\partial\!\!\!/ - m_f)\psi - \frac{g^2}{2} \int_y j_\mu^b(x) \tilde{R}_{bc}^{\mu\nu}(x-y) j_\nu^c(y) + \bar{\psi}J + J^*\psi \Big], \quad (1)$$

where the color quark current is $j_a^\mu = \bar{\psi}\lambda_a\gamma^\mu\psi$, the sums in color, flavor, and Dirac indices are implicit, $\int_x$ stands for $\int d^4x$, $a, b\ldots = 1, \ldots(N_c^2 - 1)$ stand for color in the adjoint representation, and m_f is the quark current masses matrix [16], with indices of the flavor SU(3) fundamental representation $f = u, d$ and s. The adjoint representation of flavor SU(3) will be used with indices $i, j, k = 0, 1.., N_f^2 - 1$ with the additional matrix $\lambda_0 = \sqrt{2/3}I$ to complete the U(3) algebra. In several gauges the gluon kernel is usually argued to be written in terms of the transversal and longitudinal components in momentum space, $R_T(k)$ and $R_L(k)$, as: $\tilde{R}_{ab}^{\mu\nu}(k) = \delta_{ab}[(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2})R_T(k) + \frac{k^\mu k^\nu}{k^2} R_L(k)]$. Although this decomposition may not be exact because of confinement-related effects [40], it is important to emphasize that numerical results will be calculated by considering effective gluon propagators whose contributions from other components might be parametrized into these two components. Besides that, it can be shown that contributions of other terms, of the form $\delta(k^2)$ or its derivatives, in the effective gluon propagator for the results can be expected to be much smaller. Even if other terms arise from the non-Abelian structure of the gluon sector, the quark-quark interaction (1) is a leading term of the QCD effective action. The use of a dressed (nonperturbative) gluon propagator already takes into account non-Abelian contributions that guarantee important effects. Among these, it will be assumed and required that this dressed gluon propagator provides enough strength for DChSB as obtained, for example, in [41–45]. To some extent, the present work will follow previous developments by adopting the background field method to introduce the background quarks that, dressed by gluons, give rise to the constituent quarks. A complete account of the calculations below was presented in Refs. [5,33,46,47].

To explore the flavor structure of the interaction in the action (1) that will be denoted by Ω, a Fierz transformation is performed for the quark-antiquark channel resulting in $\mathcal{F}(\Omega) = \Omega_F$. We are only interested in this work in the color-singlet scalar-pseudoscalar state sector, and vector and axial currents will be neglected. Color nonsinglet terms are suppressed by a factor of $1/N_c$, and they might give rise to higher order colorless contributions [46,48]. The following bilocal currents are needed to describe the resulting terms: $j_i^q(x,y) = \bar{\psi}(x)\lambda_i\Gamma^q\psi(y)$ where $q = s,\ p$ for scalar and pseudoscalar currents, $\Gamma_s = \lambda_i I$ (with the 4×4 Dirac identity), $\Gamma_p = i\gamma_5\lambda_i$ where λ_i are the flavor SU(3) Gell-Mann matrices ($i = 1\ldots 8$), and $\lambda_0 = I\sqrt{2/3}$. The resulting s and p nonlocal interactions are the following:

$$\Omega_F = 4\alpha g^2\{[j_S^i(x,y)j_S^i(y,x) + j_P^i(x,y)j_P^i(y,x)]R(x-y)\}, \tag{2}$$

where $\alpha = 2/9$ and $R(x-y) \equiv R = 3R_T(x-y) + R_L(x-y)$.

Next, the quark field will be split into the background field (ψ constituent quark) and the sea quark field (ψ_2) that might form light mesons and the chiral condensate. This sort of decomposition is not exclusive to the background field method (BFM) and it is found in other approaches [49]. At the one-loop BFM level it is enough to perform this splitting for the bilinears $\bar{\psi}\Gamma_q\psi$ [48,50], and it can be written that

$$\bar{\psi}\Gamma^q\psi \to (\bar{\psi}\Gamma^q\psi)_2 + (\bar{\psi}\Gamma^q\psi), \tag{3}$$

where $(\bar{\psi}\psi)_2$ will be treated in the usual way as a sea quark of the NJL model, and the full interaction Ω_F is split accordingly to $\Omega_F \to \Omega_1 + \Omega_2 + \Omega_{12}$, where Ω_{12} contains the interactions between the two components. This separation preserves chiral symmetry, and it might not be simply a low and high energy mode separation. The auxiliary field method makes it possible to introduce the light quark-antiquark chiral states, the chiral condensate, and mesons. Therefore this procedure improves the one-loop BFM since it allows one to incorporate DChSB. Because it is a standard procedure in the field, it will not be described. To make possible a clear evaluation of the effects of the resulting NJL coupling constants, the corresponding gap equations at this level, which can arise for the local limit of an auxiliary scalar field, will be considered to be those of the NJL model, given in Eq. (27) for the case of coupling constants $G_{ff} = G_0 = 10\ \mathrm{GeV}^{-2}$, as discussed below. This guarantees a clear and fair subsequent comparison of the effects of the flavor-dependent coupling constants. Otherwise, the relation between effective masses and NJL coupling constants would not be clear. The nontrivial solutions for these gap equations allow one to define the quark effective masses $M_f^* = m_f + \bar{S}_f$, where $\bar{S}_f$ ($f = u,\ d,\ s$). The quark kernel can be written as

$$S_0(x-y) = (i\partial\!\!\!/ - M_f^*)^{-1}\delta(x-y). \tag{4}$$

The aim of this work is to present corrections to the NJL-type interaction so that the meson sector in terms of auxiliary fields will be neglected. The quark determinant can then be written as

$$S_d = C_0 + \frac{i}{2}\mathrm{Tr}\ \ln\left\{\left(1 + S_0\left(\sum_q a_q\Gamma_q j_q\right)\right)^*\left(1 + S_0\left(\sum_q a_q\Gamma_q j_q\right)\right)\right\}, \tag{5}$$

where the following quantities were defined:

$$\sum_q a_q\Gamma_q j_q = \sum_q a_q\Gamma_q j_q(x,y) = 2K_0 R(x-y)[(\bar{\psi}(y)\lambda_i\psi(x)) + i\gamma_5\lambda_i(\bar{\psi}(y)i\gamma_5\lambda_i\psi(x))], \tag{6}$$

where $K_0 = \alpha g^2$ and $C_0 = \frac{i}{2}\mathrm{Tr}\ \ln[S_0^{-1}S_0^{-1}]$, which reduces to a constant in the generating functional.

A large quark mass expansion is performed with a zero order derivative expansion [51] for the local limit. The first term of the expansion is a nondegenerate mass term $M_{3,f}$ that is proportional to the masses from gap equations [47] that will not be investigated further. The second-order terms of the expansion correspond to four-fermion interactions with the chiral and flavor symmetries breaking. These terms, in the local limit, can be written as:

$$\begin{aligned}\mathcal{L} &= M_{3,f}(\bar{\psi}\psi)_f + G_{ij,s}(\bar{\psi}\lambda^i\psi)(\bar{\psi}\lambda^j\psi) + G_{ij}(\bar{\psi}i\gamma_5\lambda^i\psi)(\bar{\psi}i\gamma_5\lambda^j\psi),\\ &= M_{3,f}(\bar{\psi}\psi)_f + G_{ij}[(\bar{\psi}\lambda^i\psi)(\bar{\psi}\lambda^j\psi) + (\bar{\psi}i\gamma_5\lambda^i\psi)(\bar{\psi}i\gamma_5\lambda^j\psi)] - G_{ij}^{sb}(\bar{\psi}\lambda^i\psi)(\bar{\psi}\lambda^j\psi),\end{aligned} \tag{7}$$

where the coefficients were resolved and, after a Wick rotation for the Euclidean momentum space in the (very long-wavelength) zero momentum transfer limit, they are the following:

$$M_{3,f} = d_1 N_c K_0 \mathrm{Tr}_D \mathrm{Tr}_F \int \frac{d^4k}{(2\pi)^4} S_0(k)\lambda_i R(k), \tag{8}$$

$$G_{ij,s} = G_{ij} + G_{ij}^{sb} = d_2 N_c K_0^2 \mathrm{Tr}_D \mathrm{Tr}_F \int \frac{d^4k}{(2\pi)^4} S_0(k)\lambda_i R(k) S_0(k)\lambda_j R(k), \tag{9}$$

$$G_{ij} = d_2 N_c K_0^2 \mathrm{Tr}_D \mathrm{Tr}_F \int \frac{d^4k}{(2\pi)^4} S_0(k) R(k) i\gamma_5 \lambda_i S_0(k) R(k) i\gamma_5 \lambda_j, \tag{10}$$

where $\mathrm{Tr}_D \mathrm{Tr}_F$ are the traces in Dirac and flavor indices, $d_n = \frac{(-1)^n}{2n}$, and $S_0(k)$ is the Fourier transform of $S_0(x-y)$. If the effective gluon propagator has other terms proportional to $\delta(k^2)$, or its derivatives [40], it can be shown that their resulting contribution will be suppressed, if not disappear, with respect to the finite momenta component encoded in $R(k)$ by factors $1/(16\pi^4 M^*_{f_1} M^*_{f_2})$, at least. Eventually, these further contributions may also vanish because of explicit dependencies on internal loop momenta k_μ. To calculate these traces in flavor indices, the following strategy was adopted. Each of the quark propagators originally in the fundamental representation were written as a combination of kernels in the adjoint representation by diagonalizing them with the correct diagonal Gell-Mann matrices. This makes possible an unambiguous and straightforward calculation of the flavor traces in these equations. The quark mass matrix can be written as

$$M = M_0\sqrt{3/2}\lambda_0 + M_3\lambda_3 + M_8\sqrt{3}\lambda_8, \tag{11}$$

where M_0, M_3, M_8 are combinations of the up, down, and strange quark effective masses. The quark propagator can then be written as:

$$\begin{aligned} S_{0m}(k) = I&\left[A(i\not{k}+M_0) + 2M_8 C + \frac{2}{3}M_3 B\right] \\ &+\lambda_3[B(i\not{k}+M_0) + M_3(A+C) + M_8 B] \\ &+\lambda_8\sqrt{3}\left[C(i\not{k}+M_0) + M_8(A-C) + \frac{M_3 B}{3}\right], \end{aligned} \tag{12}$$

where

$$A = \frac{1}{3}\left(\frac{1}{R_u} + \frac{1}{R_d} + \frac{1}{R_s}\right), \tag{13}$$

$$B = \frac{1}{2}\left(\frac{1}{R_u} - \frac{1}{R_d}\right), \tag{14}$$

$$C = \frac{1}{6}\left(\frac{1}{R_u} + \frac{1}{R_d} - \frac{2}{R_s}\right). \tag{15}$$

In these equations, $R_f = (k^2 - M_f^{*2} + i\epsilon)$. Flavor traces of up to four Gell-Mann matrices were calculated, i.e., $\mathrm{Tr}_F(\lambda_m\lambda_i\lambda_n\lambda_j)$, where $m, n = 0, 3, 8$ and $i, j = 0, \ldots 8$ with all combinations. The four-point interactions above (9), (10), in the limit of degenerate quark masses $m_u = m_d = m_s$, reduce to those of Ref. [33] with a different coefficient due to the $U(3)$ group. This way of writing Eq. (7) suggests an ambiguous definition of a nearly chiral "symmetric part" of NJL interaction as G_{ij} and the chiral symmetry breaking part G_{ij}^{sb} that arises only for the scalar sector. G_{ij} is not really a chiral-symmetric interaction because of the flavor symmetry breakings for all of the flavor channels i, j. The integrals for G_{ij} have two components, one of them strongly dependent on momentum k and the other strongly dependent on the quark masses, whereas G_{ij}^{sb} is written only in terms of the second of these integrals. This difference between the integrals (9), (10) favors the above separation of the *regular* and (strongly) symmetry breaking (sb) coupling constant. Two important properties of these coupling constants is the following:

$$G_{ij} = G_{ji}, \qquad G_{ij,s} = G_{ji,s}. \tag{16}$$

All the integrals in Eqs. (9) and (10) are ultraviolet (UV) finite and infrared (IR) regular if the gluon propagator contains a parameter such as a gluon effective mass or Gribov type parameter. G_{ij}, G_{ij}^s, and G_{ij}^{sb} have dimensions of mass^{-2}. For some observables, however, it is more appropriate to define the following coupling constants between quark currents in the fundamental representation,

$$G_{ij}(\bar\psi\lambda^i\psi)(\bar\psi\lambda^j\psi) = 2G_{f_1f_2}(\bar\psi\psi)_{f_1}(\bar\psi\psi)_{f_2}, \tag{17}$$

being that none of the types of mixing terms, $G_{i\neq j}$ or $G_{f_1\neq f_2}$, will be considered in most of the present work.

III. NUMERICAL RESULTS

In the following, two types of the effective gluon propagators that incorporate the quark-gluon running coupling constant g are written and will be considered for the numerical calculations. The first effective gluon propagator is a transversal one extracted from Schwinger-Dyson equation calculations [44,45]. It can be written as:

$$D_{I,2}(k) = g^2 R_T(k) = \frac{8\pi^2}{\omega^4} D e^{-k^2/\omega^2} + \frac{8\pi^2 \gamma_m E(k^2)}{\ln[\tau + (1 + k^2/\Lambda_{\rm QCD}^2)^2]}, \quad (18)$$

where $\gamma_m = 12/(33 - 2N_f)$, $N_f = 4$, $\Lambda_{\rm QCD} = 0.234$ GeV, $\tau = e^2 - 1$, $E(k^2) = [1 - \exp(-k^2/[4m_t^2])/k^2$, $m_t = 0.5$ GeV, $D = 0.55^3/\omega$ (GeV^2), and $\omega = 0.5$ GeV.

The second type of gluon propagator is based in a longitudinal effective confining parametrization [41] that can be written as:

$$D_{II,\alpha=5,6}(k) = g^2 R_{L,\alpha}(k) = \frac{K_F}{(k^2 + M_\alpha^2)^2}, \quad (19)$$

where $K_F = (0.5\sqrt{2}\pi)^2/0.6$ GeV^2, as considered in previous works [47,48], with either a constant effective gluon mass ($M_5 = 0.8$ GeV) or a running effective mass given by $M_6 = \frac{0.5}{1 + k^2/\omega_6^2}$ GeV for $\omega_6 = 1$ GeV.

It can be noted that these effective gluon propagators exhibit different normalizations and the resulting numerical values for Eqs. (9) and (10) might be quite different. Instead of addressing specific issues on the gluon propagators and their normalizations, a more pragmatic approach was adopted so that the relevant issue is not their normalization but the overall momentum dependence that contributes to the momentum integrals that generate the flavor-dependencies of the results. Therefore a normalization procedure common to all effective gluon propagators to compare different results is needed. In addition to that, to make easier the comparison of the flavor-dependent effects, a reference coupling constant with the value $G_0 = 10$ GeV^{-2} will be considered and the following normalized quantities will be displayed in the tables below:

$$G_{ij}^n = \frac{G_{ij}}{\frac{G_{11}}{10}}, \qquad G_{ij,s}^n = \frac{G_{ij,s}}{\frac{G_{11}}{10}}. \quad (20)$$

In a first analysis, the resulting coupling constants G_{ij} and $G_{ij,s}$ are expected to be additive corrections to a constant NJL model coupling constant. Given that the overall absolute values are not determined, this sort of multiplicative normalization was chosen instead of an additive correction. Therefore, coupling constants are normalized with respect to the channel $G_{ij=11}^n = 10$ GeV^{-2} which is close to usual values adopted in the literature for the NJL model. Besides that, this way of normalizing coupling constants becomes more appropriate for investigating consequences for the differences between each of the flavor channels, particularly the light meson mass differences.

A. Flavor-dependence of coupling constants

Besides the gluon propagators that implicitly contain a quark-gluon running coupling constant, the quark constituent masses for the quark propagator are also needed. Sets of values for the parameters that will be considered are shown in Table I with the Lagrangian quark masses m_u, m_d, and m_s. These values, together with the ultraviolet (three-dimensional) cutoff Λ, were chosen because they satisfy a gap equation with an NJL coupling constant of reference G_0, and this makes possible a comparison of the contribution of the particular flavor-dependent coupling constant on the results. Besides that, it will be required that the resulting neutral pion and kaon masses are in quite good agreement with experimental values for the case of the symmetric coupling constant $G_{ij} = G_0 = 10$ GeV^{-2}. A combination of up and down quark masses, with the UV cutoff, determine neutral (or charged) pion mass and, with the additional strange quark mass, one obtains the neutral (or charged) kaon mass. Although only the effective quark masses are needed for the estimation of coupling constants, the other parameters of the Table are needed to find the meson masses. Sets of parameters have two main labels, X and Y, which correspond to different ways of dealing with the scalar meson channels below. The subscripts in labels X and Y are just numbers used to identify a set of parameters–they have no physical meaning. The numerical method used to solve the gap equations and the bound state equation, or Bethe-Salpeter equation, (BSE) is required to identify each set of values for the quark masses and cutoff and for each possible solution of the BSE. For that, a name X or Y was chosen with numbers to identify them. X stands for sets of parameters with which scalar meson masses were found by considering the same coupling constants as for the pseudoscalar mesons channel, i.e., $G_{ij} = G_{ij,s}$. This is equivalent to setting $G_{ij}^{sb} = 0$, as discussed above. Besides that, sets of parameters X will produce somewhat better results, as discussed below. Y stands for sets of parameters with which scalar mesons were found with coupling constants obtained from Eq. (9) instead of G_{ij}, i.e., $G_{ij,s} \neq G_{ij}$.

In Tables II and III, the resulting values of coupling constants G_{ij} are presented for each set of masses and for the gluon effective propagators: $D_{I,2}$, $D_{II,5}$, and $D_{II,6}$. In Table IV, the resulting values of $G_{ij,s}$ are presented for the

TABLE I. Sets of parameters used in the numerical estimations: effective and Lagrangian quark masses and ultraviolet cutoff.

Set of parameters	M_u^* MeV	M_d^* MeV	M_s^* MeV	m_u MeV	m_d MeV	m_s MeV	Λ MeV
$X_{20} = Y_{18}$	389	399	600	3	7	123	675
X_{21}	392	396	600	4	6	123	675
Y_{14}	362	362	574	5	5	123	665
Y_{19}	391	395	600	4	6	163	675

TABLE II. Numerical results for G_{ij} for the sets of parameters X and different gluon propagators. The entries for G_0 are simply defined in this Table and they correspond to fixed values independent of any gluon propagator for the sake of comparison.

Set	G^n_{11} GeV^{-2}	G^n_{33} GeV^{-2}	G^n_{44} GeV^{-2}	G^n_{66} GeV^{-2}	G^n_{88} GeV^{-2}	G^n_{00} GeV^{-2}	G^n_{03} GeV^{-2}	G^n_{08} GeV^{-2}	G^n_{38} GeV^{-2}
X_{20}-$D_{I,2}$	10.00	10.00	9.77	9.69	7.61	8.60	0.11	1.80	0.14
X_{20}-$D_{II,5}$	10.00	10.00	9.82	9.76	8.13	8.90	0.08	1.41	0.10
X_{20}-$D_{II,6}$	10.00	10.00	9.89	9.70	7.72	8.66	0.10	1.72	0.13
X_{20}-G_0	10.00	10.00	10.00	10.00	10.00	10.00	0	0	0
X_{21}-$D_{I,2}$	10.00	10.00	9.74	9.71	7.61	8.60	0.04	1.80	0.05
X_{21}-$D_{II,5}$	10.00	10.00	9.80	9.78	8.12	8.90	0.03	1.41	0.04
X_{21}-$D_{II,6}$	10.00	10.00	9.76	9.73	7.72	8.66	0.04	1.72	0.05
X_{21}-G_0	10.00	10.00	10.00	10.00	10.00	10.00	0	0	0

TABLE III. Numerical results for G_{ij} for the sets of parameters Y and different gluon propagators. The entries for G_0 are simply defined in this Table and they correspond to fixed values independent of any gluon propagator for the sake of comparison. The set of parameters Y_{18} has the same values as X_{20} in Table II.

Set	G^n_{11} GeV^{-2}	G^n_{33} GeV^{-2}	G^n_{44} GeV^{-2}	G^n_{66} GeV^{-2}	G^n_{88} GeV^{-2}	G^n_{00} GeV^{-2}	G^n_{03} GeV^{-2}	G^n_{08} GeV^{-2}	G^n_{38} GeV^{-2}
Y_{14}-$D_{I,2}$	10.00	10.00	9.69	9.69	7.52	8.53	0	2.39	0
Y_{14}-$D_{II,5}$	10.00	10.00	9.76	9.76	8.06	8.85	0	1.46	0
Y_{14}-$D_{II,6}$	10.00	10.00	9.71	9.71	7.63	8.60	0	1.04	0
Y_{14}-G_0	10.00	10.00	10.00	10.00	10.00	10.00	0	0	0
Y_{18}-$D_{I,2}$	10.00	10.00	9.77	9.69	7.61	8.60	0.11	1.80	0.14
Y_{18}-$D_{II,5}$	10.00	10.00	9.82	9.76	8.13	8.90	0.08	1.41	0.10
Y_{18}-$D_{II,6}$	10.00	10.00	9.78	9.70	7.72	8.67	0.10	1.72	0.13
Y_{18}-G_0	10.00	10.00	10.00	10.00	10.00	10.00	0	0	0
Y_{19}-$D_{I,2}$	10.00	10.00	9.74	9.71	7.61	8.60	0.04	2.04	0.06
Y_{19}-$D_{II,5}$	10.00	10.00	9.80	9.78	8.12	8.90	0.03	1.33	0.04
Y_{19}-$D_{II,6}$	10.00	10.00	9.76	9.73	7.72	8.66	0.04	1.95	0.05
Y_{19}-G_0	10.00	10.00	10.00	10.00	10.00	10.00	0	0	0

TABLE IV. Numerical results for the scalar channel normalized coupling constants $G^n_{ij,s}$, with the different sets of parameters Y.

Set	$G^n_{11,s}$ GeV^{-2}	$G^n_{33,s}$ GeV^{-2}	$G^n_{44,s}$ GeV^{-2}	$G^n_{66,s}$ GeV^{-2}	$G^n_{88,s}$ GeV^{-2}	$G^n_{00,s}$ GeV^{-2}	$G^n_{03,s}$ GeV^{-2}	$G^n_{08,s}$ GeV^{-2}	$G^n_{38,s}$ GeV^{-2}
Y_{14}-$D_{I,2}$	0.31	0.31	0.11	0.11	−0.98	−0.47	0	0.96	0
Y_{14}-$D_{II,5}$	3.03	3.03	2.87	2.87	1.25	2.00	0	1.35	0
Y_{14}-$D_{II,6}$	0.87	0.87	0.67	0.67	−0.52	0.04	0	1.04	0
Y_{14}-G_0	10.00	10.00	10.00	10.00	10.00	10.00	0	0	0
Y_{18}-$D_{I,2}$	−0.32	−0.32	−0.46	−0.52	−1.37	−0.95	0.07	0.79	0.09
Y_{18}-$D_{II,5}$	2.53	2.53	2.42	2.35	0.94	1.61	0.08	1.21	0.10
Y_{18}-$D_{II,6}$	0.28	0.29	0.15	0.09	−0.88	−0.41	0.07	0.87	0.10
Y_{18}-G_0	10.00	10.00	10.00	10.00	10.00	10.00	0	0	0
Y_{19}-$D_{I,2}$	−0.32	−0.32	−0.48	−0.50	−1.37	−0.96	0.03	0.82	0.04
Y_{19}-$D_{II,5}$	2.53	2.53	2.35	2.33	0.76	1.50	0.03	1.33	0.04
Y_{19}-$D_{II,6}$	0.28	0.28	0.09	0.06	−0.95	−0.47	0.03	0.92	0.04
Y_{19}-G_0	10.00	10.00	10.00	10.00	10.00	10.00	0	0	0

same sets Y of Table III. The set with G_0 corresponds to a constant and symmetric choice of reference, $G_{ij} = G_0\delta_{ij} = 10\delta_{ij}$ GeV^2. This set, G_0, is independent of the effective gluon propagator and it was included to make possible a clearer analysis of the effects of the flavor-dependence of the coupling constants on the quark-antiquark meson masses, as discussed above. Note that G_0 has the same value of the normalized G^n_{11} and this is important for understanding the role of the flavor-dependent coupling constants on observables. The set of parameters Y_{18} has the same values of G_{ij} as the set X_{20} in Table II, and it was included separated in this Table to make the comparison of results simpler, in particular for the scalar meson channel in the next section. Some entries were not included because they are equal to those already displayed in the Table, being CP-conserving interactions:

$$G_{22} = G_{11}, \qquad G_{55} = G_{44}, \qquad G_{77} = G_{66},$$
$$G_{22,s} = G_{11,s}, \qquad G_{55,s} = G_{44,s}, \qquad G_{77,s} = G_{66,s}. \qquad (21)$$

The interaction channels exclusively of up and down quarks G_{11} and G_{33} are seen to be close to each other, i.e., they have almost no flavor-dependent correction. In fact, in the leading order, $G_{33} - G_{11} \propto (\frac{M^*_d - M^*_u}{M^*_d + M^*_u})^2 \sim 10^{-4}$; that is very small. The channels involving strange quarks have a larger deviation from the symmetric limit because $G_{ij=4,5,8,0} - G_{\rm sym} \propto (\frac{M^*_s - M^*_d}{M^*_s + M^*_d})^n$ for $n = 1$, 2, where $G_{\rm sym}$ is obtained by the limit of equal quark masses, the flavor symmetric limit. Moreover, larger strange quark masses induce smaller values of the coupling constants. This is in qualitative agreement with [19]. There are also mixing couplings $G_{i\neq j}$ among the neutral channels from the diagonal generators of the flavor U(3) group, i.e., for i, $j = 0$, 3, 8. In the pseudoscalar channel they are all proportional to the quark mass differences in the leading order: $G_{03} \propto (M^*_d - M^*_u)/M^*$, $G_{08} \propto (M^*_s - M^*_d)/M^*$, and $G_{38} \propto (M^*_d - M^*_u)(M^*_s - M^*_d)/M^{*2}$. Note that the mixing G_{08} has the largest values and the difference between its values for the sets of parameters X_{20} and X_{21} is too small. Together with G_{03} and G_{38}, these effective coupling constants can be associated with the $\eta - \eta' - \pi_0$ mixings. These mixings, however, will not be investigated in the present work. The set of parameters Y_{14} in Table III, which contains $m_u = m_d$, yields $G_{03} = G_{38} = 0$. The set of parameters with G_0 necessarily implies $G_{i\neq j} = 0$.

The coupling constants $G_{ij,s}$, Eq. (9) of the scalar sector and Table IV, are only shown for the sets of parameters Y because for the sets of parameters X it was considered that $G_{sb} \to 0$. The gluon propagator $D_{I,2}$ yields, on average, considerably lower values of coupling constants and the normalization (20) is, at least in part, responsible for that. These coupling constants may even be negative (repulsive), and the effective gluon propagator $D_{II,5}$ yields the largest values. The relative values of the gluon propagator $G_{ij,s}$ with respect to G_{ij} are highly dependent on the quark propagator structure, similar to the problems that emerge in form factors [52]. Nevertheless the relative shift of values of $G_{ij,s}$ (and the corresponding changes in the meson masses) for each of the channels for a particular set of parameters should be meaningful. On average the scalar channel coupling constants $G_{88,s}$ and $G_{00,s}$ are somewhat lower than the others. Detailed investigations of the meson mixings and the whole scalar meson octet/nonet mass problem are outside the scope of this work.

B. Effect on some light meson masses

In this section the effect of the flavor-dependent coupling constants on the masses of light quark-antiquark pseudoscalar pions and kaons is analyzed. Besides that, the effect on some of the scalar quark-antiquark states, usually associated to the scalars a_0 and K^*, is also analyzed. This will be done according to the following quark structure [23,24]:

$$a^0_0 \sim (\bar{u}u - \bar{d}d), \qquad a^\pm_0 \sim \bar{u}d, \bar{d}u,$$
$$K^*_0, \bar{K}^*_0 \sim \bar{d}s, \bar{s}d, \qquad K^*_\pm \sim \bar{s}u, \bar{u}s. \qquad (22)$$

It must be kept in mind, however, that the scalar sector should not be expected to be fully worked out and described due to the particularities of their structures quoted in the Introduction. Problems in the description of light scalars are particularly strong for the σ that seemingly cannot be a quark-antiquark state, as reminded in the Introduction.

The Bethe-Salpeter equation, or bound state equation, for the quark-antiquark meson sector in the NJL model is usually investigated at the Born approximation level. Therefore for a constant Bethe-Salpeter kernel this can be written as $K = 2G$ [2,3,53,54] for the case of diagonal interaction G_{ii} by neglecting mixing interactions. It provides the following condition to determine a particular meson mass $P^2_0 = M$ of the channel Γ, i [$\Gamma = I, i\gamma_5$, respectively, for scalar or pseudoscalar]:

$$1 - 2G_{ii}\, I^{ii}_{\Gamma f_1 f_2}(P^2_0 = M^2, \vec{P} = 0) = 0, \qquad (23)$$

for the corresponding flavor $i = 0, \ldots 8$ of the quark-antiquark state, when written in terms of the $f_1, f_2 = u$, d, s quark flavors. Note that the dependence of the results on the effective gluon propagator is encoded in the resulting value of G_{ij}, as discussed above. The following integral, for the four-momentum P of the meson, was defined:

$$I^{ij}_{\Gamma f_1 f_2}(P) = i tr_{F,C,D} \int \frac{d^4k}{(2\pi)^4} [S_{0,f_1}(k)\Gamma\lambda_i S_{0,f_2}(k+P)\Gamma\lambda_i], \qquad (24)$$

where $tr_{F,C,D}$ stands for the traces in flavor, color, and Dirac indices. Note that the indices i, i of the Gell-Mann matrices of the adjoint representation are tied with the indices f_1, f_2 of the fundamental representation of the quark propagators for each particular channel in the integral and in the coupling constants G_{ii}. In the pseudoscalar channel, the following association appears: for the charged and neutral pions $i = 1, 2$ with $f_1, f_2 = \bar{u}, d/\bar{d}, u$ and $i = 3$ with $f_1, f_2 = \bar{u}u + \bar{d}d$, respectively; for the charged and neutral kaons $i = 4, 5$ and $f_1, f_2 = \bar{u}, s/\bar{s}, u$ and $i = 6, 7$ with $f_1, f_2 = \bar{d}s/\bar{s}d$, respectively. By the usual reduction of Eq. (24) with the GAP equation considered so far (that is, the one with the coupling constant of reference G_0, which eliminates the quadratic UV divergence), the following forms for the pseudoscalar and scalar mesons, respectively, are obtained:

$$(P^2 - (M^*_{f_1} - M^*_{f_2})^2) G_{ij} I_2^{ij} = \frac{G_{ij}}{2G_0}\left(\frac{m_{f_1}}{M^*_{f_1}} + \frac{m_{f_2}}{M^*_{f_2}}\right) + 1 - \frac{G_{ij}}{G_0},$$
$$(P^2 - (M^*_{f_1} + M^*_{f_2})^2) G_{ij} I_2^{ij} = \frac{G_{ij}}{2G_0}\left(\frac{m_{f_1}}{M^*_{f_1}} + \frac{m_{f_2}}{M^*_{f_2}}\right) + 1 - \frac{G_{ij}}{G_0}, \tag{25}$$

where the coupling constants become equal to G_0 and this equation reduces to the usual BSE with a unique coupling constant [2,3]. The Goldstone theorem is straightforwardly verified by considering the usual chiral limit for which the effective quark masses are all equal. The integral I_2^{ij} in Eq. (25) is the UV logarithmic divergent and it was solved with the same three-dimensional UV cutoff exhibited in Table I. Besides that, the pole of the scalar quark-antiquark bound state $|P_0| = M_S$, where M_S is the mass of the scalar meson, might be located in the region of external momenta larger than the sum of two quark effective masses, $P_0 > (M^*_{f_1} + M^*_{f_2})$, such that there might have additional poles in the integrals I_2^{ij} indicating instability of the bound state. An IR cutoff [55], $\Lambda_{\rm IR} = 120$ MeV, was used in this case. Its contribution for the pseudoscalar meson masses can be neglected as it usually is. The value of this cutoff is somewhat smaller than values in the literature because these larger values lead to too large a suppression of the momentum integrations modifying the scalar meson masses and their mass differences. It is also well known that these integrals might be dependent on the regularization method used [56]. However, it has been shown that the regularization method usually does not modify the light meson properties, preserving quite well the predictive power of the model [57]. Besides that, if one is interested in the influence of the flavor-dependent coupling constants on the energy/mass of the quark-antiquark meson bound state, i.e., in meson mass differences, the regularization method should not produce leading order effect. Finally, masslessness of pions and kaons is recovered in the chiral limit as described above and in the literature [2].

In Table V, results for some of the light meson masses are presented for the sets of parameters X and three effective gluon propagators presented above ($D_{I,2}$, $D_{II,5}$, and $D_{II,6}$). The case for value of reference, G_0, which is independent of the gluon propagator is also considered. There are two types of comparisons to be done for a given set X or Y below. Firstly, by reading the lines of the Tables, one can obtain the neutral-charged meson mass difference for the pseudoscalar and scalar mesons. By reading the columns of the Tables, always within a particular set of parameters X or Y, it is possible to verify the role of the flavor-dependent coupling constants (for each given effective gluon propagator). Although the masses of neutral mesons were used to choose the particular sets of parameters in Table I, the charged meson masses are obtained as a consequence of the choice of the sets of parameters, being, therefore, predictions. It is important to emphasize again that one must be

TABLE V. Masses of pseudoscalar and scalar mesons states for the sets of parameters X and the corresponding coupling constants given above without the electromagnetic mass corrections. For these sets X the bound state equation for scalars and pseudoscalars were solved with the same coupling constant G^n_{ij}. In the last line there are experimental values from [16].

Set	M_{π_0} MeV	$M_{\pi+}$ MeV	M_{K0} MeV	M_{K+} MeV	$M_{a_0^0}/M_{a_0^\pm}$ MeV	$M_{\kappa^0}/M_{\kappa^\pm}$ MeV
X_{20}-$D_{I,2}$	135.8	136.2	502	492	781/789	1009/999
X_{20}-$D_{II,5}$	135.9	136.2	501	492	780/789	1003/995
X_{20}-$D_{II,6}$	135.8	136.2	502	492	781/789	1009/999
X_{20}-G_0	135.9	136.2	496	488	781/789	1008/998
X_{21}-$D_{I,2}$	136.1	136.2	499	495	787/789	1006/1002
X_{21}-$D_{II,5}$	136.1	136.2	498	494	787/789	1006/1002
X_{21}-$D_{II,6}$	136.1	136.2	499	495	787/789	1006/1002
X_{21}-G_0	136.1	136.2	494	490	787/789	1006/1002
Experimental value	135	139.6[a]	497.6	493.7	980[b]	700[b]

[a]It has an electromagnetic contribution (~4 MeV).
[b]Comparison is valid within the assumption for the scalar mesons quark flavor structure adopted in (22).

concerned with the meson mass differences rather than the absolute values of masses. This is because slightly different absolute values for the masses are easily obtained whereas the mass difference between neutral and charged mesons are consequences of quark mass differences and also specific values of G_{ij} or $G_{ij,s}$. Furthermore, and more importantly, the comparison among the results from different effective gluon propagators (D_I, D_{II}, and G_0) within a particular set of parameters X or Y indicates the contribution of varying specifically the flavor-dependent coupling constants G_{ij} or $G_{ij,s}$. Lower values for the pion masses, of the order of 135 or 136 MeV, were chosen for the fixed cutoff, instead of larger values such as 140 MeV with larger UV cutoffs, because an electromagnetic contribution for the charged pion mass is also expected. Besides that, the mass difference between neutral and charged pions due to strong interaction is small and it depends, first of all, on the up and down Lagrangian quark mass difference $\delta_{ud} = (m_d - m_u)$. Consequently, it is dependent on $M_d^* - M_u^*$, or, more specifically, $(m^2_{\pi^\pm} - m^2_{\pi^0}) \sim (M_d^* - M_u^*)^2/(M_u^* + M_d^*)^2$, as expected [12]. The smaller δ_{ud} from the set X_{21} induces smaller neutral and charged pion mass difference. Besides that, the smallness of δ_{ud} also favors smaller charged and neutral kaon mass differences. The neutral-charged pion mass difference for X_{20} is around 0.3–0.4 MeV; slightly larger than the mass obtained by other sets, such as X_{21}. It indicates that the up and down quark masses chosen for this set are slightly larger than needed. The difference between the coupling constants G_{11} and G_{33}, however, is very small (of the order of 10^{-3} GeV, not showed in the Tables above) such that, usually, it does not cause meaningful change in the pion masses. The exception was found for the set X_{20} (the one with larger u-d mass difference), for which the shifts from the values obtained with G_0 can be of the order of 0.1 MeV, that is, slightly smaller than the mass difference between the neutral and charged pions.

The mass difference between neutral and charged kaons is also in considerably better agreement with expectations for the X_{21} set of parameters than for the X_{20} set. These mass differences are proportional to $(M^*_{f_1} - M^*_{f_2})/(M^*_{f_1} + M^*_{f_2})$ (where $f_1, f_2 = u$, d or s) and are in agreement with other works [12]. However, the most interesting comparison to be noted is the fact that different values for G_{ii}, due to different effective gluon propagators for a given set X_{20} or X_{21}, lead to different shifts in the kaon masses. The coupling constants G_{44} and G_{66} are smaller than the constant value G_0 for all the gluon propagators and therefore the kaon masses are shifted to larger values. These shifts are around 4–6 MeV, although the resulting effect of each effective gluon propagator considered might be smaller, of the order of 1–5 MeV. The shifts in kaon masses due to flavor-dependent coupling constants have nearly the same modulus as the mass difference between neutral and charged kaons. The weak decay constant will be calculated in the next section.

Concerning the scalar channel: the leading effect for the scalar meson masses are the large quark effective masses. The conditions for each of the scalar quark-antiquark (with flavors $f_1 - f_2$) bound states (25) might be written approximately as

$$M^2_{S,f_1f_2} \sim M^2_{PS,f_1f_2} + (M_{f_1} + M_{f_2})^2 + \mathcal{O}\left(\frac{1}{\tilde{G}_{ij}\tilde{I}_{f_1f_2,S}}\right), \quad (26)$$

where M_S and M_{PS} are the corresponding scalar and pseudoscalar meson masses for that particular channel (with quark-antiquark $f_1 - f_2$), $\tilde{G}_{ij} = G_{ij}$ or $G_{ij,s}$, and $\tilde{I}_{f_1f_2,S}$ is the UV logarithmic divergent integral that depends on M^2_{S,f_1f_2}. This approximate equation is in agreement with particular limits of equations from other works [27]. The largest contribution for the masses of the scalar mesons masses come from the second term, i.e., the sum of the quark and antiquark effective masses, and also the first term for the K^* with a quark structure analogous to the kaons. Besides that, the IR cutoff in the integrals for the scalar meson masses makes the integrals slightly suppressed with respect to the values obtained in the pseudoscalar channel. The contributions of the coupling constants of the last term are usually smaller than the first two terms. The resulting $a_0 - K^*$ mass hierarchy, according to the structure of (22), is inverted as it occurs in the simplest versions of the NJL model [24]. It is possible to correctly fit their masses either by introducing other Lagrangian interactions or by seeking specific values of quark masses and coupling constants G_{ij} to fit the desired values. This second procedure, however, is much too artificial from the physical point of view. Therefore the entries in the tables allow for a limited comparison that may be useful, mainly for the dependence of the meson masses on each gluon propagator, or conversely, on flavor-dependent coupling constants. It turns out, however, that it is also an interesting comparison for the mass differences of charged and neutral mesons. The overall pattern of masses is similar to the one of the pseudoscalar channel: the set of parameters X_{20} yields a larger neutral-charged mass difference than the set of parameters X_{21}. However the mass difference is of the order of 2 MeV for the $a_0^\pm - a_0^0$ isotriplet and around 4 MeV for the isoduplets $K_0^*, K_\pm^*$ in the set of parameters X_{21}. This is better than the set X_{20}, which has mass differences that are quite large. The shifts in the K^* masses due to the coupling constants are sizeable only for the set of parameters X_{20}. The experimental values for these neutral-charged meson mass differences are seemingly slightly smaller than for the case of the kaons, for example. Maybe the flavor-dependent coupling constants for the scalar channels should have the role of compensating the quark-effective mass nondegeneracy effect on the neutral-charged scalar meson mass differences. The effective gluon propagators used in this work did not provide the

TABLE VI. Masses of pseudoscalar and scalar meson states for the sets of parameters Y and the corresponding coupling constants given above without electromagnetic mass corrections. For these sets Y, the bound state equations for scalars and pseudoscalars were solved with coupling constants $G^n_{ij,s}$ and G^n_{ij}, respectively. In the last line there are experimental values from [16].

Set	M_{π_0} MeV	$M_{\pi+}$ MeV	M_{K0} MeV	M_{K+} MeV	$M_{a_0^0}/M_{a_0^\pm}$ MeV	$M_{\kappa^0}/M_{\kappa^\pm}$ MeV
Y_{14}–$D_{I,2}$	135.1	135.1	492	492	758/758	970/970
$Y_{14} - D_{II,5}$	135.1	135.1	491	491	729/729	964/964
Y_{14}–$D_{II,6}$	135.1	135.1	492	492	740/740	978/978
$Y_{14} - G_0$	135.1	135.1	487	487	725/725	946/946
Y_{18}–$D_{I,2}$	135.8	136.2	502	492	824/830	1068/1060
Y_{18}–$D_{II,5}$	135.9	136.2	501	492	788/793	1028/1017
Y_{18}–$D_{II,6}$	135.8	136.2	502	492	812/820	1067/1045
$Y_{18} - G_0$	135.9	136.2	496	488	781/789	1009/998
Y_{19}–$D_{I,2}$	136.1	136.2	499	495	831/835	1037/1037
Y_{19}–$D_{II,5}$	136.1	136.2	498	494	792/793	1024/1020
Y_{19}–$D_{II,6}$	136.1	136.2	499	495	817/820	1048/1045
$Y_{19} - G_0$	136.1	136.2	494	490	787/789	1006/1002
Experimental value	135	139.6[a]	498	494	980[b]	700[b]

[a]It has an electromagnetic contribution (~4 MeV).
[b]Comparison is valid within the assumption for the scalar mesons quark flavor structure adopted in (22).

corresponding $G_{ij,s}$ needed to reduce the neutral-charged scalar meson mass differences.

In Table VI, the masses of pseudoscalar and scalar mesons are presented for the sets of parameters Y_{14}, Y_{18}, and Y_{19}, with the different effective gluon propagators and the corresponding coupling constants given above in Tables III and IV. For these sets Y, the bound state equation for scalars and pseudoscalars were solved with different coupling constants: $G^n_{ij,s}$ and G^n_{ij}, respectively.

In the pseudoscalar channel, the trends are very similar to the previous table; results for the set of parameters Y_{18} are the same as those for X_{20} in Table V, whereas the resulting pattern of the set of parameters Y_{19} is very similar to the one for X_{21}. Therefore, similar conclusions apply. The effect of the lowering of the coupling constants G_{44} and G_{66} is to push the kaon masses to slightly larger values than the masses obtained with G_0. The effect of the change in the coupling constants on the scalar meson masses is considerably larger than this effect on the pseudoscalar masses. This might be interesting for the correct complete description of the scalar structures. In addition to these sets, there is a set of parameters Y_{14} for which $m_u = m_d$, and therefore $M^*_u = M^*_d$, that yields all pions and kaons with the same masses, as expected. Nevertheless, the coupling constants G_{44} and G_{66} are slightly, but sufficiently, different for the different gluon propagators $D_{I,2}, D_{II,5}$, and $D_{II,6}$ as much as in the other Y sets.

The main tendency presented in the scalar meson masses is in the larger meson masses. This is, on one hand, due to the contribution of the quark effective masses in Eq. (26). On the other hand, there is a nonleading effect which is the fact that the scalar coupling constants $G^n_{ij,s}$ are smaller than G^n_{ij}. The largest shifts in the scalar masses appear in the K^* states because the relative changes in the values of $G_{44,s} - G_{66,s}$ are larger than changes in the values of $G_{33,s} - G_{11,s}$, together with strange quark effective mass values. Besides that, there is almost no unique trend for the shifts in the scalar mesons masses. This is due to the very diverse values obtained for the scalar channel coupling constants $G_{ij,s}$ in Table IV. The resulting overall mass difference between the a_0 and K^*, on average, might be as large as 320 MeV, for example, for set $Y_{14} - D_{I,2}$, or 218 MeV, for example, for $Y_{14} - D_{II,6}$. The neutral-charged scalar meson mass differences are quite different for each set of parameters. This suggests, again, that both the quark effective mass and flavor-dependent coupling constants might contribute to a fine-tuning of hadron spectra and interactions. The main needed effect of inverting the mass hierarchy of a_0 and K^* does not occur.

C. Leading effects on gap equations and other observables

The effects of the flavor-dependent coupling constants G_{ij} of Table II, without the mixing couplings, on different observables are presented in this section. To have a more complete idea of these effects, observables were calculated first by considering the NJL model with the coupling constant of reference, $G_0 = 10\ \mathrm{GeV}^{-2}$, and the usual resulting solutions from the gap equations shown in Table I, and second, by recalculating effective quark masses from gap equations with the coupling constants shown in Table II.

First of all, the new or corrected gap equations in the Euclidean momentum space are written as

$$M'^{*}_{f} - m_f = 4N_c G_{ff} \int \frac{d^4k}{(2\pi)^4} \frac{M'^{*}_{f}}{k^2 + M'^{*2}_{f}}, \qquad (27)$$

where G_{ff} were extracted from Eq. (17) in the absence of both types of mixing interactions, $G_{i\neq j}$ and $G_{f_1\neq f_2}$. The resulting values are presented in the first lines of Table VII for the sets of parameters X and are compared to the initial values for G_0. The sets of parameters Y yields too small of values for coupling constants $G_{ij,s}$, which are not strong enough to allow for the DChSB from Eq. (27). These too low values of the coupling constants might be a consequence of the common multiplicative normalization adopted in Eq. (20), as discussed above. Therefore, because this normalization has shown to be extremely appropriated for the pseudoscalar channel, as discussed in the last section, and because the scalar channel is not really completely addressed in this work, this discussion will be restricted to the sets of parameters X. The reduced values, mainly of G_{00} and G_{88} (which, by the way, are mostly dependent on the strange quark mass and which provide smaller contributions for G_{ss} than for G_{uu} or G_{dd}), lead to a reduced value of the strange effective quark mass M'^{*}. Since the self-consistent way of solving the model presents some further complications, including instabilities of the solutions, in the following we present results that indicate the tendency of the observables when calculating them with flavor-dependent coupling constants and corrected effective masses. Observables predicted by the model are exhibited in Table VII, by considering two different ways of calculating them compared to experimental or expected values (e.v.).

The up, down, and strange chiral scalar quark-antiquark condensates are implicitly calculated in the gap equations and they can be written as:

$$\langle(\bar{q}q)_f\rangle \equiv -\mathrm{Tr}(S_{0,f}(k)), \qquad (28)$$

where $S_{0,f}(k)$ is the quark propagator. In Table VII, $\langle\bar{q}q\rangle_G$ stands for the quark condensates calculated with flavor-dependent coupling constants G_{ff} from Table II, but with the original quark effective masses M^{*}_{f}. Due to the larger reduction of G_{ss} (f = u, d and s) the strange quark

TABLE VII. Observables for non-self-consistent calculations by considering the sets of parameters X_{20} and X_{21} discussed above for frozen values of the mesons masses. (e.v.) corresponds to the experimental or expected values. In the last lines the reduced chi-square is presented for three different calculations and for each of the sets of parameters X_{20} and X_{21}, by considering two fitted observables: M_{π^0} and M_{K^0}. Masses, decay constants, and the chiral condensate $\langle\bar{q}q\rangle^{\frac{1}{3}}$ are written in MeV, and the coupling constants, G_{Mqq}, θ_{ps}, and χ^2 are dimensionless.

	$X_{20}^{I,2}$	$X_{20}^{II,5}$	$X_{20}^{II,6}$	X_{20}, G_0	$X_{21}^{I,2}$	$X_{21}^{II,5}$	$X_{21}^{II,6}$	X_{21}, G_0	(e.v.)
$M_u{}^*(G_0)$	389	389	389	389	392	392	392	392	
$M'_u{}^*(G_{uu})$	307	325	311	389	310	328	314	392	
$M_d{}^*(G_0)$	399	399	399	399	396	396	396	396	
$M'_d{}^*(G_{dd})$	319	336	325	399	316	333	320	396	
$M'_s{}^*(G_0)$	600	600	600	600	600	600	600	600	
$M'_s{}^*(G_{ss})$	349	400	360	600	349	400	360	600	
$-\langle\bar{u}u\rangle_G^{\frac{1}{3}}$	348	346	348	338	349	347	348	338	240–260 [58,59]
$-\langle\bar{u}u\rangle_{M'}^{\frac{1}{3}}$	322	326	322	338	322	326	323	338	
$-\langle\bar{d}d\rangle_G^{\frac{1}{3}}$	350	348	349	340	350	347	349	340	240–260 [58,59]
$-\langle\bar{d}d\rangle_{M'}^{\frac{1}{3}}$	324	328	325	340	324	328	325	340	
$-\langle\bar{s}s\rangle_G^{\frac{1}{3}}$	424	407	420	363	424	407	420	363	290–300 [60]
$-\langle\bar{s}s\rangle_{M'}^{\frac{1}{3}}$	331	340	333	363	331	340	333	363	
$G_{qq\pi}(M')$	3.28	3.40	3.31	3.83	3.28	3.40	3.31	3.83	
$G_{qqK}(M')$	3.38	3.61	3.43	4.59	3.39	3.63	3.44	4.59	
$f_\pi(M')$	95.6	97.5	96.0	103.1	95.6	97.6	96.0	103.1	92 MeV
$f_K(M')$	97.8	100.8	98.5	107.3	97.6	100.7	98.3	107.2	111 MeV
$\theta_{ps}^{(0,8)}(0)$	$-8.2°$	$-6.5°$	$-7.9°$	0.0	$-8.3°$	$-6.5°$	$-7.9°$	0.0	$-11°/-24°$ [16]
$\chi^2_{red,G}$	313	295	309	244	314	295	309	245	
$\chi^2_{red,M'}$	87	92	88	116	86	92	87	117	
$\chi^2_{red,M'}$ (no $\langle\bar{q}q\rangle$)	23	22	22	19	29	27	29	24	

condensate is increased. Second, the condensates calculated with the same flavor-dependent coupling constants but with corrected effective masses $M_f'^*$ are written as $\langle\bar{q}q\rangle_{M'}$. Their values are improved with respect to all the other values and get closer to lattice calculations. It is worth stressing that lattice results had been calculated at the energy scale of $\mu\sim 1$–2 GeV in the works quoted and the others cited therein.

The quark couplings to pions and kaons, $G_{qq,\pi}$ and $G_{qq,K}$ were obtained as the residues of the poles of the BSE at the Born level, Eq. (23) [2,3], are presented as calculated with the initial quark effective masses and with the corrected quark masses by means of the equation:

$$G_{qqPS} = \left(\frac{\partial\Pi_{ij}(P^2)}{\partial P_0^2}\right)^{-2}_{(P_0,\vec{P})\equiv 0}, \quad (29)$$

where the flavor indices are tied with the quantum numbers of the meson PS as shown above. Please note that the set of parameters G_0 does not receive correction from the flavor-dependent coupling constants and can be used for comparison. The effect of the corrected quark masses is to reduce the difference between $G_{qq\pi}$ and G_{qqK} because the effective masses M_f' are closer to each other.

The charged pseudoscalar meson (pion and kaon) weak decay constant was also calculated from [2,3]:

$$F_{ps} = \frac{N_c G_{qqPS}}{4}\int\frac{d^4q}{(2\pi)^4} \times \mathrm{Tr}_{F,D}(\gamma_\mu\gamma_5\lambda_i S_{f_1}(q+P/2)\lambda_j S_{f_2}(q-P/2)), \quad (30)$$

where f_1, f_2 correspond to the quark/antiquark of the meson and i, j are the associated flavor indices as discussed in Eq. (25). In Table VII, they are presented for the flavor-dependent coupling constants and corrected masses $M_f'^*$. The values for the sets of parameters with G_0 provide the results for the case of flavor-dependent coupling constants. Because the strange quark mass decreases with the use of the flavor-dependent interaction, the kaon decay constant has its value closer to the pion decay constant being the only observable whose behavior is not the expected one.

Finally, the pseudoscalar meson mixing that is responsible for the eta-eta′ mass difference will be shortly addressed according to the following ansatz. The logics of the auxiliary field method were adopted and the pseudoscalar flavor quark current interactions can be exchanged by auxiliary fields ($P_i\sim\bar{q}i\gamma_5\lambda_i q$). Within the auxiliary field method, the following identification, which yields the correct dimensions of each of the fields, can be done by implementing functional delta functions in the generating functional [61,62]: $\delta((\bar{q}\lambda_i q) - \frac{P_i}{G_0})$. By considering effective masses for the adjoint representation auxiliary fields, $M_{ii}^2 P_i^2$, the following terms with mixings can be written for the neutral mesons whose states are obtained from the diagonal generators of the algebra:

$$\mathcal{L}_{\rm mix} = \frac{M_{33}^2}{2}P_3^2 + \frac{M_{88}^2}{2}P_8^2 + \frac{M_{00}^2}{2}P_0^2 + \frac{G_{08}}{G_0^2}P_0P_8 + \frac{G_{03}}{G_0^2}P_0P_3 + \frac{G_{38}}{G_0^2}P_3P_8, \quad (31)$$

where M_{ii}^2 include the contributions from $G_{i=j}$ derived above. The mixing terms $G_{i\neq j}$, however, are exclusively obtained from the one-loop interactions (10). The neutral pion (P_3) mixings and mass are now neglected and by performing the usual rotation to mass eigenstates η,η' [16] it can be written:

$$|\eta\rangle = \cos\theta_{ps}|P_8\rangle - \sin\theta_{ps}|P_0\rangle,$$
$$|\eta'\rangle = \sin\theta_{ps}|P_8\rangle + \cos\theta_{ps}|P_0\rangle. \quad (32)$$

By calculating it and comparing it to the above 0–8 mixing, the following $\eta-\eta'$ mixing angle is obtained:

$$\sin(2\theta_{ps}) = \frac{2G_{08}}{G_0^2(M_\eta^2 - M_{\eta'}^2)}. \quad (33)$$

The values for θ_{ps} are shown in Table VII and they are smaller than the expected values. As remarked above, the coupling constants G_{08} are basically the same for the two different sets of parameters X_{20} and X_{21}. This new mechanism for meson mixings may not be sufficient for describing the full mixing.

In the last lines of the Table, the reduced chi-square, χ^2_{red}, is shown for the sets of parameters X for which ten observables have been taken into account, two of which are fitted parameters/observables (M_{π^0} and M_{K^0}). The first χ^2_{red} was done by using $\langle\bar{q}q\rangle_G$, the second by using $\langle\bar{q}q\rangle_{M'}$, and the third without the predictions for the quark scalar condensates. The reason is that the values for the chiral quark condensates have large (and the largest) deviations from the expected values obtained from lattice calculations, and therefore their contributions for the χ^2_{red} are too large. So the analysis of the reduced chi square can be done by considering separately the behavior of $\langle\bar{q}q\rangle$. Results show the tendency of χ^2_{red} when compared to the initial calculation, for G_0, with the contribution of the flavor-dependent coupling constants by means of the effective masses M'. Besides that, note that the numerical difference between the quark condensates calculated in lattices, LQCD, have a large deviation from the NJL prediction. Another interesting comparison of the χ^2_{red} that specifically shows the effects of the flavor-dependent coupling constant in $\chi^2_{red,M'}$ for the specific sets of parameters X, is between the set of parameters G_0 (with no flavor-dependent coupling constants effects) and the other sets $I,2,II,5$, and $II,6$.

The same comparison between G_0 and the other sets X for the reduced chi-squared $\chi^2_{red,G}$ and $\chi^2_{red,M'}$ (no $\langle \bar{q}q \rangle$) might be misleading because the chiral condensates (whose flavor-dependent corrections are larger) either are not corrected by G_{ij} or are not taken into account.

IV. FINAL REMARKS

In this work, flavor symmetry breaking corrections to the NJL-type quark interactions were derived from a quark-antiquark interaction mediated by dressed gluon exchange. All resulting coupling constants are directly proportional to the quark-gluon running coupling constant and they depend on the quark and (effective) gluon propagators. Whereas the coupling constants G_{ij}, defined as almost chiral-symmetric couplings, can be associated to the pseudoscalar channel, the coupling constants $G_{ij,s}$ of the scalar channels can be numerically quite smaller and they present stronger dependence on the gluon propagator. Different sets of coupling constants, G_{ij} and $G_{ij,s}$, were obtained from sets of quark masses by employing different effective gluon propagators. Different sets of parameters that were labeled as X or Y correspond to different solutions of an NJL gap equation for a coupling constant $G_0 = 10\ \text{GeV}^{-2}$ of reference. Although the quark effective mass differences induce the flavor-dependence of coupling constants $G_{ij}, G_{ij,s}$, the effective gluon propagator also slightly contributes to the determination of their relative strength. The effects of the flavor-dependent coupling constants were identified by comparing results obtained with them with the results for the fixed reference value G_0 in the tables. To make possible a correct assessment of the effects of the flavor-dependent coupling constants, a normalization for G_{ij} was proposed, which was defined to the pseudoscalar channel and was therefore more reasonable for pseudoscalar interactions. The channels with strangeness develop smaller values of G_{ij}, i.e., larger deviations from G_0, due to the larger strange quark mass. The mixing type interactions $G_{i\neq j}$ and $G_{i\neq j,s}$ were found to be, on average, small, being proportional to the quark effective mass differences $M_s^* - M_u^*$ and/or $M_d^* - M_u^*$. One set of parameters, Y_{14}, was defined with $m_u = m_d$ and the resulting coupling constants G_{ij} and meson masses calculated with it carry this information: $G_{0,3} = G_{38} = 0$ and also $m_{\pi^0} = m_{\pi^\pm}$, and so on. These mixings, $G_{i\neq j}$, yield light meson mixings and, although the mixing angle for the $\eta - \eta'$ mixing has been calculated, other consequences will be investigated in another work.

The charged and neutral pion mass difference was found to be very small and of the order of 0.1 MeV and it is basically due to the small up and down quark mass difference, in agreement with expectations. The effect of the coupling constants G_{11}, G_{33} is, however, still slightly smaller and it was almost not identified except for a particular set of parameters with slightly larger u-d quark mass difference. The remaining part of the pion mass difference comes from electromagnetic effects that were not calculated in this work. The neutral-charged kaon mass difference was obtained to be of the order of 4–10 MeV. Both the quark mass difference and the flavor-dependent couplings, however, yield kaon mass differences of the same order of magnitude. The flavor-dependent coupling constants G_{44} and G_{66} induce mass shifts of the order of 2–4 MeV but it could reach 6 MeV for some of the sets of parameters. Both flavor dependencies should have to be considered in the NJL model: the mass and coupling constant flavor dependence. This goes along the very idea of considering the NJL model to be an effective model for QCD, being that the initial QCD-flavor dependence, parametrized in the quark Lagrangian masses, would have consequences for all the effective parameters of the resulting effective model. This is analogous to the flavor-breaking dependence of parameters in EFT, such as ChPT.

The effects of the flavor-dependent coupling constants on some of the light scalar mesons, a_0 and K^* (or κ), follow nearly the same patterns of the pseudoscalar mesons. The shifts of the masses due to changes in the coupling constants, however, might not be as large as the changes in the quark effective masses. The largest effects due to varying $G_{ij,s}$ were found to be of the order of 30–50 MeV. The usual problem of inverted hierarchy of the scalar mesons a_0 and K^* showed up because the pattern of the values of the coupling constants does not correct it. Other contributions for these scalar channel quark-antiquark interactions are expected to correct this inverted mass hierarchy [24,27]. Nevertheless, the $K^*(890)$ meson masses might be approximately obtained by the following *ad hoc* set of parameters: $m_u = 3$ MeV, $m_d = 5$ MeV, $m_s = 143$ MeV, $\Lambda = 840$ MeV, $\Lambda_{\text{IR}} = 120$ MeV, $G_{44} = 3.65$, and $G_{66} = 3.60$, which yields values close to the experimental ones: $m_{K_0^*} = 901$ MeV and $m_{K_\pm^*} = 888$ MeV. The same type of fitting is possible for the $a_0(980)$ mesons, although the physical meaning or content is not clear. The experimental values for these neutral-charged meson mass differences, however, might be smaller than for the case of the kaons, for example. The effective gluon propagators used in this work, however, can provide the corresponding needed G_{ij} or $G_{ij,s}$ to reduce accordingly the neutral-charged scalar meson mass differences. One might expect that both nondegeneracy of quark masses values and flavor-dependent coupling constants contribute to keeping the correct experimental behavior of neutral-charge scalar mass differences.

The resulting coupling constants found above define new gap equations as presented in the last section. The corrected effective mass provides observables, as calculated in Sec. III C, that are on average in better agreement with expected or experimental values. To conclude, note that the cutoff and current quark masses were kept fixed in such a way to show clearly the effects of the flavor-dependent coupling constants. The main source of shifts of the values is the strange quark effective mass that decreases with the reduction of the coupling constant G_{ss}. The consequences in the kaon decay constant and on the strange quark condensate are clear. An interesting issue to note is the new mechanism for the meson mixings by means of the resulting coupling constants $G_{i\neq j}, G_{i\neq j,s}$, or equivalently $G_{f_1\neq f_2}$. The values found, however, were not enough to reproduce the complete $\eta - \eta'$ mass difference. Further calculations are needed and they should help to constrain further the corresponding components of the (effective) quark-antiquark interactions. The reduced chi-square was calculated for ten observables, two of them being fitted observables. The sets of parameters with the flavor-dependent coupling constants were shown to provide considerably better results. The kaon decay constant is the only observable to present worse values when receiving corrections due to G_{ij}, in the present calculation. On the other hand, the values of scalar chiral condensates are largely improved. These solutions for the corrected gap equations induce further ambiguities to define either a new cutoff or different values for Lagrangian quark masses. As such, a fully self-consistent numerical calculation may be expected for which the gap equations and the flavor-dependent coupling constants are solved at once. In this program, nonstable results easily appear since shifts in quark effective masses might be reasonably large for fixed cutoff and current quark masses. This problem may worsen if weaker NJL coupling constants ($G_0 < 10\ \mathrm{GeV}^{-2}$) are considered. This problem prevents a direct and immediate self-consistent solution for the gap equations, coupling constants, and, eventually, the BSE described above. A more complete account of the mixing interaction contributions for the light meson spectra and other observables will be treated separately in another work.

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