S-folds, string junctions, and 4D $\mathcal{N} = 2$ SCFTs

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S-folds are a nonperturbative generalization of orientifold 3-planes which figure prominently in the construction of four-dimensional (4D) $\mathcal{N} = 3$ superconformal field theories (SCFTs) and have also recently been used to realize examples of 4D $\mathcal{N} = 2$ SCFTs. In this paper, we develop a general procedure for reading off the flavor symmetry experienced by D3-branes probing 7-branes in the presence of an S-fold. We develop an S-fold generalization of orientifold projection which applies to nonperturbative string junctions. This procedure leads to a different 4D flavor symmetry algebra depending on whether the S-fold supports discrete torsion. We also show that this same procedure allows us to read off admissible representations of the flavor symmetry in the associated 4D $\mathcal{N} = 2$ SCFTs. Furthermore, this provides a prescription for how to define F-theory in the presence of S-folds with discrete torsion.

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I. INTRODUCTION

One of the important ingredients in many string theory realizations of quantum field theories is the use of singular geometries in the presence of various configurations of branes. For example, in perturbative type II string theory, all of the classical gauge groups can be realized by open strings ending on D-branes, possibly in the presence of orientifold planes. It is also possible to realize exceptional groups via the heterotic string, and with singular geometries in type II/M-/F-theory compactifications. This point of view has led to the prediction of entirely new sorts of quantum field theories in diverse dimensions.

As a striking example, stringy considerations led to the discovery of four-dimensional (4D) $\mathcal{N} = 3$ superconformal field theories (SCFTs) [1]. These $\mathcal{N} = 3$ theories are inherently strongly coupled, and many of them have a realization in string theory as a stack of D3-branes on top of an S-fold plane.[1] The S-fold is a generalization of the usual orientifold plane where the $\mathbb{Z}_2$ reflection symmetry is replaced by a $\mathbb{Z}_k$ symmetry; however, this only leads to a consistent supersymmetric field theory when the axiodilaton of type IIB string theory is locally fixed to specific k-dependent values. For additional work on $\mathcal{N} = 3$ SCFTs, see, for example, Refs. [1–27].

Of course, rather than resorting to the full machinery of string theory, one might instead ask whether general principles of self-consistency can be used to chart the landscape of possible quantum field theories. A notable example of this sort of reasoning was carried out in a series of papers [28–34], which established a complete classification of possible 4D $\mathcal{N} = 2$ SCFTs with a one-dimensional Coulomb branch (see also Refs. [35,36]). A particularly interesting feature of these results is that, at the time they were found, only some of these theories had known string theory realizations. A key feature of this analysis is the appearance of specific flavor symmetry algebras, as dictated by how the Casimir invariants of the flavor symmetry translate to deformations of the associated Seiberg-Witten curve.

Some of these 4D $\mathcal{N} = 2$ SCFTs now have known stringy realizations, both in terms of compactifications of six-dimensional (6D) SCFTs [37,38], as well as in terms of D3-brane probes of S-folded 7-branes [39]. That being said, there are still some theories predicted in Refs. [28–34] which have yet to be constructed.

Our aim in this paper will be to develop a general framework for understanding the impact of S-folds on the flavor symmetries experienced by probe D3-branes in the presence of an ambient stack of 7-branes. To this end, we develop a prescription which generalizes the standard orientifold projection construction for open strings, but now for more general S-folds acting on string junctions.

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[1]There are $\mathcal{N} = 3$ theories that come from $\mathcal{N} = 4$ super Yang-Mills with an exceptional gauge algebra that do not have a D3-brane realization[2].

Doing so, we show that the structure of the resulting flavor symmetry algebra is closely tied to the appearance of discrete torsion in the S-fold. This is quite analogous to what happens for O3-planes, where there are four distinct choices depending on whether a $\mathbb{Z}_2$ discrete torsion has been activated in either the Ramond-Ramond (RR) or Neveu-Schwarz (NS) sector. We show that the presence of discrete torsion, in tandem with the geometric $\mathbb{Z}_k$ action on the local geometry, leads to a well-defined set of rules which act on the end points of the string junction states. This in turn leads to a general quotienting procedure for the resulting flavor symmetry algebras. In fact, the string junction provides more, since we can also deduce which representations of a given flavor symmetry algebra are actually present. For earlier work on the use of string junctions and its relation to symmetries realized on a 7-brane, see, e.g., Refs. [40–44]. For earlier work on string junctions in $\mathcal{N}=3$ SCFTs, see Ref. [7].

The 4D $\mathcal{N}=2$ theories that we consider will be the following. We will start with the rank-N generalizations of the Argyres-Douglas H_0, H_1, and H_2 theories [45]; the theory of $SU(2)$ with four fundamentals; and the Minahan-Nemeschansky E_6, E_7, and E_8 theories [46,47]. These theories will be labeled as the "parent" theories, and they are related to each other via mass deformations from the E_8 Minahan-Nemeschansky theory. Furthermore, each of these parent theories has a realization as a world volume theory on a stack of D3-branes in a 7-brane background (see, e.g., Refs. [48,49]). We will consider the "S-fold descendant theories," or simply "descendants," as the theories obtained by further inclusion of an S-fold plane on top of the D3-brane stack, either with or without discrete torsion.

One of the main results of our analysis is that the resulting flavor symmetry depends on the discrete torsion of the S-fold. In particular, we find that when no torsion is switched on there is a simple geometric picture available which matches to a quotient of the associated F-theory geometry for the 7-branes. When a discrete torsion is present on the S-fold, we find that the resulting flavor symmetry of a probe D3-brane is also different. In these cases, the standard F-theory geometry is not valid, but we can instead deduce its structure from the corresponding Seiberg-Witten curve of the 4D $\mathcal{N}=2$ SCFT.

Indeed, using this procedure, we show how to match each possible S-fold quotient of 7-branes to a corresponding theory appearing in the list of rank-1 4D $\mathcal{N}=2$ SCFTs appearing in Refs. [28–34], where the rank-1 theories are classified by the associated Kodaira fiber type obtained from the Seiberg-Witten curve. In matching to our 7-brane realization, we can visualize this process in terms of an overall quotienting/smoothing deformation. See Table I for a summary of this correspondence, and Fig. 1 for a summary of how these different theories are related by mass deformations and discrete quotients. Implicit in our

TABLE I. For each possible discrete quotient of an F-theory Kodaira fiber as associated with a probe D3-brane in the presence of a 7-brane and an S-fold with or without discrete torsion, we find a corresponding interacting rank-1 theory as given in Table 1 of Ref. [32].

Quotient	Rank-1 4D $\mathcal{N}=2$ SCFTs
$\mathrm{IV}^*/\mathbb{Z}_2$	$[\mathrm{II}^*, F_4]$
$\mathrm{I}_0^*/\mathbb{Z}_2$	$[\mathrm{III}^*, B_3]$
$\mathrm{IV}/\mathbb{Z}_2$	$[\mathrm{IV}^*, A_2]$
$\mathrm{I}_0^*/\mathbb{Z}_3$	$[\mathrm{II}^*, G_2]$
$\mathrm{III}/\mathbb{Z}_3$	$[\mathrm{III}^*, A_1]$
$\mathrm{IV}/\mathbb{Z}_4$	$[\mathrm{II}^*, B_1]$
$\mathrm{IV}^*/\hat{\mathbb{Z}}_2$	$[\mathrm{II}^*, C_5]$
$\mathrm{I}_0^*/\hat{\mathbb{Z}}_2$	$[\mathrm{III}^*, C_3 C_1]$
$\mathrm{IV}/\hat{\mathbb{Z}}_2$	$[\mathrm{IV}^*, C_2 U_1]$
$\mathrm{I}_0^*/\hat{\mathbb{Z}}_3$	$[\mathrm{II}^*, A_3 \rtimes \mathbb{Z}_2]$
$\mathrm{III}/\hat{\mathbb{Z}}_3$	$[\mathrm{III}^*, A_1 U_1 \rtimes \mathbb{Z}_2]$
$\mathrm{IV}/\hat{\mathbb{Z}}_4$	$[\mathrm{II}^*, A_2 \rtimes \mathbb{Z}_2]$

considerations is that if we remove all the 7-branes then we realize $\mathcal{N}=3$ theories and discrete quotients thereof. An additional comment here is that there are a few theories from Refs. [28–34] which do not appear to have a simple 7-brane realization. We take this to mean that the resulting quotients used to construct these additional theories may not arise from purely geometric ingredients present in the ultraviolet but may instead involve structures which only emerge in the infrared.

The theories we construct include some notably subtle cases such as theories with F_4 flavor symmetry. Indeed, an important point in this case is that there are some putative 4D $\mathcal{N}=2$ SCFTs with F_4 global symmetry which are now known to be inconsistent [50,51]. These inconsistent cases are those in which the Higgs branch of the 4D theory would have coincided with the instanton moduli space of F_4 gauge theory. Our brane realization makes clear that we are dealing with a different theory since in our case we have a bulk E_6 7-brane in the presence of a codimension-4S-fold with no discrete torsion. A D3-brane sitting on top of the S-fold sees an F_4 flavor symmetry, while moving it inside the 7-brane but off the S-fold results in an E_6 flavor symmetry. This is also in line with the fact that the anomalies of Ref. [32] are different from the ones of the putative (and sick) F_4 theory ruled out in Ref. [51]. As an additional comment, in F-theory, there are no 7-branes with eight-dimensional (8D) gauge group F_4, in line with the feature that such an object does not exist either from the standpoint of F-theory, nor from that of generalized Green-Schwarz anomalies [52].

Turning the discussion around, we can also see how the emergent Seiberg-Witten geometry for these $\mathcal{N}=2$ theories provides an operational definition of F-theory in S-fold backgrounds with discrete torsion. As a point of clarification, we note that in the single D3-brane case there can

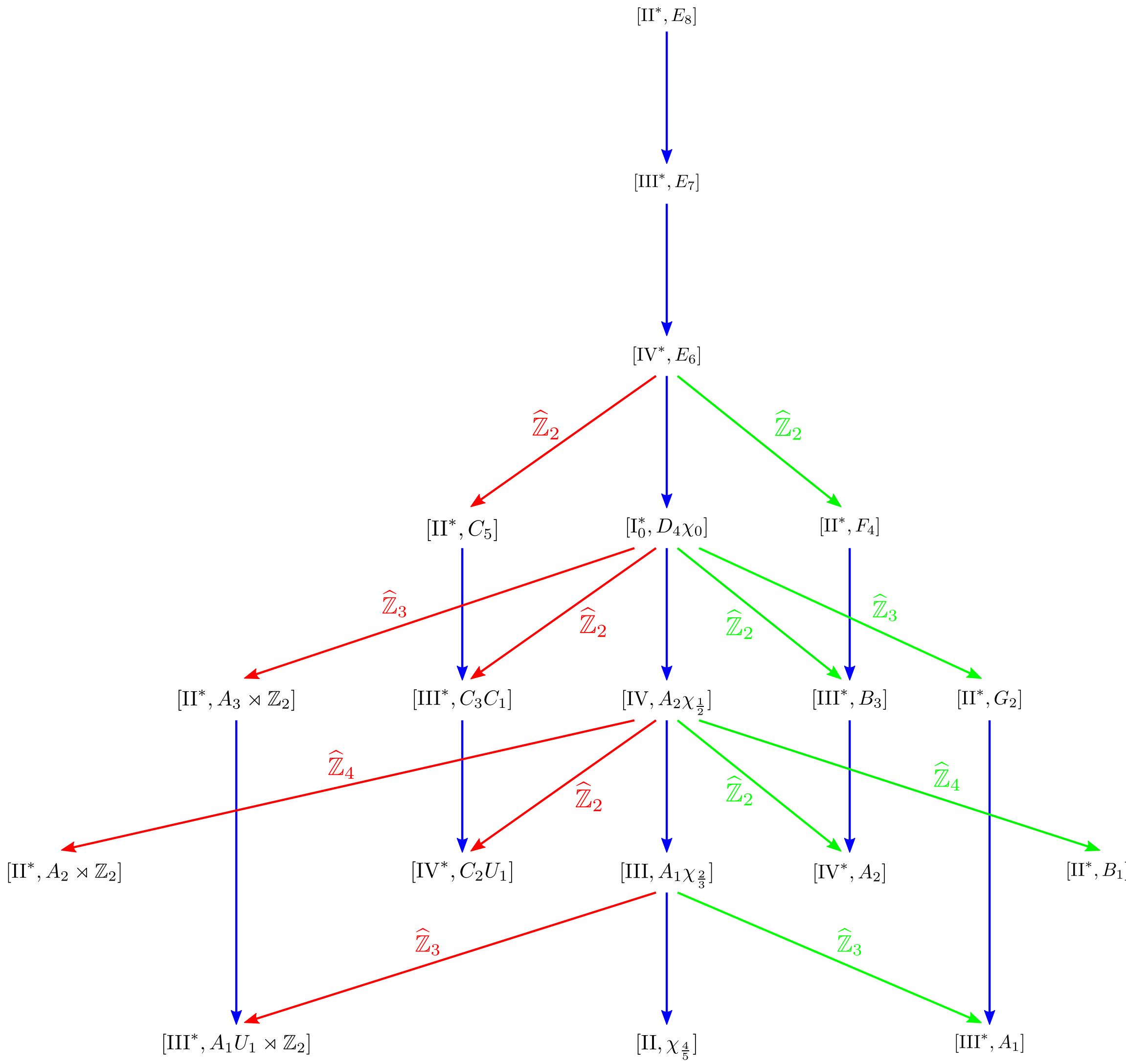


FIG. 1. Realization of the different rank-1 4D $\mathcal{N} = 2$ SCFTs starting from the E_8 Minahan-Nemeschansky theory, written as $[\mathrm{II}^*, E_8]$. We can perform mass deformations (as indicated by downward blue arrows), or we can act by a discrete twist by an outer automorphism of an algebra, possibly composed with an inner automorphism. All of the different choices can be realized by a suitable choice of S-fold projection with (diagonal red arrows and $\hat{\mathbb{Z}}_k$) or without (diagonal green arrows and $\mathbb{Z}_k$) discrete torsion. Here, we use the conventions of Refs. [28–34], which labels a given theory by its Kodaira fiber type as well as the associated flavor symmetry algebra. We note that, while this notation does not necessarily uniquely specify a particular 4D SCFT, it does so for the theories listed here. The notation χ_a refers to the fact that the theory has a chiral deformation parameter which has scaling dimension a. The theories connected to the $[\mathrm{II}^*, E_8]$ theory by blue arrows will be referred to as parent theories, and the theories determined via the red/green arrows from a given parent will be referred to as the descendants of that parent.

be additional enhancements in the flavor symmetry. The F-theory geometry is then obtained by performing a mass deformation to the generic flavor symmetry and performing a further rescaling in the local coordinates.

The rest of this paper is organized as follows. First, in Sec. II, we present a brief review of S-folds. In Sec. III, we discuss the specific case of S-folds without discrete torsion and their realization in F-theory compactifications. In Sec. IV, we present a general prescription for reading off the flavor symmetry of D3-branes probing an S-folded 7-brane. We then use this to provide a geometric proposal for F-theory geometry in the presence of discrete torsion in Sec. V. As a further check on our proposal, we also compute the leading-order contributions to the conformal anomalies a and c in the limit of a large number of probe D3-branes in Sec. VI. Section VII presents our conclusions. Some additional details on brane motions in the presence of S-folds are presented in the Appendix A, and an explicit example of string junction projections is worked out in Appendix B.

II. *S*-FOLDS

In this section, we present a brief review of S-folds. In particular, we emphasize that these objects can sometimes carry a discrete torsion. S-fold planes are a generalization of orientifold planes introduced in Ref. [1] and further studied in Ref. [5]. Initially, they were used to build four-dimensional $\mathcal{N}=3$ supersymmetric field theories on the world volume of D3-branes in the proximity of an S-fold. This was generalized in Ref. [39] by adding 7-branes on top of the S-fold, thus producing $\mathcal{N}=2$ theories. In this section, we will review the construction of Ref. [1] and discuss various properties of S-folds that we shall need in the following. We will discuss the inclusion of 7-branes in Sec. III.

A. *S*-fold quotients

S-folds arise from particular terminal singularities in F-theory backgrounds [1]. The singularity is produced by an orbifold action that acts simultaneously on the base and elliptic fiber. This implies that the geometric quotient on the base is accompanied by an $SL(2,\mathbb{Z})$ action on the elliptic curve, thus explaining the name of these objects. More concretely, we consider an F-theory solution on $\mathbb{C}^3_{(z_1,z_2,z_3)}\times\mathbf{T}^2_w$ quotiented by a $\mathbb{Z}_k$ action with generator σ_k acting on the coordinates as

$$\sigma_k\colon\ (z_1,z_2,z_3,w)\to(\zeta_k z_1,\zeta_k^{-1}z_2,\zeta_k z_3,\zeta_k^{-1}w). \qquad (2.1)$$

Here, ζ_k is a kth primitive root of unity. The singularity produced is terminal as it does not admit any crepant resolution [53,54]. One important observation is that in order to have a well-defined action on the torus the only allowed values of k are $k=2$, 3, 4, 6. Compatibility with the quotient fixes the value of the complex structure τ of the torus when $k>2$, while leaving it a free parameter for $k=2$. The allowed values of τ as well as the $SL(2,\mathbb{Z})$ action ρ on the elliptic fiber are collected in Table II. This background preserves 12 supercharges for all values of $k>2$, and adding D3-branes probing the singularity does not further break any additional supersymmetry (see, e.g., Ref. [55]). The $k=2$ case preserves 16 supercharges and therefore produces an $\mathcal{N}=4$ supersymmetric theory, and the S-fold in this case simply corresponds to the usual O3$^-$-plane. Let us note that for $k=3$ we have chosen to use the value $\tau=\exp(2\pi i/3)$, which is, under a T-transformation of $SL(2,\mathbb{Z})$, equivalent to taking $\exp(2\pi i/6)$, the "standard" value in the fundamental domain. This has no material effect on any statements we make about the flavor symmetry algebra since we can always conjugate all $SL(2,\mathbb{Z})$ generators by this T-transformation anyway. The reason for this choice is to make the $\mathbb{Z}_k$ action of the S-fold more manifest.

TABLE II. Allowed values of the type IIB axiodilaton τ and $SL(2,\mathbb{Z})$ monodromies for various S-folds when no 7-branes are present.

	τ	ρ
$k=2$	Free	$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$
$k=3$	$e^{\frac{2\pi i}{3}}$	$\begin{pmatrix}-1 & -1\\ 1 & 0\end{pmatrix}$
$k=4$	i	$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$
$k=6$	$e^{\frac{2\pi i}{6}}$	$\begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}$

B. Discrete torsion

As in the case of orientifold 3-planes, it is possible to construct different variants of S-folds by considering trapped 3-form fluxes at the singularity, i.e., discrete torsion. To understand the different allowed possibilities for discrete torsion, it is helpful to consider the asymptotic profile of the spacetime far from the singularity, as captured by a quotient of S^5. As in Refs. [5,56], it suffices to consider N D3-branes probing a $\mathbb{Z}_k$ S-fold plane. The holographic dual in the large N limit is given by type IIB string theory on $\mathrm{AdS}_5\times S^5/\mathbb{Z}_k$. To understand which fluxes can be introduced, it is necessary to study the cohomology of $S^5/\mathbb{Z}_k$, in particular the third cohomology group which corresponds to the introduction of 3-form fluxes. In type IIB, we have two possible choices of 3-form fluxes, and in the following, the first component will be the Neveu-Schwarz-Neveu-Schwarz flux, and the second one will be the RR flux. Usually, we would simply need to compute the cohomology with coefficients in $\mathbb{Z}\oplus\mathbb{Z}$; however, due to the fact that the S-fold action is nontrivial on the fluxes, it is necessary to take cohomology with coefficients in $(\mathbb{Z}\oplus\mathbb{Z})_\rho$ where ρ is the $SL(2,\mathbb{Z})$ element listed for every S-fold in Table II. This computation was done in Ref. [5] where it was shown that $H^3(S^5/\mathbb{Z}_k,(\mathbb{Z}\oplus\mathbb{Z})_\rho)$ is the cokernel of the map $(\mathbf{id}-\rho)\colon\mathbb{Z}^2\to\mathbb{Z}^2$. The resulting cohomology groups are

$$H^3(S^5/\mathbb{Z}_2,(\mathbb{Z}\oplus\mathbb{Z})_\rho)=\mathbb{Z}_2\oplus\mathbb{Z}_2, \qquad (2.2)$$

$$H^3(S^5/\mathbb{Z}_3,(\mathbb{Z}\oplus\mathbb{Z})_\rho)=\mathbb{Z}_3, \qquad (2.3)$$

$$H^3(S^5/\mathbb{Z}_4,(\mathbb{Z}\oplus\mathbb{Z})_\rho)=\mathbb{Z}_2, \qquad (2.4)$$

$$H^3(S^5/\mathbb{Z}_6,(\mathbb{Z}\oplus\mathbb{Z})_\rho)=I. \qquad (2.5)$$

The $k=2$ case reproduces the well-known example of the four different O3-planes [56]. We list here all the inequivalent choices of discrete torsion for the various S-fold planes:

$$k=2,\qquad \{(0,0),(1,0),(0,1),(1,1)\}, \qquad (2.6)$$

$$k = 3, \qquad \{(0,0),(1,0),(2,0)\}, \tag{2.7}$$

$$k = 4, \qquad \{(0,0),(1,0)\}, \tag{2.8}$$

$$k = 6, \qquad \{(0,0)\}. \tag{2.9}$$

One final piece of information that will be useful in the following is the D3-brane charge carried by the S-fold plane. The charge of the $\mathbb{Z}_k$ S-fold plane is [5]

$$\epsilon_{\mathrm{D3}} = \pm \frac{1-k}{2k}, \tag{2.10}$$

where the plus sign refers to the case without discrete torsion and the minus sign refers to the case with discrete torsion.

III. F-THEORY AND S-FOLDS WITHOUT TORSION

Having reviewed some basic features of S-folds, we now turn to the structure of local F-theory models in the presence of an S-fold. Here, we study how this is detected by the world volume theory of a spacetime filling D3-brane. Recall that in F-theory the appearance of 7-branes is encoded in the local profile of the type IIB axiodilaton. Strictly speaking, this geometric correspondence between the Coulomb branch of the D3-brane moduli space and the F-theory geometry is only valid in the purely geometric phase of F-theory, where no discrete torsion is present. Indeed, in Sec. V, we will later turn the discussion around and argue that the associated Seiberg-Witten curve provides an operational *definition* of F-theory in such backgrounds.

The rest of this section is organized as follows. First, we discuss the action of S-folds on a local Weierstrass model. These local Weierstrass models are chosen such that they correspond to an F-theory background for the parent theories, to wit, the rank-N generalizations of the Argyres-Douglas, $SU(2)$ with four flavors, and Minahan-Nemeschansky theories. After this, we turn to an explicit analysis of the various possible S-fold quotients of such geometries, organizing our discussion by the corresponding $\mathbb{Z}_2$, $\mathbb{Z}_3$, and $\mathbb{Z}_4$ group action. In the case of $\mathbb{Z}_6$, the admissible minimal Kodaira fibers are trivial, and we get an $\mathcal{N} = 3$ theory from D3-branes probing such a singularity. Following this procedure, we show how to recover some examples of the Seiberg-Witten geometries, and thus physical data like the flavor symmetry algebras, for 4D $\mathcal{N} = 2$ SCFTs of the sort predicted in Refs. [28–34]. As a point of clarification, the flavor symmetry which is really detected in this way is the generic one present for multiple D3-branes probing the S-fold. There is also an $SU(2)$ flavor symmetry as associated with the rotational group in the world volume of the 7-brane (but transverse to the D3-brane), and in the special case of a single D3-brane, there can be an "accidental" enhancement in the infrared. In the world volume theory of the D3-brane, z will refer to the Coulomb branch coordinate in the covering space, and u will refer to the Coulomb branch coordinate in the quotient geometry. The M_i will refer to a degree i Casimir invariant built from the mass deformations of the 7-brane flavor symmetry algebra.

A. Weierstrass models

To understand which kinds of 7-brane configurations are allowed in the presence of an S-fold plane, it is convenient to understand the F-theory Weierstrass model on the orbifolded base. Specifically, we consider F-theory on the base $B = \mathbb{C}^3_{(z_1,z_2,z_3)}/\mathbb{Z}_k$ where the generator of $\mathbb{Z}_k$ acts on the coordinates of the base as in (2.1). For additional details on the procedure, see, for example, Ref. [57]. The Weierstrass model on such a base is given as usual by the polynomial

$$y^2 = x^3 + f(z_1,z_2,z_3)x + g(z_1,z_2,z_3). \tag{3.1}$$

However, due to the orbifold action on the base coordinates, f and g become $\mathbb{Z}_k$-equivariant polynomials. By the condition that the elliptic fibration be a Calabi-Yau variety, the cofficients of the Weierstrass model, f and g, are required to be sections of $\mathcal{O}(-4K_B)$ and $\mathcal{O}(-6K_B)$, respectively. Homogeneity fixes x to be a section of $\mathcal{O}(-2K_B)$ and y to be a section of $\mathcal{O}(-3K_B)$. For an orbifold, a section of $\mathcal{O}(-lK_B)$ must transform with a factor $\det(\gamma)^l$ where γ is the matrix representation of any orbifold group element acting on the coordinates. To write down possible Weierstrass models, it is convenient to expand f and g as polynomials in the variables z_i,

$$f = \sum_{a,b,c\geq 0} f_{abc} z_1^a z_2^b z_3^c, \tag{3.2}$$

$$g = \sum_{a,b,c\geq 0} g_{abc} z_1^a z_2^b z_3^c. \tag{3.3}$$

Requiring f and g to transform appropriately under the orbifold action puts restrictions on the allowed polynomial coefficients f_{abc} and g_{abc}. We list in the following the possible choices for the different S-fold planes:

(i) $k = 2$.—In this case, both f and g are invariant under the orbifold action. This fixes $f_{abc} = g_{abc} = 0$ for $a - b + c \neq 0 \bmod 2$. The lowest-order terms are the constant ones giving generically a smooth elliptic curve with constant complex structure over $\mathbb{C}^3$.[2]

[2]Note that this does not mean that the orbifold action is trivial on the elliptic curve. Indeed, the coordinate y changes sign under the action of the generator of $\mathbb{Z}_2$.

(ii) $k = 3$.—In this case the orbifold action implies that g is invariant and $f \to e^{2\pi i/3} f$. This fixes $f_{abc} = 0$ for $a - b + c \neq 1$ mod 3 and $g_{abc} = 0$ for $a - b + c \neq 0$ mod 3.

(iii) $k = 4$.—In this case, the orbifold action implies that f is invariant and $g \to -g$. This fixes $f_{abc} = 0$ for $a - b + c \neq 0$ mod 4 and $g_{abc} = 0$ for $a - b + c \neq 2$ mod 4.

(iv) $k = 6$.—In this case, the orbifold action implies that g is invariant and $f \to e^{4\pi i/3} f$. This fixes $f_{abc} = 0$ for $a - b + c \neq 4$ mod 6 and $g_{abc} = 0$ for $a - b + c \neq 0$ mod 6.

In the following, we will be interested in a restricted class of Weierstrass models that preserve $\mathcal{N} = 2$ supersymmetry. This can be achieved by taking all 7-branes to wrap the (z_1, z_2)-plane, implying that f and g will only depend on z_3. Moreover, to simplify the notation, we shall denote by z the coordinate z_3 in the covering space.

We exclusively focus on Weierstrass models where the axiodilaton is constant so that we can realize a SCFT on the world volume of the D3-brane. F-theory constructions with constant coupling were discussed in Ref. [58]. Additionally, we require that the singularity type remains minimal, which imposes the further condition that the degrees of f and g as polynomials in z are $\deg(f) < 4$ and $\deg(g) < 6$. For each possible S-fold quotient, we list the covering space theory prior to the quotient in Table III. Note that the $k = 6$ quotient does not allow any dependence on z in the Weierstrass model without incurring non-minimal Kodaira fibers, and thus there can be no 7-branes present. This implies that the theory will have enhanced $\mathcal{N} \geq 3$ supersymmetry.

A careful comparison of Tables II and III also reveals that the correlation of values of k with τ are different in the presence or absence of 7-branes. This is to be expected because the presence of 7-branes impacts the profile of the axiodilaton.

The relevance of the Weierstrass model is that it will allow us to read off the Seiberg-Witten curve of the resulting $\mathcal{N} = 2$ theory for the case of a single D3-brane probe. Indeed, in this case, the Seiberg-Witten curve can be identified with the elliptic fiber of the F-theory model, and the coordinate z becomes the Coulomb branch parameter of the theory. In the following, we will discuss each possible case leading to a rank-1 SCFT writing down the Seiberg-Witten curve and match the results to the ones known in the literature. We would like to stress that the procedure works only in the case *without* discrete torsion, and in the presence of discrete torsion, we do not have a procedure to read off the Seiberg-Witten curve from the geometry. We will confirm the various identifications via a string junction analysis in Sec. IV, in which we will also be able to identify the theories on the probe D3-branes in the presence of discrete torsion. Before turning to the discussion of each case separately, we would like to point out that in the above

TABLE III. Allowed values of S-fold projection compatible with a specified minimal Kodaira fiber type. Here, we drop all higher-order singularities and focus on the specific situation where the axiodilaton is constant.

Quotient	Weierstrass model	Kodaira fiber type	τ
$k = 2$	$y^2 = x^3 + z^4$	IV^*	$e^{\frac{2\pi i}{3}}$
$k = 2$	$y^2 = x^3 + z^2 x$	I_0^*	i
$k = 2$	$y^2 = x^3 + z^2$	IV	$e^{\frac{2\pi i}{6}}$
$k = 3$	$y^2 = x^3 + z^3$	I_0^*	$e^{\frac{2\pi i}{6}}$
$k = 3$	$y^2 = x^3 + zx$	III	i
$k = 4$	$y^2 = x^3 + z^2$	IV	$e^{\frac{2\pi i}{6}}$
$k = 6$	$y^2 = x^3 + g_0$	$\varnothing$	$e^{\frac{2\pi i}{6}}$

we have been using the covering space coordinates. It is also helpful to work directly in terms of a local coordinate in the quotient geometry. In general, for a $\mathbb{Z}_k$ quotient, we would need to use $u = z^k$, which is invariant under the quotient. To find the appropriate invariant combinations for x and y, we can use the fact that under the general rescaling [32,39,59]

$$x \mapsto \lambda^2 x, \qquad y \mapsto \lambda^3 y, \tag{3.4}$$

which modifies f and g as

$$f \mapsto \lambda^{-4} f, \qquad g \mapsto \lambda^{-6} g, \tag{3.5}$$

and the elliptic fibration is left invariant. By choosing $\lambda = z^{1-k}$, the rescaled x and y variables will be invariant under the $\mathbb{Z}_k$ quotient.

Using this information, we will be able to write down the Seiberg-Witten curves for the various rank-1 theories.

Before going into the specifics of each example, we would like to point out that in Refs. [32,60] a $\mathbb{Z}_5$ quotient of type II fiber was considered. This kind of quotient is different from the others we considered so far in that it is not possible to remove the 7-branes off the singularity produced by the orbifold and therefore does not admit a limit to an $\mathcal{N} = 3$ theory. Indeed, the Seiberg-Witten curve for the theory is

$$y^2 = x^3 + z, \tag{3.6}$$

with no mass deformation allowed by compatibility with the orbifold action. The resulting theory on probe D3-branes has no flavor symmetry either before or after the quotient, and therefore it is not amenable to being studied using string junctions. Because of this, we will not consider this theory further in the rest of the paper.

B. $\mathbb{Z}_2$ quotients

In this subsection, we turn to $\mathbb{Z}_2$ quotients of an F-theory model. This sort of quotient can be taken for parent theories with an E_6 7-brane, as realized by a type IV* fiber; a D_4 7-brane, as realized by a type I_0^* fiber; and an H_2 7-brane, as realized by a type IV fiber.

1. Quotient of E_6

The Weierstrass model for an E_6 singularity can be written as

$$y^2 = x^3 + z^4. \tag{3.7}$$

Homogeneity fixes the scaling dimension of z to be $\Delta(z) = 3$. The maximal deformation of the singularity compatible with the $\mathbb{Z}_2$ quotient involves introducing the following M_i:

$$y^2 = x^3 + x(M_8 + M_2 z^2) + z^4 + M_6 z^2 + M_{12}. \tag{3.8}$$

Here, we chose the convention to label the mass deformations of the 4D $\mathcal{N} = 2$ SCFT as degree i Casimir invariants M_i where the scaling dimension is $\Delta(M_i) = i$. We can now move to the quotient space by performing the aforementioned rescaling. Let us be explicit in this first case. The scaling is

$$x \to z^{-2} x, \qquad y \to z^{-3} y, \tag{3.9}$$

which leads to an overall factor on the y^2 and x^3 terms in the Weierstrass equation of z^{-6}. Removing this denominator is equivalent to the rescaling

$$f \to z^4 f, \qquad g \to z^6 g, \tag{3.10}$$

as described in the general case in Ref. [59]. After this rescaling, we perform the replacement with the quotiented coordinate, u, via $u = z^2$. The resulting model becomes

$$y^2 = x^3 + x(M_8 u^2 + M_2 u^3) + u^5 + M_6 u^4 + M_{12} u^3, \tag{3.11}$$

where we have used the same notation x and y for before and after the rescaling. In this case, turning off all mass deformations, we obtain a II* singular fiber at the origin. Comparing with Ref. [29], we see that this Weierstrass model matches the Seiberg-Witten curve of the $[\mathrm{II}^*, F_4]$ theory.

2. Quotient of D_4

The D_4 singularity admits two different minimal Weierstrass presentations, one of which is compatible with the $\mathbb{Z}_2$ quotient and the other which is compatible with the $\mathbb{Z}_3$ quotient. For the $\mathbb{Z}_2$ quotient, we have the Weierstrass model

$$y^2 = x^3 + x z^2. \tag{3.12}$$

Homogeneity fixes the scaling dimension of z to be $\Delta(z) = 2$, and the deformation of the singularity compatible with the $\mathbb{Z}_2$ quotient is given by the introduction of the Casimirs M_2, M_4, and M_6:

$$y^2 = x^3 + x(M_4 + z^2) + M_2 z^2 + M_6. \tag{3.13}$$

Again, we move to the quotient space by performing the rescaling, as described above. After rescaling, the model becomes

$$y^2 = x^3 + x(M_4 u^2 + u^3) + M_2 u^4 + M_6 u^3. \tag{3.14}$$

In this case, turning off all mass deformations, we obtain a III* singular fiber at the origin, and if we compare with Ref. [29], we see that this Weierstrass model matches the Seiberg-Witten curve of the $[\mathrm{III}^*, B_3]$ theory listed therein.

3. Quotient of H_2

The Weierstrass model for an H_2 singularity, also known as a type IV fiber, is

$$y^2 = x^3 + z^2. \tag{3.15}$$

As usual, the scaling dimension of z is fixed by homogeneity of the Weierstrass equation. We have $\Delta(z) = 3/2$. The singularity can be deformed in such a way that is compatible with a $\mathbb{Z}_2$ quotient by introducing M_2 and M_3 as follows:

$$y^2 = x^3 + x M_2 + z^2 + M_3. \tag{3.16}$$

The resulting model in the quotient space is obtained by performing the now-familiar rescaling:

$$y^2 = x^3 + x M_2 u^2 + u^4 + M_3 u^3. \tag{3.17}$$

We can see that, turning off all mass deformations, we obtain a IV* singular fiber at the origin. Comparing with Ref. [29], we see that this Weierstrass model is precisely giving the Seiberg-Witten curve of the $[\mathrm{IV}^*, A_2]$ theory.

C. $\mathbb{Z}_3$ quotients

We next turn to $\mathbb{Z}_3$ quotients of a local F-theory geometry. This can be carried out for a D_4 7-brane, as realized by a type I_0^* fiber, and an H_1 7-brane, as realized by a type III fiber.

1. Quotient of D_4

The other Weierstrass model for the I_0^* singularity, the one compatible with the $\mathbb{Z}_3$ symmetry, is

$$y^2 = x^3 + z^3, \tag{3.18}$$

and homogeneity fixes the scaling dimension of z to be $\Delta(z) = 2$. The deformation of the singularity compatible with the $\mathbb{Z}_3$ quotient is

$$y^2 = x^3 + M_2 xz + M_6 + z^3. \tag{3.19}$$

We can now move to the quotient space by performing the aforementioned rescaling. The resulting model becomes

$$y^2 = x^3 + xM_2 u^3 + u^5 + M_6 u^4. \tag{3.20}$$

In this case, turning off all mass deformations, we obtain a II* singular fiber at the origin. Comparing with Ref. [29], we see that this Weierstrass model matches the Seiberg-Witten curve of the $[\mathrm{II}^*, G_2]$ theory.

2. Quotient of H_1

The Weierstrass model for an H_1 singularity, or type III fiber, compatible with the $\mathbb{Z}_3$ symmetry is

$$y^2 = x^3 + xz. \tag{3.21}$$

Homogeneity fixes the scaling dimension of z to be $\Delta(z) = 4/3$. The deformation of the singularity compatible with the $\mathbb{Z}_3$ quotient is

$$y^2 = x^3 + xz + M_2. \tag{3.22}$$

As usual, we can move to the quotient space by performing the rescaling described above. The resulting model becomes

$$y^2 = x^3 + xu^3 + M_2 u^4. \tag{3.23}$$

In this case, turning off all mass deformations, we obtain a III* singular fiber at the origin, and a comparison with Ref. [29] shows that this Weierstrass model reproduces the Seiberg-Witten curve of the $[\mathrm{III}^*, A_1]$ theory.

D. $\mathbb{Z}_4$ quotient of H_2

Finally, we turn to the case of $\mathbb{Z}_4$ quotients. In this case, there is only a single choice available, as given by an H_2 7-brane, namely, a type IV fiber. Recall that the Weierstrass model for an H_2 singularity is

$$y^2 = x^3 + z^2 \tag{3.24}$$

and that homogeneity of the polynomial fixes the scaling dimension of z to be $\Delta(z) = 3/2$. The deformation of the singularity compatible with the $\mathbb{Z}_4$ quotient allows the introduction of only a single Casimir invariant M_2:

$$y^2 = x^3 + M_2 x + z^2. \tag{3.25}$$

We can pass to the quotient space geometry by performing the aforementioned rescaling. The resulting Weierstrass model is

$$y^2 = x^3 + M_2 xu^3 + u^5. \tag{3.26}$$

Turning off all mass deformations, we obtain a II* singular fiber at the origin. Comparing with Ref. [29], we see that this Weierstrass model matches the Seiberg-Witten curve of the $[\mathrm{II}^*, B_1]$ theory.

IV. STRING JUNCTIONS

In the previous section, we presented a general analysis of how to read off the Seiberg-Witten curve for the world volume theory of a probe D3-brane in the presence of a 7-brane and an S-fold without discrete torsion. Geometrically, this provides a satisfying picture for how to realize a subset of possible 4D $\mathcal{N} = 2$ SCFTs, but it also leaves open the question as to whether we can also understand quotients with discrete torsion. An additional issue is that in all cases the information of the flavor symmetry is encoded indirectly in the Seiberg-Witten curve via the unfolding of the singularity.

To provide a systematic analysis of cases with and without discrete torsion, we now analyze the spectrum of string junctions in the presence of an S-fold. The rules we develop lead to a different quotienting procedure for the flavor symmetry algebra, and the available options are all contained in the options predicted in Refs. [28–34]. Again, we must add the caveat that our analysis really leads to a derivation of the generic flavor symmetry, namely, the one which is present for multiple probe D3-branes.

To better understand how S-fold projection works, we first review the standard case of orientifold projection for oriented perturbative strings; we follow this with the rules for S-fold projection in the case of $\mathbb{Z}_2$, $\mathbb{Z}_3$, and $\mathbb{Z}_4$ quotients. We then turn to the explicit S-fold projections for string junctions attached to 7-branes.

In what follows, we will find it useful to arrange the bound states of $[p, q]$ 7-branes so that the group action amounts to a simple rearrangement operation on these stacks. We refer to these branes according to the resulting $SL(2, \mathbb{Z})$ monodromy on the axiodilaton, writing the monodromy as

$$M_{[p,q]} = \begin{bmatrix} 1 + pq & -p^2 \\ q^2 & 1 - pq \end{bmatrix}, \tag{4.1}$$

for a $[p, q]$ 7-brane. We will frequently refer to the branes:

$$A = M_{[1,0]} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \qquad B = M_{[1,-1]} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix},$$
$$C = M_{[1,1]} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}, \qquad D = M_{[0,1]} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$
$$X = M_{[2,-1]} = \begin{bmatrix} -1 & -4 \\ 1 & 3 \end{bmatrix}, \qquad Y = M_{[2,1]} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}. \tag{4.2}$$

We will also need to rearrange our branes to make the S-fold quotient more manifest. We accomplish this with different brane arrangements (see Appendix A). This includes

$$E_6\colon\ A^5BC^2 \sim A^6XC \sim AAACAAAC \tag{4.3}$$

$$D_4\colon\ A^4BC \sim AACAAC \sim AABBDD \tag{4.4}$$

$$H_2\colon\ A^3C \sim ACAC \sim AYAY \sim DADA \tag{4.5}$$

$$H_1\colon\ A^2C \sim ABD. \tag{4.6}$$

These 7-branes correspond to the F-theory backgrounds that give rise to the parent theories on the probe D3-branes, when there is no S-fold. We again stress that the symmetry algebra obtained when we include the S-fold is the one enjoyed by the probe D3-branes.

The utility in introducing these different brane systems is that we can then read off the corresponding root system as well as representations from string junctions stretched between these different constituent branes. As a point of notation, we write a_i to denote weights associated with A-branes, with similar conventions for the B, C, and D branes, and where the presence of a minus sign indicates the orientation of the string. For example, the roots of $SU(N)$ for a stack A^N would then be represented as $(a_i - a_j)$ for $i, j = 1, \ldots, N$ and $i \neq j$. A junction with end points on different types of branes is represented similarly by an oriented graph with weights. Elements of the Cartan subalgebra correspond to string junctions which begin and end on the same branes.

A. Orientifold projection

Before delving into how the S-fold planes act on the string junctions stretching between 7-branes, we first review how the usual orientifold planes that appear in perturbative string theory act on string states. Recall that in the presence of a stack of $2N$ D-branes, open string states containing a vector are labeled by Chan-Paton factors λ_{ij} for $i, j = 1, \ldots, 2N$. Each λ_{ij} state is an open string stretching between the ith and the jth brane. When the stack of D-branes sits on top of an orientifold plane, it is necessary to specify the action of world sheet orientation reversal on these states. The general action is

$$\Omega\colon\ \lambda \mapsto -M\lambda^T M^{-1}. \tag{4.7}$$

The minus sign appears because of the effect of world sheet parity on the open string oscillators, and transposition appears because the end points of an open string are interchanged. M is an additional conjugation on the end points, and consistency fixes it to be either symmetric or antisymmetric. When M is chosen to be symmetric, the resulting Lie algebra on the stack of D-branes will be D_N, and when M is antisymmetric, the Lie algebra will be C_N. Given this, we will label the symmetric choice M_{SO} and the antisymmetric one M_{Sp}. In the following, we will choose[3]

$$M_{\mathrm{SO}} = \begin{pmatrix} 0 & J_N \\ J_N & 0 \end{pmatrix}, \tag{4.8}$$

$$M_{\mathrm{Sp}} = \begin{pmatrix} 0 & iJ_N \\ -iJ_N & 0 \end{pmatrix}. \tag{4.9}$$

Here, $J_N = \delta_{i+j,N+1}$ with $i, j = 1, \ldots, N$, namely, the anti-diagonal matrix in which nonzero entries are all equal to 1. We can therefore explicitly write the action of Ω on the various string states, which we label as $|ij\rangle$ for a string stretching between the ith and jth branes. We will find it convenient to use the notation $i' = 2N + 1 - i$. The map is

$$\Omega|ij\rangle = \gamma_\Omega |j'i'\rangle. \tag{4.10}$$

Here, the choice of phase factor is specified via (see Fig. 2):

$$\text{Sp projection} \to \gamma_\Omega = 1 \tag{4.11}$$

$$\text{SO projection} \to \begin{cases} \gamma_\Omega = -1, & \text{string crosses orientifold} \\ \gamma_\Omega = -1, & i = j \\ \gamma_\Omega = 1, & \text{otherwise.} \end{cases} \tag{4.12}$$

Finally, it is important to understand which projection corresponds to which orientifold plane. The system that more closely resembles the ones we will study in the following is a stack of D7-branes on top of an orientifold 3-plane. Recall that there exist four different orientifold planes usually called $\mathrm{O3}^-$, $\widetilde{\mathrm{O3}}^-$, $\mathrm{O3}^+$, and $\widetilde{\mathrm{O3}}^+$. In terms of the discrete torsion introduced in Sec. II B, they have torsion (0,0), (0,1), (1,0), and (1,1), respectively. The first two give a D-type algebra on a stack of D3-branes and a C-type algebra on a stack of D7-branes; the last two give a C-type algebra on a stack of D3-branes and a D-type algebra on a stack of D7-branes. The action on other kinds

[3]Note that it is customary in the literature to choose M_{SO} to be the identity matrix. Our choice will give isomorphic algebras after projection and leads to a simpler geometric picture in terms of branes probing the orientifold plane.

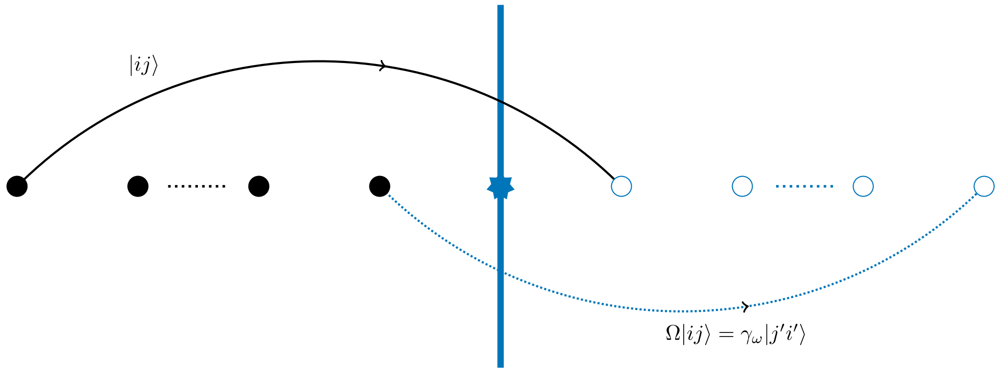


FIG. 2. Illustration of orientifold projection acting on perturbative open strings. We denote the orientifold image branes by open shapes and image strings by dashed blue lines.

of 7-branes can be obtained via $SL(2,\mathbb{Z})$ conjugation knowing that the $O3^-$-plane is invariant under $SL(2,\mathbb{Z})$ and that the action of $SL(2,\mathbb{Z})$ for the other planes can be inferred by looking at the action on the plane's discrete torsion. For example, an $O3^+$-plane will give a C-type algebra on a stack of [0, 1] 7-branes. With this information, we can easily infer that when a string junction of charge $[p,q]$ crosses an orientifold 3-plane of discrete torsion (a,b) world sheet parity will produce a sign $(-1)^{ap-bq}$ on the string state. In the following, we will generalize this to other S-folds. As a final comment, we note that when mutually nonlocal 7-branes are present we find that all that matters is whether discrete torsion is switched on or not; this is different from the situation with all 7-branes mutually local. In particular, when all 7-branes are mutually local, then the spectrum is "blind" to some sector of discrete torsion; for example, when all 7-branes are mutually local D7-branes, then the Ramond-Ramond component of the discrete torsion cannot be detected by the 7-branes.

B. *S*-fold projection

In the following, we will consider different $\mathbb{Z}_k$ projections on the set of string junctions. To get invariant states, we will call Π_k the generator of the $\mathbb{Z}_k$ action on the string state, and we will sum over the $\mathbb{Z}_k$ images to get the states after projection, meaning that we shall consider the combination

$$\frac{1}{k}\left(\mathbb{I}+\sum_{l=1}^{k-1}\Pi_k^l\right). \tag{4.13}$$

This action is considered over the generators of the complexified Lie algebra, not on the root vectors. In particular, Lie algebra generators that are mapped to themselves may be projected out due to some phases in Π_k. Indeed, since the only requirement for Π_k is that its kth power is the identity, it is possible to twist it by some $\mathbb{Z}_k$ phases corresponding to different choices of discrete torsion; these choices were reviewed in Sec. II B. What needs to be fixed is the phase that is acquired by the various junctions in the presence of discrete torsion. Note that this information is relevant only for junctions whose root vectors are invariant under the S-fold projection as the addition of these phases may project them out. We will write down the phase for a $[p,q]$-string crossing the S-fold with torsion (a,b). The phase is fixed by requiring invariance under the torsion equivalence relations described in Sec. II B. The various cases are

$$k=2, \qquad (-1)^{ap-bq}, \tag{4.14}$$

$$k=3, \qquad e^{\frac{2\pi i}{3}(ap-bp-aq+bq)}, \tag{4.15}$$

$$k=4, \qquad (-1)^{ap-bp-aq+bq}, \tag{4.16}$$

where we omit the case $k=6$ since no discrete torsion is available for this value. See Fig. 3 for a depiction of S-fold projection on string junction states.

In the above discussion, we have made reference to a specific duality frame. Given that we are working at strong coupling, it is natural to ask about the behavior of our S-fold projection under $SL(2,\mathbb{Z})$ duality transformations. Note that, while the expression for the phase is invariant under global $SL(2,\mathbb{Z})$ transformations for $k=2$, for $k>2$, it is necessary to conjugate the pairing between junction charges and discrete torsion under global $SL(2,\mathbb{Z})$ transformations in order to ensure that the phase is unchanged.[4] This should not come as a surprise as for $k>2$ we are implicitly referring to a specific choice of an $SL(2,\mathbb{Z})$ frame when discussing the torsional fluxes; indeed, the equivalence relations among discrete torsion discussed in Sec. II B refers to a matrix ρ that is not invariant under global $SL(2,\mathbb{Z})$ transformations. Given that the product appearing in the phase is fixed by requiring compatibility with these equivalence relations, it will necessarily be

[4] In practice, we will conjugate the pairing for all values of k.

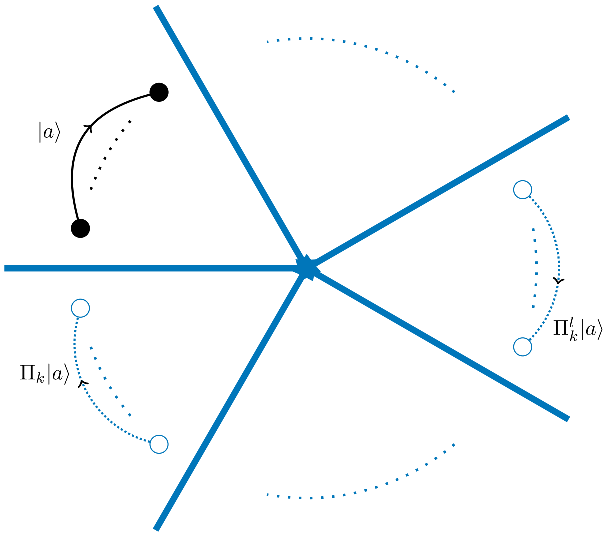


FIG. 3. Projection rules for S-fold planes acting on string junctions. We denote the orientifold image branes by open shapes and image strings by dashed blue lines.

different when going to a new $SL(2,\mathbb{Z})$ frame in order to ensure that the new equivalence relations are respected.

To proceed further, we now examine the different choices of S-fold projections on different stacks of 7-branes.

C. $\mathbb{Z}_2$ quotients of E_6

We now turn to an analysis of $\mathbb{Z}_2$ quotients of E_6; namely, we consider the action of O3-planes on string junctions attached to an E_6 7-brane. We start by writing E_6 in a $\mathbb{Z}_2$-symmetric fashion. The usual brane configuration A^6XC [41] can be permuted to a configuration A^3CA^3C, see Fig. 4. We discuss the permutations in Appendix A. The set of 72 junctions giving the roots of E_6 is

$$\begin{aligned}
&\pm(a_i-a_j), \quad 1\le j<i\le 6,\\
&\pm\left(\sum_{i=1}^{3}a_i-\sum_{j=4}^{6}a_j-a_k+a_l+c_1-c_2\right), \quad 1\le k\le 3, 4\le l\le 6,\\
&\pm(a_i-a_j+c_1-c_2), \quad 1\le k\le 3, 4\le l\le 6,\\
&\pm\left(\sum_{i=1}^{3}a_i-\sum_{j=4}^{6}a_j+c_1-c_2\right),\\
&\pm\left(\sum_{i=1}^{3}a_i-\sum_{j=4}^{6}a_j+2c_1-2c_2\right),\\
&\pm(c_1-c_2).
\end{aligned} \tag{4.17}$$

A set of simple roots is given by

$$\{a_1-a_2, a_2-a_3, a_3-a_4, a_4-a_5, a_5-a_6, c_1-c_2\}. \tag{4.18}$$

We will now turn to studying the effects of the S-fold projection, both without and with discrete torsion (for all possible choices) turned on.

1. $\mathbb{Z}_2$ quotient without discrete torsion

Consider first the case without any discrete torsion. After the projection, 48 string junctions survive (see Appendix B for a fully worked example of which string junctions are projected out for the quotients of E_6). Given the symmetry of the system, we can write all junctions, specifying only the charges on half the set of branes for the sake of convenience. The remaining junctions after projection are

$$\begin{aligned}
&\pm\frac{1}{2}(a_i-a_j), \qquad 1\le j<i\le 3,\\
&\pm\frac{1}{2}(a_i+a_j), \qquad 1\le j<i\le 3,\\
&\pm a_i, \qquad 1\le i\le 3,\\
&\pm(a_i+c_1), \qquad 1\le i\le 3,\\
&\pm\left(\sum_{i=1}^{3}a_i-a_j+c_1\right), \qquad 1\le j\le 3,\\
&\pm\frac{1}{2}\left(\sum_{i=1}^{3}a_i-a_j+2c_1\right), \qquad 1\le j\le 3,\\
&\pm\frac{1}{2}\left(\sum_{i=1}^{3}a_i+a_j+2c_1\right), \qquad 1\le j\le 3,\\
&\pm c_1,\\
&\pm\left(\sum_{i=1}^{3}a_i+c_1\right),\\
&\pm\left(\sum_{i=1}^{3}a_i+2c_1\right).
\end{aligned} \tag{4.19}$$

This gives in total 48 junctions, as expected for F_4. One choice of simple roots is

$$\left\{\frac{1}{2}(a_1-a_2), \frac{1}{2}(a_2-a_3), a_3, c_1\right\}. \tag{4.20}$$

It is possible to check that, using the intersection matrix of the brane system of E_6, one obtains the Cartan matrix of F_4, thus indicating that the resulting algebra is F_4. The simple roots after the quotient are depicted in Fig. 5.

From the above considerations, we conclude that D3-branes probing this S-folded 7-brane configuration will enjoy an F_4 global symmetry. At first glance, this would appear to be at odds with Ref. [51], which demonstrated that for 4D $\mathcal{N}=2$ SCFTs with Higgs branch given by the single instanton moduli space of F_4 gauge theory there is a global inconsistency in the anomalies of the associated theory. An important point to emphasize here, however, is

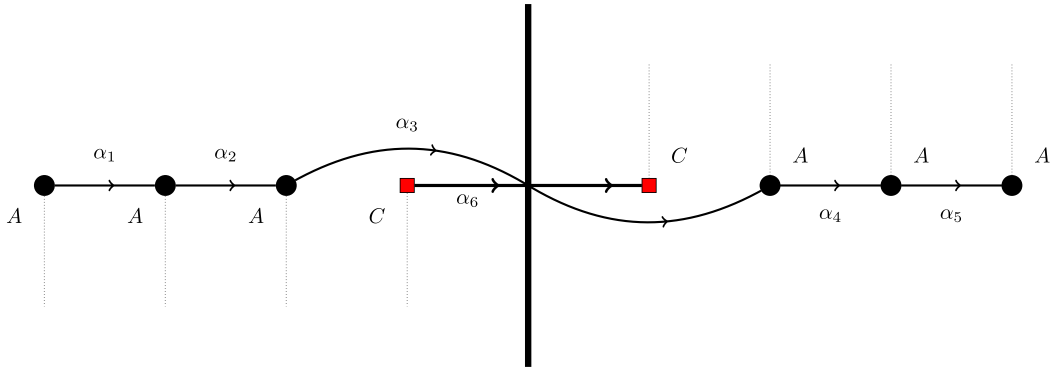


FIG. 4. $\mathbb{Z}_2$ symmetric configuration for E_6 theory.

that the same class of assumptions also allows one to extract the values of various anomalies including $\kappa_F = 5$, $a = 4/3$ and $c = 5/3$, which is rather different from the values of Refs. [28–34], which have $\kappa_F = 6$, $a = 41/24$, and $c = 13/6$. Our analysis is compatible with these considerations and indicate that the structure of the Higgs branch is more subtle. Indeed, this is in line with the fact that, moving the D3-brane off the S-fold but still inside the E_6 7-brane, the local spectrum of string junction states is actually E_6. The brane picture indicates that it is more appropriate, then, to view the Higgs branch moduli space for the D3-brane as an instanton in an E_6 gauge theory but in the presence of a codimension-4S-fold defect.

2. $\mathbb{Z}_2$ quotient with discrete torsion

Consider next the case of an orientifold projection with discrete torsion for string junctions attached to an E_6 7-brane. The $\mathbb{Z}_2$ symmetric configuration for the E6 7-brane is depicted in Fig. 4. We find that in all cases the junctions that are not invariant under the $\mathbb{Z}_2$ action are not affected by the torsion. These are the ones with $1/2$ factors in the formulas written in (4.19). For all choices of nontrivial discrete torsion, we find that 16 additional junctions are projected out, though which ones in particular depends on the choice of the discrete torsion. This gives in all cases a set of 32 junctions after projection. We illustrate the different string junction configurations which survive the S-fold projection in Fig. 6. This shows that, although different string junctions survive for each choice of discrete torsion, the actual flavor symmetry algebra realized in all these cases is the same. Moreover, this analysis establishes that in all cases the root system is the one of C_4.

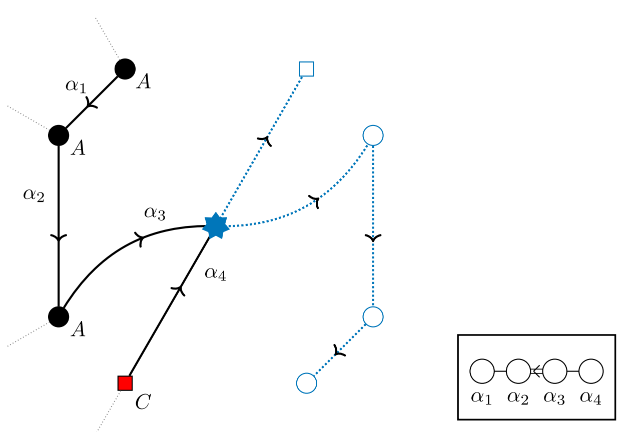


FIG. 5. String junctions for the S-fold projection of E_6 to F_4. We denote the orientifold image branes by open shapes and image strings by dashed blue lines.

D. $\mathbb{Z}_2$ quotients of D_4

Consider next $\mathbb{Z}_2$ quotients of D_4. Recall that in F-theory this is associated with a type I_0^* fiber. In this case, it is helpful to use the fact that the E_6 stack can be written as $AAACAAAC$, so removing an A-brane from each grouping, we arrive at $AACAAC$, the $\mathbb{Z}_2$ symmetric grouping for D_4. Therefore, one obtains the roots of D_4 by selecting the E_6 junctions without a_3 and a_6. Moreover, for notational simplicity, we shall rename a_{i+3} as a_{i+2} for $i = 1$, 2. There are 24 remaining junctions (as expected), and we list them here:

$$\begin{aligned} &\pm(a_i - a_j), \qquad 1 \le j < i \le 4, \\ &\pm(a_1 + a_2 - a_3 - a_4 + c_1 - c_2), \\ &\pm(a_i - a_j + c_1 - c_2), \qquad 1 \le i \le 2, 3 \le j \le 4, \\ &\pm(c_1 - c_2). \end{aligned} \tag{4.21}$$

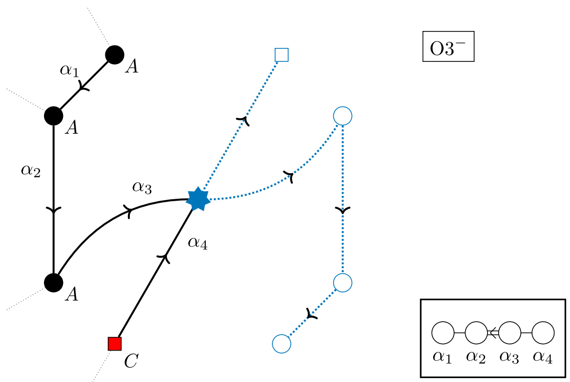
A set of simple roots is given by

$$\{a_1 - a_2, a_2 - a_3, a_3 - a_4, c_1 - c_2\}. \tag{4.22}$$

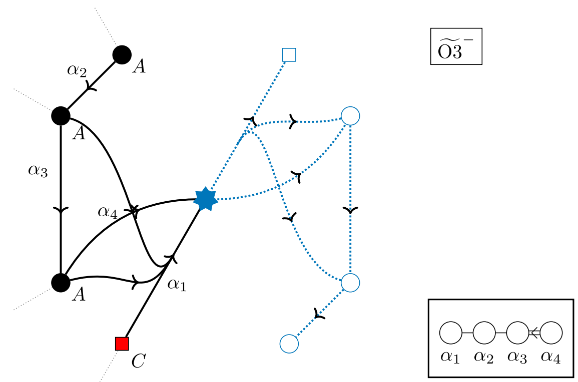
Let us now turn to the different S-fold (really orientifold) projections in this case.

1. $\mathbb{Z}_2$ quotient without discrete torsion

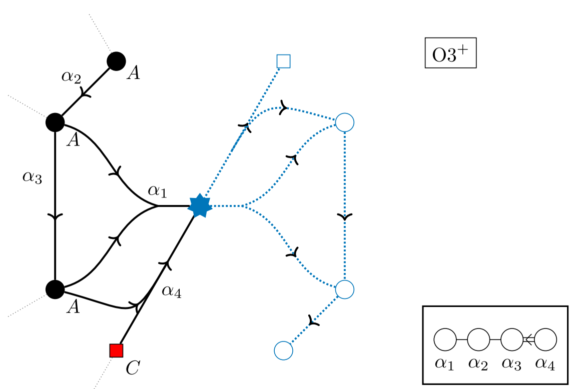
Consider first the S-fold projection of D_4 without discrete torsion. In this case, we find 18 junctions which survive, and this is the dimension of the root system of both B_3 and C_3. As before, after the quotient, we can write the



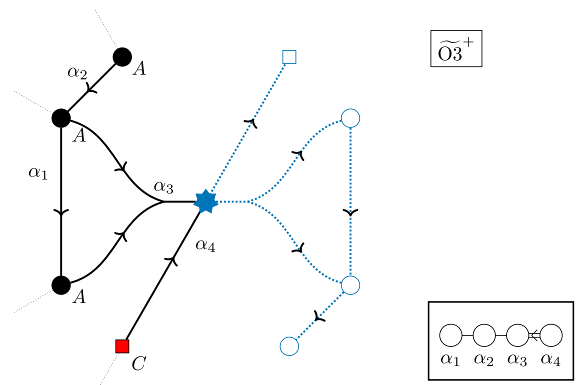
(a) F_4 obtained with $\mathrm{O3}^-$. The simple roots are: $\{\frac{a_1-a_2}{2}, \frac{a_2-a_3}{2}, a_3, c_1\}$.



(b) C_4 obtained with $\widetilde{\mathrm{O3}}^-$. The simple roots are: $\{\frac{a_2+a_3}{2}+c_1, \frac{a_1-a_2}{2}, \frac{a_2-a_3}{2}, a_3\}$.



(c) C_4 obtained with $\mathrm{O3}^+$. The simple roots are: $\{\frac{a_2+a_3}{2}, \frac{a_1-a_2}{2}, \frac{a_2-a_3}{2}, a_3+c_1\}$.



(d) C_4 obtained with $\widetilde{\mathrm{O3}}^+$. The simple roots are: $\{\frac{a_2-a_3}{2}, \frac{a_1-a_2}{2}, \frac{a_2+a_3}{2}, c_1\}$.

FIG. 6. Depiction of the different S-fold projections for an E_6 stack of 7-branes. Applying this projection results in two physically distinct configurations, the one without discrete torsion (a) and the ones with discrete torsion (b,c,d). We denote the orientifold image branes by open shapes and image strings by dashed blue lines.

junction specifying the charges on only half the set of branes. The junctions after the projection are

$$\begin{aligned}
&\pm\frac{1}{2}(a_1-a_2),\\
&\pm\frac{1}{2}(a_1+a_2),\\
&\pm a_i, \qquad 1\le i\le 2,\\
&\pm(a_i+c_1), \qquad 1\le i\le 2,\\
&\pm(a_1+a_2+c_1),\\
&\pm\frac{1}{2}(a_1+a_2+2c_1),\\
&\pm c_1.
\end{aligned} \tag{4.23}$$

Computing the Cartan matrix, we find it corresponds to the Lie algebra B_3. One choice of simple roots is

$$\left\{\frac{1}{2}(a_1-a_2), a_2, c_1\right\}. \tag{4.24}$$

For an additional comment, we observe that the above brane construction can be viewed as specifying a mass deformation from a theory with F_4 global symmetry to one with B_3 symmetry. This is indeed precisely the sort of deformation observed from purely bottom up considerations in Ref. [29]. One can see this mass deformation as a blue arrow between the $[\mathrm{II}^*, F_4]$ and the $[\mathrm{III}^*, B_3]$ theories in Fig. 1.

2. $\mathbb{Z}_2$ quotient with discrete torsion

Consider next the case of D_4 7-branes in the presence of an orientifold (i.e., $\mathbb{Z}_2$ S-fold) with discrete torsion. The result is that after the projection ten string junctions survive for all different choices of discrete torsion other than the trivial one. In all cases, the resulting algebra is $C_2\oplus A_1$.

E. $\mathbb{Z}_2$ quotients of H_2

The final case allowed with the $\mathbb{Z}_2$ S-fold is an H_2 7-brane, namely, a type IV fiber at the origin. We can obtain it by starting from the $AACAAC$ realization of the D_4 case and dropping an A-brane from both stacks, resulting in the configuration $ACAC$. The junctions can thus be obtained from the D_4 ones, and this yields

$$\begin{aligned} &\pm (a_1 - a_2), \\ &\pm (a_1 - a_2 + c_1 - c_2), \\ &\pm (c_1 - c_2). \end{aligned} \tag{4.25}$$

A set of simple roots is given by

$$\{a_1 - a_2, c_1 - c_2\}. \tag{4.26}$$

So, we get six junctions as expected for H_2, giving an A_2 algebra. Let us now turn to S-fold projections of this flavor symmetry.

1. $\mathbb{Z}_2$ quotient without discrete torsion

Consider first the $\mathbb{Z}_2$ quotient without discrete torsion of an H_2 flavor 7-brane. In this case, it is interesting to note that all string junctions are invariant under the $\mathbb{Z}_2$ action when there is no discrete torsion. Consequently, we retain the same flavor symmetry algebra. In the context of 4D $\mathcal{N} = 2$ SCFTs [29], we observe that we can also consider the associated flow, via mass deformation, from $[\mathrm{III}^*, B_3]$ to $[\mathrm{IV}^*, A_2]$, as in Fig. 1, which is compatible with our brane picture.

2. $\mathbb{Z}_2$ quotients with discrete torsion

We next consider the $\mathbb{Z}_2$ projection with discrete torsion of the H_2 theory. The result is that after the projection two string junctions survive for all different choices of discrete torsion other than the trivial one. In all cases, the resulting algebra is $A_1 \oplus U(1)$. Here, we observe the appearance of a $U(1)$ factor in the symmetry algebra. We see this since there are string junctions stretched to just the C brane of the configuration A^3C realizing H_2 and its subsequent $\mathbb{Z}_2$ quotient. This is also in accord with the quotient group action on the symmetry algebra of the parent theory.

F. $\mathbb{Z}_3$ quotients of D_4

As we already saw in Sec. III, the D_4 configuration of 7-branes also admits a $\mathbb{Z}_3$ S-fold quotient. Here, we study the resulting algebras both in the absence and in the presence of discrete torsion. To proceed, we observe that the $\mathbb{Z}_3$ symmetric choice of branes is $AABBDD$ where D is a [0, 1]-brane. In this presentation, the junctions giving the root system of D_4 are

$$\begin{aligned} &\pm (a_1 - a_2), \qquad \pm(b_1 - b_2), \qquad \pm(d_1 - d_2), \\ &\pm (a_i - b_j - d_k), \quad i = 1, 2, \quad j = 1, 2, \quad k = 1, 2, \\ &\pm (a_1 + a_2 - b_1 - b_2 - d_1 - d_2). \end{aligned} \tag{4.27}$$

One choice of simple roots is

$$\{-a_1 + a_2, a_1 - b_1 - d_1, d_1 - d_2, b_1 - b_2\}. \tag{4.28}$$

1. $\mathbb{Z}_3$ quotient without discrete torsion

With this in place, we are ready to discuss $\mathbb{Z}_3$ S-fold projections of D_4 7-branes. Consider first the case of S-fold projections without discrete torsion. The $\mathbb{Z}_3$ action maps the branes as follows:

$$a_i \to -b_i, \qquad b_i \to d_i, \qquad d_i \to -a_i. \tag{4.29}$$

After the projection, the remaining junctions are

$$\begin{aligned} &\pm (a_i - b_i - d_i), \quad i = 1, 2, \\ &\pm \frac{1}{3}(-a_1 + a_2 + b_1 - b_2 + d_1 - d_2), \\ &\pm (a_1 + a_2 - b_1 - b_2 - d_1 - d_2), \\ &\pm \frac{1}{3}(2a_1 + a_2 - 2b_1 - b_2 - 2d_1 - d_2), \\ &\pm \frac{1}{3}(a_1 + 2a_2 - b_1 - 2b_2 - d_1 - 2d_2). \end{aligned} \tag{4.30}$$

The simple roots after projection can be chosen to be

$$\left\{\frac{1}{3}(-a_1 + a_2 + b_1 - b_2 + d_1 - d_2), a_1 - b_1 - d_1\right\}, \tag{4.31}$$

whose intersection gives the Cartan matrix of G_2, which matches to the $[\mathrm{II}^*, G_2]$ theory of Ref. [29].

2. $\mathbb{Z}_3$ quotients with discrete torsion

We next consider the $\mathbb{Z}_3$ projection with discrete torsion of the D_4 theory. In this case, the reason why some junctions may be projected out is that after summing over the Π_3 images they get a factor $1 + \zeta + \zeta^2 = 0$, where ζ is a primitive third root of unity. One can check that for both choices of discrete torsion the junctions $\pm(a_i - b_i - d_i)$ for $i = 1, 2$ and $\pm(a_1 + a_2 - b_1 - b_2 - d_1 - d_2)$ are projected out. This leaves in total six junctions giving the A_2 algebra.[5]

[5]Going from G_2 to A_2 follows because the root system of G_2 is nothing but the root system of A_2 with the addition of the weights of the $\mathbf{3}$ and $\bar{\mathbf{3}}$ representations. Including discrete torsion projects out these vectors, leaving only A_2 behind.

TABLE IV. Simple roots of $\mathbb{Z}_2$ S-folds (i.e., orientifold projection) with all possible choices of discrete torsion.

S-fold	$E_6/\mathbb{Z}_2$	$D_4/\mathbb{Z}_2$	$H_2/\mathbb{Z}_2$
$\mathrm{O3}^-$	F_4: $\{\frac{a_1-a_2}{2}, \frac{a_2-a_3}{2}, a_3, c_1\}$	B_3: $\{\frac{a_1-a_2}{2}, a_2, c_1\}$	A_2: $\{a_1, c_1\}$
$\widetilde{\mathrm{O3}}^-$	C_4: $\{\frac{a_2+a_3}{2}+c_1, \frac{a_1-a_2}{2}, \frac{a_2-a_3}{2}, a_3\}$	$C_2 \oplus A_1$: $\{\frac{a_1-a_2}{2}, a_2\} \oplus \{\frac{a_1+a_2}{2}+c_1\}$	$A_1 \oplus U(1)$: $\{a_1\}$
$\mathrm{O3}^+$	C_4: $\{\frac{a_2+a_3}{2}, \frac{a_1-a_2}{2}, \frac{a_2-a_3}{2}, a_3+c_1\}$	$C_2 \oplus A_1$: $\{\frac{a_1-a_2}{2}, c_1+a_2\} \oplus \{\frac{a_1+a_2}{2}\}$	$A_1 \oplus U(1)$: $\{a_1+c_1\}$
$\widetilde{\mathrm{O3}}^+$	C_4: $\{\frac{a_2-a_3}{2}, \frac{a_1-a_2}{2}, \frac{a_2+a_3}{2}, c_1\}$	$C_2 \oplus A_1$: $\{\frac{a_1-a_2}{2}, -a_1-a_2-c_1\} \oplus \{\frac{a_1-a_2}{2}\}$	$A_1 \oplus U(1)$: $\{c_1\}$

G. $\mathbb{Z}_3$ quotients of H_1

Let us now turn to $\mathbb{Z}_3$ quotients of the H_1 stack of 7-branes. We can use our analysis of the D_4 stack of 7-branes to aid in this analysis. To this end, we begin with the realization of the D_4 algebra using the $\mathbb{Z}_3$ symmetric stack $AABBDD$. We get to the H_1 stack by removing one A brane, one B brane, and one D brane. The remaining junctions are

$$\pm(a-b-d), \tag{4.32}$$

thus giving an A_1 algebra.

1. $\mathbb{Z}_3$ quotient without discrete torsion

Consider first the $\mathbb{Z}_3$ S-fold projection in the absence of discrete torsion. This junction is already invariant under the $\mathbb{Z}_3$ quotient, suggesting that the theory can be identified with the $[\mathrm{III}^*, A_1]$ of Ref. [29]. Indeed, there is a flow $[\mathrm{II}^*, G_2] \to [\mathrm{III}^*, A_1]$ for the corresponding 4D $\mathcal{N}=2$ SCFTs.

2. $\mathbb{Z}_3$ quotients with discrete torsion

Next, consider the $\mathbb{Z}_3$ S-fold projection with discrete torsion. In both cases of $\mathbb{Z}_3$ discrete torsion, there are no junctions surviving, leaving only one single Cartan generator behind. The flavor symmetry is therefore simply $U(1)$.

H. $\mathbb{Z}_4$ quotients of H_2

We next turn to the $\mathbb{Z}_4$ S-fold projection of the H_2 stack of 7-branes. The brane system can be conjugated to a $DADA$ system where again D is a [0, 1]-brane. The junctions giving the roots are

$$\pm(a_1-a_2), \qquad \pm(d_1-d_2), \qquad \pm(a_1-a_2+d_1-d_2). \tag{4.33}$$

1. $\mathbb{Z}_4$ quotient without discrete torsion

Consider first the $\mathbb{Z}_4$ S-fold projection without discrete torsion on the H_2 stack of 7-branes. The $\mathbb{Z}_4$ projection maps

$$a_1 \to d_1, \quad d_1 \to -a_2, \quad a_2 \to d_2, \quad d_2 \to -a_1. \tag{4.34}$$

After projection, one finds only the junctions

$$\pm(a_1+d_1-a_2-d_2). \tag{4.35}$$

The algebra is therefore A_1, thus giving the $[\mathrm{II}^*, B_1]$ theory.[6]

2. $\mathbb{Z}_4$ quotient with discrete torsion

In the case of the $\mathbb{Z}_4$ S-fold projection with discrete torsion of the H_2 stack of 7-branes, we find by a similar analysis that the algebra is A_1, i.e., there is no distinction in the flavor symmetry algebras for the cases with and without discrete torsion.

I. Collection of flavor symmetry algebras

In this section, we collect our results on the resulting flavor symmetry algebras. First, we remind the reader that the particular nonzero values of the discrete torsion are irrelevant; the spectrum of physical states, as determined from the string junctions, is identical for all cases with nonzero discrete torsion. We then summarize the different algebras and a choice of root system in Tables IV, V, and VI. In Table VII, we summarize the relevant patterns, indicating quotients without discrete torsion as $\mathbb{Z}_k$ and those with discrete torsion as $\hat{\mathbb{Z}}_k$.

The aforementioned flavor algebras are always realized on the world volume of the 7-branes and for all ranks of the SCFT. However, it is expected that in the case of rank-1 theories a quotient with discrete torsion can result in an enhancement of the geometric flavor symmetry and that realized by a 7-brane. This geometric symmetry is $SU(2)$ for $\hat{\mathbb{Z}}_2$ quotients and $U(1)$ for the other $\hat{\mathbb{Z}}_k$ quotients. We can determine that there is likely an enhancement when

[6] Note that at the level of Lie algebras we have $A_1 \simeq B_1$.

TABLE V. Simple roots of $\mathbb{Z}_3$ S-folds with all possible choices of discrete torsion.

S-fold	$D_4/\mathbb{Z}_3$	$H_1/\mathbb{Z}_3$
Trivial torsion	$G_2: \{\frac{1}{3}(-a_1+a_2+b_1-b_2+d_1-d_2), a_1-b_1-d_1\}$	$A_1: \{a-b-d\}$
Nontrivial torsion	$A_2: \{\frac{1}{3}(-a_1+a_2+b_1-b_2+d_1-d_2), \frac{1}{3}(2a_1+a_2-2b_1-b_2-2d_1-d_2)\}$	$U(1)$

TABLE VI. Simple roots of $\mathbb{Z}_4$ S-folds with all possible choices of discrete torsion. Here, having nontrivial torsion does not affect the gauge algebra or the simple root system.

S-fold	$H_2/\mathbb{Z}_4$
Trivial torsion	$A_1: \{a_1+d_1-a_2-d_2\}$
Nontrivial torsion	$A_1: \{a_1+d_1-a_2-d_2\}$

TABLE VII. Summary of symmetry algebras obtained from an S-fold projection of a parent stack of 7-branes. We find that there are two qualitative quotients, based on $\mathbb{Z}_k$ without discrete torsion and based on $\hat{\mathbb{Z}}_k$ with discrete torsion.

Parent	$\mathbb{Z}_2$	$\hat{\mathbb{Z}}_2$	$\mathbb{Z}_3$	$\hat{\mathbb{Z}}_3$	$\mathbb{Z}_4$	$\hat{\mathbb{Z}}_4$
E_8						
E_7						
E_6	F_4	C_4				
D_4	B_3	$C_2 \oplus A_1$	G_2	A_2		
H_2	A_2	$A_1 \oplus U(1)$			A_1	A_1
H_1			A_1	U_1		
H_0						

the level of the $SU(2)$ and the level of the 7-brane flavor symmetry (both of which we can calculate) match. The expected enhancements [32] are

(i) For the $\hat{\mathbb{Z}}_2$ quotient of E_6, the rank-1 theory is expected to have C_5 flavor symmetry.
(ii) For the $\hat{\mathbb{Z}}_2$ quotient of D_4, the rank-1 theory is expected to have $C_3 \oplus A_1$ flavor symmetry.
(iii) For the $\hat{\mathbb{Z}}_2$ quotient of H_2, the rank-1 theory is expected to have $C_2 \oplus U_1$ flavor symmetry.
(iv) For the $\hat{\mathbb{Z}}_3$ quotient of D_4, the rank-1 theory is expected to have $A_3 \rtimes \mathbb{Z}_2$ flavor symmetry.
(v) For the $\hat{\mathbb{Z}}_3$ quotient of H_1, the rank-1 theory is expected to have $A_1 \oplus U_1 \rtimes \mathbb{Z}_2$ flavor symmetry.
(vi) For the $\hat{\mathbb{Z}}_4$ quotient of H_2, the rank-1 theory is expected to have $A_2 \rtimes \mathbb{Z}_2$ flavor symmetry.

J. Admissible representations

So far, we have focused on the structure of the Lie algebra of the flavor symmetry. The string junction picture also allows us to access the admissible representations. We will discuss only the cases where the center of the simply connected group of a given Lie algebra is nontrivial. We begin by first discussing S-fold projections without discrete torsion and then turn to the case of examples with discrete torsion. If there happen to be other sources of flavor symmetries, this can lead to additional global structure. For example, E_8 has an $E_6 \times SU(3)/\mathbb{Z}_3$ subgroup but also has representations in the $(\mathbf{27}, \mathbf{3})$. If we ignore the $SU(3)$ factor, then we would loosely refer to this as realizing an E_6 group. In the probe D3-brane theories, we also know that there is an $SU(2)$ flavor symmetry associated with symmetries internal to the 7-brane but transverse to the D3-brane, so determining the full structure of the 4D flavor symmetry must reference this feature as well. We leave this determination for future work. What we can assert from the string junction picture is whether we see evidence for a given type of representation, and so to indicate this information, we will mildly abuse terminology and refer to G_{rep} as specifying the "the flavor group" and its admissible representations.

1. S-fold projections without discrete torsion

We now turn to S-fold projections without discrete torsion in which, for a given Lie algebra, the associated simply connected Lie group has a nontrivial center. This limits us to the following cases:

(i) The $\mathbb{Z}_2$ quotient of a D_4 stack of 7-branes without discrete torsion yields a B_3 algebra, which means that the flavor group is either $Spin(7)$ or $Spin(7)/\mathbb{Z}_2 \simeq SO(7)$. One quick way to check which representations are allowed is to use the fact that the B_3 theory descends from the F_4 theory. Decomposing the adjoint of F_4, one finds

$$F_4 \to Spin(7) \otimes SO(2) \tag{4.36}$$

$$\mathbf{52} \to \mathbf{1}_0 \oplus \mathbf{7}_2 \oplus \mathbf{7}_{-2} \oplus \mathbf{21}_0 \oplus \mathbf{8}_1 \oplus \mathbf{8}_{-1}. \tag{4.37}$$

Note that the $\mathbf{8}$ is the spinor representation of $Spin(7)$, so indeed the flavor group is $Spin(7)$.

(ii) The $\mathbb{Z}_2$ quotient of a H_2 stack of 7-branes without discrete torsion yields an A_2 algebra, which means that the flavor group is either $SU(3)$ or $SU(3)/\mathbb{Z}_3 \simeq PSU(3)$. Similarly to the previous case, we can use the fact that the A_2 theory descends from the B_3 theory. Decomposing the adjoint of $Spin(7)$, one finds

$$Spin(7) \to SU(3) \otimes U(1) \tag{4.38}$$

$$\mathbf{21} \to \mathbf{1}_0 \oplus \mathbf{8}_0 \oplus \mathbf{3}_4 \oplus \bar{\mathbf{3}}_{-4} \oplus \mathbf{3}_2 \oplus \bar{\mathbf{3}}_{-2}. \quad (4.39)$$

Since the **3** representation of A_2 is present, this fixes the flavor symmetry group to be $SU(3)$.

(iii) The $\mathbb{Z}_3$ quotient of an H_1 theory without discrete torsion gives the flavor algebra A_1, which means that the flavor group could be either $SU(2)$ or $SU(2)/\mathbb{Z}_2 \simeq SO(3)$. We can follow the logic outlined before, noting that this theory comes from the G_2 theory. Decomposing the adjoint of G_2, we find

$$G_2 \to SU(2) \otimes SU(2), \quad (4.40)$$

$$\mathbf{14} \to (\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus (\mathbf{4},\mathbf{2}). \quad (4.41)$$

It is possible to check by computing the charges of the junctions that after breaking G_2 the junctions lie in the **4** representation of the unbroken group, implying that this group is $SU(2)$ rather than $SO(3)$, given that the **4** is charged under the center.

2. S-fold projections with discrete torsion

Let us now turn to the related case of S-fold projections with discrete torsion. Again, we confine our analysis to those Lie algebras which have a simply connected Lie group with nontrivial center. The relevant cases are as follows:

(i) The $\mathbb{Z}_2$ quotient of the E_6 theory with discrete torsion gives the flavor algebra C_4, which means that the flavor group can be either $USp(8)$ or $USp(8)/\mathbb{Z}_2$. In this case, we note that all junctions must descend from junctions of the parent E_6 theory and its weight lattice is generated by the junctions giving the **27** representation. Decomposing it, we find

$$E_6 \to USp(8), \quad (4.42)$$

$$\mathbf{27} \to \mathbf{27}. \quad (4.43)$$

The **27** of $USp(8)$ is the two-index antisymmetric representation which is not charged under the center. This implies that no junctions charged under the center can be generated, implying that the flavor group is $USp(8)/\mathbb{Z}_2$.

(ii) The $\mathbb{Z}_2$ quotient of the D_4 theory with discrete torsion gives the flavor algebra $C_2 \oplus A_1$. Here, there are various possibilities for the global structure of the gauge group. Knowing that this theory descends from the C_4 theory, we can decompose the adjoint of C_4,

$$USp(8) \to USp(4) \otimes SU(2) \otimes U(1),$$
$$\mathbf{36} \to (\mathbf{4},\mathbf{2})_1 \oplus (\mathbf{4},\mathbf{2})_{-1} \oplus (\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{1},\mathbf{3})_2 \oplus (\mathbf{1},\mathbf{3})_{-2} \oplus (\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{10},\mathbf{1})_0. \quad (4.44)$$

We see that the only representations charged under the center of $USp(4)$ and $SU(2)$ appear together, which suggests that the group is $(USp(4) \otimes SU(2))/\mathbb{Z}_2$. Note that other quotients like, for instance, $USp(4)/\mathbb{Z}_2 \otimes SU(2)/\mathbb{Z}_2$ are not compatible with the representations appearing, given that the fundamental representations of $USp(4)$ and $SU(2)$ appear in the previous decomposition. Following a similar logic starting from the **27** representation of $USp(8)$, which is the smallest representation available, confirms this result.

(iii) The $\mathbb{Z}_2$ quotient of the H_2 theory with discrete torsion gives the flavor algebra $A_1 \oplus U_1$. In this case, we can decompose the adjoint of $C_2 \oplus A_1$ as

$$USp(4) \otimes SU(2) \to SU(2) \otimes U(1)_a \otimes U(1)_b, \quad (4.45)$$

$$(\mathbf{10},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \to \mathbf{1}_{(0,0)} \oplus \mathbf{1}_{(0,0)} \oplus \mathbf{3}_{(0,0)} \oplus (\mathbf{2}_{(1,1)} \oplus \mathbf{1}_{(2,2)} \oplus \mathbf{1}_{(2,-2)} \oplus \mathrm{H.c.}). \quad (4.46)$$

The broken generator is $U(1)_b$, leaving $SU(2) \otimes U(1)_a$. Therefore, the flavor symmetry group seems to be $(SU(2) \otimes U(1))/\mathbb{Z}_2$. The conclusion does not change when looking at other representations of $(USp(4) \otimes SU(2))/\mathbb{Z}_2$.

(iv) The $\mathbb{Z}_3$ quotient of the D_4 theory with discrete torsion gives the flavor algebra A_2, which means that the flavor group can be either $SU(3)$ or $SU(3)/\mathbb{Z}_3 \simeq PSU(3)$. In this case, we note that all junctions must descend from junctions of the parent E_6 theory and its weight lattice is generated by the junctions giving the $\mathbf{8_s}$, the $\mathbf{8_c}$, and the $\mathbf{8_v}$ representations. Decomposing them, we find

$$Spin(8) \to SU(3), \quad (4.47)$$

$$\mathbf{8_s} \to \mathbf{8}, \quad (4.48)$$

$$\mathbf{8_c} \to \mathbf{8}, \quad (4.49)$$

$$\mathbf{8_v} \to \mathbf{8}. \quad (4.50)$$

The **8** representation of A_2 is of course the adjoint which is uncharged under the center. This means that no representation charged under the center is present, giving the flavor symmetry $PSU(3)$.

(v) The $\mathbb{Z}_4$ quotient of the H_2 theory with discrete torsion gives the flavor algebra A_1, which means that the flavor group can be either $SU(2)$ or $SU(2)/\mathbb{Z}_2 \simeq SO(3)$. In this case, we note that all junctions must descend from junctions of the parent H_2 theory and its weight lattice is generated by the junctions giving the **3** representation. Decomposing them, we find

$$SU(3) \to SU(2), \tag{4.51}$$

$$\mathbf{3} \to \mathbf{3}. \tag{4.52}$$

The **3** representation of A_1 is of course the adjoint which is uncharged under the center. This means that no representation charged under the center is present, giving the flavor symmetry $SO(3)$.

V. F-THEORY AND S-FOLDS WITH DISCRETE TORSION

One useful application of the F-theory construction is that it allows one to read off the Seiberg-Witten curve from the geometry for the rank-1 theories. However, as we stressed before, this procedure works only in the absence of discrete torsion. Given this identification between geometry and the low-energy field theory data, it is tempting to push this identification beyond the case without discrete torsion. We propose that the F-theory geometry in the presence of discrete torsion is the Seiberg-Witten curve of the theory on a single probe D3-brane. In this section, we will list all the maximally mass deformed Seiberg-Witten curves from Ref. [29] for the various theories we obtained in the presence of discrete torsion. One subtle point is that in the case of a single D3-brane there can be additional enhancements in the flavor symmetry relative to the case of multiple D3-branes. In these cases, we interpret the F-theory geometry as the one obtained by taking a mass deformation of the enhanced symmetry algebra, which takes us to the generic flavor symmetry, and then taking a further scaling limit so that the terms with the mass deformation are scaled out. In all cases, this is associated with the degree-2 Casimir invariants of the flavor symmetry algebra. In what follows, we leave this operation implicit in our discussion. With notation as earlier, we use the Coulomb branch parameter u to indicate the directions transverse to the 7-brane in the quotiented geometry:

(i) The Seiberg-Witten curve for the $\mathbb{Z}_2$ quotient with discrete torsion of the E_6 theory is

$$\begin{aligned} y^2 = {}& x^3 + 3x[2u^3M_2 + u^2(M_4^2 - 2M_8) \\ & + 2uM_4M_{10} - M_{10}^2] \\ & + 2[u^5 + u^4M_6 + u^3(2M_4^3 - 3M_4M_8 - 3M_2M_{10}) \\ & + 3u^2M_8M_{10} - 3uM_4M_{10}^2 + M_{10}^3]. \end{aligned} \tag{5.1}$$

(ii) The Seiberg-Witten curve for the $\mathbb{Z}_2$ quotient with discrete torsion of the D_4 theory is

$$\begin{aligned} y^2 = {}& x^3 + x[12u^3 - u^2(M_4 + 4M_2^2) + 12uM_2M_6 - 3M_6^2] \\ & - 12u^4(2M_2 + 3\tilde{M}_2) + 2u^3(M_2M_4 + 6M_6) \\ & - u^2(16M_2^2 + M_4)M_6 + 12uM_2M_6^2 - 2M_6^3. \end{aligned} \tag{5.2}$$

Note the presence of two independent degree-2 Casimirs, M_2 and $\tilde{M}_2$. This occurs whenever the flavor symmetry is semisimple; in this case, it is $C_3 \oplus A_1$.

(iii) The Seiberg-Witten curve for the $\mathbb{Z}_2$ quotient with discrete torsion of the H_2 theory is

$$\begin{aligned} y^2 = {}& x^3 - x[3u^2(M_2 + M_1^2) + 12uM_1M_4 + 3M_4^2] \\ & - 864u^4 + 2u^3M_1(M_1^2 - 3M_2) \\ & - 3u^2(5M_1^2 + M_2)M_4 - 12uM_1M_4^2 - 2M_4^3. \end{aligned} \tag{5.3}$$

(iv) The Seiberg-Witten curve for the $\mathbb{Z}_3$ quotient with discrete torsion of the D_4 theory is

$$\begin{aligned} y^2 = {}& x^3 + 3xu^2(2uM_2 - M_4^2) \\ & + 2u^3(u^2 + M_4^3 + uM_6). \end{aligned} \tag{5.4}$$

This was identified in Ref. [30] and reproduces the curve already found in Ref. [61].

(v) The Seiberg-Witten curve for the $\mathbb{Z}_3$ quotient with discrete torsion of the H_1 theory is

$$y^2 = x^3 + 3x(u^3 - u^2\tilde{M}_2^2) + 2(u^4M_2 + u^3\tilde{M}_2^3). \tag{5.5}$$

(vi) The Seiberg-Witten curve for the $\mathbb{Z}_4$ quotient with discrete torsion of the H_2 theory is

$$\begin{aligned} y^2 = {}& x^3 - \frac{1}{8}x(2u - M_6)^3M_2 \\ & - \frac{1}{8}(2u - M_6)^4(u + 2M_6). \end{aligned} \tag{5.6}$$

For an additional comment, we note that here we have mainly focused on the situation where we treat the M_i as mass parameters. Of course, since the S-fold introduces a codimension-4 defect in the world volume of the 7-brane, we can also include additional position dependence in these mass parameters. Doing so would produce F-theory backgrounds which we can characterize as elliptically fibered Calabi-Yau threefolds in the presence of discrete torsion.

VI. ANOMALIES

For a further check on our proposal, in this section, we study the scaling of the conformal anomalies a and c in the limit of large N, that is, when we have a large number of probe D3-branes. We shall also determine the flavor symmetry anomaly κ_G associated with two flavor currents and an R-symmetry current, namely, $\mathrm{Tr}(\mathcal{R}GG)$, where $\mathcal{R}$ denotes the current for the $U(1)_{\mathcal{R}}$ factor of the R-symmetry $SU(2)\times U(1)_{\mathcal{R}}$ of a 4D $\mathcal{N}=2$ SCFT and G refers to a flavor symmetry current associated with a 7-brane. Since we are dealing with topological features of the theory, we will extrapolate our results back to small values of N, much as in Ref. [62]. From our analysis, we can read off both the order-N^2 and order-N contributions to the conformal anomalies; however, we will not be able to access the $\mathcal{O}(N^0)$ contributions via these methods. This will allow us to compare with the results of Ref. [38], which studies certain 4D SCFTs from T^2 compactifications of six-dimensional (6D) $\mathcal{N}=(1,0)$ SCFTs, as well as with Ref. [39], which studies some examples of D3-brane probes of S-folds with discrete torsion. In the rank-1 case, $N=1$, we will find consistency with the rank-1 theories of Ref. [32], though in those cases, we will have to subtract a free hypermultiplet to match with the interacting SCFT.

The computation is done using holography as in Ref. [62]. The large N dual of the background we are considering is type IIB on $\mathrm{AdS}_5\times S^5/\mathbb{Z}_k$ with 7-branes. We will separate the various terms appearing in the central charges according to their N scaling, with leading order being N^2:

(i) $\mathcal{O}(N^2)$.—This term comes from the total D3-brane charge induced by the background. The general formula is

$$a|_{\mathcal{O}(N^2)}=c|_{\mathcal{O}(N^2)}=\frac{M^2\pi^3}{4V_5},\tag{6.1}$$

where M is the D3-brane charge and V_5 is the volume of the internal 5-manifold. In our case, $M=N+\varepsilon$, where $\varepsilon=\pm(1-k)/2k$ is the charge of the S-fold plane[7] and $V_5=\pi^3/k\Delta$. The reason for the last identification is that the volume of the 5-sphere is reduced by a factor of k by the S-fold quotient [5,39] and by a factor of Δ due to the deficit angle of the 7-branes [62].[8]

(ii) $\mathcal{O}(N)$.—This term comes from the Chern-Simons terms on the 7-branes. The general formula is

$$a|_{\mathcal{O}(N)}=\frac{M(\Delta-1)}{2},\tag{6.2}$$

$$c|_{\mathcal{O}(N)}=\frac{3M(\Delta-1)}{4}.\tag{6.3}$$

As before, $M=N+\varepsilon$. Notice that there is no dependence on k. This is because both the volume wrapped by the 7-branes and the volume of the sphere are both affected in the same way by the quotient (the Chern-Simons action is proportional to the ratio of these volumes). Moreover, these terms disappear whenever $\Delta=1$, that is, in the case when there are no 7-branes.[9]

While we have, in principle, been determining the terms at quadratic and linear orders in N, we in fact have determined contributions at $\mathcal{O}(1)$ from the ε terms in M. We will disregard these terms, as we cannot determine the $\mathcal{O}(1)$ terms anyway, and we are in fact required to subtract these terms if the central charges are to match those occurring for the $\mathcal{N}\geq 3$ theories [5]. Adding the quadratic and linear terms together, we get

$$a=\frac{k\Delta}{4}N^2+\frac{(k\Delta\varepsilon+\Delta-1)}{2}N,\tag{6.4}$$

$$c=\frac{k\Delta}{4}N^2+\frac{(2k\Delta\varepsilon+3\Delta-3)}{4}N.\tag{6.5}$$

Recall that $\varepsilon=\pm(1-k)/2k$. We can use these formulas and can check that they agree with the known results for rank-1 4D SCFTs [32], although in these cases, we need to subtract a center of mass hypermultiplet. In addition, we are able to compute κ_G, the anomaly associated with $\mathrm{Tr}(\mathcal{R}GG)$, with G the flavor symmetry generated by the 7-branes in the presence of the S-fold. The results for the cases with discrete torsion are in Refs. [38,39], and here we focus on the cases without discrete torsion. In general, following Ref. [62], one finds that the central charge for the flavor symmetry G on the 7-branes and the geometric $SU(2)$ flavor symmetry are

$$\kappa_G=2N\Delta,\qquad \kappa_{SU(2)}=kN^2\Delta-N(\Delta-1-2k\Delta\varepsilon).\tag{6.6}$$

Let us note that in the special case where $N=1$ we always find that either $\kappa_{SU(2)}=0$ or that there is an accidental enhancement in the infrared where the $SU(2)$ merges with the 7-brane flavor symmetry. We tabulate the values that we get for all cases without discrete torsion writing both the rank-N and rank-1 values, indicating as well the Kodaira fiber type prior to the quotient. As expected, these are the same values displayed in Ref. [32] (for the rank-N case, the

[7] Recall that the plus sign corresponds to the case without discrete torsion, and the minus sign corresponds to that with discrete torsion, regardless of the particular choice of the discrete torsion.

[8] Δ is both the deficit angle and the dimension of the Coulomb branch operator. The values of Δ are $\Delta=6$ for the E_8 theory, $\Delta=4$ for the E_7 theory, $\Delta=3$ for the E_6 theory, $\Delta=2$ for the D_4 theory, $\Delta=3/2$ for the H_2 theory, $\Delta=4/3$ for the H_1 theory, and $\Delta=6/5$ for the H_0 theory.

[9] The number of 7-branes is $n_7=12(\Delta-1)/\Delta$.

results here match with Ref. [38], worked out from compactifications of a 6D SCFT):

S-fold Quotient	$24a$	$12c$	κ_G	$(24a, 12c, \kappa_G)\vert_{N=1}$
$\mathrm{IV}^*/\mathbb{Z}_2$	$36N^2 + 6N$	$18N^2 + 9N$	$6N$	(42,27,6)
$\mathrm{I}_0^*/\mathbb{Z}_2$	$24N^2$	$12N^2 + 3N$	$4N$	(24,15,4)
$\mathrm{IV}/\mathbb{Z}_2$	$18N^2 - 3N$	$9N^2$	$3N$	(15,9,3)
$\mathrm{I}_0^*/\mathbb{Z}_3$	$36N^2 - 12N$	$18N^2 - 3N$	$4N$	(24,15,4)
$\mathrm{III}/\mathbb{Z}_3$	$24N^2 - 12N$	$12N^2 - 5N$	$8N/3$	(12,7,8/3)
$\mathrm{IV}/\mathbb{Z}_4$	$36N^2 - 21N$	$18N^2 - 9N$	$3N$	(15,9,3)

Here, we denoted the theories using the fiber type before taking the quotient and the type of quotient applied. All the values obtained match with Ref. [32]. Note that the formulas for a and c match the $\mathcal{N} = 3$ case (obtained when $\Delta = 1$), provided that the $\mathcal{O}(1)$ term coming from the center of mass of the system of D3-branes is added back. For completeness, we can also list the same information in the cases with discrete torsion, again focusing on the rank-1 case. As expected, these are the same values displayed in Ref. [32] (see also Refs. [38,39]). We can determine these values in the following manner. We use the formulas in (6.4) to determine the leading and subleading contributions in N. The $\mathcal{O}(1)$ terms were determined in Ref. [39], in which it was argued that the parent theory should include $k(\Delta - 1)$ additional free hypermultiplets before the quotient, and we include them here verbatim.

S-fold Quotient	$24a$	$12c$	κ_G	$(24a, 12c, \kappa_G)\vert_{N=1}$
$\mathrm{IV}^*/\hat{\mathbb{Z}}_2$	$36N^2 + 42N + 4$	$18N^2 + 27N + 4$	$6N + 1$	(82,49,7)
$\mathrm{I}_0^*/\hat{\mathbb{Z}}_2$	$24N^2 + 24N + 2$	$12N^2 + 15N + 2$	$(4N + 1, 8N)$	(50,29,(5,8))
$\mathrm{IV}/\hat{\mathbb{Z}}_2$	$18N^2 + 15 + 1$	$9N^2 + 9N + 1$	$3N + 1$	(34,19,(4,-))
$\mathrm{I}_0^*/\hat{\mathbb{Z}}_3$	$36N^2 + 36N + 3$	$18N^2 + 21N + 3$	$12N + 2$	(75,42,14)
$\mathrm{III}/\hat{\mathbb{Z}}_3$	$24N^2 + 20N + 1$	$12N^2 + 11N + 1$	-	(45,24,-)
$\mathrm{IV}/\hat{\mathbb{Z}}_4$	$36N^2 + 33N + 2$	$18N^2 + 18N + 2$	$12N + 2$	(71,38,14)

In the above, we have included a "–" in some entries to reflect the fact that our present methods do not fix the level of the $U(1)$ flavor current.

VII. CONCLUSIONS

S-folds are a nonperturbative generalization of O3-planes which figure in the stringy construction of novel 4D quantum field theories. In this paper, we have proposed a procedure for how S-fold projection acts on the spectrum of string junctions attached to a stack of 7-branes and probe D3-branes. We have developed a general prescription for reading off the resulting flavor symmetry algebra under S-fold projection. This procedure leads to new realizations of many of the rank-1 4D $\mathcal{N} = 2$ SCFTs which arise from mass deformations and/or discrete gaugings of the rank-1 E_8 Minahan-Nemeschansky theory. We have also argued that the Seiberg-Witten curves associated with some of these theories provide an operational definition of F-theory in the presence of an S-fold background with discrete torsion. In the remainder of this section, we discuss some avenues for future investigation.

An interesting feature of our analysis is that there is a close correspondence between possible S-fold quotients of 7-branes, and admissible rank-1 4D $\mathcal{N} = 2$ SCFTs. That being said, there are a few examples which appear in Ref. [32], which seem to involve some additional ingredients. The Kodaira fiber types and flavor symmetries for these cases are $[\mathrm{II}^*, C_2]$, $[\mathrm{III}^*, C_1]$, $[\mathrm{IV}_1^*, \varnothing]$, and $[\mathrm{II}^*, C_1]$. In some cases, we can understand the origin of these theories as arising from a mass deformation of another theory, followed by an additional discrete quotient. That being said, it remains to be understood whether these operations can be fully realized purely in geometric terms.

There are in principle other ways to generate the same class of rank-1 4D $\mathcal{N} = 2$ SCFTs. In particular, compactifications of 6D SCFTs with suitable discrete twists provide an alternative way to realize many such examples (see, e.g., Ref. [38]). Since there is now a classification of possible F-theory backgrounds which can generate 6D SCFTs (see, e.g., Refs. [63–65] for a review), it would be interesting to systematically classify all possible ways of incorporating such discrete effects, thus providing a complementary viewpoint on many of the same questions.

In this paper, we have mainly focused on structures associated with 4D $\mathcal{N} = 2$ SCFTs. It would be quite natural to investigate the structure of related systems with only 4D $\mathcal{N} = 1$ supersymmetry. For example, starting from a 4D $\mathcal{N} = 2$ SCFT, deformations by nilpotent mass deformations often trigger flows to such theories [66–68].

O3-planes often play an important role in the construction of consistent type IIB string vacua. Having analyzed the effect of S-fold projection on the flavor symmetries of probe D3-branes in the vicinity of 7-branes, it is also natural to consider possible ways in which such ingredients might be used in compact F-theory models.

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APPENDIX A: BRANE MOTIONS

In this Appendix, we present an illustrative example for how to rearrange various $[p,q]$ 7-branes so that S-fold projection acts geometrically on the associated string junction states. This is best illustrated via pictures, so we mainly display the relevant figures here. Our starting point is an E_6 stack written as $A^5BC^2 \sim A^6XC \sim AAACAAAC$ (see Fig. 7), a D_4 stack written as $A^4BC \sim AACAAC$ (see Fig. 8), and an H_2 stack written as $A^3C \sim ACY^2 \sim ACAC \sim DADA$ (see Fig. 9).

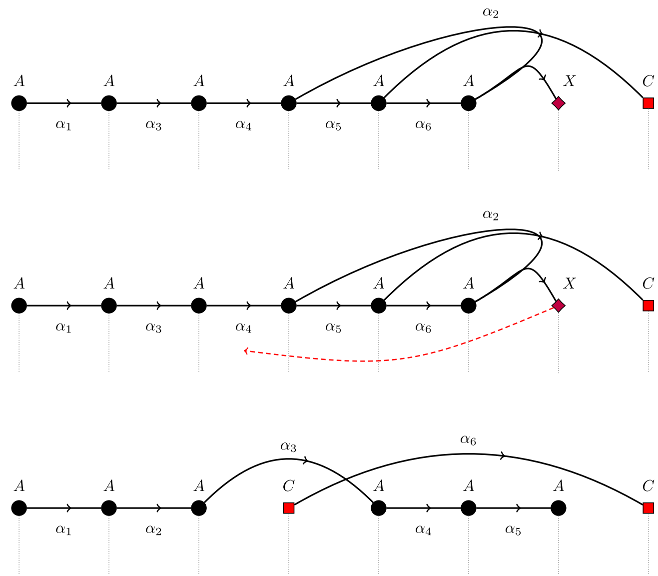

FIG. 7. Brane motion for E_6 7-branes to a configuration which is $\mathbb{Z}_2$ symmetric and thus amenable to a $\mathbb{Z}_2$ S-fold projection, i.e., an orientifold projection. In the figure, we also indicate how the X-brane is moved to accomplish this rearrangement to the $\mathbb{Z}_2$ symmetric configuration $AAACAAAC$.

APPENDIX B: EXPLICIT $\mathbb{Z}_2$ QUOTIENT OF E_6 WITHOUT TORSION

In this Appendix, we give the explicit root system of $\mathfrak{e}_6$ and show how only 48 roots survive the $\mathbb{Z}_2$ quotient (without torsion), corresponding exactly to the roots of an $\mathfrak{f}_4$ algebra. The roots of E_6, which are given in line (4.17), can be written as

$$\begin{aligned}
&\pm\{(-1,1,0,0,0,0,0,0),(0,0,0,-1,0,0,0,1),(0,-1,1,0,0,0,0,0),(0,0,-1,0,1,0,0,0),\\
&(0,0,0,0,-1,1,0,0),(0,0,0,0,0,-1,1,0),(-1,0,1,0,0,0,0,0),(0,0,-1,-1,1,0,0,1),\\
&(0,-1,0,0,1,0,0,0),(0,0,-1,0,0,1,0,0),(0,0,0,0,-1,0,1,0),(-1,0,0,0,1,0,0,0),\\
&(0,-1,0,-1,1,0,0,1),(0,0,-1,-1,0,1,0,1),(0,-1,0,0,0,1,0,0),(0,0,-1,0,0,0,1,0),\\
&(-1,0,0,-1,1,0,0,1),(-1,0,0,0,0,1,0,0),(0,-1,0,-1,0,1,0,1),(0,0,-1,-1,0,0,1,1),\\
&(0,-1,0,0,0,0,1,0),(-1,0,0,-1,0,1,0,1),(-1,0,0,0,0,0,1,0),(0,-1,-1,-1,1,1,0,1),\\
&(0,-1,0,-1,0,0,1,1),(-1,0,-1,-1,1,1,0,1),(-1,0,0,-1,0,0,1,1),(0,-1,-1,-1,1,0,1,1),\\
&(-1,-1,0,-1,1,1,0,1),(-1,0,-1,-1,1,0,1,1),(0,-1,-1,-1,0,1,1,1),\\
&(-1,-1,0,-1,1,0,1,1),(-1,0,-1,-1,0,1,1,1),(-1,-1,0,-1,0,1,1,1),\\
&(-1,-1,-1,-1,1,1,1,1),(-1,-1,-1,-2,1,1,1,2)\},
\end{aligned} \tag{B1}$$

where the vectors follow the order of the branes of Fig. 4. Namely, for instance, the highest root $(1,1,1,2,-1,-1,-1,-2)$ corresponds to the string junction $(a_1+a_2+a_3+2c_1-a_4-a_5-a_6-2c_2)$. For the projection, we define the matrix

$$Z = -\begin{pmatrix} 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0 \end{pmatrix}. \tag{B2}$$

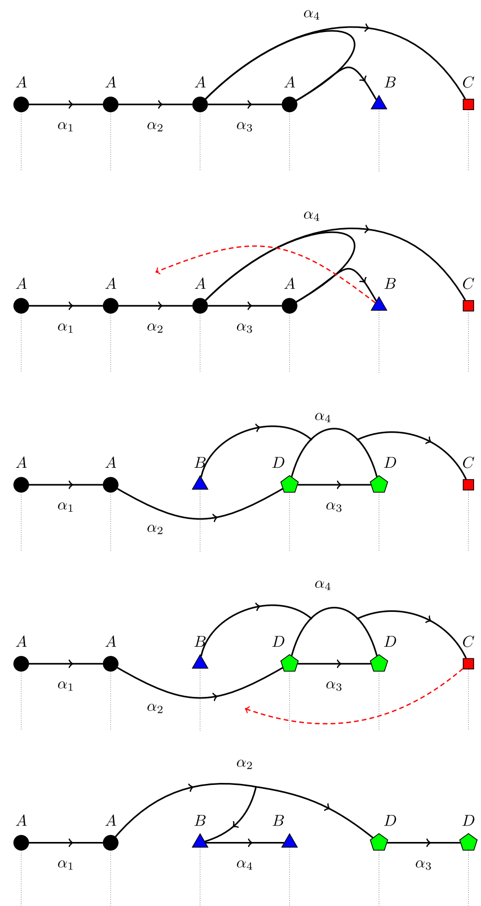

FIG. 8. Brane motion for D_4 7-branes to a configuration which is $\mathbb{Z}_3$ symmetric and thus amenable to a $\mathbb{Z}_3$ S-fold projection. In the figure, we start with the presentation of this brane system as the bound state A^4BC, which we then split up into three stacks of branes which are permuted under the $\mathbb{Z}_3$ group action.

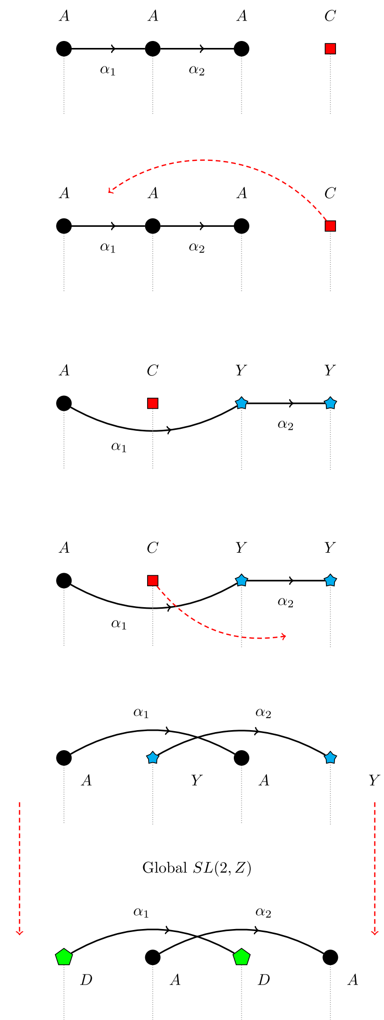

FIG. 9. Brane motion for H_2 7-branes to the configuration $DADA$ which is $\mathbb{Z}_4$ symmetric and thus amenable to an S-fold projection. In the last step of rearrangement, we apply an $SL(2,\mathbb{Z})$ transformation as indicated by Y, with notation as in Eq. (4.1).

We then map every root r in (B1) to $\frac{1}{2}(r + Z\cdot r)$. This results in the following 48 roots:

$$\pm\left\{(0,0,0,-1,0,0,0,1),(0,0,-1,0,0,0,1,0),\left(0,-\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},-\frac{1}{2},0\right),\right.$$
$$\left(-\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},-\frac{1}{2},0,0\right),(0,0,-1,-1,0,0,1,1),\left(0,-\frac{1}{2},-\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},0\right),$$
$$\left(-\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,-\frac{1}{2},0\right),(0,-1,0,0,0,1,0,0),\left(0,-\frac{1}{2},-\frac{1}{2},-1,0,\frac{1}{2},\frac{1}{2},1\right),$$
$$\left(-\frac{1}{2},0,-\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0\right),(0,-1,0,-1,0,1,0,1),\left(-\frac{1}{2},0,-\frac{1}{2},-1,\frac{1}{2},0,\frac{1}{2},1\right),$$
$$\left(-\frac{1}{2},-\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},0,0\right),(-1,0,0,0,1,0,0,0),(0,-1,-1,-1,0,1,1,1),$$

$$\left(-\frac{1}{2},-\frac{1}{2},0,-1,\frac{1}{2},\frac{1}{2},0,1\right),(-1,0,0,-1,1,0,0,1),\left(-\frac{1}{2},-\frac{1}{2},-1,-1,\frac{1}{2},\frac{1}{2},1,1\right),$$
$$(-1,0,-1,-1,1,0,1,1),\left(-\frac{1}{2},-1,-\frac{1}{2},-1,\frac{1}{2},1,\frac{1}{2},1\right),\left(-1,-\frac{1}{2},-\frac{1}{2},-1,1,\frac{1}{2},\frac{1}{2},1\right),$$
$$(-1,-1,0,-1,1,1,0,1),(-1,-1,-1,-1,1,1,1,1),(-1,-1,-1,-2,1,1,1,2)\bigg\}. \quad \text{(B3)}$$

From there, we can extract the four simple roots of F_4,

$$\left\{\left(\frac{1}{2},-\frac{1}{2},0,0,-\frac{1}{2},\frac{1}{2},0,0\right),\left(0,\frac{1}{2},-\frac{1}{2},0,0,-\frac{1}{2},\frac{1}{2},0\right),(0,0,1,0,0,0,-1,0),(0,0,0,1,0,0,0,-1)\right\}, \quad \text{(B4)}$$

corresponding exactly to the simple roots chosen in Eq. (4.20).

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