

Correspondence between the twisted $N = 2$ super-Yang-Mills and conformal Baulieu-Singer theories

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We characterize the correspondence between the twisted $N = 2$ super-Yang-Mills theory and the Baulieu-Singer topological theory quantized in the self-dual Landau gauges. While the first is based on an on-shell supersymmetry, the second is based on an off-shell Becchi-Rouet-Stora-Tyutin symmetry. Because of the equivariant cohomology, the twisted $N = 2$ in the ultraviolet regime and Baulieu-Singer theories share the same observables, the Donaldson invariants for 4-manifolds. The triviality of the Gribov copies in the Baulieu-Singer theory in these gauges shows that working in the instanton moduli space on the twisted $N = 2$ side is equivalent to working in the self-dual gauges on the Baulieu-Singer one. After proving the vanishing of the β function in the Baulieu-Singer theory, we conclude that the twisted $N = 2$ in the ultraviolet regime, in any Riemannian manifold, is correspondent to the Baulieu-Singer theory in the self-dual Landau gauges—a conformal gauge theory defined in Euclidean flat space.

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I. INTRODUCTION

Throughout the 1980s, based on the self-dual Yang-Mills equations introduced by A. Belavin *et al.* in their study of instantons [1], S. K. Donaldson discovered and described topological structures of polynomial invariants for smooth 4-manifolds [2–4]. The connection between the Floer theory for 3-manifolds [5,6] and Donaldson invariants for 4-manifolds with a nonempty boundary, i.e., that assumes values in Floer groups, has led to Atiyah’s conjecture [7,8]. In this conjecture, he proposed that the Floer homology must lead to a relativistic quantum field theory. This conjecture was the motivation for Witten’s topological quantum field theory (TQFT) in four dimensions, as Witten himself admits [8]. In Ref. [7], Atiyah showed that Floer’s results [6] can be seen as a version of a supersymmetric gauge theory. Answering Atiyah’s conjecture, Witten found a relativistic formulation of Ref. [7], capable of reproducing the Donaldson polynomials in the weak coupling limit of the twisted $N = 2$ Super-Yang-Mills (SYM) theory. This TQFT is commonly referred to as

the Donaldson-Witten theory (DW) in the Wess-Zumino gauge [9].

In practice, TQFTs have the power to reproduce topological invariants of the basis manifold as observables. The first one to obtain topological invariants from a quantum field theory was A. S. Schwarz in 1978 [10]. He showed that the Ray-Singer analytic torsion [11] can be represented as a partition function of the Abelian Chern-Simons (CS) action, which is invariant by diffeomorphisms. The Schwarz topological theory was the prototype of Witten theories in the 1980s. Indeed, the well-known Witten paper, in which he reproduces the Jones polynomials of knot theory [12], is the non-Abelian generalization of Schwarz’s results [10]. In this work, Witten is actually able to represent topological invariants of 3-manifolds as the partition function of the non-Abelian CS theory.

After Witten’s result [8], L. Baulieu and I. M. Singer (BS) showed in Ref. [13] that the same topological observables can be obtained from a gauge-fixed topological action. In such an approach, the Becchi-Rouet-Stora-Tyutin (BRST) symmetry [14–16] plays a fundamental role. It is not built through a linear transformation of a supersymmetric gauge theory, like Witten’s TQFT. It is built through a gauge-fixing procedure of a topological-invariant action, in such a way that the BRST operator naturally appears as nilpotent without requiring the use of equations of motion. The geometric interpretation of the BS theory is that the non-Abelian topological theory lies in a universal space graded as a sum of the ghost number and the form degree,

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where the vertical direction of this double complex is determined by the ghost number and the horizontal one is determined by the form degree. In this space, the topological BRST transformations are written in terms of a universal connection, and its curvature naturally explains the BS approach as a topological Yang-Mills theory with the same global observables of Witten's TQFT.

From the physical point of view, the motivation to study TQFTs comes from the mathematical tools of such theories, capable of revealing the topological structure of field theories that are independent of variations of the metric, and consequently of the background choice. One of the major obstacles to constructing a quantum theory of gravity is the integration over all metrics. The introduction of a topological phase in gravity would have the power to make a theory of gravity arise from a symmetry breaking mechanism of a background independent topological theory¹ [8,17]. On the other hand, we can investigate conformal properties of field theories via topological models. In three dimensions, for instance, the connection between the three-dimensional Chern-Simons theory and two-dimensional conformal theories is well known [12]. In four dimensions, TQFTs are intimately connected with the AdS/CFT correspondence [18,19]. More recently, motivated by string dualities, a topological gravity phase in the early Universe was proposed [20]. Such a phase could explain some puzzles concerning early Universe cosmology.

The fact that DW theory at the UV regime and BS theories share the same observables is a well-known result in the literature [8,13,21–23]. In this paper, we characterize the correspondence between DW TQFT and a conformal BS gauge theory at quantum level. While Witten's theory is based on the *twisted* version of the $N = 2$ super-Yang-Mills theory, the mentioned conformal theory is based on the Baulieu-Singer BRST gauge-fixing approach to a topological action [13]. In recent works [24–26], the existence of an extra bosonic symmetry was proved in the case of self-dual Landau gauges.² This bosonic symmetry relates the Faddeev-Popov and the topological ghost fields. Together with the known vector supersymmetry [27] and the vanishing three-level gauge propagator, one observes that the BS theory at the self-dual Landau gauges is indeed tree-level exact [26]. Essentially, the proof of this property is diagrammatic with some help of algebraic renormalization techniques [16]. This remarkable property inevitably implies a vanishing β function, since it does not receive quantum corrections. Nevertheless, an entire algebraic

proof was still lacking until now. It turns out that, for a complete proof of the vanishing of the β function of the BS theory in the self-dual Landau gauges, one extra property must be considered: the fact that the Gribov copies are inoffensive to the self-dual BS theory [28]. This property establishes that the self-dual BS theory is conformal, as it allows us to recover some discrete symmetries. The use of these symmetries makes it possible to eliminate the renormalization ambiguities discussed in Ref. [25]. With this information, we were able to establish the correspondence between self-dual BS theories (a conformal gauge theory defined in Euclidean spaces) for any value of the coupling constant and DW theory at the deep UV.

The paper is organized as follows. Section II contains an overview of the main properties of DW and BS theories. We introduce the main aspects of each approach, explaining how each one is constructed from different quantization schemes. As the quantum properties of the Witten's TQFT is well known in literature, we dedicate Sec. III to discussing the quantum properties of the BS theory in the self-dual Landau gauges. In Sec. IV, we analyze and compare the corresponding β functions of each model, after proving that the self-dual BS is conformal. Finally, in Sec. V, we describe the quantum correspondence between Witten and self-dual BS topological theories. Section VI contains our concluding remarks.

II. TOPOLOGICAL QUANTUM FIELD THEORIES

A topological quantum field theory on a smooth manifold is a quantum field theory which is independent of the metric on the basis manifold. Such a theory has no dynamics nor local degrees of freedom and is only sensitive to topological invariants which describe the manifold in which the theory is defined. The observables of a TQFT are naturally metric independent. The latter statement leads to the main property of topological field theories, namely, the metric independence of the observable correlation functions of the theory,

$$\frac{\delta}{\delta g_{\mu\nu}} \langle \mathcal{O}_{\alpha_1}(\varphi_i) \mathcal{O}_{\alpha_2}(\varphi_i) \cdots \mathcal{O}_{\alpha_p}(\varphi_i) \rangle = 0, \quad (2.1)$$

with

$$\begin{aligned} & \langle \mathcal{O}_{\alpha_1}(\varphi_i) \mathcal{O}_{\alpha_2}(\varphi_i) \cdots \mathcal{O}_{\alpha_p}(\varphi_i) \rangle \\ &= \mathcal{N} \int [D\varphi_i] \mathcal{O}_{\alpha_1}(\varphi_i) \mathcal{O}_{\alpha_2}(\varphi_i) \cdots \mathcal{O}_{\alpha_p}(\varphi_i) e^{-S[\varphi]}, \end{aligned} \quad (2.2)$$

where $g_{\mu\nu}$ is the metric tensor, $\varphi_i(x)$ are quantum fields, \mathcal{O}_α are the functional operators of the fields composing global observables, $S[\varphi]$ is the classical action, and \mathcal{N} is the appropriate normalization factor. A typical operator \mathcal{O}_α is integrated over the whole space in order to capture the

¹We must say that the introduction of such a topological phase is one of the intricate problems in topological quantum field theories, since one should develop a mechanism to break the topological symmetry.

²For simplicity, we will refer to the (anti-)self-dual Landau gauges, defined by instantons and anti-instantons configurations, see gauge condition (3.3.), only by the denotation *self-dual gauges*.

global structures of the manifold. Since there are no particles, the only nontrivial observables are of global nature [29,30].

As a particular result of (2.1), the partition function of a topological theory is itself a topological invariant,

$$\frac{\delta}{\delta g_{\mu\nu}} Z[J] = 0, \quad (2.3)$$

insofar as $Z[J]$ represents the expectation value of the vacuum in the presence of a external source, $Z[J] = \langle 0|0 \rangle_J$. As discussed in Ref. [31], if the action is explicitly independent of the metric, the topological theory is said to be of *Schwarz type*; otherwise, if the variation of the action with respect to the metric gives a “BRST-like”-exact term, one says the theory is of *Witten type*. More precisely, because δ is an infinitesimal transformation that denotes the symmetry of the action S which characterizes the observables of the model, then, if the following properties are satisfied,

$$\delta \mathcal{O}_\alpha(\varphi_i) = 0, \quad T_{\mu\nu}(\varphi_i) = \delta G_{\mu\nu}(\varphi_i), \quad (2.4)$$

where $T_{\mu\nu}$ is the energy-momentum tensor of the model,

$$\frac{\delta}{\delta g_{\mu\nu}} S = T_{\mu\nu}, \quad (2.5)$$

and $G_{\mu\nu}$ some tensor, the quantum field theory can be regarded as topological. Obviously, in this case, Eq. (2.3) is also satisfied, since the expectation value of the δ -exact term vanishes³ [8,13]. In fact, by using (2.4) and (2.5), and assuming that the measure $[D\varphi_i]$ is invariant under δ ,

$$\begin{aligned} & \frac{\delta}{\delta g_{\mu\nu}} \langle \mathcal{O}_{\alpha_1}(\varphi_i) \mathcal{O}_{\alpha_2}(\varphi_i) \cdots \mathcal{O}_{\alpha_p}(\varphi_i) \rangle \\ &= - \int [D\varphi_i] \mathcal{O}_{\alpha_1}(\varphi_i) \mathcal{O}_{\alpha_2}(\varphi_i) \cdots \mathcal{O}_{\alpha_p}(\varphi_i) T_{\mu\nu} e^{-S} \\ &= \langle \delta [\mathcal{O}_{\alpha_1}(\varphi_i) \mathcal{O}_{\alpha_2}(\varphi_i) \cdots \mathcal{O}_{\alpha_p}(\varphi_i) G_{\mu\nu}] \rangle \\ &= 0. \end{aligned} \quad (2.6)$$

In the above equation, we assumed that all \mathcal{O}_α are metric independent. Nevertheless, this is not a requirement of the theory. It is also possible to have a more general theory in which

$$\frac{\delta}{\delta g_{\mu\nu}} \mathcal{O}_\alpha = \delta \mathcal{Q}_{\mu\nu} \neq 0, \quad (2.7)$$

³The nilpotent δ operator works precisely as a BRST operator, and it is well known that expectation values of BRST-exact terms vanish. For a further analysis concerning renormalization properties, and the definition of physical observables, see Refs. [14,16,32].

which preserves the topological structure of $\delta_{g_{\mu\nu}} \langle \mathcal{O}_{\alpha_1} \cdots \mathcal{O}_{\alpha_p} \rangle = \langle \delta(\cdots) \rangle = 0$ [31]. Analogously to the BRST operator, Eq. (2.6) only makes sense if the δ operator is nilpotent.⁴

A. Donaldson-Witten theory

As mentioned in the Introduction, Witten constructed in Ref. [8] a four-dimensional generalization of Ref. [7], capable of reproducing the Donaldson invariants [2–4] in the weak coupling limit. Such a construction can be obtained from the twist transformation of the $N = 2$ SYM. Let us quickly revise some important features of such approach.

1. Twist transformation

The eight supersymmetric charges $(Q_\alpha^i, \bar{Q}_{j\dot{\alpha}})$ of $N = 2$ SYM theories obey the supersymmetry (SUSY) algebra

$$\begin{aligned} \{Q_\alpha^i, \bar{Q}_{j\dot{\alpha}}\} &= \delta_j^i (\sigma_\mu)_{\alpha\dot{\alpha}} \partial_\mu, \\ \{Q_\alpha^i, Q_{j\alpha}\} &= \{\bar{Q}_{i\dot{\alpha}}, \bar{Q}_{j\dot{\alpha}}\} = 0, \end{aligned} \quad (2.8)$$

where all indices $(i, j, \alpha, \dot{\alpha})$ run from 1 to 2. The indices (i, j) denote the internal $SU(2)$ symmetry of the $N = 2$ SYM action, and $(\alpha, \dot{\alpha})$ are Weyl spinor indices: α denotes right-handed spinors, and $\dot{\alpha}$ denotes left-handed ones. The fact that both indices equally run from 1 to 2 suggests the identification between spinor and supersymmetry indices,

$$i \equiv \alpha. \quad (2.9)$$

The $N = 2$ SYM action theory possesses a gauge group symmetry given by

$$SU_L(2) \times SU_R(2) \times SU_I(2) \times U_R(1), \quad (2.10)$$

where $SU_L(2) \times SU_R(2)$ is the rotation group, $SU_I(2)$ is the internal supersymmetry group labeled by i , and $U_R(1)$ is the so-called \mathcal{R} -symmetry defined by the supercharges $(Q_\alpha^i, \bar{Q}_{j\dot{\alpha}})$, which are assigned eigenvalues $(+1, -1)$, respectively. The identification performed in Eq. (2.9) amounts to a modification of the rotation group,

$$SU_L(2) \times SU_R(2) \rightarrow SU_L(2) \times SU_R(2)', \quad (2.11)$$

where $SU_R(2)'$ is the diagonal sum of $SU_R(2)$ and $SU_I(2)$. The twisted global symmetry of $N = 2$ SYM takes the form $SU_L(2) \times SU_R(2)' \times U_R(1)$, with the corresponding twisted supercharges

$$Q_\alpha^i \rightarrow \bar{Q}_{\alpha\dot{\alpha}}, \quad \bar{Q}_{i\dot{\alpha}} \rightarrow \bar{Q}_{\alpha\dot{\alpha}}, \quad (2.12)$$

⁴In Donaldson-Witten theory, for instance, such an operator is on-shell nilpotent, i.e., $\delta^2 = 0$ by using the equations of motion.

which can be rearranged as

$$\frac{1}{\sqrt{2}}\epsilon^{\alpha\beta}Q_{\alpha\beta}\equiv\delta, \quad (2.13)$$

$$\frac{1}{\sqrt{2}}\bar{Q}^{\dot{\alpha}\dot{\alpha}}(\sigma_\mu)^{\dot{\alpha}\alpha}\equiv\delta_\mu, \quad (2.14)$$

$$\frac{1}{\sqrt{2}}(\sigma_{\mu\nu})^{\dot{\alpha}\alpha}Q_{\dot{\alpha}\alpha}\equiv d_{\mu\nu}, \quad (2.15)$$

where we adopt the conventions for $\epsilon^{\alpha\beta}$, $(\sigma^\mu)^{\dot{\alpha}\alpha}$, and $(\sigma_{\mu\nu})^{\dot{\alpha}\alpha}$ as the same as in Ref. [33]. The operator $d_{\mu\nu}$ is manifestly self-dual due to the structure of $\sigma_{\mu\nu}$,

$$d_{\mu\nu}=\frac{1}{2}\epsilon_{\mu\nu\lambda\rho}d^{\lambda\rho}, \quad (2.16)$$

reducing to 3 the number of its independent components. The operators δ , δ_μ , and $\delta_{\mu\nu}$ possess eight independent components into which the eight original supercharges ($Q_{\beta\alpha}$, $\bar{Q}_{\dot{\alpha}\dot{\alpha}}$) are mapped. These operators obey the following twisted supersymmetry algebra:

$$\delta^2=0, \quad (2.17)$$

$$\{\delta, \delta_\mu\}=\partial_\mu, \quad (2.18)$$

$$\{\delta_\mu, \delta_\nu\}=\{d_{\mu\nu}, \delta\}=\{d_{\mu\nu}, d_{\lambda\rho}\}=0, \quad (2.19)$$

$$\{\delta_\mu, d_{\lambda\rho}\}=-(\epsilon_{\mu\lambda\rho\sigma}\partial^\sigma+g_{\mu\lambda}\partial_\rho-g_{\mu\rho}\partial_\lambda). \quad (2.20)$$

The nilpotent scalar supersymmetry charge δ defines the cohomology of Witten's TQFT, as its observables appear as cohomology classes of δ , which is invariant under a generic differential manifold. It is implicit in the anticommutation relation (2.18) the topological nature of the model, as it allows us to write the common derivative as a δ -exact term.

The gauge multiplet of the $N=2$ SYM in Wess-Zumino gauge is given by the fields

$$(A_\mu, \psi_\alpha^i, \bar{\psi}_{\dot{\alpha}}^i, \varphi, \bar{\varphi}), \quad (2.21)$$

where ψ_α^i is a Majorana spinor (the supersymmetric partner of the gauge connection A_μ) and φ is a scalar field, all of them belonging to the adjoint representation of the gauge group. The *twist transformation* is defined by the identification (2.9) and thus only acts on the fields $(\psi_\mu^i, \bar{\psi}_\mu^i)$, leaving the bosonic fields $(A_\mu, \varphi, \bar{\varphi})$ unaltered. Explicitly, the twist transformation is given by the linear transformations⁵

⁵Notation: $\Phi_{(\alpha\beta)}=\Phi_{\alpha\beta}+\Phi_{\beta\alpha}$ and $\Phi_{[\alpha\beta]}=\Phi_{\alpha\beta}-\Phi_{\beta\alpha}$.

$$\psi_\beta^i\rightarrow\psi_{\alpha\beta}=\frac{1}{2}(\psi_{(\alpha\beta)}+\psi_{[\alpha\beta]}), \quad (2.22)$$

$$\bar{\psi}_\alpha^i\rightarrow\bar{\psi}_{\dot{\alpha}\dot{\alpha}}\rightarrow\psi_\mu=(\sigma_\mu)^{\dot{\alpha}\alpha}\bar{\psi}_{\dot{\alpha}\alpha}, \quad (2.23)$$

together with

$$\psi_{(\alpha\beta)}\rightarrow\chi_{\mu\nu}=(\sigma_{\mu\nu})^{\alpha\beta}\psi_{(\alpha\beta)}, \quad (2.24)$$

$$\psi_{[\alpha\beta]}\rightarrow\eta=\epsilon^{\alpha\beta}\psi_{[\alpha\beta]}. \quad (2.25)$$

The *twist* consists of a mapping of degrees of freedom. The field $\bar{\psi}_{\dot{\alpha}\dot{\alpha}}$ has four independent components as $(\alpha, \dot{\alpha})=\{1, 2\}$ and is mapped into the field ψ_μ that also has four independent components of the path integral, as the Lorentz index $\mu=\{1, 2, 3, 4\}$ in four dimensions. In the other mappings (2.22), (2.24) and (2.25), the same occurs, as the symmetric part of $\psi_{\alpha\beta}$, i.e., $\psi_{(\alpha\beta)}$ has three independent components mapped into the self-dual field $\chi_{\mu\nu}$, and the antisymmetric part, $\psi_{[\alpha\beta]}$, with only one independent component, into η , a scalar field. We must note that $(\psi_\mu, \chi_{\mu\nu}, \eta)$ are anticommuting field variables due to their spinor origin.

Because it is a linear transformation, the twist simply corresponds to a change of variables with trivial Jacobian that could be absorbed in the normalization factor; in other words, both theories (before and after the twist) are perturbatively indistinguishable. Finally, twisting the $N=2$ SYM action ($S_{\text{SYM}}^{N=2}$) [8,34], in flat Euclidean space, we obtain the Witten four-dimensional topological Yang-Mills action (S_W),

$$S_{\text{SYM}}^{N=2}[A_\mu, \psi_\alpha^i, \bar{\psi}_{\dot{\alpha}}^i, \varphi, \bar{\varphi}]\rightarrow S_W[A_\mu, \psi_\mu, \chi_{\mu\nu}, \bar{\varphi}, \varphi], \quad (2.26)$$

where

$$S_W=\frac{1}{g^2}\text{Tr}\int d^4x\left(\frac{1}{2}F_{\mu\nu}^+F^{+\mu\nu}-\chi_{\mu\nu}(D_\mu\psi_\nu-D_\nu\psi_\mu)^+\right. \\ \left.+ \eta D_\mu\psi^\mu - \frac{1}{2}\bar{\varphi}D_\mu D^\mu\varphi + \frac{1}{2}\bar{\varphi}\{\psi_\mu, \psi_\mu\} \right. \\ \left. - \frac{1}{2}\varphi\{\chi_{\mu\nu}, \chi_{\mu\nu}\} - \frac{1}{8}[\varphi, \eta]\eta - \frac{1}{32}[\varphi, \bar{\varphi}][\varphi, \bar{\varphi}]\right), \quad (2.27)$$

wherein $F_{\mu\nu}^+$ is the self-dual field⁶

$$F_{\mu\nu}^+=F_{\mu\nu}+\tilde{F}_{\mu\nu}, \quad (\tilde{F}_{\mu\nu}^+=F_{\mu\nu}^+), \quad (2.28)$$

with $\tilde{F}_{\mu\nu}=\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}$, and, analogously,

⁶Following Refs. [8,34], we are considering the positive sign, that corresponds to anti-instantons in the vacuum. A similar construction can be done for instantons, only by changing the sign.

$$(D_\mu \psi_\nu - D_\nu \psi_\mu)^+ = D_\mu \psi_\nu - D_\nu \psi_\mu + \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} (D_\alpha \psi_\beta - D_\beta \psi_\alpha), \quad (2.29)$$

with $D_\mu \equiv \partial_\mu - g[A_\mu, \cdot]$ being the covariant derivative in the adjoint representation of the Lie group G , with g being the coupling constant. The Witten action⁷ (2.27) possesses the usual Yang-Mills gauge invariance, denoted by⁸

$$\delta_{\text{gauge}}^{\text{YM}} S_W = 0. \quad (2.30)$$

The theory, however, does not possess gauge anomalies [36]. The symmetry that defines the cohomology of the theory, also known as *equivariant cohomology*, is the fermionic scalar supersymmetry

$$\begin{aligned} \delta A_\mu &= -\varepsilon \psi_\mu, & \delta \varphi &= 0, & \delta \bar{\varphi} &= 2i\varepsilon \eta, & \delta \eta &= \frac{1}{2} \varepsilon [\varphi, \bar{\varphi}], \\ \delta \psi_\mu &= -\varepsilon D_\mu \varphi, & \delta \chi_{\mu\nu} &= \varepsilon F^+, \end{aligned} \quad (2.31)$$

where ε is the supersymmetry fermionic parameter that carries no spin, ensuring that the propagating modes of commuting and anticommuting fields have the same helicities.⁹ This symmetry relates bosonic and fermionic degrees of freedom, which are identical—an inheritance of the supersymmetric original action.¹⁰ The price of working in Wess-Zumino gauge is the fact that, disregarding gauge

⁷Technically, the Witten action (2.27) is the four-dimensional generalization of the nonrelativistic topological quantum field theory [7], whose construction is based on the Floer theory for 3-manifolds \mathcal{M}_{3D} , in which the Chern-Simons action is taken as a Morse function on \mathcal{M}_{3D} ; see Floer's original paper [5]. In a few words, the critical points of CS action (W_{CS}) yield the curvature free configurations, as $\frac{\delta W_{\text{CS}}}{\delta A_i^a} = -\frac{1}{2} \varepsilon^{ijk} F^{jk}$, where F^{jk} is the 2-form curvature in three dimensions, which defines the gradient flows of a Morse function; see Ref. [17]. In the supersymmetric formulation of Ref. [7], the Hamiltonian (H) is obtained via the “supersymmetric charges” d_i and d_i^* , from the well-known relation $d_i d_i^* + d_i^* d_i = 2H$, see Ref. [35], whereby $d_i = e^{-tW_{\text{CS}}} d e^{tW_{\text{CS}}}$ and $d_i^* = e^{tW_{\text{CS}}} d^* e^{-tW_{\text{CS}}}$, for a real number t , with d being the exterior derivative on the space of all connections \mathcal{A} , according to the transformation $\delta A_i^a = \psi_i^a$, and d^* being its dual. Before identifying the twist transformation, this formulation (in four dimensions) was employed by Witten in his original paper [12] to obtain the relativistic topological action (2.27).

⁸The typical Yang-Mills transformations of all fields are implicit in this notation, where the gauge field transforms as $A_\mu^a = S^{-1} A_\mu^a S + S^{-1} \partial_\mu S$ with $S \in SU(N)$.

⁹Precisely, the propagating modes of A_μ have helicities $(1, -1)$. For the propagating modes of $(\varphi, \bar{\varphi})$, the helicities are $(0, 0)$. And the fermionic fields (η, ψ, χ) carry helicities $(1, -1, 0, 0)$.

¹⁰The action S_W is also invariant under global scaling with dimensions $(1, 0, 2, 2, 1, 2)$ for $(A, \varphi, \bar{\varphi}, \eta, \psi, \chi)$, respectively, and preserves an additive U symmetry for the assignments $(0, 2, -2, -1, 1, -1)$. In the BRST formalism, the latter is naturally recognized as ghost numbers, as we will see later on.

transformations, one needs to use the equations of motion to recover the nilpotency of δ [30]. This characterizes the DW theory as an *on-shell approach*. One can easily verify that (see Ref. [8])

$$\delta^2 \Phi = 0, \quad \text{for } \Phi = \{A, \psi, \varphi, \bar{\varphi}, \eta\}, \quad (2.32)$$

but

$$\delta^2 \chi = \text{equations of motion}. \quad (2.33)$$

Considering the result of Eq. (2.33), hereafter, we will say that the Witten fermionic symmetry is *on-shell* nilpotent. This symmetry is associated to an on-shell nilpotent “BRST charge,” Q , according to the definition of the δ variation of any functional \mathcal{O} as a transformation on the space of all functionals of field variables, namely,

$$\delta \mathcal{O} = -i\varepsilon \cdot \{Q, \mathcal{O}\}, \quad \text{such that } Q^2|_{\text{on-shell}} = 0. \quad (2.34)$$

To verify that Witten theory is valid in curved spacetimes, it is worth noting that the commutators of covariant derivatives always appear acting in the scalar field φ , like in $\delta \text{Tr}\{D_\mu \psi_\nu \cdot \bar{\chi}_{\mu\nu}\} = \frac{1}{2} \text{Tr}([D_\mu, D_\nu] \varphi \cdot \bar{\chi}^{\mu\nu})$, so the Riemann tensor does not appear, and the theory could be extended to any Riemannian manifold. In practice, one can simply take

$$\int d^4x \rightarrow \int d^4x \sqrt{g}, \quad (2.35)$$

in order to work in a curved spacetime. Such a theory has the property of being invariant under infinitesimal changes in the metric. This property characterizes the Witten model as a topological quantum field theory. Such a feature is verified by the fact that the energy-momentum tensor can be written as the anticommutator

$$T_{\mu\nu} = \{Q, V_{\mu\nu}\}, \quad (2.36)$$

which means that $T_{\mu\nu}$ is an on-shell BRST-exact term,

$$T_{\mu\nu} = \delta V_{\mu\nu}, \quad \delta^2|_{\text{on-shell}} = 0, \quad (2.37)$$

with

$$\begin{aligned} V_{\mu\nu} &= \frac{1}{2} \text{Tr} \left\{ F_{\mu\sigma} \chi_\nu^\sigma + F_{\nu\sigma} \chi_\mu^\sigma - \frac{1}{2} g_{\mu\nu} F_{\sigma\rho} \chi^{\sigma\rho} \right\} \\ &\quad + \frac{1}{4} g_{\mu\nu} \text{Tr} \eta [\varphi, \bar{\varphi}] \\ &\quad + \frac{1}{2} \text{Tr} \{ \psi_\mu D^\nu \bar{\varphi} + \psi_\nu D^\mu \bar{\varphi} - g_{\mu\nu} \psi_\sigma D^\sigma \bar{\varphi} \}. \end{aligned} \quad (2.38)$$

Equation (2.37) together with $\delta S_W = 0$ means that Witten theory satisfies (on-shell) the second condition displayed in Eq. (2.4), which allows us to say that S_W

automatically leads to a four-dimensional topological field model. In other words,

$$\begin{aligned} \frac{\delta}{\delta g_{\mu\nu}} Z_W &= \int \mathcal{D}\Phi \left(-\frac{\delta}{\delta g_{\mu\nu}} \mathcal{S}_W \right) \exp(-\mathcal{S}_W) \\ &= -\frac{1}{g^2} \left\langle \left\{ \mathcal{Q}, \int_M d^4x \sqrt{g} V_{\mu\nu} \right\} \right\rangle = 0, \end{aligned} \quad (2.39)$$

as all expected values of a δ -exact term vanish. It remains to know which kind of topological/differential invariants can be represented by the Feynman path integral of Witten's TQFT. As we know, it will naturally reproduce the Donaldson invariants for four-manifolds.

2. Donaldson polynomials in the weak coupling limit

An important feature of the twisted $N = 2$ SYM is the fact that the theory can be interpreted as quantum fluctuations around classical instanton configurations. To find the nontrivial classical minima, one must note that the pure gauge field terms in S_W are

$$S_W^{\text{gauge}}[A] = \frac{1}{2} \text{Tr} \int d^4x (F_{\mu\nu} + \tilde{F}_{\mu\nu})(F^{\mu\nu} + \tilde{F}^{\mu\nu}), \quad (2.40)$$

which is positive semidefinite and only vanishes if the field strength $F_{\mu\nu}$ is anti-self-dual,

$$F_{\mu\nu} = -\tilde{F}_{\mu\nu}, \quad (2.41)$$

the same nontrivial vacuum configuration that minimizes the Yang-Mills action in the case of anti-instantons fields. Hence, Witten's action has a nontrivial classical minima for $F = -\tilde{F}$ and $\Phi_{\text{other fields}} = 0$. Being precise, the evaluation of the twisted $N = 2$ SYM path integral computes quantum corrections to classical anti-instantons solutions.

Another important property of Witten's theory is the invariance under infinitesimal changes in the coupling constant. The variation of Z_W with respect to g^2 yields, for similar reasons as in (2.39),

$$\delta_{g^2} Z_W = \delta_{g^2} \left(-\frac{1}{g^2} \right) \langle \{ \mathcal{Q}, X \} \rangle = 0, \quad (2.42)$$

where

$$X = \frac{1}{4} \text{Tr} F_{\mu\nu} \chi^{\mu\nu} + \frac{1}{2} \text{Tr} \psi_\mu D^\mu \bar{\varphi} - \frac{1}{4} \text{Tr} \eta [\varphi, \bar{\varphi}]. \quad (2.43)$$

The Witten partition function, Z_W , is independent of the gauge coupling g^2 ; therefore, we can evaluate Z_W in the weak coupling limit, i.e., in the regime of very small g^2 , in which Z_W is dominated by the classical minima.

The instanton moduli space $\mathcal{M}_{k,N}$ is defined to be the space of all solutions to $F = \tilde{F}$ for an instanton with a

giving winding number k and gauge group $SU(N)$. By perturbing $F = \tilde{F}$ nearby the solution A_μ via a gauge transformation $A_\mu \rightarrow A_\mu + \delta A_\mu$, we obtain the self-duality equation

$$D_\mu \delta A_\nu + D_\nu \delta A_\mu + \varepsilon_{\mu\nu\alpha\beta} D^\alpha \delta A^\beta = 0. \quad (2.44)$$

The solutions of the above equation are called zero modes. Requiring the orthogonal gauge-fixing condition,¹¹ $D_\mu A_\mu = 0$, one gets

$$D_\mu (\delta A_\mu) = 0. \quad (2.45)$$

The Atiyah-Singer index theorem [37,38] counts the number of solutions to Eqs. (2.44) and (2.45). In Euclidean space-times, for instance, the index theorem gives, see Ref. [39],

$$\dim(\mathcal{M}) = 4kN, \quad (2.46)$$

where the modes due to global gauge transformations of the group were included. Looking at fermion zero modes, the χ equation for S_W gives

$$D_\mu \psi_\nu + D_\nu \psi_\mu + \varepsilon_{\mu\nu\alpha\beta} D^\alpha \psi^\beta = 0, \quad (2.47)$$

and from the η equation,

$$D_\mu \psi^\mu = 0. \quad (2.48)$$

These are the same equations obtained for the gauge perturbation around an instanton in the orthogonal gauge fixing, so the number of ψ zero modes is also given by $\mathcal{M}_{k,N}$.¹² To get a nonvanishing partition function, Witten assumed that the moduli space consists of discrete, isolated instantons. Precisely, he assumed that the dimension of the moduli space vanishes.¹³

In expanding around an isolated instanton, in the weak coupling limit $g^2 \rightarrow 0$, the action is reduced to quadratic terms,

¹¹This condition is equivalent to the Landau gauge, as $D_\mu A_\mu = \partial_\mu A_\mu$. It is important to note that one can promote ∂_μ to D_μ in this case, in order to show that A_μ and ψ_μ obey the same equations.

¹²As Witten himself admits in his paper [8], "this relation between the fermion equations and the instanton moduli problem was the motivation for introducing precisely this collection of fermions."

¹³Otherwise, a net violation of the $U(1)$ global symmetry of S_W occurs, and Z_W vanishes due to the fermion zero modes; see Refs. [8,40]. The dimension of the instanton moduli spaces depends on the bundle, E , and the manifold, M . In the $SU(2)$ gauge theory, it can be written as $\dim(\mathcal{M}) = 8k(E) - \frac{3}{2}(\chi(M) + \sigma(M))$, where $k(E)$ is the first Pontryagin (or winding) number of the bundle E and $\chi(M)$ and $\sigma(M)$ are the Euler characteristic and signature of M [38]. [For $M = R^4$, $\chi(R^4) = \sigma(R^4) = 0$.] Thus, one can choose a suitable E and M in order to get a vanishing dimension, $\dim(\mathcal{M}) = 0$.

$$S_W^{(2)} = \int_M d^4x \sqrt{g} (\Phi^{(b)} D_B \Phi^{(b)} + i \Psi^{(f)} D_F \Psi^{(f)}), \quad (2.49)$$

where $\Phi^{(b)} \equiv \{A, \varphi, \bar{\varphi}\}$ are the bosonic fields and $\Psi^{(f)} \equiv \{\eta, \psi, \chi\}$ are the fermionic ones. The Gaussian integral over all fields gives

$$Z_W|_{g^2 \rightarrow 0} = \frac{\text{Pfaff}(D_F)}{\sqrt{\det(D_B)}}, \quad (2.50)$$

where $\text{Pfaff}(D_F)$ is the Pfaffian of D_F , i.e., the square root of the determinant of D_F up to a sign. The supersymmetry relates the eigenvalues of the operators D_B and D_F . The relation is a standard result in instanton calculus [41], which yields

$$Z_W|_{g^2 \rightarrow 0} = \pm \prod_i \frac{\lambda_i}{\sqrt{|\lambda_i|^2}}, \quad (2.51)$$

with i running over all nonzero eigenvalues of D_B (D_F). Therefore, for the k th isolated instanton, $Z_W^{(k)} = (-1)^{n_k}$, where $n_k = 0$ or 1 according to the orientation convention of the moduli space (Donaldson proved the orientability of the moduli space, i.e., that the definition of the sign of $\text{Pfaff}(D_F)$ is consistent, without global anomalies [4,8]). In the end, summing over all isolated instantons,

$$Z_W|_{g^2 \rightarrow 0} = \sum_k (-1)^{n_k}, \quad (2.52)$$

which is precisely one of topological invariant for 4-manifolds described by Donaldson.

The other metric independent observables are constructed in the context of Eq. (2.7). These observables can be generated by exploring the descent equations defined by the equivariant cohomology, i.e., the supersymmetry δ cohomology. For that, with U_i being the global charge of the operator \mathcal{O}_i (see footnote 10), it must be understood that, for the observable $\prod_i \mathcal{O}_i$, $\dim(\mathcal{M}) = \sum_i U_i$.¹⁴ The simplest δ -invariant operator, which does not depend explicitly on the metric and cannot be written as $\delta(X) = \{\mathcal{Q}, X\}$ (due to the scaling dimensions), is

$$W_0(x) = \frac{1}{2} \text{Tr} \varphi^2(x), \quad U(W_0) = 4. \quad (2.53)$$

Although W_0 is not a δ -exact operator, taking the derivative of W_0 with respect of the coordinates, we find

$$\frac{\partial}{\partial x_\mu} W_0 = i \{\mathcal{Q}, \text{Tr} \varphi \psi_\mu\}, \quad (2.54)$$

which is δ exact. Using the exterior derivative, d , we can rewrite (2.54) as

$$dW_0 = i \{\mathcal{Q}, W_1\}, \quad (2.55)$$

where W_1 is the 1-form $\text{Tr}(\varphi \psi_\mu) dx^\mu$. A straightforward calculation gives

$$dW_1 = i \{\mathcal{Q}, W_2\}, \quad dW_2 = i \{\mathcal{Q}, W_3\}, \quad (2.56)$$

$$dW_3 = i \{\mathcal{Q}, W_4\}, \quad dW_4 = 0, \quad (2.57)$$

with

$$W_2 = \text{Tr} \left(\frac{1}{2} \psi \wedge \psi + i \varphi \wedge F \right), \quad (2.58)$$

$$W_3 = i \text{Tr} \psi \wedge F, \quad (2.59)$$

$$W_4 = -\frac{1}{2} \text{Tr} F \wedge F, \quad (2.60)$$

where \wedge is the wedge product; the total charge is $U = 4 - k$ for each W_k ; and φ , ψ , and F are 0-, 1-, and 2-forms on M , respectively. F is the field strength in the p -form formalism,¹⁵ defined in Eq. (2.73). Considering now the integral

$$I(\gamma) = \int_\gamma W_k, \quad (2.61)$$

with γ being a k -dimensional homology cycle on M , we have

$$\{\mathcal{Q}, I\} = \int_\gamma \{\mathcal{Q}, W_k\} = i \int_\gamma dW_{k-1} = 0. \quad (2.62)$$

It proves that $I(\gamma)$ is δ -invariant and, then, a possible observable. To be a global observable of the topological theory, we just have to prove that $I(\gamma)$ is BRST exact, which can be immediately verified by taking γ as the boundary $\partial\beta$ and applying the Stokes theorem,

$$I(\gamma) = \int_{\partial\beta} W_k = \int_\beta dW_k = i \left\{ \mathcal{Q}, \int_\beta W_{k+1} \right\}. \quad (2.63)$$

We conclude, from Eqs. (2.62) and (2.63), that $I(\gamma)$ are the global observables of the model as their expectation values produce metric independent quantities, i.e., topological invariants for 4-manifolds. Finally, the general path integral representation of Donaldson invariants in Witten's TQFT takes the form

¹⁴To construct topological invariants, the net U charge must equal the dimension of the moduli space; see Refs. [8,17].

¹⁵For the definitions and conventions concerning the p -form formalism used here, see Sec. II B 2.

$$Z(\gamma_1, \dots, \gamma_r) = \int \mathcal{D}\Phi \left(\prod_i \int_{\gamma_i} W_{k_i} \right) e^{-S_W} \\ = \left\langle \prod_i \int_{\gamma_i} W_{k_i} \right\rangle, \quad (2.64)$$

with moduli space dimension

$$\dim(\mathcal{M}) = \sum_i^r (4 - k_r). \quad (2.65)$$

One of the beautiful results is the appearing of W_4 in the descent equations. Up to a sign, the observable

$$\int_{\gamma} W_4 = -\frac{1}{2} \int_{\gamma} F \wedge F \quad (2.66)$$

is the Pontryagin action written in the formalism of p-forms. The Pontryagin action, a well-known topological invariant of 4-manifolds, naturally appears as one of the Donaldson polynomials, with a trivial winding number in this case, since $U(W_4) = 0$, and consequently the dimension of the moduli space vanishes.

B. Baulieu-Singer off-shell approach

Let us now turn to the main properties Baulieu-Singer approach for TQFTs [13], which is based on an off-shell BRST symmetry, built from the gauge fixing of an original action composed of topological invariants.

1. BRST symmetry in topological gauge theories

The four-dimensional spacetime is assumed to be Euclidean and flat.¹⁶ The non-Abelian topological action $S_0[A]$ in four-dimensional spacetime representing the topological invariants is the Pontryagin action¹⁷

$$S_0[A] = \frac{1}{2} \int d^4x F_{\mu\nu}^a \tilde{F}_{\mu\nu}^a, \quad (2.67)$$

which labels topologically inequivalent field configurations, as $S_0[A] = 32\pi^2 k$, in which k is the topological charge known as the winding number. We must note that the Pontryagin action has two different gauge symmetries to be fixed, these are:

(i) the gauge field symmetry,

$$\delta A_{\mu}^a = D_{\mu}^{ab} \omega^b + \alpha_{\mu}^a, \quad (2.68)$$

(ii) the topological parameter symmetry,

$$\delta \alpha_{\mu}^a = D_{\mu}^{ab} \lambda^b, \quad (2.69)$$

where $D_{\mu}^{ab} \equiv \delta^{ab} \partial_{\mu} - g f^{abc} A_{\mu}^c$ are the components of the covariant derivative in the adjoint representation of the Lie group G ; f^{abc} are the structure constants of G ; and ω^a , α_{μ}^a , and λ^a are the infinitesimal G -valued gauge parameters. As a consequence of (2.69), the field strength also transforms as a gauge field,¹⁸

$$\delta F_{\mu\nu}^a = -g f^{abc} \omega^b F_{\mu\nu}^c + D_{[\mu}^{ab} \alpha_{\nu]}^b. \quad (2.70)$$

The first parameter (ω^a) reflects the usual Yang-Mills symmetry of $S[A]$, whereas the second one (α_{μ}^a) is the topological shift associated to the fact that $S[A]$ is a topological invariant, i.e., invariant under continuous deformations. The third gauge parameter (λ^a) is due to an internal ambiguity present in the gauge transformation of the gauge field (2.68). The transformation of the gauge field is composed by two independent symmetries. If the space has a boundary, the parameter $\alpha_{\mu}^a(x)$ must vanish at this boundary but not $\omega^a(x)$, which explains the internal ambiguity described by (2.69) in which $\alpha_{\mu}^a(x)$ is absorbed into $\omega^a(x)$, and not the other way around [13].

Following the BRST quantization procedure, the gauge parameters present in the gauge transformations (2.68)–(2.70) are promoted to ghost fields: $\omega^a \rightarrow c^a$, $\alpha_{\mu}^a \rightarrow \psi_{\mu}^a$, and $\lambda^a \rightarrow \varphi^a$; c^a is the well-known Faddeev-Popov (FP) ghost; ψ_{μ}^a is a topological fermionic ghost; and φ^a is a bosonic ghost. The corresponding BRST transformations are

$$s A_{\mu}^a = -D_{\mu}^{ab} c^b + \psi_{\mu}^a, \\ s c^a = \frac{g}{2} f^{abc} c^b c^c + \varphi^a, \\ s \psi_{\mu}^a = g f^{abc} c^b \psi_{\mu}^c + D_{\mu}^{ab} \varphi^b, \\ s \varphi^a = g f^{abc} c^b \varphi^c, \quad (2.71)$$

from which one can easily check the nilpotency of the BRST operator,

$$s^2 = 0, \quad (2.72)$$

by applying two times the BRST operator s on the fields. Naturally, $S_0[A]$ is invariant under the BRST transformations (2.71). The nilpotency property of s defines the

¹⁶Throughout this work, we consider flat Euclidean spacetime. Although the topological action is background independent, the gauge-fixing term entails the introduction of a background. Ultimately, background independence is recovered at the level of correlation functions due to BRST symmetry [13,42,43].

¹⁷It is worth mentioning that the action $S_0[A]$ could encompass a wide range of topological gauge theories. The Pontryagin action is the most common case because it can be defined for all semisimple Lie groups. Nevertheless, other cases can also be considered. For instance, Gauss-Bonnet and Nieh-Yang topological gravities can be formulated for orthogonal groups [44].

¹⁸The antisymmetrization index notation here employed means that, for a generic tensor, $S_{[\mu\nu]} = S_{\mu\nu} - S_{\nu\mu}$.

cohomology of the theory, which allows for the gauge fixing of the Pontryagin action in a BRST-invariant fashion. Furthermore, such a property is related to the geometric structure of the off-shell BRST transformations in non-Abelian topological gauge theories.

2. Geometric interpretation

To simplify equations in the following sections, we will employ again the formalism of differential forms. In this formalism, the fields c and φ are 0-forms, ψ is the 1-form $\psi_\mu dx_\mu$, and F is the 2-form

$$F = dA + A \wedge A = \frac{1}{2} F_{\mu\nu} dx_\mu \wedge dx_\nu, \quad (2.73)$$

where A is the 1-form $A_\mu dx^\mu$, \wedge is the *wedge product* which indicates that the tensor product is completely antisymmetric, and d is the exterior derivative.¹⁹ With this, we can then write the BRST transformations in the form

$$\begin{aligned} sA &= Dc + \psi, \\ sc &= \frac{1}{2}[c, c] + \varphi, \\ s\psi &= D\varphi + [c, \psi], \\ s\varphi &= [c, \varphi]. \end{aligned} \quad (2.74)$$

The geometric meaning of the topological BRST transformations of (2.74) is revealed from the definition of the extended exterior derivative, \tilde{d} , as the sum of the ordinary exterior derivative with the BRST operator,

$$\tilde{d} = d + s, \quad (2.75)$$

and the generalized connection

$$\tilde{A} = A + c. \quad (2.76)$$

By direct inspection, one sees that the BRST transformations can be written in terms of the generalized curvature²⁰

$$\mathcal{F} = F + \psi + \varphi, \quad (2.77)$$

¹⁹The exterior derivative operation in the space of smooth p -forms, Λ_p , $d: \Lambda_p \rightarrow \Lambda_{p+1}$, on a generic p -form ω_p , $\omega_p = \omega_{i_1, i_2, \dots, i_p} dx^{i_1} \wedge dx^{i_2} \cdots \wedge dx^{i_p}$, is locally defined by $d\omega_p = \frac{\partial \omega_{i_1, i_2, \dots, i_p}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge dx^{i_2} \cdots \wedge dx^{i_p}$, with ω_p being a p -form and $d\omega_p$ being a $(p+1)$ -form. It follows that the exterior derivative is nilpotent, $d^2 = 0$, due to the antisymmetric property of the indices. One assumes that s anticommutes with d , $\{s, d\} = 0$.

²⁰The nature of φ as the “curvature” in the instanton moduli space direction is implicit in the BRST transformation of the FP ghost, which can be rewritten in the geometric form $sc + \frac{1}{2}[c, c] = \varphi$.

such that

$$\mathcal{F} = \tilde{d}\tilde{A} + \frac{1}{2}[\tilde{A}, \tilde{A}], \quad (2.78)$$

with the Bianchi identity

$$\tilde{D}\mathcal{F} = \tilde{d}\mathcal{F} + [\tilde{A}, \mathcal{F}] = 0. \quad (2.79)$$

Here, the space is graded as a sum of form degree and ghost number, in which the BRST operator is the exterior differential operator in the moduli space direction \mathcal{A}/\mathcal{G} , where the gauge fields that differ by a gauge transformation are identified. The whole space is then $M \times \mathcal{A}/\mathcal{G}$, with M being a four-dimensional manifold. According to the gauges worked out in this paper, M will be a Euclidean flat space.

In the definition (2.76) and following equations, we are adding quantities with different form degrees and ghost numbers as though they were of the same nature. Obviously, this is not being done directly. We must see Eqs. (2.78) and (2.79) as an expansion in form degrees and ghost numbers in which the elements with the same nature on both sides have to be compared. The relevant cohomology is defined by the cohomology of $M \times \mathcal{A}/\mathcal{G}$, $\tilde{d}^2 = 0$, being valid without requiring equations of motion. Such a geometric structure reveals the BRST *off-shell* character of the BS approach.²¹ We will discuss in Sec. II B 5 how the universal curvature \mathcal{F} generates the same global observables of Witten theory, i.e., the Donaldson polynomials.

3. Doublet theorem and gauge fixing: Baulieu-Singer gauges

Let us recall the *doublet theorem* [16], which will be indispensable for fixing the gauge ambiguities without changing the physical content of the theory. Consider a theory that contains a pair of fields or sources that form a doublet, i.e.,

$$\begin{aligned} \hat{\delta}\mathcal{X}_i &= \mathcal{Y}_i, \\ \hat{\delta}\mathcal{Y}_i &= 0, \end{aligned} \quad (2.80)$$

where i is a generic index and $\hat{\delta}$ is a fermionic nilpotent operator. The field (source) \mathcal{X}_i is assumed to be fermionic. As the operator $\hat{\delta}$ increases the ghost number in one unity, by definition, and if \mathcal{X}_i is an anticommuting quantity, \mathcal{Y}_i is a commuting one. The doublet structure of $(\mathcal{X}_i, \mathcal{Y}_i)$ in Eq. (2.80) implies that such fields (or sources) belong to the trivial part of the cohomology of $\hat{\delta}$. The proof is as follows. First, we define the operators

²¹For a detailed study on the geometric interpretation of the universal fiber bundle and its curvature, we suggest, for instance, Refs. [21,45].

$$\hat{N} = \int dx \left(\mathcal{X}_i \frac{\partial}{\partial \mathcal{X}_i} + \mathcal{Y}_i \frac{\partial}{\partial \mathcal{Y}_i} \right), \quad (2.81)$$

$$\hat{A} = \int dx \mathcal{X}_i \frac{\partial}{\partial \mathcal{Y}_i}, \quad (2.82)$$

$$\hat{\delta} = \mathcal{Y}_i \frac{\partial}{\partial \mathcal{X}_i}, \quad (2.83)$$

which obey the commutation relations

$$\{\hat{\delta}, \hat{A}\} = \hat{N}, \quad (2.84)$$

$$[\hat{\delta}, \hat{N}] = 0, \quad (2.85)$$

where $\hat{\delta}$ is a nilpotent operator as it is fermionic, $\hat{\delta}^2 = 0$. The operator \hat{N} is the counting operator. With Δ being a polynomial in the fields, sources, and parameters, the cohomology of the nilpotent operator $\hat{\delta}$, as we know, is given by the solutions of

$$\hat{\delta}\Delta = 0, \quad (2.86)$$

which is not exact, i.e., which cannot be written in the form

$$\Delta = \hat{\delta}\Sigma. \quad (2.87)$$

The general expression of Δ is then

$$\Delta = \tilde{\Delta} + \hat{\delta}\Sigma, \quad (2.88)$$

where $\tilde{\Delta}$ belongs to the nontrivial part of the cohomology; in other words, it is closed, $\hat{\delta}\tilde{\Delta} = 0$, but not exact, $\tilde{\Delta} \neq \hat{\delta}\tilde{\Sigma}$. One can expand Δ in eigenvectors of \hat{N} ,

$$\Delta = \sum_{n \geq 0} \Delta_n, \quad (2.89)$$

such that $\hat{N}\Delta_n = n\Delta_n$, where n is the total number of \mathcal{X}_i and \mathcal{Y}_i in Δ_n . Such an expansion is consistent as each Δ_n is a polynomial in \mathcal{X}_i and \mathcal{Y}_i , and $\hat{\delta}\Delta_n = 0$ for $\forall n \geq 1$, according to (2.80) and the commuting properties of \mathcal{X}_i and \mathcal{Y}_i . Finally, rewriting (2.89) as

$$\Delta = \Delta_0 + \sum_{n \geq 1} \frac{1}{n} \hat{N} \Delta_n, \quad (2.90)$$

and then using the commuting relation (2.84), we get

$$\Delta = \Delta_0 + \hat{\delta} \left(\sum_{n \geq 1} \frac{1}{n} \hat{A} \Delta_n \right), \quad (2.91)$$

which shows that all terms which contain at least one field (source) of the doublet never enter the nontrivial part of the

cohomology of $\hat{\delta}$, being thus nonphysical—for a more complete analysis, see for instance Refs. [16,46].

To fix the three gauge symmetries of the non-Abelian topological theory (2.68)–(2.70), we introduce the three BRST doublets

$$\begin{aligned} s\bar{c}^a &= b^a, & sb^a &= 0, \\ s\bar{\chi}_{\mu\nu}^a &= B_{\mu\nu}^a, & sB_{\mu\nu}^a &= 0, \\ s\bar{\varphi}^a &= \bar{\eta}^a, & s\bar{\eta}^a &= 0, \end{aligned} \quad (2.92)$$

where $\bar{\chi}_{\mu\nu}^a$ and $B_{\mu\nu}^a$ are (anti-)self-dual fields according to the (negative) positive sign; see (2.95) below. The \mathcal{G} -valued Lagrange multiplier fields b^a , $B_{\mu\nu}^a$, and $\bar{\eta}$ have, respectively, ghost numbers 0, 0, and -1 , while the antighost fields \bar{c}^a , $\bar{\chi}_{\mu\nu}^a$, and $\bar{\varphi}^a$ have ghost numbers -1 , -1 , and -2 . (For completeness and further use, the quantum numbers of all fields are displayed in Table I.)

Working in Baulieu-Singer gauges amounts to considering the constraints [13]

$$\partial_\mu A_\mu^a = -\frac{1}{2}\rho_1 b^a, \quad (2.93)$$

$$D_\mu^{ab} \psi_\mu^a = 0, \quad (2.94)$$

$$F_{\mu\nu}^a \pm \tilde{F}_{\mu\nu}^a = -\frac{1}{2}\rho_2 B_{\mu\nu}^a, \quad (2.95)$$

where ρ_1 and ρ_2 are real gauge parameters. In a few words, beyond the gauge fixing of the topological ghost (2.94), we must interpret the requirement of two extra gauge fixings due to the fact that the gauge field possesses two independent gauge symmetries. In this sense, condition (2.93) fixes the usual Yang-Mills symmetry $\delta A_\mu^a = D_\mu^{ab} \omega^b$, and the second one (2.95) fixes the topological shift $\delta A_\mu^a = \alpha_\mu^a$. The (anti-)self-dual condition for the field strength (in the limit $\rho_2 \rightarrow 0$) is convenient to identify the well-known observables of topological theories for 4-manifolds (see Ref. [17]) given by the Donaldson invariants [2,3], which are described in terms of the instantons.

The partition functional of the topological action in BS gauges (2.93) takes the form

$$Z_{\text{BS}} = \int \mathcal{D}c \mathcal{D}\bar{c} \mathcal{D}\psi_\mu \mathcal{D}\bar{\chi}_{\mu\nu} \mathcal{D}B_{\mu\nu} \mathcal{D}\varphi \mathcal{D}\bar{\varphi} \mathcal{D}\eta e^{-S_{\text{BS}}}, \quad (2.96)$$

whereby

TABLE I. Quantum numbers of the fields.

Field	A	ψ	c	φ	\bar{c}	b	$\bar{\varphi}$	$\bar{\eta}$	$\bar{\chi}$	B
Dim	1	1	0	0	2	2	2	2	2	2
Ghost n ^o	0	1	1	2	-1	0	-2	-1	-1	0

$$S_{\text{BS}} = S_0[A] + S_{gf}^{\text{BS}}, \quad (2.97)$$

with S_{gf}^{BS} being the gauge-fixing action which belongs to trivial part of the cohomology, given by

$$\begin{aligned} S_{gf}^{\text{BS}} &= s\text{Tr} \int d^4x \left[\bar{\chi}_{\mu\nu} \left(F_{\mu\nu} \pm \tilde{F}_{\mu\nu} + \frac{1}{2} \rho_2 B_{\mu\nu} \right) + \bar{\varphi} D_\mu \psi_\mu + \bar{c} \left(\partial_\mu A_\mu - \frac{1}{2} \rho_1 b \right) \right] \\ &= \text{Tr} \int d^4x \left[B_{\mu\nu} \left(F_{\mu\nu} \pm \tilde{F}_{\mu\nu} + \frac{1}{2} \rho_2 B_{\mu\nu} \right) + \bar{\chi}_{\mu\nu} \left(D_{[\mu} \psi_{\nu]} \pm \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} D_{[\alpha} \psi_{\beta]} \right) \right. \\ &\quad \left. - \bar{\chi}_{\mu\nu} [c, F_{\mu\nu} \pm \tilde{F}_{\mu\nu}] + \eta D_\mu \psi_\mu + \bar{\varphi} [\psi_\mu, \psi_\mu] + \bar{\varphi} D_\mu D_\mu \varphi - b \left(\partial_\mu A_\mu - \frac{1}{2} \rho_1 b \right) - \bar{c} \partial_\mu D_\mu c - \bar{c} \partial_\mu \psi_\mu \right]. \end{aligned} \quad (2.98)$$

A key observation is that, for $\rho_1 = \rho_2 = 1$, one can eliminate the topological term $S_0[A]$, i.e., the Pontryagin action, by integrating out the field $B_{\mu\nu}$, such that

$$\begin{aligned} &\text{Tr} \left\{ B_{\mu\nu} (F_{\mu\nu} + \tilde{F}_{\mu\nu}) + \frac{1}{2} B_{\mu\nu} B_{\mu\nu} \right\} \\ &\rightarrow \text{Tr} \{ F_{\mu\nu} F_{\mu\nu} + F_{\mu\nu} \tilde{F}_{\mu\nu} \}. \end{aligned} \quad (2.99)$$

In this case, we obtain a classical topological action which is equivalent to a Yang-Mills action plus ghost interactions. Such an action, however, does not produce local observables as the cohomology of the theory remains the same, as we will discuss in more detail later in Sec. II B 5.

Another important property is that the Green's functions of local observables in (2.96) do not depend on the choice of the background metric [13]. Let S_{BS}^g be an action with metric choice $g_{\mu\nu}$ and $S_{\text{BS}}^{g+\delta g}$ be the same action up to the change of $g_{\mu\nu}$ into $g_{\mu\nu} + \delta g_{\mu\nu}$. As the only terms depending on the metric belong to the trivial part of cohomology, we conclude immediately that S_{BS}^g and $S_{\text{BS}}^{g+\delta g}$ only differ by a BRST-exact term,

$$S_{\text{BS}}^g - S_{\text{BS}}^{g+\delta g} = s \int d^4x \Delta^{(-1)}, \quad (2.100)$$

where $\Delta^{(-1)}$ is a polynomial of the fields, with ghost number -1 . It means that the expectation values of local operators are the same if computed with a background metric $g_{\mu\nu}$ or $g_{\mu\nu} + \delta g_{\mu\nu}$,

$$\frac{\delta}{\delta g_{\mu\nu}} \left\langle \prod_p \mathcal{O}_{a_p}(\varphi_i) \right\rangle = 0, \quad (2.101)$$

where $\mathcal{O}_{a_p}(\varphi_i)$ are functional operators of the quantum fields $\varphi_i(x)$; see Eq. (2.6). An anomaly in the topological BRST symmetry would break the above equation. However, there is no 4-form with ghost number 1, $\Delta_{4\text{-form}}^{(1)}$, defined modulo s - and d -exact terms which obeys (cf. Ref. [13])

$$s\Delta_{4\text{-form}}^{(1)} + d\Delta_{3\text{-form}}^{(2)} = 0. \quad (2.102)$$

Therefore, radiative corrections which could break the topological property (2.101) at the quantum level are not expected. The formal proof of the absence of gauge anomalies to all orders in the topological BS theory is achieved by employing the isomorphism described in Refs. [22,47].

4. Absence of gauge anomalies

The proof of the absence of gauge anomalies for the Slavnov-Taylor identity,

$$\mathcal{S}(S) = 0, \quad (2.103)$$

consists in proving that the cohomology of \mathcal{S} is empty. In the equation above, S is the classical action for a given gauge choice, and

$$\mathcal{S} = \int d^4x (s\Phi^I) \frac{\delta}{\delta \Phi^I}, \quad (2.104)$$

where Φ^I represents all fields. As \mathcal{S} is a Ward identity, in the absence of anomalies, the symmetry (2.103) is also valid at the quantum level, i.e., $\mathcal{S}(\Gamma) = 0$, with Γ being the quantum action in loop expansion. In Eq. (2.104), $s\Phi^I$ represents the BRST transformation of each field Φ^I . The fields \bar{c} , b , $\bar{\chi}_{\mu\nu}$, $B_{\mu\nu}$, $\bar{\varphi}$, and $\bar{\eta}$ transform as doublets, cf. Eq. (2.92). Changing the variables according to the redefinitions

$$\begin{aligned} \psi &\rightarrow \psi' = \psi - Dc, \\ \varphi &\rightarrow \varphi' = \varphi - \frac{1}{2} [c, c], \end{aligned} \quad (2.105)$$

the BRST transformations (2.74) are reduced to the doublet transformations

$$\begin{aligned}
sA &= \psi', \\
s\psi' &= 0, \\
sc &= \varphi', \\
s\varphi' &= 0.
\end{aligned} \tag{2.106}$$

It configures a reduced transformation in which the nonlinear part of the BRST transformations in the Slavnov-Taylor identity was eliminated. The complete transformation in this space is given by the reduced operator

$$\mathcal{S}_{\text{doublet}} = \int d^4x (s\Phi'^I) \frac{\delta}{\delta\Phi'^I}, \tag{2.107}$$

where $\Phi' = \{A, \psi', c, \varphi', \bar{c}, b, \bar{\chi}_{\mu\nu}, B_{\mu\nu}, \bar{\varphi}, \eta\}$, which is composed of five doublets. It means that $\mathcal{S}_{\text{doublet}}$ has vanishing cohomology (H),

$$H(\mathcal{S}_{\text{doublet}}) = \emptyset, \tag{2.108}$$

in other words, that any polynomial of the fields Φ' , $\Delta(\Phi')$, which satisfies

$$\mathcal{S}_{\text{doublet}}(\Delta(\Phi')) = 0, \tag{2.109}$$

belonging to the trivial part of the cohomology of $\mathcal{S}_{\text{doublet}}$ (see the doublet theorem in the previous section). The crucial point here is the fact that the cohomology of \mathcal{S} in the space of local integrated functionals in the fields and sources is isomorphic to a subspace of $H(\mathcal{S}_{\text{doublet}})$. Consequently, \mathcal{S} has also vanishing cohomology [22,29,47],

$$H(\mathcal{S}) = \emptyset. \tag{2.110}$$

The result (2.110) shows that there is no room for an anomaly in the Slavnov-Taylor identity (2.103). All counterterms at the quantum level will belong to the trivial part of cohomology, and the condition (2.102) for the existence of an anomaly capable of breaking the topological property (2.101) never occurs. Because of the algebraic structure of the theory, Eq. (2.110) proves that all Ward identities are free of gauge anomalies, cf. Ref. [22]. As a consequence of this result, the background metric independence is valid to all orders in perturbation theory.

The second point, and not least, is the conclusion that the BS theory has no local observables. Because of its vanishing cohomology (2.110), all BRST-invariant quantities must belong to the nonphysical (or trivial) part of the cohomology of s , and the only possible observables are the global ones, i.e., topological invariants for 4-manifolds. Such observables are characterized by the cohomology of s [29,48], in which the observables are globally defined in agreement with the supersymmetric formulation of J. H. Horne [49]. A simple way to identify these observables is accomplished by studying the cohomology of the extended

space $M \times \mathcal{A}/\mathcal{G}$, where the metric independent observables, known as *Chern classes*, are constructed in terms of the universal curvature \mathcal{F} (2.80). The Donaldson polynomials are naturally recovered, characterized by the so-called equivariant cohomology, which relates the BS approach to Witten's theory at the level of observables.

5. Equivariant cohomology and global observables

Witten's topological theory is constructed without fixing its remaining ordinary Yang-Mills gauge symmetry. The theory is developed in the instanton moduli space \mathcal{A}/\mathcal{G} . A generic observable of his theory, $\mathcal{O}_{\alpha_i}^{(W)}$, is naturally gauge invariant under Yang-Mills gauge transformations,

$$s_{\text{YM}}\mathcal{O}_{\alpha_i}^{(W)} = 0, \tag{2.111}$$

where s_{YM} is the nilpotent BRST operator related to the ordinary Yang-Mills symmetry, i.e., without including the topological shift, namely,

$$\begin{aligned}
s_{\text{YM}}A_\mu &= D_\mu c, \\
s_{\text{YM}}\Phi_{\text{adj}} &= [c, \Phi_{\text{adj}}],
\end{aligned} \tag{2.112}$$

where Φ_{adj} is a generic field in adjoint representation. We conclude that we can add an ordinary Yang-Mills gauge transformation (in the \mathcal{A}/\mathcal{G} direction) to Witten fermionic symmetry based on the “topological shift” $\delta A_\mu \sim \psi_\mu$,

$$\delta \rightarrow \delta_{\text{eq}} = \delta + s_{\text{YM}}, \tag{2.113}$$

in such a way that the descent equations for $\delta \sim \{\mathcal{Q}, \cdot\}$ will remain the same; see (2.34) and (2.56)–(2.60). The operator δ_{eq} is nilpotent when acting on gauge-invariant quantities under Yang-Mills transformations, defining thus a cohomology in a space where the fields that differ by a Yang-Mills gauge transformations are identified, known as equivariant cohomology. Such a property indicates that there is a link between Witten's theory and the BS approach in which the BRST operator, s , is naturally defined by taking into account the topological shift and the ordinary Yang-Mills transformation in a single formalism.

To prove the link between both approaches, we must remember that the universal curvature in the space $M \times \mathcal{A}/\mathcal{G}$ is given by the sum $\mathcal{F} = F + \psi + \varphi$. The difference between the on-shell BRST operator, s , and the Witten fermionic symmetry, δ , for $\mathbb{X} = (F, \psi, \varphi)$ is of the form

$$s\mathbb{X} = \delta\mathbb{X} + [\mathbb{X}, c]; \tag{2.114}$$

in other words, in the space of the fields (F, ψ, φ) , s and δ differ by an ordinary Yang-Mills transformation, as (F, ψ, φ) transform in the adjoint representation of the gauge group. These fields are the only ones we need to

obtain the Donaldson polynomials as the observables of the BS theory, since in the space $M \times \mathcal{A}/\mathcal{G}$ they are constructed in terms of \mathcal{F} . This allows for identifying the equivariant operator with the BRST one, $\delta_{\text{eq}} \equiv s$, according to the construction of the observables in Witten's and BS theories.

To understand the above statement, we must invoke the n th Chern class, $\tilde{\mathcal{W}}_n$, defined in terms of the universal curvature by

$$\tilde{\mathcal{W}}_n = \text{Tr}(\underbrace{\mathcal{F} \wedge \mathcal{F} \wedge \cdots \wedge \mathcal{F}}_{n \text{ times}}), \quad (2.115)$$

where $n = \{1, 2, 3, \dots\}$ is the number of wedge products. $\tilde{\mathcal{W}}_n$ represents the most general observables of BS theory.²² The Weyl theorem [21] ensures that $\tilde{\mathcal{W}}_n$ is closed with respect to the extended differential operator $\tilde{d} = d + s$ [13,50], i.e.,

$$\tilde{d}\tilde{\mathcal{W}}_n = 0. \quad (2.116)$$

If we choose the first Chern class

$$\tilde{\mathcal{W}}_1 = \text{Tr}(\mathcal{F} \wedge \mathcal{F}), \quad (2.117)$$

the expansion in ghost numbers of Eq. (2.116) yields

$$s\text{Tr}(F \wedge F) = d\text{Tr}(-2\psi \wedge F), \quad (2.118)$$

$$s\text{Tr}(\psi \wedge F) = d\text{Tr}\left(-\frac{1}{2}\psi \wedge \psi - \varphi F\right), \quad (2.119)$$

$$s\text{Tr}(\psi \wedge \psi + 2\varphi F) = d\text{Tr}(2\psi\varphi), \quad (2.120)$$

$$s\text{Tr}(\psi\varphi) = d\text{Tr}\left(-\frac{1}{2}\varphi\varphi\right), \quad (2.121)$$

$$s\text{Tr}(\varphi\varphi) = 0, \quad (2.122)$$

which are the same descent equations obtained in (2.56)–(2.60) following Witten analysis, only replacing δ (or δ_{eq}) by s , proving that Baulieu-Singer and Witten's topological

theories possess the same observables given by the Donaldson invariants (2.64).

The fact that the observables in the BS approach are naturally invariant under ordinary Yang-Mills symmetry should not seem surprising, as the n th Chern class is Yang-Mills invariant itself (2.115) since \mathcal{F} transforms in the adjoint representation of the gauge group. Equation (2.116) provides a powerful tool to obtain Donaldson polynomials for any ghost number. One must note that we do not have to worry about with the independence of Faddeev-Popov ghosts in order to construct the observables in the BS approach. Although the gauge-fixed BS action has FP ghosts due to the gauge fixing of the Yang-Mills ambiguity, the (c, \bar{c}) independence of $\tilde{\mathcal{W}}_n$ is a direct consequence of the fact that the universal curvature of the space $M \times \mathcal{A}/\mathcal{G}$ does not depend on FP ghosts, but only on F , and the ghosts ψ and φ .

In the weak coupling limit of the twisted $N = 2$, the observables of both theories are undoubtedly the same: the topological Donaldson invariants [21–23]. We might ask if the quantum behavior is also compatible, once BS and Witten actions does not differ by a BRST-exact term,

$$S_{\text{BS}} - S_W = \Sigma_{\mathcal{G}} \neq s(\cdots). \quad (2.123)$$

The relation above does not depend on the gauge choice. Consequently, we cannot say, in principle, that BS and Witten partition functions are equivalent at quantum level, since

$$Z_{\text{BS}} = \int \mathcal{D}\Phi e^{-S_{\text{BS}}} = \int \mathcal{D}\Phi e^{-S_W - \Sigma_{\mathcal{G}}}, \quad (2.124)$$

wherein $\Sigma_{\mathcal{G}}$ is not s exact. At first view, $Z_{\text{BS}} \neq Z_W = \int \mathcal{D}\Phi e^{-S_W}$. The fact that $\Sigma_{\mathcal{G}} \neq s(\cdots)$ opens the possibility for both theories to have different quantum properties. The one-loop exactness of twisted $N = 2$ SYM β function, for instance, is a well-known result in the literature [34]. We will now analyze the Ward identities of the BS theory in self-dual Landau gauges, in order to compare the quantum properties of the DW and BS theories.

III. QUANTUM PROPERTIES OF BS THEORY IN THE SELF-DUAL LANDAU GAUGES

In this section, we will summarize the quantum properties of BS theory in the self-dual Landau (SDL) gauges.²³ Extra details can be found in Refs. [24–27,51].

²²It is not possible to construct topological observables using the Hodge product, as it is metric dependent. For this reason, we never obtain Yang-Mills terms of the type $\{\text{Tr}(F_{\mu\nu}F^{\mu\nu}), \text{Tr}(F_{\mu\nu}F^{\nu\sigma}F^{\mu}_{\sigma}), \dots\}$, without Levi-Civita tensors in the internal product, in the place of metric tensors. Moreover, the Wilson loop $W_p^{(C)} = \text{Tr}\{\mathcal{P}e^{i\oint_C A_{\mu}dx_{\mu}}\}$ is not an observable in the non-Abelian topological BS case, as it is not gauge invariant due to the topological shift symmetry. In any case, it does not make sense to discuss confinement in the BS theory, as it is not confining to any energy scale. Thence, the only possibilities for topological invariants are the wedge products in $\tilde{\mathcal{W}}_n$.

²³For simplicity, throughout the text, we will refer to the Baulieu-Singer theory in the self-dual Landau gauges as self-dual BS theory.

A. Absence of radiative corrections

Working in the self-dual Landau gauges amounts to considering the constraints [27]

$$\partial_\mu A_\mu^a = 0, \quad (3.1)$$

$$\partial_\mu \psi_\mu^a = 0, \quad (3.2)$$

$$F_{\mu\nu}^a \pm \tilde{F}_{\mu\nu}^a = 0. \quad (3.3)$$

Through the introduction of the three BRST doublets described in Eq. (2.92), the complete gauge-fixed topological action in SDL gauges takes the form

$$S[\Phi] = S_0[A] + S_{gf}[\Phi], \quad (3.4)$$

with $S_0[A]$ standing for the Pontryagin action and

$$\begin{aligned} S_{gf}[\Phi] &= s \int d^4z \left[\bar{c}^a \partial_\mu A_\mu^a + \frac{1}{2} \bar{\chi}_{\mu\nu}^a (F_{\mu\nu}^a \pm \tilde{F}_{\mu\nu}^a) + \bar{\varphi}^a \partial_\mu \psi_\mu^a \right] \\ &= \int d^4z \left[b^a \partial_\mu A_\mu^a + \frac{1}{2} B_{\mu\nu}^a (F_{\mu\nu}^a \pm \tilde{F}_{\mu\nu}^a) + (\bar{\eta}^a - \bar{c}^a) \partial_\mu \psi_\mu^a + \bar{c}^a \partial_\mu D_\mu^{ab} c^b + \right. \\ &\quad - \frac{1}{2} g f^{abc} \bar{\chi}_{\mu\nu}^a c^b (F_{\mu\nu}^c \pm \tilde{F}_{\mu\nu}^c) - \bar{\chi}_{\mu\nu}^a \left(\delta_{\mu\alpha} \delta_{\nu\beta} \pm \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \right) D_\alpha^{ab} \psi_\beta^b + \bar{\varphi}^a \partial_\mu D_\mu^{ab} \varphi^b + \\ &\quad \left. + g f^{abc} \bar{\varphi}^a \partial_\mu (c^b \psi_\mu^c) \right]. \end{aligned} \quad (3.5)$$

This action possesses a rich set of symmetries; see Ref. [24] and Appendix. To control the nonlinearity of the Slavnov-Taylor identity [Eq. (A1)] and the bosonic symmetry \mathcal{T} [Eq. (A13)], we have to introduce external sources given by the following three BRST doublets [27]:

$$\begin{aligned} s\tau_\mu^a &= \Omega_\mu^a, & s\Omega_\mu^a &= 0, \\ sE^a &= L^a, & sL^a &= 0, \\ s\Lambda_{\mu\nu}^a &= K_{\mu\nu}^a, & sK_{\mu\nu}^a &= 0. \end{aligned} \quad (3.6)$$

The respective external action, written as a BRST-exact contribution preserving the physical content of theory, takes the form

$$\begin{aligned} S_{\text{ext}} &= s \int d^4z \left(\tau_\mu^a D_\mu^{ab} c^b + \frac{g}{2} f^{abc} E^a c^b c^c + g f^{abc} \Lambda_{\mu\nu}^a c^b \bar{\chi}_{\mu\nu}^c \right) \\ &= \int d^4z \left[\Omega_\mu^a D_\mu^{ab} c^b + \frac{g}{2} f^{abc} L^a c^b c^c + g f^{abc} K_{\mu\nu}^a c^b \bar{\chi}_{\mu\nu}^c + \tau_\mu^a (D_\mu^{ab} \varphi^b + g f^{abc} c^b \psi_\mu^c) \right. \\ &\quad + g f^{abc} E^a c^b \varphi^c + g f^{abc} \Lambda_{\mu\nu}^a c^b B_{\mu\nu}^c - g f^{abc} \Lambda_{\mu\nu}^a \varphi^b \bar{\chi}_{\mu\nu}^c \\ &\quad \left. - \frac{g^2}{2} f^{abc} f^{bde} \Lambda_{\mu\nu}^a \bar{\chi}_{\mu\nu}^c c^d c^e \right], \end{aligned} \quad (3.7)$$

with the corresponding quantum number of the external sources displayed in Table II. Therefore, the full classical action to be quantized is

$$\Sigma[\Phi] = S_0[A] + S_{gf}[\Phi] + S_{\text{ext}}[\Phi]. \quad (3.8)$$

TABLE II. Quantum numbers of the external sources.

Source	τ	Ω	E	L	Λ	K
Dim	3	3	4	4	2	2
Ghost n°	-2	-1	-3	-2	-1	0

The introduction of the external action does not break the original symmetries, and the physical limit is obtained by setting the external sources to zero [16].

One of the symmetries is of particular interest to us: the vector supersymmetry described by Eq. (A12), cf. Refs. [24,27]. By applying BRST-algebraic renormalization techniques [16], and disregarding Gribov ambiguities, it was proved in Ref. [24], with the help of Feynman diagrams, that all two-point functions are tree-level exact, as a consequence of the Ward identities of the model. In particular, as a consequence of the vector supersymmetry

(A11), the gauge field propagator vanishes to all orders in perturbation theory,

$$\langle A_\mu^a(p) A_\nu^b(q) \rangle = 0. \quad (3.9)$$

In Ref. [26], this result was generalized: not only are the two-point functions of the self-dual BS theory tree-level

exact, but all n -point Green functions of the model do not receive any radiative corrections. This is a direct consequence of the null gauge propagator (3.9) together with the vertex structure of the full action (3.8). Following the Feynman rules notation of Ref. [26], we represent the relevant propagators by

$$\begin{aligned} \langle AA \rangle &= \text{-----} , & \langle c\bar{c} \rangle &= \text{.....} , & \langle \bar{\chi}\psi \rangle &= \text{~~~~~} , \\ \langle AB \rangle &= \text{=====} , & \langle \phi\bar{\phi} \rangle &= \text{:.....:} . \end{aligned} \quad (3.10)$$

The relevant vertices are represented by

$$\quad (3.11)$$

Using these diagrams, one identifies a kind of *cascade effect* in which the number of internal A legs always increases when trying to construct loop Feynman diagrams, according to the diagram below,

$$\quad (3.12)$$

This makes it impossible to close loops without using the $\langle AA \rangle$ propagator,²⁴ which vanishes by means of (3.9). Note that, internally, the A leg always propagates to the vertex BAA . Looking at the full action (3.8), the only vertex which possesses A legs is $\bar{\phi}c\psi$. However, the $\bar{\phi}$ leg could only propagate to the vertex $\bar{\phi}A\phi$ through $\langle \bar{\phi}\phi \rangle$; the c leg could only propagate to $\bar{c}Ac$ through $\langle \bar{c}c \rangle$; and the ψ leg propagates to the vertices $\bar{\chi}A\psi$, $\bar{\chi}cA$, or $\bar{\chi}cAA$ through $\langle \psi\bar{\chi} \rangle$ ($\langle \bar{\eta}\psi \rangle$ is not considered because there is no vertex $\bar{\eta}$). All possible branches produce at least one remaining internal A leg, and the cascade effect is not avoided, as represented by the diagrams

²⁴The formal proof of this result can be found in Ref. [26].

The apparently only nonzero correlation functions are of the type

$$\langle BBB...bb \rangle = \langle s\bar{\chi}BB...bb \rangle = \langle s(\bar{\chi}BB...bb) \rangle, \quad (3.14)$$

i.e., with external $B_{\mu\nu}^a$ or b^a fields. But (3.14) automatically vanishes as it is BRST exact.

In a few words, using perturbative techniques, one sees that the tree-level exactness of the BS in the self-dual gauges is a consequence of the vector supersymmetry and BRST symmetry.

B. Renormalization ambiguity

Once we have at our disposal all Ward identities, we are able to construct the most general counterterm Σ^c that can absorb the divergences arising in the evaluation of Feynman graphs. Because of the triviality of the BRST cohomology [24,27], Σ^c belongs to trivial part of the BRST cohomology. The fact that the BS theory is quantum stable is a well-known result in the literature [24,27,51]. Reference [24] introduced an extra nonlinear bosonic symmetry that relates the topological ghost with the Faddeev-Popov one (among other transformations involving other fields) through the transformation

$$\delta\psi_\mu^a \mapsto D_\mu^{ab} c^b, \quad (3.15)$$

described by the Ward identity \mathcal{T} in Eq. (A13). Taking into account this extra symmetry, from the multiplicative redefinition of the fields, sources, and parameters of the model,

$$\begin{aligned} \Phi_0 &= Z_\Phi^{1/2} \Phi, \\ \Phi_0 &= \{A_\mu^a, \psi_\mu^a, c^a, \bar{c}^a, \varphi^a, \bar{\varphi}^a, b^a, \bar{\eta}^a, \bar{\chi}_{\mu\nu}^a, B_{\mu\nu}^a\}, \\ \mathcal{J}_0 &= Z_\mathcal{J} \mathcal{J}, \quad \mathcal{J} = \{\tau_\mu^a, \Omega_\mu^a, E^a, L^a, \Lambda_{\mu\nu}^a, K_{\mu\nu}^a\}, \\ g_0 &= Z_g g, \end{aligned} \quad (3.16)$$

one proves the quantum stability of the BS theory in self-dual gauges with only one independent renormalization parameter, i.e., that the quantum action $\Gamma \equiv \Sigma(\Phi_0, \mathcal{J}_0, g_0)$ at one loop is of the form

$$\Sigma(\Phi_0, \mathcal{J}_0, g_0) = \Sigma(\Phi, \mathcal{J}, g) + \epsilon \Sigma^c(\Phi, \mathcal{J}, g), \quad (3.17)$$

with

$$\Sigma^c = a \int d^4x (B_{\mu\nu}^a F_{\mu\nu}^a - 2\bar{\chi}_{\mu\nu}^a D_\mu^{ab} \psi_\nu^b - g f^{abc} \bar{\chi}_{\mu\nu}^a c^b F_{\mu\nu}^c), \quad (3.18)$$

whereby the resulting Z factors obey the system of equations

$$\begin{aligned} Z_A^{1/2} &= Z_b^{-1/2} = Z_g^{-1}, \\ Z_c^{1/2} &= Z_{\bar{\eta}}^{1/2} = Z_\psi^{-1/2} = Z_\Omega = Z_c^{-1/2}, \\ Z_{\bar{\varphi}}^{1/2} &= Z_\varphi^{-1/2} = Z_\tau = Z_L = Z_g^{-1} Z_c^{-1}, \\ Z_E &= Z_g^{-2} Z_c^{-3/2}, \\ Z_K &= Z_g^{-1} Z_c^{-1/2} Z_{\bar{\chi}}^{-1/2}, \\ Z_\Lambda &= Z_g^{-2} Z_c^{-1} Z_{\bar{\chi}}^{-1/2}, \\ Z_B^{1/2} Z_A^{1/2} &= Z_{\bar{\chi}}^{1/2} Z_c^{1/2} = 1 + \epsilon a, \end{aligned} \quad (3.19)$$

with the independent renormalization parameter denoted by a . Because of the recursive nature of algebraic renormalization [16], the results (3.19) show that the model is renormalizable to all orders in perturbation theory.

From the algebraic analysis so far, we cannot prove that $Z_g = 1$, as suggested by the tree-level exactness obtained via the study of the Feynman diagrams. The system of Z factors (3.19) is undetermined. As we can easily see, the number of equations n and the number of variables z (the Z factors) are related by $z = n + 2$, indicating that there is a kind of freedom in the choice of two of the Z factors.

In Ref. [25], the origin of such an ambiguity was explained: it is due to the absence of a kinetic gauge field term out from the trivial BRST cohomology and due to the absence of discrete symmetries involving the ghost fields. The symmetries of the SDL gauges eliminate the kinetic term of the Faddeev-Popov ghost in the counterterm, i.e.,

$$Z_c Z_{\bar{c}} = 1. \quad (3.20)$$

Moreover, from the gauge-ghost vertex ($\bar{c}Ac$), which is also absent in the counterterm, we achieve

$$Z_g Z_A^{1/2} = 1. \quad (3.21)$$

The two relations (3.20) and (3.21) are decoupled; in other words, only by determining Z_c or $Z_{\bar{c}}$, we do not get any information about Z_g or Z_A . As there are no kinetic terms for the gauge field in the classical action (3.8), the independent determination of Z_A becomes impossible. The same analysis can be performed for the bosonic and topological ghosts; see Ref. [25].

Extra information is then required in order to determine the system (3.19). In the ordinary Yang-Mills theories (quantized in the Landau gauge), $Z_c = Z_{\bar{c}}$, which relies on the discrete symmetry

$$\begin{aligned} c^a &\rightarrow \bar{c}^a, \\ \bar{c}^a &\rightarrow -c^a. \end{aligned} \quad (3.22)$$

This condition, together with the Faddeev-Popov ghost kinetic term, is sufficient to determine Z_c and $Z_{\bar{c}}$. It is easy to see that the action (3.8) does not obey such a symmetry. Discrete symmetries between the other ghosts of topological Yang-Mills theories (φ^a and $\bar{\varphi}^a$; ψ_μ^a and $\bar{\chi}_{\mu\nu}^a$) are also not present in (3.8), which explains the second ambiguity. In Witten's theory, such an ambiguity will not appear by this reasoning since Witten's action contains discrete symmetries ensured by the time-reversal symmetry (3.22) in Landau gauge, together with

$$\begin{aligned} \varphi &\rightarrow \bar{\varphi}, & \bar{\varphi} &\rightarrow \varphi, \\ \psi_\mu &\rightarrow \chi_\mu, & \chi_\mu &\rightarrow \psi_\mu, \end{aligned} \quad (3.23)$$

whereby the components of χ_μ are defined as

$$\chi_0 \equiv \eta, \quad \chi_i \equiv \chi_{0i} = \frac{1}{2} \varepsilon_{ijk} \chi_{jk}, \quad (3.24)$$

implying a “particle-antiparticle” relationship between \bar{c} and c , $\bar{\varphi}$ and φ , and ψ_μ and χ_μ , as demonstrated in Ref. [52].

This ambiguity is also present in a generalized class of renormalizable gauges [25]. In fact, one could relate this ambiguity with the fact that all local degrees of freedom are nonphysical (e.g., the gauge field propagator is totally gauge dependent). In self-dual Landau gauges, where the vector supersymmetry is present, the Feynman diagram structure indicates that imposing $Z_c = Z_{\bar{c}}$ and $Z_\varphi = Z_{\bar{\varphi}}$ is consistent with the model. Hence, the Z-factor system (3.19) would naturally yield $Z_g = 1$, in accordance with the absence of radiative corrections in this gauge choice. However, without recovering the discrete symmetries between the ghosts, such an imposition seems to be artificial. As we will see later, the renormalization ambiguity can be solved in the SDL gauges, i.e., the discrete symmetries can be reconstructed, due to the triviality of the Gribov copies [28], which allows for a nonlocal

transformation with trivial Jacobian, capable of recovering such symmetries.

IV. PERTURBATIVE β FUNCTIONS

Our aim in this section is to compare the DW and BS β functions to prove that these topological gauge theories are not completely equivalent at the quantum level and then identify in which energy regimes the correspondence could occur. The DW β function is well known [34,52], as we will briefly describe. It still lacks the task of determining the self-dual BS one to perform the comparison.

A. Twisted $N = 2$ super-Yang-Mills theory

In Ref. [52], the authors have computed the one-loop β function of the DW theory. Later, the authors of Ref. [34] employed the algebraic renormalization techniques to also study DW theory and prove that the β function of twisted $N = 2$ SYM ($\beta_g^{N=2}$) is one-loop exact. The reason is that the composite operator $\text{Tr} \varphi^2(x)$ does not renormalize [34]. For that, they considered the fact that the operator $d_{\mu\nu}$, defined in expression (2.15), is redundant [53]. Thence, the definition of an extended BRST operator, namely,

$$\mathcal{S} = s_{\text{YM}} + \omega \delta + \varepsilon_\mu \delta_\mu, \quad (4.1)$$

could be employed. In expression (4.1), ω and ε_μ are global ghosts, and δ and δ_μ were defined in Eqs. (2.13) and (2.14). The relevant property of the operator \mathcal{S} is that it is on-shell nilpotent in the space of integrated local functionals, since

$$\mathcal{S}^2 = \omega \varepsilon_\mu \partial_\mu + \text{equations of motion}. \quad (4.2)$$

We point out that this extended BRST construction requires the equations of motion to obtain a nilpotent BRST operator—a standard behavior of Witten's theory, representing a different quantization scheme of the BS theory. Considering the nonrenormalization of $\text{Tr} \varphi^2$ and the on-shell cohomology of the operator defined in Eq. (4.2), the result is that the β function only receives contributions to one-loop order and is given by

$$\beta_g^{N=2} = -K g^3, \quad (4.3)$$

with K being a constant. The computation of $\beta_g^{N=2}$ via Feynman diagrams is performed in Ref. [52] by evaluating the one-loop contributions to the gauge field propagator [where the Landau gauge was used to fix the Yang-Mills symmetry of Witten action (2.30)]. The behavior of one-loop exactness of the $N = 2$ β function had been usually understood in terms of the analogous Adler-Bardeen theorem for the $U(1)$ axial current in the $N = 2$ SYM [9].

Despite the independence of the Witten partition function under changes in the coupling constant, the result (4.3) should not be surprising. In the twisted version, we can see

that the trace of the energy momentum is not zero but given by (see Ref. [8])

$$g_{\mu\nu}T^{\mu\nu} = \text{Tr} \left\{ D_\mu \varphi D^\mu \bar{\varphi} - 2i D_\mu \eta \psi^\mu + 2i \bar{\varphi} [\psi_\mu, \psi^\mu] + 2i \varphi [\eta, \eta] + \frac{1}{2} [\varphi, \bar{\varphi}]^2 \right\}, \quad (4.4)$$

meaning that S_W is not conformally invariant under the transformation

$$\delta g_{\mu\nu} = h(x) g_{\mu\nu}, \quad (4.5)$$

for an arbitrary real function $h(x)$ on M . Nonetheless, the trace of the energy-momentum tensor can be written as a total covariant divergence,

$$g_{\mu\nu}T^{\mu\nu} = D_\mu [\text{Tr}(\bar{\varphi} D^\mu \varphi - 2i \eta \psi^\mu)], \quad (4.6)$$

which means that S_W is invariant under a global rescaling of the metric, $\delta g_{\mu\nu} = w g_{\mu\nu}$, with w constant [8]. The $N = 2$ β function only vanishes if we take the weak coupling limit $g^2 \rightarrow 0$,

$$\beta_g^{N=2}(g^2 \rightarrow 0) = 0. \quad (4.7)$$

In this limit, the possibility of loop corrections to the effective action is eliminated, and the Donaldson polynomials are reproduced as the observables of the theory. There is no Ward identity, or a particular property of the vertices and propagators of S_W , capable of eliminating these quantum corrections for an arbitrary energy regime—this situation is distinct from the BS theory in the self-dual Landau gauges.

B. Baulieu-Singer topological theory

As suggested by the tree-level exactness of the BS theory in the self-dual Landau gauges, according to the analysis of the Feynman diagrams performed in Sec. III, we will formally prove that the self-dual BS theory is conformal. Before proving the vanishing of the BS β function in this gauge, we will first discuss the nonphysical character of the coupling constant in this off-shell approach, since g is introduced in the BS theory as a gauge parameter, in the trivial part of the BRST cohomology.

1. Nonphysical character of the β function in the off-shell approach

In Ref. [52], Brooks *et al.* argued that only one counterterm is required in the on-shell Witten theory, specifically for the Yang-Mills term $\text{Tr} F_{\mu\nu}^2$. In any case, the Donaldson invariants are described by DW theory in the weak coupling limit $g^2 \rightarrow 0$, where the theory is dominated by the classical minima. On the other hand, it is evident that the BS theory

is distinct from Brooks *et al.* construction because their methods are based on different BRST quantization schemes, with different cohomological properties. We do not expect a similar result in the BS theory. According to the cohomology of the BS model, if the β_g^{BS} is not zero, we should find that it is $\text{Tr}(F_{\mu\nu} \pm \tilde{F}_{\mu\nu})^2$ rather than $\text{Tr} F_{\mu\nu}^2$ which is renormalized.²⁵ In this way, the minima of the effective action preserves the instanton configuration at the quantum level, according to the global degrees of freedom of the instantons, which defines the observables of the BS theory—the Donaldson invariants.

A possible discrepancy between β functions for the BS approach in different gauge choices cannot be attributed to a gauge anomaly, since it is forbidden in these models due to the trivial BRST cohomology [54], cf. Eq. (2.102). For instance, if we would have chosen the gauge $D_\mu^{ab} \psi_\mu^b = 0$ for the topological ghost, with the covariant derivative instead of the ordinary one, the vector supersymmetry would be broken, and the gauge propagator would not vanish to all orders anymore. In ordinary Yang-Mills theories, the β function is an on-shell gauge-invariant physical quantity. Nonetheless, in gauge-fixed BRST topological theories of BS type, *the coupling constant is nonphysical*, introduced in the trivial part of the cohomology, together with the gauge-fixing action. In these terms, it is not contradictory that the β function is gauge dependent as it computes the running of a nonphysical parameter. We must observe that the physical observables of the theory, the Donaldson invariants, naturally do not depend on the gauge coupling. So that, there is no inconsistency that the observables of this kind of theory, described by topological invariants (exact numbers) do not depend on the coupling constant, and consequently on its running. Thence, g is an unobservable quantity.

As DW and BS theories possess the same observables, we should then consider the instanton configuration not as a gauge-fixing condition but as a physical requirement in order to obtain the correct degrees of freedom that correspond to the description of all global observables. Furthermore, the Atiyah-Singer index theorem [37] determines the dimension of the instanton moduli space, in which the Donaldson invariants are defined—see Refs. [56,57] for some exact instanton solutions, whose properties cannot be attributed to gauge artifacts.

2. Conformal structure of the self-dual gauges

To prove that the algebraic renormalization is in harmony with the Feynman diagram analysis in the self-dual Landau gauges, which shows that the BS model does not receive radiative corrections in this gauge, we must invoke a result

²⁵See Ref. [54], where Birmingham *et al.* employ the Batalin-Vilkovisky algorithm [55]—a similar quantization to the BS approach, i.e., with similar cohomological properties.

recently published in Ref. [28]. In this work, it was demonstrated that the Gribov ambiguities [58,59] are inoffensive in the self-dual BS theory.²⁶ The quantization of this model in a local section of the field space where the eigenvalues of the Faddeev-Popov determinant are positive is equivalent to its quantization in the whole field space. In other words, the introduction of the Gribov horizon does not affect the dynamics of the BS theory in SDL gauges, as its correspondent gap equation forbids the introduction of a Gribov massive parameter in the gauge field propagator. This result also suggests that the fiber bundle structure of the BS theory is trivial [61].

Let us quickly recall the Gribov procedure in the quantization of non-Abelian gauge theories [58,59]. It essentially consists in eliminating remaining gauge ambiguities usually present in non-Abelian gauge theories, known as Gribov copies, which are not eliminated in the FP procedure [62,63]. In Yang-Mills theories, the FP gauge-fixing procedure results in the well-known functional generator

$$\begin{aligned} Z_{\text{YM}} &= \mathcal{N} \int \mathcal{D}A |\det[-\partial_\mu D_\mu^{ab}]| \delta(\partial_\mu A_\mu) e^{-S_{\text{YM}}} \\ &= \mathcal{N} \int \mathcal{D}A \mathcal{D}\bar{c} \mathcal{D}c e^{-(S_{\text{YM}} + S_{\text{gf}})}, \end{aligned} \quad (4.8)$$

whereby S_{gf} is the well-known gauge-fixing action given by

$$S_{\text{gf}} = \int d^4x \left(\bar{c}^a \partial_\mu D_\mu^{ab} c^b - \frac{1}{2\alpha} (\partial_\mu A_\mu^a)^2 \right). \quad (4.9)$$

In (4.9), the limit $\alpha \rightarrow 0$ must be taken in order to reach the Landau gauge,

$$\partial_\mu A_\mu = 0. \quad (4.10)$$

Consider a gauge orbit,²⁷

$$A_\mu^U = U A_\mu U^\dagger - \frac{i}{g} (\partial_\mu U) U^\dagger, \quad (4.11)$$

with $U = e^{-igT^a\theta^a(x)} | U \in SU(N)$, with $\theta^a(x)$ being the local gauge parameters of the non-Abelian symmetry and T^a being the generators of the gauge group. The FP hypothesis [62,63] is that there is only one gauge configuration in the orbit (4.11) obeying the Landau gauge condition (4.10). In his seminal work [58], V. N. Gribov demonstrated that this hypothesis fails at the YM low-energy regime because one

can always find two configurations \tilde{A} and A obeying the Landau gauge condition and yet being related by a gauge transformation. At the infinitesimal level, the condition for a configuration A to have a Gribov copy \tilde{A} is that the FP operator develops zero modes through

$$-\partial_\mu D_\mu \theta = 0, \quad (4.12)$$

with θ^a taken as an infinitesimal parameter, $U \approx 1 - \theta^a T^a$. Equation (4.12) is recognized as the Gribov copies equation in the Landau gauge (and also in linear covariant gauges—see Refs. [64–67]). Equation (4.12) can be seen as an eigenvalue equation for the FP operator where θ is the zero mode of the operator. In Landau gauge, this operator is Hermitian, and thus its eigenvalues are real. For values of A_μ sufficiently small, the eigenvalues of the FP operator will be positive, as $-\partial^2$ only has positive eigenvalues.²⁸ As A_μ increases, it will attain a first zero mode (4.12). Such a region in which the FP operator has its first vanishing eigenvalue is called Gribov horizon (see also Ref. [59]). Gribov's proposal was to restrict the path integral domain to the region Ω (the Gribov region) defined by

$$\Omega = \{A_\mu^a; \partial_\mu A_\mu = 0, -\partial D > 0\}. \quad (4.13)$$

Such restriction ensures the elimination of all infinitesimal copies and also guarantees that no independent field is eliminated [68].

The implementation of the restriction to the Gribov region Ω is accomplished by the introduction of a step function $\Theta(-\partial D)$ in the Feynman path integral, which leads to the well-known *no-pole condition* for the FP ghost propagator $\langle (\partial D)^{-1} \rangle$, which explodes at when a zero mode is attained. The main result of introducing the restriction of the Feynman path integral domain to the Gribov region is a modified gluon propagator, due to the emergence of a massive parameter for the gauge field, the so-called Gribov parameter γ . In the presence of the Gribov horizon, the gluon propagator takes the form

$$\langle A_\mu^a(k) A_\nu^b(k) \rangle = \delta^{ab} \delta(p+k) \frac{k^2}{k^4 + \gamma^4} P_{\mu\nu}(k), \quad (4.14)$$

where $P_{\mu\nu}(k) = \delta_{\mu\nu} - k_\mu k_\nu / k^2$, and γ is fixed by the gap equation [60,69],

$$\frac{\partial \Gamma}{\partial \gamma^2} = 0. \quad (4.15)$$

²⁶The result was proved to be valid to all orders in perturbation theory by making use of the Zwanziger's approach [60] to the Gribov problem [58].

²⁷The gauge orbit is the equivalence class of gauge field configurations that only differ by a gauge transformation, representing thus the same physics according to the gauge principle.

²⁸In Abelian theories, such as QED, $-\partial^2$ is the “FP operator,” and the copy equation only possesses trivial solutions in the thermodynamic limit, meaning that the Gribov copies are inoffensive in this case.

According to Zwanziger's generalization [60], the gap equation above is valid to all orders in perturbation theory—see Refs. [70,71], in which the all-order proof of the equivalence between Gribov and Zwanziger methods was worked out. An important feature of the Gribov parameter is that it only affects the infrared dynamics. The matching between Gribov-Zwanziger theory and recent lattice data is achieved through the introduction of two-dimensional condensates; see Ref. [72]. The introduction of the Gribov horizon in the action explicitly breaks the BRST symmetry. This is usually an unwanted result, as the BRST symmetry is necessary to prove the unitarity, to ensure the renormalizability to all orders, and to define the physical gauge-invariant observables of the theory [73–75]. This breaking, however, brought to light the physical meaning of the infrared γ parameter and its intrinsic nonperturbative character. One can prove that the BRST breaking is proportional to γ^2 , in other words, the BRST symmetry is restored in the perturbative regime. One says that the BRST symmetry is only broken in a *soft* way, cf. Refs. [75–78]. Only more recently, a universal, gauge-independent, (nonperturbative) BRST-invariant way to introduce the Gribov horizon was developed [66,67,79–81].

In the self-dual topological BS theory, it was proved in Ref. [28] that all topological gauge copies associated to the gauge ambiguities (2.68) and (2.69) are eliminated through the introduction of the usual Gribov restriction, in which the path integral domain is restricted to the region Ω —see Eq. (4.13). Moreover, because of the triviality of the gap equation, it was verified that the Gribov copies do not affect the infrared dynamics of the self-dual BS theory because $\gamma_{\text{BS}} = 0$ is the only possible solution of the gap equation.²⁹ Thus, no mass parameter seems to emerge in the BS theory, preserving its conformal character at quantum level. Specifically, the tree-level exactness in SDL gauges is preserved. Such a behavior can be inferred from the absence of radiative corrections, which ensures the semi-positivity of all two-point functions. The FP ghost propagator, for instance, reads

$$\langle \bar{c}^a(k) c^b(k) \rangle = \delta^{ab} \frac{1}{k^2}, \quad (4.16)$$

which is valid to all orders, demonstrating that the FP operator will remain positive definite at quantum level, consistent with the inverse of the FP propagator being positive, thus proving that we are inside the Gribov region. Moreover, the gauge two-point function remains trivial, i.e., $\langle A_\mu^a(k) A_\nu^b(k) \rangle = 0$ to all orders.

²⁹A similar situation occurs in the $N = 4$ SYM, which possesses a vanishing β function, indicating the conformal structure of the self-dual BS. The absence of an invariant scale makes it impossible to attach a dynamical meaning to the Gribov parameter [82].

Exploring the positive-definite-ness of the FP ghost propagator, we are able to safely perform the following shifts:

$$\begin{aligned} \bar{\eta}^a &\mapsto \bar{\eta}^a + \bar{c}^a, \\ \varphi^b &\mapsto \varphi^b - g f^{cde} (\partial_\nu D_\nu^{bc})^{-1} \partial_\mu (c^d \psi_\mu^e), \\ \bar{c}^a &\mapsto \bar{c}^a - \frac{1}{2} g f^{cde} \bar{\chi}_{\mu\nu}^d (F_\pm^e)_{\mu\nu} (\partial_\nu D_\nu^{ca})^{-1}. \end{aligned} \quad (4.17)$$

It is worth noting that these shifts generate a trivial Jacobian. Calling $\frac{1}{2}\rho_1 = \alpha$ and $\frac{1}{2}\rho_2 = \beta$, and implementing the BS gauge constraints (2.93) and (2.95), together with $\partial_\mu \psi_\mu = 0$, the final gauge-fixing action, integrating out the auxiliary fields b and B in the action (2.98), is

$$\begin{aligned} S_{gf}(\alpha, \beta) = \int d^4x &\left[-\frac{1}{2\alpha} (\partial A)^2 - \frac{1}{4\beta} F_\pm^2 \right] \\ &- \int d^4x \left[(\bar{\eta}^a - \bar{c}^a) \partial_\mu \psi_\mu^a + \bar{c}^a \partial_\mu D_\mu^{ab} c^b \right. \\ &- \frac{1}{2} g f^{abc} \bar{\chi}_{\mu\nu}^a c^b (F_{\mu\nu}^c \pm \tilde{F}_{\mu\nu}^c) \\ &- \bar{\chi}_{\mu\nu}^a \left(\delta_{\mu\alpha} \delta_{\nu\beta} \pm \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \right) D_\alpha^{ab} \psi_\beta^b \\ &\left. + \bar{\varphi}^a \partial_\mu D_\mu^{ab} \varphi^b + g f^{abc} \bar{\varphi}^a \partial_\mu (c^b \psi_\mu^c) \right], \end{aligned} \quad (4.18)$$

where $F_\pm = F \pm \tilde{F}$ and $D_\pm \equiv (\delta_{\mu\alpha} \delta_{\nu\beta} - \delta_{\nu\alpha} \delta_{\mu\beta} \pm \epsilon_{\mu\nu\alpha\beta}) D_\alpha^{ab}$. The self-dual Landau gauges is recovered by setting $\alpha, \beta \rightarrow 0$. Then, applying the shifts (4.17) on the action $S_{gf}(\alpha, \beta)$, one gets

$$\begin{aligned} S_{gf}^{\text{shifted}}(\alpha, \beta) = \int d^4x &\left[-\frac{1}{2\alpha} (\partial A)^2 - \frac{1}{4\beta} F_\pm^2 \right] \\ &- \int d^4x \left[\bar{\eta}^a \partial_\mu \psi_\mu^a + \bar{c}^a \partial_\mu D_\mu^{ab} c^b \right. \\ &- \bar{\chi}_{\mu\nu}^a \left(\delta_{\mu\alpha} \delta_{\nu\beta} \pm \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \right) D_\alpha^{ab} \psi_\beta^b \\ &\left. + \bar{\varphi}^a \partial_\mu D_\mu^{ab} \varphi^b \right]. \end{aligned} \quad (4.19)$$

As the Jacobian of the shifts that performs $S_{gf}(\alpha, \beta) \rightarrow S_{gf}^{\text{shifted}}(\alpha, \beta)$ is trivial, the quantization of both actions are perturbatively equivalent, cf. Ref. [34]. Such a Jacobian only generates a number that can be absorbed by the normalization factor. This shows that the discrete symmetries (3.22) and (3.23) (present in the Witten theory) can be recovered, which naturally impose the relations

$$Z_c = Z_{\bar{c}} \quad \text{and} \quad Z_\varphi = Z_{\bar{\varphi}} \quad (4.20)$$

to be valid in the BS theory. Hence, combining (4.20) with the Z-factor system (3.19), one obtains

$$Z_g = 1, \quad (4.21)$$

which proves that the algebraic analysis is in harmony with the result obtained via the study of the Feynman diagrams in the presence of the vector supersymmetry, i.e., that the topological BS theory (following the self-dual Landau gauges) is conformal, in accordance with the absence of radiative corrections.

The algebraic renormalization and the Feynman diagram analysis consist of two independent methods. In the loop expansion, used to construct the diagrams in Sec. III A, we expand around the trivial $A = 0$ sector, i.e., around an instanton with winding number zero. One may wonder if it is physically relevant, thinking about the importance of instanton configurations in the topological theory, in order to construct the Donaldson invariants. Exactly the topological nature of the off-shell BS theory saves the day here. Let us first remark that it is possible to write down a BRST-invariant version of the Gribov restriction, that is, if γ were to be nonzero, while preserving equivalence with the above formalism³⁰; see Refs. [66,79,81] for details. As already reminded before, the topological partition function does not depend on the coupling g . This means all observables can be computed in the $g \rightarrow 0$ limit. Expanding around a nontrivial instanton background rather than around $A = 0$ would lead to corrections of the type e^{-1/g^2} into the effective action, but the latter vanishes exponentially fast once $g \rightarrow 0$ is considered, which only represents a liberty of the theory; i.e., it would not affect the global observables (see [28]). As such, we can *a priori* work around $A = 0$, without losing the generality of the result, which will be unaltered for a generic instanton background.

V. CHARACTERIZATION OF THE DW/BS CORRESPONDENCE

We will characterize in this section the quantum correspondence between the twisted $N = 2$ SYM in the ultra-violet regime and the conformal Baulieu-Singer theory in the SDL gauges.

A. Quantum equivalence between DW and self-dual BS theories

The result obtained in (4.21) in the SDL gauges proves that the self-dual BS β function vanishes. This result is completely different from the twisted $N = 2$ SYM which receives one-loop corrections, and possesses a nonvanishing β function given by (4.3). The correspondence between the BS and $N = 2$ β -functions occurs when we take the weak coupling limit ($g^2 \rightarrow 0$) on the $N = 2$ side. In this limit, $\beta_g^{N=2} \rightarrow 0$. On the BS side, however, the vanishing of the β function is valid for an arbitrary coupling constant, and not only for a weak coupling, with the conformal

property being a consequence of the vector supersymmetry which forbids radiative corrections. In DW theory, such a property is obtained by taking $g^2 \rightarrow 0$ as small as we want (as long as $g^2 \neq 0$), as a consequence of the property that shows that the observables of DW theory are insensitive under changes of g . That is how Witten computed its partition function that naturally reproduces the Donaldson invariants for four manifolds, i.e., by eliminating the influence of the vertices at $g^2 \rightarrow 0$, and taking only the quadratic part of the action. The BS theory is also invariant under changes of g , as it only appears in the trivial part of the BRST cohomology, but the tree-level exactness is a general property of the BS theory in self-dual Landau gauges; i.e., it is valid for an arbitrary perturbative regime.

The characterization of the correspondence between the twisted $N = 2$ SYM and a conformal field theory is now complete. The fact that the twisted $N = 2$ SYM in the weak coupling limit and the BS theory share the same global observables is a well-known result in literature [22,23,83]. In the DW theory, the Donaldson invariants are defined by the δ supersymmetry (2.31) according to the bidescent equations encoded in (2.64). In the BS one, the same bidescent equations appears, constructed from the n th Chern class $\tilde{\mathcal{W}}_n$ defined in terms of the universal curvature in the extended space $M \times \mathcal{A}/\mathcal{G}$. Such an equivalence is ensured by the equivariant cohomology that allows for the replacement $s \rightarrow \delta$, as $\tilde{\mathcal{W}}_n$ is invariant under ordinary Yang-Mills transformations. We are now defining in which energy regimes such an equivalence occurs when we employ the self-dual Landau in the BS and are formally proving the correspondence between the twisted $N = 2$ and a conformal gauge theory. The fact that the observables are the same, as a consequence of the equivariant cohomology, does not characterize the correspondence at quantum level (we will provide a counterexample in the next section). The correspondence between the DW and BS observables, given by the equivalence

$$\begin{aligned} & \langle \mathcal{O}_{\alpha_1}^{\text{DW}}(\varphi_i) \mathcal{O}_{\alpha_2}^{\text{DW}}(\varphi_i) \cdots \mathcal{O}_{\alpha_p}^{\text{DW}}(\varphi_i) \rangle_{g^2 \rightarrow 0} \\ &= \langle \mathcal{O}_{\alpha_1}^{\text{BS}}(\varphi'_i) \mathcal{O}_{\alpha_2}^{\text{BS}}(\varphi'_i) \cdots \mathcal{O}_{\alpha_p}^{\text{BS}}(\varphi'_i) \rangle_{\text{SDL}}, \end{aligned} \quad (5.1)$$

is independent of the gauge choice. The field content that defines the observables is the same in both theories, $\varphi_i \equiv \varphi'_i$, since the observables are independent of the FP ghosts (which appear in the gauge-fixed BS action). In a few words, $\mathcal{O}_{\alpha}^{\text{DW}}(\varphi_i) \equiv \mathcal{O}_{\alpha}^{\text{BS}}(\varphi'_i)$, represented by the product in Eq. (2.64). As demonstrated in Sec. II B 5, the BS observables naturally do not depend on (c, \bar{c}) , due to the invariance of $\tilde{\mathcal{W}}_n$ under s_{YM} (the Yang-Mills BRST operator). The BS reproduces the Donaldson polynomials only as a consequence of the structure of the off-shell BRST transformations (2.71). Witten works exclusively in the moduli space \mathcal{A}/\mathcal{G} , i.e., without fixing the gauge, its observables being naturally independent of the FP ghost.

³⁰In the sense that all correlation functions will be identical.

TABLE III. Characterization of the DW/BS correspondence.

Twisted $N = 2$ SYM	BS in self-dual Landau gauges
On-shell δ supersymmetry	Off-shell BRST + vector supersymmetry
Donaldson invariants (δ)	Donaldson invariants ($s \rightarrow \delta$)
$g^2 \rightarrow 0$	Arbitrary g
Any Riemannian manifold, $g_{\mu\nu}$	Euclidean spaces, $\delta_{\mu\nu}$
\mathcal{A}/\mathcal{G}	Gauge fixed $ \gamma_{\text{Gribov}}^4 = 0$
$\beta_g^{N=2} \rightarrow 0$	$\beta_g^{\text{BS}}(g) = 0$

Another crucial point is that the gauge-fixing term in which the FP ghosts are introduced in the self-dual BS theory does not allow for the influence of Gribov copies. For this reason, working in the moduli space, \mathcal{A}/\mathcal{G} in the DW theory are completely correspondent to work in the BS theory in SDL gauges for an arbitrary g , since $\gamma^4 = 0$. Fixing the remaining YM gauge symmetry of Witten's action (2.30), instead of working in \mathcal{A}/\mathcal{G} , would not break such a correspondence since the Gribov copies could only affect the nonperturbative regime, being inoffensive at the ultraviolet limit $g^2 \rightarrow 0$. The quantum equivalence is illustrated by the agreement between the β functions, $\beta_g^{N=2}(g^2 \rightarrow 0) = \beta_g^{\text{BS}}(g) = 0$.

Finally, because of the property of Witten's theory (2.35), which ensures that Witten's theory can be extended to any Riemannian manifold, the DW/BS correspondence is characterized as follows: the twisted $N = 2$ SYM at $g^2 \rightarrow 0$, in any Riemannian manifold (that can be continuously deformed into each other, including \mathbb{R}^4),³¹ defined in the instanton moduli space \mathcal{A}/\mathcal{G} , is correspondent to the topological BS theory in the self-dual Landau gauges in Euclidean spaces, in an arbitrary perturbative regime. Such a BS theory consists of a conformal field theory, where the gauge copies are inoffensive in the infrared, since the massive infrared parameter originated from the gauge copies vanishes in this gauge—see Table III.

We emphasize that we use perturbative techniques to prove the conformal property of the self-dual BS theory. The fact that the self-dual BS theory in the strong limit $g^2 \rightarrow \infty$ is also correspondent to Witten's TQFT defined at $g^2 \rightarrow 0$ can be conjectured by means of the cohomological structure of the off-shell BRST symmetry. Changing g in the BS theory is equivalent to adding a BRST-exact term in the action; i.e., it is equivalent to performing a change in the gauge choice. Moreover, the global observables of BS theory, described by the Chern classes \mathcal{W}_n , do not depend on the gauge choice, having only the power of reproducing

the Donaldson invariants for 4-manifolds. Also, the Gribov ambiguities are irrelevant to the BS model (at least in the self-dual Landau gauges), a property that should remain valid at the strong regime.

B. Considerations about the gauge dependence and possible generalizations

Because of the exact nature of the topological Donaldson invariants, which are given by exact numbers, we can consider the supposition that quantum corrections should not affect the tree-level results and that the description of the Donaldson invariants by the gauge-fixed BS approach should not depend on the gauge choice. Although intuitive, this argument is not sufficient or complete. As a counterexample, we invoke the β function obtained by Birmingham *et al.* in Ref. [54], in which the Batalin-Vilkovisky (BV) algorithm [55] was employed. Such a model possesses cohomological properties similar to those in the BS theory. For a particular configuration of auxiliary fields used in Ref. [54], the BV method recovers the BS gauges with $\rho_1 = \rho_2 = 0$, together with the constraint $D_\mu \psi_\mu = 0$ —see Eq. (2.93). This constraint on the topological ghost, with the covariant derivative instead of the ordinary one, breaks the vector supersymmetry, allowing for quantum corrections. Consequently, the β function computed by Birmingham *et al.* is not zero. As noted by the authors of Ref. [54], it is $\text{Tr}(F_{\mu\nu} \pm \tilde{F}_{\mu\nu})^2$ rather than $\text{Tr}F_{\mu\nu}^2$ which is renormalized, meaning that the vacuum configurations are preserved. As expected, the structure of the instanton moduli space, that defines the Donaldson invariants, is not altered.

About the gauge dependence of the β function in off-shell topological gauge models, see Sec. IV B 1. The coupling constant in this model is nonphysical, belonging to the trivial part of cohomology. Any change in the unobservable coupling constant only leads to a BRST-exact variation. The only observables are the global ones, and we must expect that, for different gauge choices, the global observables are not affected. According to the result of Birmingham *et al.* in Ref. [54], it is possible to obtain nontrivial quantum corrections without destroying the topological properties of the underlying theory, preserving the observables. Analogously, we can consider the possibility in which the fields could also be consistently

³¹This is the only requirement that guarantees that the observables of both sides are correspondent, as the conformal BS is defined in Euclidean spaces. In the DW theory, spaces that can be continuously deformed, one into the other, represent the same Donaldson invariants for a class of manifolds, since a continuous variation of the metric is equivalent to adding a δ -exact term to the action, which does not alter the observables.

renormalized, i.e., in such a way that the bidescent equations, that define the Donaldson invariants, are not altered. This reasoning shows that the renormalization of topological gauge theories, consistent with the global observables, is not a trivial issue.

The vector supersymmetry, that leads to the conformal BS theory, is a particular symmetry of the self-dual Landau gauge. One must note that $\partial_\mu A_\mu = D_\mu A_\mu$, due to antisymmetric property of f^{abc} . To impose $\partial_\mu \psi_\mu = 0$ or $D_\mu \psi_\mu = 0$ automatically preserves the instanton moduli space, where A_μ and ψ_μ obey the same equations of motion, according to the Atiyah-Singer theorem that correctly defines its dimension. The preservation of the instanton moduli space is then the only requirement of the topological theory, being the conformal property a particular feature of the self-dual Landau gauges. The dimension of the instanton moduli space should not depend on the gauge choice, being protected by the Atiyah-Singer theorem.

The second generalization that can be worked out is in the direction of developing a model in which the BS theory can be constructed for a generic $g_{\mu\nu}$. Again, any change on the Euclidean metric to a generic one corresponds to the addition of a BRST-exact term in the BS theory. This means that such a variation can be interpreted as a change in the gauge choice, and the previous discussion can be also applied here. The vector supersymmetry is easily defined in flat spaces. To reproduce the conformal properties of the SDL gauges in Euclidean space for a generic $g_{\mu\nu}$, we will face the task of finding a class of metrics whose corresponding action possesses a rich set of Ward identities (\mathbf{W}_I), capable of reproducing the same effect of the self-dual ones, see Appendix, given by the 11 functional operators $\mathbf{W}_I^{\text{BS}} \equiv \{\mathcal{S}, \mathcal{W}_\mu, \mathcal{W}_1^a, \mathcal{W}_2^a, \mathcal{W}_3^a, \mathcal{W}_4^a, \mathcal{G}_\varphi^a, \mathcal{G}_1^a, \mathcal{G}_2^a, \mathcal{T}, \mathcal{G}_3\}$.

Besides that, we will face another inconvenient task of having to study the Gribov copies in curved spacetimes, which is a highly nontrivial problem. The vanishing of the Gribov parameter in the self-dual BS in Euclidean spaces ensures that the DW/BS correspondence is valid for a generic coupling constant on the BS side.

VI. CONCLUSIONS

We perform a comparative study between Donaldson-Witten TQFT [8] and the Baulieu-Singer approach [13]. While DW theory is obtained via the twist transformation of the $N = 2$ SYM, BS theory is based on the BRST gauge fixing of an action composed of topological invariants—see Secs. II A and II B. Besides that, Witten's theory is defined by an on-shell supersymmetry, according to the fermionic symmetry, see Eq. (2.31), whose associated charge is only nilpotent if we use the equations of motion. Such a symmetry defines the observables of the theory, given by the Donaldson polynomials. The BS approach, in turn, consists of an off-shell BRST construction, which enjoys the topological BRST symmetry (2.71), whose observables

are also given by the Donaldson invariants, due to the equivariant cohomology—defined by Witten's fermionic symmetry—which also characterizes the BS observables that can be written as Chern classes for the curvature in the extended space $M \times \mathcal{A}/\mathcal{G}$, where M is a Riemannian manifold and \mathcal{A}/\mathcal{G} is the instanton moduli space; see Sec. II B 5. Despite sharing the same observables, we note that these theories do not necessarily have the same quantum properties, as Witten and BS actions do not differ by a BRST-exact term, cf. Eq. (2.123). In a few words, the BRST quantization schemes of Witten's and BS theories are not equivalent.

In harmony with the quantum properties of BS approach in the self-dual Landau gauges, see Sec. III, we formally prove that the topological self-dual BS theory is conformal. According to the Feynman diagram analysis performed in Ref. [26], we proved the absence of quantum corrections in the BS theory in the presence of the vector supersymmetry. In Sec. IV B 1, we discussed the nonphysical character of the coupling constant in the off-shell BS approach. Then, to construct an algebraic proof that the self-dual BS is conformal, we first solved the renormalization ambiguity in topological Yang-Mills theories described in Ref. [25], using a nonlocal transformation which recovers discrete symmetries between ghost and antighost fields. Such nonlocal transformations showed to be a freedom of the self-dual BS theory due to the triviality of the Gribov copies in the SDL gauges [28]; see Sec. IV B 2. As a consequence of this triviality, using the Ward identities of the model—together with the symmetry between the topological and Faddeev-Popov ghosts introduced in Ref. [24]—and employing algebraic renormalization techniques, we conclude that $Z_g = 1$, i.e., that the self-dual BS β function indeed vanishes.

We observed that these theories do not possess the same quantum structure, by comparing the β function of each model; see Sec. IV. From this analysis, we characterized the correspondence between the twisted $N = 2$ SYM and BS theories at quantum level, defining in which regimes such a correspondence occurs; see Sec. V. In spite of having distinct BRST constructions, we conclude that working in the instanton moduli space \mathcal{A}/\mathcal{G} on the DW side is completely equivalent to working in the self-dual Landau gauges on the BS one, since the Gribov copies do not affect its infrared dynamics. In a few words, the twisted $N = 2$ SYM in any Riemannian manifold (that can be continuously deformed into $M = \mathbb{R}^4$), in the weak coupling limit $g^2 \rightarrow 0$, is correspondent to the BS theory in the self-dual Landau gauges in an arbitrary perturbative regime, which consists of a conformal gauge theory defined in Euclidean flat space; see Table III. Such a characterization could shed some light on the connection between supersymmetry, topology, off-shell BRST symmetry, and non-Abelian conformal gauge theories in four dimensions.

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APPENDIX: BS WARD IDENTITIES IN THE SELF-DUAL LANDAU GAUGES

The BS action in the self-dual Landau gauges (3.8) enjoys the following Ward identities:

- (i) The Slavnov-Taylor identity, which expresses the BRST invariance of the action (3.8),

$$\mathcal{S}(\Sigma) = 0, \quad (\text{A1})$$

where

$$\begin{aligned} \mathcal{S}(\Sigma) = \int d^4z \left[\left(\psi_\mu^a - \frac{\delta \Sigma}{\delta \Omega_\mu^a} \right) \frac{\delta \Sigma}{\delta A_\mu^a} + \frac{\delta \Sigma}{\delta \tau_\mu^a} \frac{\delta \Sigma}{\delta \psi_\mu^a} + \left(\varphi^a + \frac{\delta \Sigma}{\delta L^a} \right) \frac{\delta \Sigma}{\delta c^a} + \frac{\delta \Sigma}{\delta E^a} \frac{\delta \Sigma}{\delta \varphi^a} \right. \\ \left. + b^a \frac{\delta \Sigma}{\delta \bar{c}^a} + \bar{\eta}^a \frac{\delta \Sigma}{\delta \bar{\varphi}^a} + B_{\mu\nu}^a \frac{\delta \Sigma}{\delta \tilde{\chi}_{\mu\nu}^a} + \Omega_\mu^a \frac{\delta \Sigma}{\delta \tau_\mu^a} + L^a \frac{\delta \Sigma}{\delta E^a} + K_{\mu\nu}^a \frac{\delta \Sigma}{\delta \Lambda_{\mu\nu}^a} \right]. \end{aligned} \quad (\text{A2})$$

- (ii) Ordinary Landau gauge fixing and the Faddeev-Popov antighost equation:

$$\begin{aligned} \mathcal{W}_1^a \Sigma &= \frac{\delta \Sigma}{\delta b^a} = \partial_\mu A_\mu^a, \\ \mathcal{W}_2^a \Sigma &= \frac{\delta \Sigma}{\delta \bar{c}^a} - \partial_\mu \frac{\delta \Sigma}{\delta \Omega_\mu^a} = -\partial_\mu \psi_\mu^a. \end{aligned} \quad (\text{A3})$$

- (iii) Topological Landau gauge fixing and the bosonic antighost equation:

$$\begin{aligned} \mathcal{W}_3^a \Sigma &= \frac{\delta \Sigma}{\delta \bar{\eta}^a} = \partial_\mu \psi_\mu^a, \\ \mathcal{W}_4^a \Sigma &= \frac{\delta \Sigma}{\delta \bar{\varphi}^a} - \partial_\mu \frac{\delta \Sigma}{\delta \tau_\mu^a} = 0. \end{aligned} \quad (\text{A4})$$

- (iv) The bosonic ghost equation,

$$\mathcal{G}_\varphi^a \Sigma = \Delta_\varphi^a, \quad (\text{A5})$$

where

$$\begin{aligned} \mathcal{G}_\varphi^a &= \int d^4z \left(\frac{\delta}{\delta \varphi^a} - g f^{abc} \bar{\varphi}^b \frac{\delta}{\delta b^c} \right), \\ \Delta_\varphi^a &= g f^{abc} \int d^4z (\tau_\mu^b A_\mu^c + E^b c^c + \Lambda_{\mu\nu}^b \tilde{\chi}_{\mu\nu}^c). \end{aligned} \quad (\text{A6})$$

- (v) The ordinary Faddeev-Popov ghost equation,

$$\mathcal{G}_1^a \Sigma = \Delta^a, \quad (\text{A7})$$

where

$$\begin{aligned}\mathcal{G}_1^a &= \int d^4z \left[\frac{\delta}{\delta c^a} + g f^{abc} \left(\bar{c}^b \frac{\delta}{\delta b^c} + \bar{\varphi}^b \frac{\delta}{\delta \bar{\eta}^c} + \bar{\chi}_{\mu\nu}^b \frac{\delta}{\delta B_{\mu\nu}^c} + \Lambda_{\mu\nu}^b \frac{\delta}{\delta K_{\mu\nu}^c} \right) \right], \\ \Delta^a &= g f^{abc} \int d^4z (E^b \varphi^c - \Omega_\mu^b A_\mu^c - \tau_\mu^b \psi_\mu^c - L^b c^c + \Lambda_{\mu\nu}^b B_{\mu\nu}^c - K_{\mu\nu}^b \bar{\chi}_{\mu\nu}^c).\end{aligned}\quad (\text{A8})$$

(vi) The second Faddeev-Popov ghost equation,

$$\mathcal{G}_2^a \Sigma = \Delta^a, \quad (\text{A9})$$

where

$$\mathcal{G}_2^a = \int d^4z \left[\frac{\delta}{\delta c^a} - g f^{abc} \left(\bar{\varphi}^b \frac{\delta}{\delta \bar{c}^c} + A_\mu^b \frac{\delta}{\delta \psi_\mu^c} + c^b \frac{\delta}{\delta \varphi^c} - \bar{\eta}^b \frac{\delta}{\delta b^c} + E^b \frac{\delta}{\delta L^c} \right) \right]. \quad (\text{A10})$$

(vii) Vector supersymmetry,

$$\mathcal{W}_\mu \Sigma = 0, \quad (\text{A11})$$

where

$$\begin{aligned}\mathcal{W}_\mu &= \int d^4z \left[\partial_\mu A_\nu^a \frac{\delta}{\delta \psi_\nu^a} + \partial_\mu c^a \frac{\delta}{\delta \varphi^a} + \partial_\mu \bar{\chi}_{\nu\alpha}^a \frac{\delta}{\delta B_{\nu\alpha}^a} + \partial_\mu \bar{\varphi}^a \left(\frac{\delta}{\delta \bar{\eta}^a} + \frac{\delta}{\delta \bar{c}^a} \right) \right. \\ &\quad \left. + (\partial_\mu \bar{c}^a - \partial_\mu \bar{\eta}^a) \frac{\delta}{\delta b^a} + \partial_\mu \tau_\nu^a \frac{\delta}{\delta \Omega_\nu^a} + \partial_\mu E^a \frac{\delta}{\delta L^a} + \partial_\mu \Lambda_{\nu\alpha}^a \frac{\delta}{\delta K_{\nu\alpha}^a} \right].\end{aligned}\quad (\text{A12})$$

(viii) Bosonic nonlinear symmetry,

$$\mathcal{T}(\Sigma) = 0, \quad (\text{A13})$$

where

$$\mathcal{T}(\Sigma) = \int d^4z \left[\frac{\delta \Sigma}{\delta \Omega_\mu^a} \frac{\delta \Sigma}{\delta \psi_\mu^a} - \frac{\delta \Sigma}{\delta L^a} \frac{\delta \Sigma}{\delta \varphi^a} - \frac{\delta \Sigma}{\delta K_{\mu\nu}^a} \frac{\delta \Sigma}{\delta B_{\mu\nu}^a} + (\bar{c}^a - \bar{\eta}^a) \left(\frac{\delta \Sigma}{\delta \bar{c}^a} + \frac{\delta \Sigma}{\delta \bar{\eta}^a} \right) \right].$$

(ix) Global ghost supersymmetry,

$$\mathcal{G}_3 \Sigma = 0, \quad (\text{A14})$$

where

$$\mathcal{G}_3 = \int d^4z \left[\bar{\varphi}^a \left(\frac{\delta}{\delta \bar{\eta}^a} + \frac{\delta}{\delta \bar{c}^a} \right) - c^a \frac{\delta}{\delta \varphi^a} + \tau_\mu^a \frac{\delta}{\delta \Omega_\mu^a} + 2E^a \frac{\delta}{\delta L^a} + \Lambda_{\mu\nu}^a \frac{\delta}{\delta K_{\mu\nu}^a} \right]. \quad (\text{A15})$$

The last two symmetries are the new ones introduced in Ref. [24]. The nonlinear bosonic symmetry (vii) is precisely the one discussed in Sec. III. B, see Eq. (3.15), which relates the FP and topological ghosts. We remark that the Faddeev-Popov ghost equations (A7) and (A9) can be combined to obtain an exact global supersymmetry,

$$\Delta \mathcal{G}^a \Sigma = 0, \quad (\text{A16})$$

where

$$\Delta\mathcal{G}^a = \mathcal{G}_1^a - \mathcal{G}_2^a = \int d^4z f^{abc} \left[(\bar{c}^b - \bar{\eta}^b) \frac{\delta}{\delta b^c} + \bar{\varphi}^b \left(\frac{\delta}{\delta \bar{\eta}^c} + \frac{\delta}{\delta \bar{c}^c} \right) + A_\mu^b \frac{\delta}{\delta \psi_\mu^c} + \bar{\chi}_{\mu\nu}^b \frac{\delta}{\delta B_{\mu\nu}^c} + c^b \frac{\delta}{\delta \varphi^c} + \Lambda_{\mu\nu}^b \frac{\delta}{\delta K_{\mu\nu}^c} + \tau_\mu^b \frac{\delta}{\delta \Omega_\mu^c} + E^b \frac{\delta}{\delta L^c} \right]. \quad (\text{A17})$$

We observe the similarity of Eq. (A16) with the vector supersymmetry (A11). It is also worth mentioning that, even though the ghost number of the operator (A17) is -1 , resembling an anti-BRST symmetry, it is not a genuine anti-BRST symmetry—see, for instance, Ref. [84] for the explicit anti-BRST symmetry in topological gauge theories.

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