Chiral vortical effect for vector fields

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We consider the photonic vortical effect, i.e., the difference of the flows of left- and right-handed photons along the vector of angular velocity in rotating photonic medium. Two alternative frameworks to evaluate the effect are considered, both of which have already been tried in the literature: first, the standard thermal field theory and, alternatively, Hawking-radiation-type derivation. In our earlier attempt to compare the two approaches, we found a crucial factor of 2 difference. Here, we revisit the problem, paying more attention to details of infrared regularizations. We find out that introduction of an infinitesimal mass of the vector field brings the two ways of evaluating the chiral vortical effect into agreement with each other. Some implications, on both the theoretical and phenomenological sides, are mentioned.

I. INTRODUCTION

We will consider thermodynamics of media whose constituents are massless particles of nonzero spin S. The best-studied case is $S = 1/2$, and, as a starting point, we quote some results obtained for spin-1/2 constituents. However, we are mainly interested in properties of photonic media, consisting of left- and right-handed photons. Recently, the number of papers devoted to this case has also been growing; see, in particular, Refs. [1–9].

The chiral vortical effect was first evaluated by Vilenkin [10], who considered gas of noninteracting spin-1/2 fermions in a rotating coordinate system. It was demonstrated that there exists a current of the particle number flowing along the vector of the angular velocity $\vec{\Omega}$. Numerically, in the case of a single right-handed Weyl spinor, the current is given by

$$\vec{J}_N(S = 1/2) = \left(\frac{\mu_R^2}{4\pi^2} + \frac{T^2}{12}\right)\vec{\Omega}, \qquad (1.1)$$

where μ_R is the chemical potential and T is the temperature. The term with μ_R^2 in the expression (1.1) is, in fact, proportional to k_F^2, where k_F is the radius of the Fermi sphere (of massless chiral fermions). A microscopic derivation of the current in terms of the Fermi sphere in all details can be found in Ref. [11].

Nowadays, the literature on the chiral effects is huge, and we cannot even briefly review the subject. Here, we mention only the pioneering paper [12] which opened the chapter on theory of the chiral effects in the regime of strong coupling. It turns out that in the hydrodynamic approximation and in the absence of dissipation one can derive chiral effects without exploiting the noninteracting gas approximation. Moreover, the magnitude of the chiral effects is determined uniquely in terms of the corresponding chiral anomaly of the fundamental theory underlying the phenomenological hydrodynamic approach; for a review, see Ref. [13].

In particular, in the case of the chiral vortical effect (1.1), the term proportional to the chemical potential squared is indeed related to the chiral anomaly. There is a simple substitution which allows one to generate chiral hydrodynamic effects from the standard chiral anomaly [14]:

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$$eA_\alpha \to eA_\alpha + \mu u_\alpha, \qquad (1.2)$$

where A_α is the electromagnetic potential, e is the charge of the fundamental constituents, and u_α is the 4-velocity of an element of the medium. One can readily check that the substitution (1.2) does reproduce the μ_R^2 term in Eq. (1.1).

On the other hand, any field-theoretic interpretation of the T^2 term in Eq. (1.1) had been missing until the recent paper [15], which relates it to the gravitational chiral anomaly (for earlier attempts in the same direction, see [16,17]). In the case of massless spin-1/2 particles interacting with an external gravitational field, the anomaly reads

$$\nabla_\mu J_N^\mu = \frac{1}{384\pi^2} R_{\mu\nu\kappa\lambda}\tilde{R}^{\mu\nu\kappa\lambda}, \qquad (1.3)$$

where $R_{\mu\nu\kappa\lambda}$ is the Riemann tensor and $\tilde{R}^{\mu\nu\kappa\lambda} = \frac{1}{2\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma}R_{\rho\sigma}^{\kappa\lambda}$.

To bridge (1.3) to the chiral vortical effect (1.1), one exploits the construction similar to the one introduced first to relate the Hawking radiation from a black hole to the field-theoretic anomalies [18–20]. Namely, one considers a space-time with a horizon. What is specific for the horizon is that there is a flow of particles from the horizon and absorption of the particles falling onto the horizon. One can say there is a right-left asymmetry at the horizon. The rate of the particle production at the horizon can be fixed in terms of the anomalies of the field theory or of the gravitational field on the horizon. In more detail, the relevant anomaly looks as

$$\nabla_\mu T^{\mu\nu} = -\frac{c}{96\pi}\frac{\epsilon^{\nu\alpha}}{\sqrt{|g|}}\partial_\alpha R, \qquad (1.4)$$

where $\epsilon^{\mu\alpha}$ is the 2D antisymmetric tensor, c is the central charge, and R is the Ricci scalar. On the other hand, far off from the horizon, the flow of the particles can be compared to the thermal radiation. It was demonstrated [18–20] that the matching of the two expressions for the flow of the particles reproduces the Hawking or Unruh temperature. The calculation is to be performed for each spherical wave separately and cumbersome technically.

The paper in Ref. [15] evaluates the chiral vortical effect in a similar way. There is a significant simplification, however. Namely, the metric introduced is not a solution of the Einstein equations but rather imitates rotation of a fluid:

$$ds^2 = -f(z)\frac{(dt - \Omega r^2 d\phi)^2}{(1 - \Omega^2 r^2)} + \frac{1}{f(z)}dz^2$$
$$+ dr^2 + \frac{r^2(d\phi - \Omega dt)^2}{(1 - \Omega^2 r^2)}. \qquad (1.5)$$

At large distances z, the function $f(z)$ tends to unit, and the metric (1.5) reduces to that of the flat space in cylindrical coordinates:

$$ds^2 \to -dt^2 + dz^2 + dr^2 + r^2 d\phi^2. \qquad (1.6)$$

Integrating the rhs of Eq. (1.3), one evaluates the flow of particles at large z which is to be identified with the chiral thermal vortical effect, by the logic of the construction. The translation from the field theory to the thermal physics is achieved through the identification

$$\frac{a_{\text{horizon}}}{2\pi} \to T, \qquad \Omega_{\text{horizon}} \to \Omega, \qquad (1.7)$$

where a_{horizon} and Ω_{horizon} are the gravitational acceleration and angular velocity on the horizon, respectively while T and Ω are the flat-space values of the temperature and of angular velocity, respectively, as measured at large z. See Eq. (1.5).

It was demonstrated [15] that in this way one reproduces the T^2 term in Eq. (1.1), which is an amusing success of the modern ideas on the relation between gravitational acceleration and thermal physics.

In Ref. [8], it was suggested to extend the checks of the theory [15] by considering quantum particles of higher spin S. In particular, we concentrate on the $S = 1$ case, or photons. The corresponding gravitational anomaly was introduced in Ref. [21]. Using the machinery just described, one can turn the knowledge of the gravitational anomaly into a prediction of the magnitude of the chiral vortical effect for photons. Moreover, this prediction can be compared with the results of direct calculations by means of various techniques within the thermal field theory; see, in particular, [1–7].

There is a disagreement of a factor of 2 between the two ways of evaluating the chiral vortical effect for photons [8]. Here, we revisit the problem of comparing various results for the photonic vortical effect. The main point we are emphasizing now is that the evaluation both of the gravitational anomaly and of the chiral vortical effect involve regularization procedures. For our pure theoretical purposes, we need identical regularizations on both sides (gravitational and flat-space ones). Our overall conclusion here is that, in the sense indicated, there is no direct contradiction between the two ways of evaluating the chiral vortical effect for photons.

II. CHIRAL VORTICAL EFFECT IN EQUILIBRIUM

A. Massless spin-1/2 particles

There are different ways of evaluating the chiral vortical effect in the one-loop approximation. The most straightforward way is to find energy levels, evaluate the current for each mode, and weight the results with the Fermi or Bose

(whichever is relevant) distribution of the levels. In particular, this was the strategy adopted first in the pioneering work in Ref. [10]. In this section, we review briefly the evaluation of the photonic chiral effect in equilibrium. In the next section, we comment on the applications of the Kubo relations.

We begin with quoting the results obtained first in Ref. [10] in the form most suitable for generalizations to higher-spin cases. For the sake of normalization, let us remind the reader that the energy density for a single Weyl fermion in terms of Fermi distribution is written as

$$\rho_F^{Weyl} = \frac{1}{8\pi^3} \int_{-\infty}^{+\infty} d\epsilon (4\pi\epsilon^3) \left(\frac{1}{e^{\beta\epsilon}+1} - \theta(-\epsilon) \right). \quad (2.1)$$

The degree of freedom with a positive energy describes a fermion polarized along its momentum. The second, subtraction term in the rhs of Eq. (2.1) is introduced to ensure vanishing of the density of the energy at $T, \mu_R = 0$. Alternatively, the Fermi distribution (2.1) can be written in terms of states with positive energies, which unifies particles and antiparticles:

$$\rho_F^{Weyl} = 2\frac{1}{8\pi^3} \int_0^\infty d\epsilon (4\pi\epsilon^3) \frac{1}{e^{+\beta\epsilon}+1}, \quad (2.2)$$

where we keep the overall factor of 2 in the rhs to emphasize that there are two degenerate levels for each energy.

Now, the statistically averaged matrix element of the vortical current can be represented in the following form [10,15] convenient for interpretation:

$$J_N(s = 1/2) = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \epsilon^2 d\epsilon \left(\frac{1}{1 + e^{\beta(\epsilon-(\mu_R+\Omega/2))}} - \frac{1}{1 + e^{\beta(\epsilon-(\mu_R-\Omega/2))}} \right), \quad (2.3)$$

where J_N is written, as usual for the case of a single right-handed Weyl fermion, and we restore a nonvanishing chemical potential, $\mu_R \neq 0$. Upon integration over the energy, Eq. (2.3) reduces to Eq. (1.1).

Comparing (2.3) and (1.1), we see that the vanishing of (1.1) at $\Omega \to 0$ is a consequence of the oddness of the current in (2.3): $J_N(-\Omega) = -J_N(\Omega)$. If we consider another limit $T \to 0$, then (2.3) will give [22]

$$J_N(s = 1/2) = \frac{1}{12\pi^2} \left\{ \theta\left(\mu_R + \frac{\Omega}{2}\right) \left|\mu_R + \frac{\Omega}{2}\right|^3 - \theta\left(\mu_R - \frac{\Omega}{2}\right)\left|\mu_R - \frac{\Omega}{2}\right|^3 + \theta\left(-\mu_R + \frac{\Omega}{2}\right)\left|\mu_R - \frac{\Omega}{2}\right|^3 \right.$$
$$\left. - \theta\left(-\mu_R - \frac{\Omega}{2}\right)\left|\mu_R + \frac{\Omega}{2}\right|^3 \right\} = \frac{\mu_R^2}{4\pi^2}\Omega + \frac{1}{48\pi^2}\Omega^3. \quad (2.4)$$

The linear term in (2.4) matches (1.1) at $T \to 0$.

The interpretation of Eq. (2.3) is straightforward. Indeed, by introducing the rotation we remove the twofold degeneracy of all the levels and get two levels split by the energy $\Delta E = \Omega$. Indeed, in equilibrium, one introduces an effective interaction:

$$\delta\hat{H}_{\text{eff}} = -\vec{\Omega} \cdot \hat{\vec{M}}, \quad (2.5)$$

where $\hat{\vec{M}}$ is the angular momentum operator for spin $S = 1/2$. Note that such an interpretation assumes that the quantization is performed in cylindrical coordinates [while Eq. (2.1) can be derived, say, in Cartesian coordinates as well]. Therefore, the energy levels now correspond to the states which have a definite projection of the momentum on z axis p_z and projection of the angular momentum L_z.

B. Chiral photonic current

The notion of chirality for photons is well known. Namely, the left- and right-handed polarized photons are chiral states, since they correspond to a certain projection of the spin of the photon on its momentum, $S_p = \pm 1$. The chiral current, therefore, can be defined as

$$K^\mu = \frac{1}{\sqrt{-g}} \epsilon^{\mu\nu\rho\sigma} A_\nu \partial_\rho A_\sigma, \quad (2.6)$$

where A_μ is the vector potential of the electromagnetic field and we also reserved for a nontrivial determinant g of the metric tensor. The current (2.6) is defined in such a way that the eigenvalues of the associated charge are indeed ± 1.

Note, however, that the current (2.6) is not gauge invariant, and we should be careful to associate observables only with a kind of gauge-invariant projections of K^μ. In particular, a well-known example of such a gauge-invariant observable is the charge corresponding to the current (2.6):

$$Q_{\text{magnetic helicity}} = \int d^3x \epsilon^{0ijk} A_i \partial_j A_k = \int d^3x \vec{H} \cdot \vec{A}, \quad (2.7)$$

where $\vec{H}$ is the magnetic field. Then, under the gauge transformation, $\delta_{\text{gauge}} A_i = \partial_i \Lambda$, the variation of the magnetic-helicity charge density is given by

$$\delta_{\text{gauge}}(\vec{H} \cdot \vec{A}) = \vec{H} \cdot \vec{\nabla}\Lambda = \vec{\nabla}(\vec{H}\Lambda) - (\vec{\nabla} \cdot \vec{H})\Lambda. \quad (2.8)$$

Here, $\vec{\nabla} \cdot \vec{H} = 0$ by virtue of the equations of motion, while the integral over d^3x from the first term becomes a boundary term and can be neglected. This completes the proof that the $Q_{\text{magnetic helicity}}$ is gauge invariant. Note that the momentum space definition of the charge assumes $q_i \equiv 0, q_0 \to 0$ limiting procedure [where (q_0, q_i) is the 4-momentum carried by the current].

Note also that the matrix element of the operator of the magnetic-helicity charge, $\hat{Q}_{\text{magnetic helicity}}$, counts the difference between the numbers of right- and left-handed photons $n_{L,R}$:

$$\left\langle \left| \int d^3x \epsilon^{0ijk} A_i \partial_j A_k \right| \right\rangle = n_R - n_L. \quad (2.9)$$

In other words, the normalization of the chiral currents for spins $S = 1/2, 1$ is similar.

Now, to evaluate the photonic vortical effect, we need to consider the spatial component $\vec{K}$ of the current (2.6). Moreover, since we are interested in the physics of equilibrium, we have to consider the static (or stationary) limit with no time dependence. In the momentum space, as is first emphasized by Ref. [17], we are interested in the limiting procedure, $q_0 \equiv 0, q_i \to 0$. Let us check that the current $\vec{K} = A_0\vec{H} - \vec{A} \times \vec{E}$ is gauge invariant in this limit. Under the gauge transformation,

$$\delta_{\text{gauge}}\vec{K} = (\partial_0\Lambda)\vec{H} + (\vec{\nabla}\Lambda) \times \vec{E}. \quad (2.10)$$

As a result, we get the local term proportional to

$$\delta_{\text{gauge}}\vec{K} \sim \Lambda \cdot (\partial_0\vec{H} + \vec{\nabla} \times \vec{E}),$$

which vanishes because of the equations of motion, plus total derivatives which become boundary terms upon the integration over the volume, or time.

In summary, the apparent gauge dependence of the chiral photonic current does not imply, generally speaking, that the chiral photonic current in equilibrium is gauge dependent. Moreover, if we start with the matrix element of the original axial current over a two-photon state, then

$$j_\mu^5 = (\text{const}) \frac{q_\mu}{q^2} \epsilon^{\alpha\beta\gamma\delta} e_\alpha^{(1)} k_\beta^{(1)} e_\gamma^{(2)} k_\delta^{(2)}, \quad (2.11)$$

where q_μ is the 4-momentum carried in by the current and $e_\alpha^{(1)}$, $k_\alpha^{(1)}$, $e_\alpha^{(2)}$, and $k_\alpha^{(2)}$ are 4-vectors of polarization and

momentum of the two photons. The current j_μ^5 is manifestly Lorentz covariant and gauge invariant.

There is a reservation, however, that the problem considered is infrared sensitive. In particular, considering uniform rotation everywhere in the space is inconsistent with finiteness of the speed of light; for a discussion, see, for example, [6]. Thus, neglecting the boundary terms just discussed above might be in conflict with some other constraints on the behavior of the fields on the boundaries.

It might worth emphasizing that the current (2.6) is not a Noether current and its conservation is not automatic:

$$\nabla_\mu K^\mu = \frac{1}{2} F_{\alpha\beta}\tilde{F}^{\alpha\beta}, \quad (2.12)$$

where $\tilde{F}^{\mu\nu} = \frac{1}{2\sqrt{-g}} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$. However, on the mass shell, or for the electromagnetic plane waves, the rhs of Eq. (2.12) vanishes, and the current (2.6) is conserved for free photons. What is even more fascinating, one can introduce nontrivial dynamics through interaction of the photons with external gravitational fields, and the chiral photonic current is still (naively) conserved. We come back to discuss this point further later.

C. Photonic vortical effect

After these preliminary remarks, we are set to consider a direct, *à la* Vilenkin evaluation of the chiral vortical effect for photons. There are no general reasons to believe that such a calculation is less reliable than the showcase [10,15] of spin-1/2 massless fermions. Probably, the naive expectation would be that the final result for the photons is very similar to Eq. (1.1), with some obvious changes (that is, increasing the splitting, due to the rotation, between the levels by a factor of 2, due to the spin of the photon, and replacing the Fermi distribution by the Bose distribution). The hard work of quantizing photons in cylindrical coordinates, finding the levels and the corresponding wave functions, and evaluating the magnetic-helicity current on the modes has been done in Ref. [6], with the following result:

$$|\vec{K}| = \mathbf{2/3} \frac{1}{8\pi^3} \int_0^\infty d\epsilon (4\pi\epsilon^2) \left(\frac{1}{e^{+\beta(\epsilon-\Omega)} - 1} - \frac{1}{e^{+\beta(\epsilon+\Omega)} - 1} \right).$$

Almost everything looks like what we expected to find naively except for the bald-faced overall factor of 2/3. Thus, in the rest of this section, we will look for a convincing interpretation of this factor. For further discussion of the relation between the cases of massless and nearly massless photons, see also the Appendix.

We introduced the current K^μ, with an idea that it provides us with a unique definition of the chirality of the photon. However, it is actually well known that there exist various currents which in the case of massless

particles measure their spin or chirality; see [21] and references therein.

The best-known spin-related 4-vector has been introduced by Pauli and Lubanski [23,24]. This vector Γ^μ is defined in the most general terms of the generators of the Poincaré group:

$$\Gamma^\mu = \epsilon^{\mu\nu\kappa\lambda} P_\nu M_{\kappa\lambda}, \qquad (2.13)$$

where P_ν and $M_{\kappa\lambda}$ are the generators of the coordinate shifts and of the Lorentz rotations, respectively. For the one-particle state of a massless field, $p^2 = 0$, of spin S, we have

$$\Gamma_\mu |p_\alpha, \lambda\rangle = p_\mu \lambda |p_\alpha, \lambda\rangle, \qquad (2.14)$$

where λ is the helicity $\lambda = \pm S$. The helicity plays a crucial role, because it determines the energy splitting induced by the rotation. Indeed, we already quoted the expression (2.5) for the effective interaction introduced in the statistical physics to describe equilibrium. Once we quantize the projection of the angular momentum onto the vector $\vec{\Omega}$, the energy induced by the effective interaction becomes equal to

$$\delta E = \lambda \Omega, \qquad (2.15)$$

where λ is the helicity.

In view of this, it is useful to introduce a current j^μ_{helicity}, such that the corresponding charge is equal to the helicity of the one-particle states. The explicit expression for the j^μ_{helicity} is

$$j^\mu_{\text{helicity}} = \frac{i}{3!} \epsilon^{\mu\nu\rho\sigma} S_{\nu\rho\sigma}, \qquad (2.16)$$

where $S_{\nu\rho\lambda}$ is the spin part of the density of the angular momentum density in the Lagrangian formalism:

$$S_{\nu\rho\lambda} = \frac{\delta L}{\delta(\partial^\nu \phi^a)} (\Sigma_{\rho,\lambda})_{ab} \phi^b, \qquad (2.17)$$

where $(\Sigma_{\rho\lambda})$ is the representation of the generators of the Lorentz rotations on the fields ϕ_a:

$$(\Sigma_{\rho\lambda})_{ab} = i(g_{\rho a} g_{\lambda b} - g_{\rho b} g_{\lambda a}). \qquad (2.18)$$

The crucial point is that for one-particle massless states the values of K^μ and of j^μ_{helicity} are proportional to each other but are not identical. In particular, for a 4-vector field:

$$(j_\mu)_{\text{helicity}} = \frac{2}{3} K_\mu \qquad \text{4-vector field,} \qquad (2.19)$$

while in case of the Dirac field the matrix elements of the currents j^μ_{helicity} and of K^μ coincide with each other.

Apparently, the factors of 2/3 in Eqs. (2.3) and (2.19) are of pure geometric origin and related to each other. But at the moment we are not aware of any clear derivation of such a relation. Note also that these states do not coincide with the propagating states. To construct a propagator in terms of the equilibrium states, one needs to derive reexpansion of one complete set of functions over the other set. A similar recent analysis can be found in Ref. [25].

Coming back to our main problem of evaluating the photonic vortical effect, our next step is the introduction of the infrared regularization by a finite photon mass $m_\gamma \neq 0$.

III. FINITE PHOTON MASS

A. Chiral anomaly and infrared regularization

Introduction of $m_\gamma \neq 0$ is a logical step, within our approach. Indeed, we are going to compare predictions for the photonic vortical effect obtained in two different ways, namely, in terms of the gravitational chiral anomaly for the K_μ current and in terms of the statistically averaged matrix element of the same current. As we remind the reader next, a finite photon mass is introduced to regularize in the infrared the gravitational anomaly [21]. Therefore, we are invited to consider the statistical-theory approach at a finite photon mass as well.

To substantiate the point, let us reiterate basic steps of derivation of the photonic gravitational anomaly [21]. As we already mentioned [see Eq. (2.12)], there is no conservation of the K_μ current off-mass shell. However, for electromagnetic waves or on-mass shell, $\vec{E} \cdot \vec{H} = 0$ and the current is conserved. It is only natural then that, upon inclusion of interaction of photons with an external gravitational field, we expect covariant conservation, $\nabla_\mu K^\mu = 0$.

However, this "naive" expectation is to be checked against the possibility of the existence of an anomaly. To uncover the anomaly, one considers the matrix element of transition of the K_μ current into two gravitons, in the annihilation channel. If the gravitons are on the mass shell, the matrix element is defined in terms of a single form factor $f(q^2)$:

$$\langle 0|K_\mu|2g\rangle = f(q^2) q_\mu R_{\alpha\beta\gamma\delta} \tilde{R}^{\alpha\beta\gamma\delta}, \qquad (3.1)$$

where q_μ is the 4-momentum, carried in by the K_μ current.

The next step is to use dispersion relations to evaluate $f(q^2)$. The imaginary part $\text{Im} f(q^2)$ is given by tree graphs, and, naively, it respects all the symmetries of the problem. However, an attempt to explicitly evaluate the imaginary part by means of the Cutkosky cutting rules reveals a problem, because one of the propagators of intermediate particles has a kinematic pole which—for all the particles being massless—falls onto the physical region of integration. To regularize the calculation, one introduces then an infinitesimal photon mass. As a result, $\text{Im} f(q^2)$ does not

vanish but is proportional to the mass squared. The crucial point is that the dispersive integral over $\text{Im}f(q^2)$ turns to be saturated in the infrared and produces a factor m^{-2} so that the real part of $f(q^2)$ is finite in the limit of the vanishing mass. And finite $\text{Re}f(q^2)$ represents the anomaly. As a result,

$$\text{Im}f(q^2) = \lim_{m^2 \to 0}\left(\frac{1}{128\pi q^2}v^2(1-v^2)\ln\frac{1+v}{1-v}\right) = \frac{1}{96\pi}\delta(q^2),$$
(3.2)

where v is the velocity of the intermediate photons in the c.m. system.

Finally, the real part of $f(q^2)$ corresponding to (3.2) is given by

$$\langle\nabla_\alpha K^\alpha\rangle = \frac{1}{96\pi^2}R_{\mu\nu\kappa\delta}\tilde{R}^{\mu\nu\kappa\delta}.$$
(3.3)

Equation (3.3) is nothing else but the photonic gravitational anomaly.

At first glance, introducing an infinitesimal photon mass seems to be a very controversial step. However, radiative corrections associated, for example, with the bremsstrahlung in QED are typically infrared divergent. Commonly, one introduces an infinitesimal photon mass $m_\gamma \neq 0$ to regularize the infrared divergences. Then, the results typically contain factors $ln(E_\gamma/m_\gamma)$ and, at face value, are model dependent. The way out was found by Bloch and Nordsieck, who showed that the probability of inclusive processes depends on the energy resolution Δ which replaces m_γ in the log factors if $\Delta \gg m_\gamma$. The final result for inclusive processes does not depend then on m_γ and gauge invariant.

Following then Ref. [15] with the expressions (1.3) and (3.3) in mind, we conclude that the photonic vortical current is predicted to be

$$\vec{K} = \vec{J}_N(s=1) = 4\cdot\frac{T^2}{12}\vec{\Omega},$$
(3.4)

or 4 times larger than that for massless spin-1/2 particles [8].

Turn now to the statistical-theory approach. Introduction of a finite photon mass simplifies the evaluation of the chiral vortical effect greatly. The reason is that, for massive photons, we recover the factorization property which makes Eq. (2.3) look so simple. Namely, we start with noninteracting gas of massive photons in the absence of the rotation. Each level is degenerate three times, since projection of the spin of the massive photon is now $S_z = \pm 1, 0$. Account for the rotation splits the levels so that the energies now are $\epsilon + \Omega$, ϵ, and $\epsilon - \Omega$. These energy differences are readily calculable in the rest frame of the massive photon and are invariant under the boosts along the

z axis. In the case of massless photons, there is no rest frame for photons, and this makes the calculation much more involved; for further comments, see, in particular, [9].

Thus, for massive photons but in the massless limit, the vortical current is given by

$$|\vec{K}|_{m_\gamma\to 0} = \frac{1}{8\pi^3}\int_0^\infty d\epsilon(4\pi\epsilon^2)\left(\frac{1}{e^{\beta(\epsilon-\Omega)}-1} - \frac{1}{e^{\beta(\epsilon+\Omega)}-1}\right)$$
$$= 4\frac{T^2}{12}\Omega + \mathcal{O}(\Omega^2),$$
(3.5)

in agreement with the prediction (3.4).

This coincidence of the results obtained in the case of massive photons within the thermal field theory and via the gravitational anomaly is our main result in these notes. In view of this, we will check it against calculations of the vortical effect by means of the Kubo relations.

B. Kubo-type relation in the case of massive photons

We are interested to evaluate the coefficient σ_V entering the definition of the vortical current $J^{5\mu}$:

$$J^{5\mu} = \frac{\sigma_V}{2}\epsilon^{\mu\nu\rho\sigma}u_\nu\partial_\rho u_\sigma,$$
(3.6)

where u_μ is the 4-velocity of an element of the fluid. The Kubo-type relation fixes the coefficient σ_V in terms of the correlator between the spatial components of the current K^i and the T^{0j} component of the energy-momentum tensor [1,2,16]:

$$\lim_{p_k\to 0}\langle K^i, T^{0j}\rangle|_{\omega\equiv 0} = \sigma_V(S=1)\frac{i}{2}\epsilon^{ijk}p_k + \mathcal{O}(p^2),$$
(3.7)

where p_k is the momentum brought in by the current K^i, $K^i = \epsilon^{i\nu\rho\sigma}A_\nu\partial_\rho A_\sigma$, and

$$T^{0j} = -(\partial_k A^0 - \partial^0 A_k)(\partial^k A^j - \partial^j A^k) + m_\gamma^2 A^0 A^j,$$
(3.8)

where A_μ is now the field describing massive photons.

The propagator of the massive vector field in the momentum space is given by

$$\langle A_\mu, A_\nu\rangle = \frac{g_{\mu\nu} - \frac{q_\mu q_\nu}{m_\gamma^2}}{q^2 - m_\gamma^2}.$$
(3.9)

In the case of massless photons, the correlator (3.7) was calculated in Refs. [1,2], with the result quoted above, $\sigma_V(S=1) = T^2/6$. We are calculating now the change in $\sigma_V(S=1)$ due to $m_\gamma \neq 0$. It turns out that in the limit $m_\gamma \ll T$ there is a finite jump in the value of $\sigma_V(S=1)$ which stems from the cancellation of the factor m_γ^{-2} in the propagators (3.9) and of the factor m_γ^2 in the component T^{0j}; see Eq. (3.8).

The final result is

$$\sigma_V(S = 1, m_\gamma \neq 0) = \frac{T^2}{3}. \qquad (3.10)$$

This result is in full agreement with the expectations (3.4) based on the gravitational anomaly and with the evaluation (3.5) of the vortical effect within the statistical approach for massive photons.

C. Vector meson chirality and baryon polarization

One may ask whether the chirality of massive vector particles has any phenomenological implications. The answer is provided by the relation of axial charge to the average polarization of baryons which is the way to implement the quark-hadron duality in this problem [26] which may be realized in both kinetic [27] and hydrodynamic [28] calculations. There is a natural explanation of the fact the $\bar{\Lambda}$ polarization is larger than that of Λ as the same (C-even) chiral charge is distributed between the smaller number of particles. For a quantitative description of the effect, it is mandatory [29] to take into account the axial charge carried by K^* mesons. Therefore, their chirality is implicitly present here. The role of the numerical factors studied here depends on the assumptions on the distribution of chirality or axial charge between baryons and mesons and remains to be studied.

It is interesting whether meson chirality can affect the measured tensor polarization of vector mesons [30]. One should stress that, contrary to baryon polarization, it is a P-even quantity and may emerge due to the product of quark polarizations as well as due to their spin correlations [31]:

$$\rho_{00} = \frac{1 - Tr_{||}(C) + Tr_{\perp}(C)}{1 + 3\text{Tr}(C)}, \qquad (3.11)$$

where enter the parallel and orthogonal to quantization axis components of tensor C:

$$C_{ij} = \langle P_i^q P_j^{\bar{q}} \rangle = \langle P_i^q \rangle \langle P_j^{\bar{q}} \rangle + \langle \mathbf{P_i^q P_j^{\bar{q}}} \rangle - \langle \mathbf{P_i^q} \rangle \langle \mathbf{P_j^{\bar{q}}} \rangle, \quad (3.12)$$

containing contributions of average quark polarizations and (boldfaced) correlations. The relative smallness of the first term is implied by its relation to squared baryon polarization and squared vorticity [32] so that the terms probing the entanglement of quark spins may play the dominant role.

At the same time, the role of squared vorticity may be overtaken by the square (and higher even powers) of the magnetic field [33,34]. The emerging longitudinal polarization is related [35] to the conductivity in the magnetic field [36,37] and supports its growth.

Let us finally note that vector (related to chirality) and tensor polarizations are mixed in the positivity constraints and invariants of density matrix [38,39], providing another possible direction of experimental investigations.

IV. DISCUSSION AND CONCLUSIONS

As is noticed in Ref. [8], knowing the gravitational chiral anomaly and following the logic of Ref. [15], one can predict the value of the chiral vortical effect for any spin S of the massless constituents:

$$\sigma_V(S) = \frac{T^2}{12}(-1)^{2S}4(2S^3 - S), \qquad (4.1)$$

where σ_V is defined in Eq. (3.6). Note that there are no free parameters in this prediction.

Alternatively, one can calculate σ_V within the framework of the thermal field theory and, in this sense, test Eq. (4.1). First, and with great success, the prediction (4.1) was tested in the original paper [15] in the case of massless spin-1/2 constituents. This case is remarkable for the fact that $\sigma_V(S = 1/2)$ was evaluated in a few ways within the statistical approach, and the value $\sigma_V(S = 1/2) = T^2/(12)$ is well established and noncontroversial.

Proceeding to the case $S = 1$ [1–9], we notice $\sigma_V(S = 1)$ obtained in the literature [1,2] on the basis of the Kubo relation differs from (4.1) by a factor of 2. Moreover, these results themselves are not without controversy—see [1–9]—which is not easy to resolve. Finally, consideration of the limit of large spin S apparently brings Eq. (4.1) to a qualitative disagreement with the thermal field theory.

It is on this background that we have to appreciate the significance of the new observation that for a massive vector field there is full agreement of results for $\sigma_V(S = 1, m_\gamma \neq 0)$ obtained within the field-theoretic and statistical approaches.

As is mentioned above, the main argument against the duality between the statistical and anomaly-based approaches is an apparent conflict between the predictions for the chiral vortical effect obtained within the two frameworks in the limit of large spin S. The finding that in the case of the vector field introduction of a finite mass $m_\gamma \neq 0$ brings the consequences from the Kubo-type relation and from the anomaly into agreement with each other does not settle by itself the issue of the violation of the duality for large spin S. However, increasing spin S generically makes the theory more and more dependent on details of the infrared regularization [8]. In particular, the rotational vacuum becomes unstable at spin $S \geq 3/2$. This infrared instability does not affect directly the derivation [40] of the gravitational anomaly. However, beginning with $S = 3/2$, one has to assume that the infrared issues are settled somehow without changing prediction (4.1) which, in the language of the thermal field theory, is expected to be saturated by contribution of high energies of the order of the temperature.

The results for the chiral vortical effect obtained at $m_\gamma \neq 0$ make the validity of this extra assumption more questionable. Indeed, we have demonstrated that introduction of $m_\gamma \ll T$ results in a finite jump in the value of $\sigma_V(S = 1)$.

Explicit evaluation of σ_V in the case of $S = 3/2$ becomes a crucial step to be made. (For a recent discussion of the theory of massless charged spin-3/2 particles, see [41]).

We conclude this section with a remark on possible phenomenological implications of the evaluation of $\sigma_V(S = 1)$. The point is that in the case of superfluidity the chiral vortical effect can be manifested through polarization or spin of heavy particles (for details and references, see [42,43]). In particular, in the case of superfluidity, the average value of the vortical current $\langle \vec{J}^5 \rangle$ is equal to the spin density carried by the cores of the vortices, which, in turn, is equal to the spin density carried by heavy particles $\langle \vec{\sigma}_{\text{heavy}} \rangle$ [44,45]:

$$\langle \vec{J}^5 \rangle \approx \langle \vec{\sigma}_{\text{heavy}} \rangle. \qquad (4.2)$$

In view of the low viscosity of the quark-gluon plasma, such a relation might work well in the case of heavy-ion collisions [45]. Usually, the relation (4.2) is used in the case of spin-1/2 constituents and applied to hyperons, as heavy particles. Since $\sigma_V(S = 1) = 4\sigma_V(S = 1/2)$, one can speculate that the contribution of heavy mesons is not less important than the contribution of hyperons.

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APPENDIX: MASSLESS VS NEARLY MASSLESS PHOTON

In the main body of the paper, we considered the cases of strictly massless photons and of photons with infinitesimally small mass. The predictions for the photonic vortical effect differ by a finite factor of 2/3; see Eq. (2.13). Nevertheless, it is rather obvious that the two results are absolutely consistent with each other. To appreciate this, we should be more careful to formulate the question to be answered.

Let us start with no rotation. Then, depending on whether m_γ is strictly zero or small, we have energy densities which differ from each other by a factor of 2/3:

$$\rho_\gamma(m_\gamma \equiv 0) = 2\frac{1}{8\pi^3}\int_0^\infty d\epsilon (4\pi\epsilon^3)\frac{1}{e^{\beta\epsilon} - 1},$$

$$\rho_\gamma(m_\gamma \neq 0) = 3\frac{1}{8\pi^3}\int_0^\infty d\epsilon (4\pi\epsilon^3)\frac{1}{e^{\beta\epsilon} - 1}. \qquad (A1)$$

If we switch on the effect of the rotation, the number of levels is not changed. Moreover, the distribution between the levels with $L_z = +1, 0, -1$ apparently is the same for massless photons and photons with infinitesimal mass (i.e., $m_\gamma \ll T$). For this reason, the factor of 2/3 which reflects the difference in the total number of levels goes through to the final answer for the chiral vortical effect. Thus, the factor of 2/3 in Eq. (2.13) reflects so to say renormalization of the total amount of thermal energy stored in the system.

On the technical side, we argued that the simple, "factorized" form of distribution of levels is valid for massive particles and is not valid for strictly massless particles. This is in accord with the theoretical expectations. The difference between massive and strictly massless cases goes back to the fact that for massive particles the 4-momentum vector is orthogonal to the Pauli-Lubanski vector while for strictly massless cases the two vectors are parallel to each other; for a related discussion, see [9].

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