Black hole scalarization in Gauss-Bonnet extended Starobinsky gravity

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We propose a class of higher-derivative gravities that can be viewed as the Gauss-Bonnet extension of the Starobinsky model. The theory admits the Minkowski spacetime vacuum whose linear spectrum consists of the graviton and a massive scalar mode. In addition to the usual Schwarzschild black hole, we use numerical analysis to establish that in some suitable mass range, new black holes carrying the massive scalar hair can emerge. The new black hole serves as a "wall" separating the naked spacetime singularity and wormholes in the parameter space of the scalar hair. Our numerical results also indicate that although the new hairy black hole and the Schwarzschild black hole have different spacetime geometry, their entropy and temperature are the same for the same mass.

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I. INTRODUCTION

A natural extension to Einstein's theory of general relativity is to include higher-order Riemann tensor polynomial invariants. The simplest example is the quadratic extension and the theory was proven renormalizable in four spacetime dimensions, at the price of having massive spin-2 ghostlike excitations in the spectrum [1]. The origin of ghost is due to the fact that the field equations can involve up to fourth order derivatives, leading to new massive scalar and massive spin-2 modes in the linearized spectrum, in addition to the usual spin-2 massless graviton. There have been various proposals to deal with the ghost excitations. One is to consider critical values of the couplings such that the massive spin-2 mode becomes massless [2–5], but it typically leads to logarithmic ghost modes [6,7]. An alternative is to consider combinations of the Riemann tensor polynomials such that the field equations remain in second order. These include the Gauss-Bonnet or more generally the Lovelock series, which are nontrivial only in dimensions $D \geq 5$. The absence of massive modes in the linearized spectrum in maximally symmetric vacua can be achieved in four dimensions. This class of massless gravity theories include quasitopological gravities and their variants; see, e.g., [8–11].

In higher-derivative gravities, while the massive spin-2 modes are inevitably ghostlike, the massive scalar mode can be unitary. The ghost-free condition can thus be achieved by decoupling the massive spin-2 mode, which amounts to a single condition on the coupling constants in the theory. The simplest such example is the celebrated Starobinsky $R + R^2$ model [12]. The theory admits the Schwarzschild black hole, but no static black holes can carry the scalar hair [13,14]. By contrast, new black holes with massive spin-2 hair do exist in quadratically extended gravity [14]. This leads to an obvious question whether the no-scalar-hair theorem is a general phenomenon in higher-derivative gravities or only a special case for Starobinsky gravity.

The absence of the scalar hairy black holes in the Starobinsky model is analogous to the no-hair theorem in Einstein gravity minimally coupled to a free scalar. Recently, it was shown by numerical approach that scalar hairy black holes can be constructed when the free massless scalar field is further coupled to the Gauss-Bonnet term [15–19]. Analogous construction for a massive scalar field was subsequently obtained [20,21]. These motivate us to consider extending the Starobinsky theory with the Gauss-Bonnet combination in an appropriate way such that its massive scalar mode can be excited by the black hole curvature. The resulting theory remains pure gravity and ghost-free, constructed from the Ricci scalar and the Gauss-Bonnet term.

The advantage of studying the black hole scalarization in the Gauss-Bonnet extended Starobinsky model is that the theory is pure gravity and we do not need to introduce a matter scalar that may not exist. In fact, the prevalence of the scalar mode in higher-derivative gravities makes the study of black hole scalarization an unavoidable topic. Furthermore, the scalar mode in the Starobinsky model is

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necessarily massive and its effect is invisible in the long range. The scalarization however implies that it can nontrivially affect the black hole horizon.

II. NO-HAIR THEOREM AND ITS EVASION

The simplest higher-order extension to Einstein gravity is to include the quadratic curvature invariants. In four dimensions, owing to the fact that the Gauss-Bonnet combination is a total derivative, the quadratic invariants of the Riemann tensor have two independent structures, the squared Ricci scalar and Weyl tensor, which excite, respectively, a massive scalar mode and a massive spin-2 mode in the linear spectrum. This is because Weyl squared gravity in four dimensions is conformal, with no scalar excitation. The extended gravity admits the usual Schwarzschild black hole; in addition, new black holes carrying massive spin-2 hair were constructed [14]. However, a no-scalar-hair theorem can be established, for which it is sufficient to consider only the R^2 extension, namely, the Starobinsky model

$$\mathcal{L} = \sqrt{-g}(R + \alpha R^2), \qquad \alpha \geq 0. \tag{1}$$

The trace of the Einstein equation is simply $6\alpha\Box R = R$. Since R acts like a free massive scalar field here, it is straightforward to prove that R must vanish for black hole solutions [14].

The Starobinsky gravity is the simplest example of $f(R)$ gravity and it is an effective scalar-tensor theory where the scalar equation is algebraic. We can equivalently express (1) as

$$\mathcal{L} = \sqrt{-g}\left(R + \phi R - \frac{1}{2}\mu^2\phi^2\right), \qquad \mu^2 = \frac{1}{2\alpha} > 0. \tag{2}$$

Inspired by [16–18], we extend the theory with the Gauss-Bonnet invariant, i.e.,

$$\mathcal{L} = \sqrt{-g}\left(R + \phi R - \frac{1}{2}\mu^2\phi^2 + U(\phi)E^{\mathrm{GB}}\right),$$
$$E^{\mathrm{GB}} = R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}. \tag{3}$$

The Einstein equation is

$$\begin{aligned} R_{\mu\nu} &- \frac{1}{2}Rg_{\mu\nu} + \phi R_{\mu\nu} - \nabla_\mu\nabla_\nu\phi + \Box\phi g_{\mu\nu} \\ &- \frac{1}{2}\phi R g_{\mu\nu} + \frac{1}{4}\mu^2\phi g_{\mu\nu} - 2R\nabla_\mu\nabla_\nu U \\ &- 4\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\Box U + 8R^\rho{}_{(\mu}\nabla_{\nu)}\nabla_\rho U \\ &- 4R^{\rho\sigma}\nabla_\rho\nabla_\sigma U g_{\mu\nu} + 4R_\mu{}^\rho{}_\nu{}^\sigma\nabla_\rho\nabla_\sigma U = 0. \end{aligned} \tag{4}$$

Combining the algebraic scalar field equation, i.e.,

$$R - \mu^2\phi + U'(\phi)E^{\mathrm{GB}} = 0, \tag{5}$$

with the trace of (4), we have

$$\begin{aligned} 3\Box\phi = \mu^2\phi &- (1+\phi)U'(\phi)E^{\mathrm{GB}} - 2R\Box U(\phi) \\ &+ 4R^{\mu\nu}\nabla_\mu\nabla_\nu U(\phi). \end{aligned} \tag{6}$$

When the coupling function U vanishes, the equation describes a standard free massive scalar and the no-hair theorem applies for black hole solutions. The no-hair theorem is no longer applicable when the scalar Gauss-Bonnet term is included, making it possible for scalar hairy black holes.

It is important that the scalar equation in the extended theory is algebraic and hence can be integrated out to give pure gravity. We present two concrete examples. For $U = \beta\phi$, we have $\phi = 2\alpha(R + \beta E^{\mathrm{GB}})$ and hence,

$$\mathcal{L} = \sqrt{-g}(R + \alpha(R + \beta E^{\mathrm{GB}})^2). \tag{7}$$

For $U = \frac{1}{2}\beta\phi^2$, we have $\phi = 2\alpha R/(1 - 2\alpha\beta E^{\mathrm{GB}})$ and hence,

$$\mathcal{L} = \sqrt{-g}\left(R + \frac{\alpha R^2}{1 - 2\alpha\beta E^{\mathrm{GB}}}\right). \tag{8}$$

Both above Gauss-Bonnet extensions of the Starobinsky model are pure gravity theories. As we shall discuss next, we focus on the second example, for which $\phi = 0$ is a solution, giving rise to the Schwarzschild black hole. We shall construct new scalar hairy black holes in this model. By contrast, the first model (7) does not admit the Schwarzschild black hole as a solution; in fact, any black hole with nonvanishing E^{GB} curvature necessarily involves the scalar hair. We shall relegate the study of these black holes to a future work.

III. NUMERICAL BLACK HOLE SOLUTIONS

We construct static and spherically symmetric black hole solutions that are asymptotic to the Minkowski spacetime. We consider the Lagrangian (3) with $U = \frac{1}{2}\beta\phi^2$, corresponding to the Starobinsky model extended with the Gauss-Bonnet invariant (8). The general ansatz is

$$ds^2 = -h(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$
$$\phi = \phi(r). \tag{9}$$

Since $\phi = 0$ is clearly a solution of (6), the theory admits the usual Schwarzschild black hole $h = f = 1–2M/r$. Here we investigate whether there can exist new black holes that carry the massive scalar hair. At asymptotic infinity, the scalar has the Yukawa falloff

$$\phi = \frac{\phi_0}{r} e^{-\frac{\mu}{\sqrt{3}} r} + \cdots . \quad (10)$$

This modifies the metric functions, with leading falloffs

$$h = 1 - \frac{2M}{r} - \frac{\phi_0}{r} e^{-\frac{\mu}{\sqrt{3}} r} + \cdots ,$$
$$f = 1 - \frac{2M}{r} + \phi_0 \left(\frac{1}{r} + \frac{\mu}{\sqrt{3}} \right) e^{-\frac{\mu}{\sqrt{3}} r} + \cdots . \quad (11)$$

The Arnowitt-Deser-Misner mass is M, independent of the scalar hair ϕ_0, or the couplings (μ, β) of the theory. Consequently, the massive scalar is effectively invisible in the long range. However, it can alter the horizon structure significantly. The leading-order expansions in the vicinity of the horizon $r = r_+$ are

$$h = h_1 (r - r_+) + \cdots , \qquad f = f_1 (r - r_+) + \cdots ,$$
$$\phi = \phi_+ + \phi_1 (r - r_+) + \cdots , \quad (12)$$

with

$$f_1 = \frac{(3 + 3\phi_+ - \beta\mu^2\phi_+^3) r_+}{12\beta\phi_+(\phi_+ + 1)}$$
$$- \frac{1}{48\beta\phi_+(\phi_+ + 1)} [16 r_+^2 (-\beta\mu^2\phi_+^3 + 3\phi_+ + 3)^2$$
$$- 96\beta\phi_+(\phi_+ + 1)(4\beta\mu^2\phi_+^3 - 3\mu^2\phi_+^2 r_+^2$$
$$- 2\phi_+(\mu^2 r_+^2 - 6) + 12)]^{\frac{1}{2}},$$
$$\phi_1 = \frac{4(1 - f_1 r_+)(1 + \phi_+) - \mu^2 \phi_+^2 r_+^2}{2 f_1 (4\beta\phi_+ + r_+^2)} . \quad (13)$$

The near-horizon expansion has to be in integer powers so that the metric does not suffer from the branch-cut singularities. We find that all the coefficients can be solved order by order, and the horizon is characterized by two integration constants, the horizon radius r_+ and the horizon scalar hair ϕ_+, matching the asymptotic M and ϕ_0. The constant h_1 can be in principle arbitrary owing to the time scaling invariance; it is determined by requiring the resulting h approach unit asymptotically. We can then read off the temperature and entropy

$$T = \frac{\sqrt{h_1 f_1}}{4\pi}, \qquad S = \pi r_+^2 (1 + \phi_+) + 2\pi\beta\phi_+^2 . \quad (14)$$

Here the temperature is obtained by the standard geometric approach of finding the period of Euclidean time on the horizon. The entropy follows from the Wald entropy formula [22].

In the absence of exact solutions, we use a numerical approach to connect the horizon data (r_+, ϕ_+) to asymptotic ones (M, ϕ_0). The theory itself contains two coupling parameters (μ, β) and they will be fixed to some appropriate values. One way is to integrate the solution from the horizon to large r. Since the equations are singular on the horizon, we can perform the Taylor expansions and shift the initial integration point slightly out from the horizon. However, the complexity of the expressions for (f_1, ϕ_1) implies that we cannot analytically push the Taylor series to much higher orders to improve the accuracy. Furthermore, generic (r_+, ϕ_+) values will in general excite the divergent $e^{+\mu r/\sqrt{3}}$ mode in the scalar, making the numerical analysis very unstable. Only the extremely fine-tuned balance between (r_+, ϕ_+) may lead to a black hole solution. This is typical for solutions involving a massive mode, as in the case of the hairy black holes constructed in [14].

Alternatively, we can integrate the solution from large r to the middle. Since the scalar mode falls off exponentially, the higher-order corrections can be ignored numerically. For appropriately chosen (M, ϕ_0) parameters, a black hole is characterized by the fact that the functions (h, f) will vanish simultaneously somewhere that can be identified as the horizon. In practice, this is difficult to achieve precisely since the equations are singular on the horizon. One can nevertheless establish the existence of the scalar hairy black holes. For a concrete example, we consider $\mu = 1/100$, $\beta = 50$, and $M = 5$. Its Schwarzschild radius is $r_+^s = 10$. For negative values of ϕ_0, we find that the function f approaches zero at some $r_0 \geq r_+^s$ before h, giving rise to a solution that describes half of a wormhole. The wormhole throat r_0 increases as ϕ_0 becomes more negative. For positive and small ϕ_0, we find that naked singularity develops where the function h reaches zero at $r_0 < r_+^s$, and f diverges. As ϕ_0 increases, r_0 decreases and the divergence of f becomes less and less severe until ϕ_0 reaches a critical value $\phi_0^* = 0.397$ where f and h vanish simultaneously at $r_+ = 9.487$. This leads to a new scalar hairy black hole. If we keep on increasing ϕ_0, we find that f approaches zero at smaller r_0 before h, giving rise to wormholes again. The metric functions (h, f) for these three types of spacetime are plotted in Fig. 1. Thus, for appropriately given mass M, in the line of scalar hair ϕ_0, the Schwarzschild black hole and new black holes are “walls” separating wormholes from naked singularities. However, analogous transitions at $\phi_0 = -2.177$ and $\phi_0 = 1.474$ are like “cliffs” with no black holes. We sketch this in Fig. 2.

While the mass of the Schwarzschild black hole can be all positive values, new scalar hairy ones emerge only in the restricted mass region. For $\mu = 1/100$ and $\beta = 50$, we find that $M_{\max} = 5.505$ and $M_{\min} = 4.699$. The mass/horizon dependence is plotted in the left panel of Fig. 3. It is of interest to note that for the same r_+, the new black hole has bigger mass than that of the Schwarzschild black hole, indicating the condensation of the scalar hair contributes to the mass. In contrast, the mass would be smaller when the black hole carries the ghostlike massive spin-2 hair [14].

The black hole solutions have three parameters, the couplings (μ, β) and mass M, which are all dimensionful.

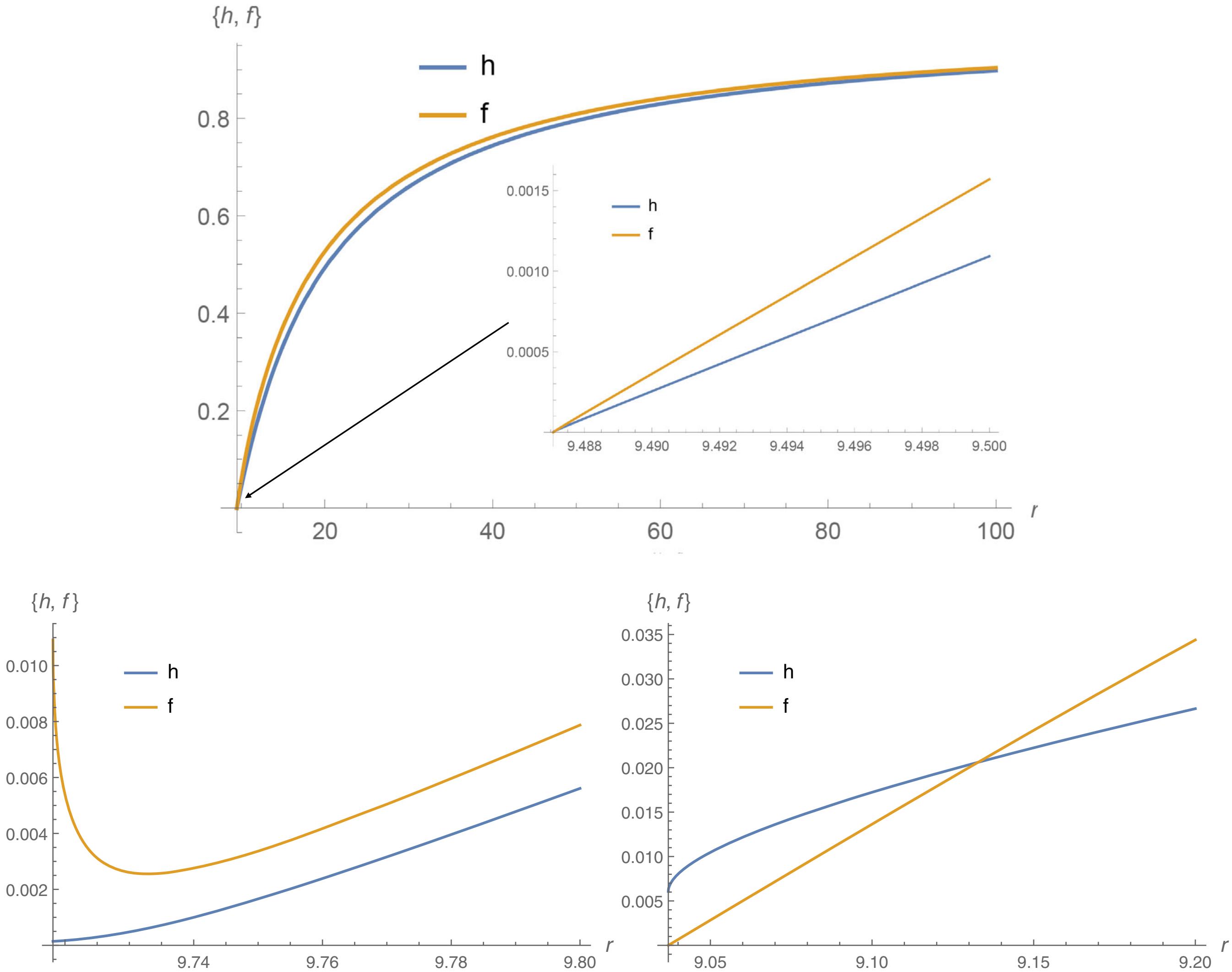

FIG. 1. The metric functions (h, f) for three asymptotically-flat spacetimes with scalar hair are plotted. All the three solutions have the same coupling parameters $\mu = 1/100$, $\beta = 50$, and the same mass $M = 5$. The first plot illustrates a black hole at critical $\phi_0^* = 0.397$, for which h and f vanish simultaneously on the horizon $r_+ = 9.487$, which is smaller than the Schwarzschild black hole horizon of the same mass. When $\phi = 0.2$, as depicted in the bottom left panel, the function f has no zero, indicating a naked singularity. The bottom right panel corresponds to $\phi_0 = 0.9$, where h is nonvanishing when f vanishes, characteristics of half a wormhole.

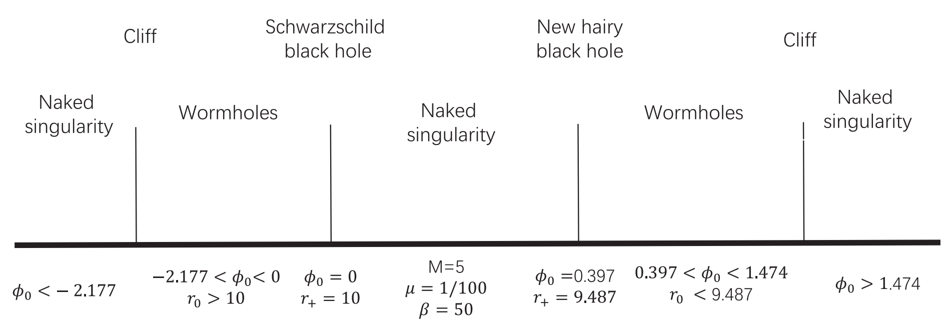

FIG. 2. This figure sketches the existence of a new scalar hairy black hole of mass $M = 5$ for the $\mu = 1/100$ and $\beta = 50$ theory. The Schwarzschild black hole ($r_+^s = 10$) is hairless ($\phi_0 = 0$) and a new black hole with $r_+ = 9.487$ emerges at $\phi_0 = 0.397$. Both black holes are at the boundaries transiting from naked singularity to wormholes. Analogous transitions at $\phi_0 = -2.177, 1.474$, however, are like cliffs and yield no black hole.

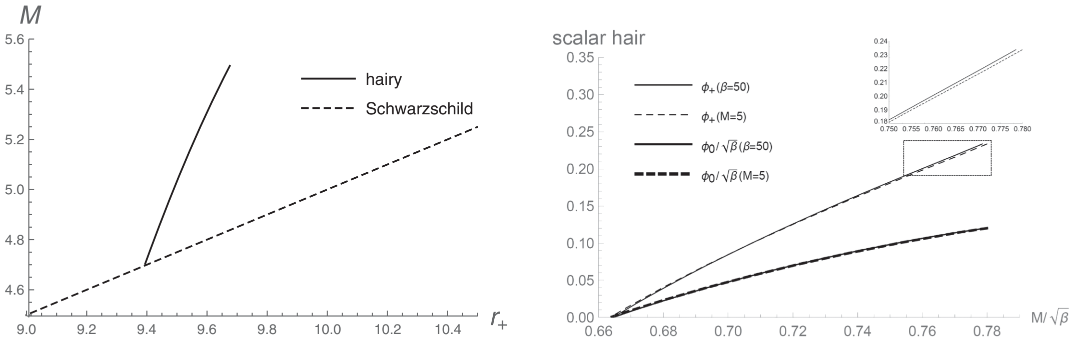

FIG. 3. The left panel shows the range of the allowed mass M for the new black holes with $(\mu = 1/100, \beta = 50)$ and their mass dependence on the horizon radius compared to the Schwarzschild black hole. The right panel shows that the (dimensionless) scalar hair parameters $(\phi_+, \phi_0/\sqrt{\beta})$ are not independent, but functions of $\tilde{M} = M/\sqrt{\beta}$, as well as either $\tilde{\mu} = \mu\sqrt{\beta}$ or $\hat{\mu} = \mu M$ for a different choice of parametrization. The solutions all have $\mu = 1/100$, and hence, the solid lines are for fixed $\tilde{\mu}$ and the dashed lines are for fixed $\hat{\mu}$. The extra μ parameter in the theory makes it no longer equivalent to vary M while fixing β versus to vary β while fixing M and the difference can be seen from the solid and dashed scalar hair functions.

We can form dimensionless parameters $\tilde{M} = M/\sqrt{\beta}$, $\tilde{\mu} = \mu\sqrt{\beta}$, or $\hat{\mu} = \mu M$. All dimensionless quantities of the black hole must be functions of either the pair $(\tilde{M}, \tilde{\mu})$ or the alternative but equivalent choice $(\tilde{M}, \hat{\mu})$. The dimensionless scalar hair parameter ϕ_+ on the horizon and $\phi_0/\sqrt{\beta}$ of the asymptotic are shown in the right panel of Fig. 3.

We now examine the black hole thermodynamics for the new hairy solutions. It should be pointed out that the massive scalar hair ϕ_0 cannot enter the first law of black

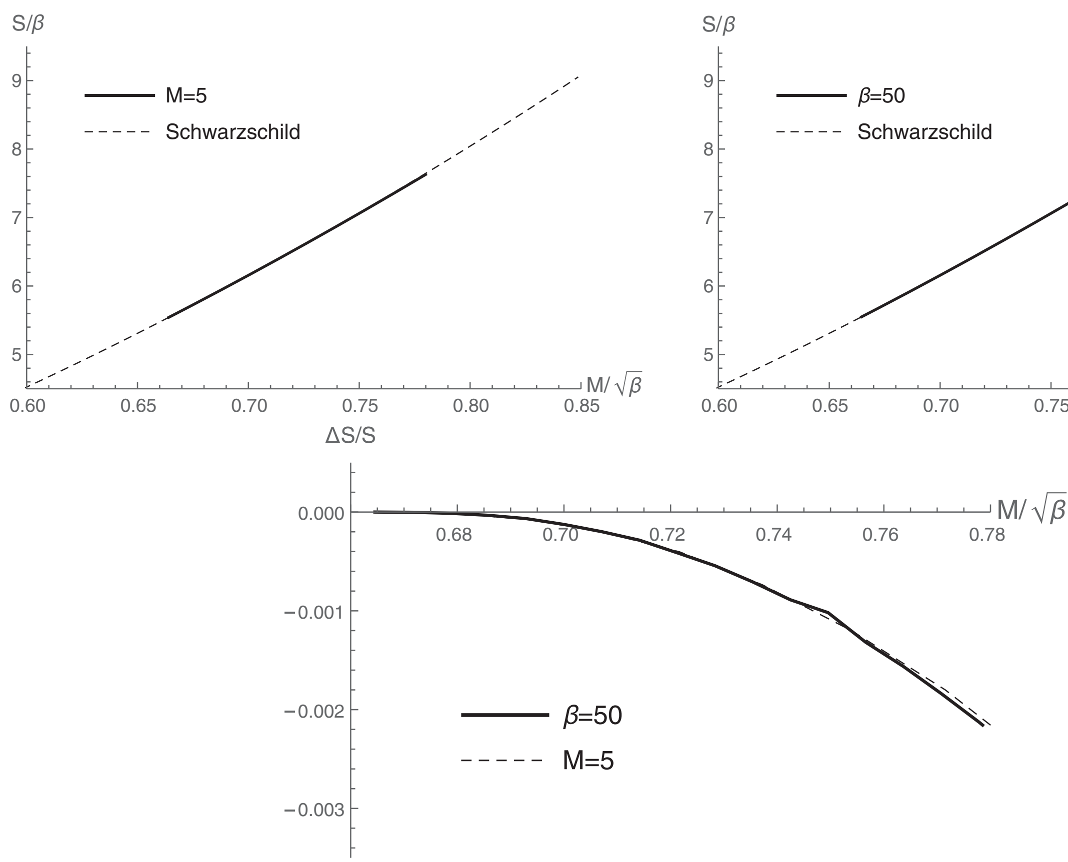

FIG. 4. The first two graphs show the (dimensionless) entropy/mass relation for the new hairy black holes, indistinguishable from that of the Schwarzschild black hole, for the $\mu = 1/100$. The left panel keeps $\hat{\mu}$ fixed, while the right panel keeps $\tilde{\mu}$ fixed. The last graph shows the numerical deviation of the new black hole entropy from the Schwarzschild one and the numerical deviation is within 0.3%, where $\Delta S/S \equiv (S_{\rm hairy} - S_{\rm Sch})/S_{\rm hairy}$.

hole thermodynamics since its thermodynamical conjugate is associated with divergent mode in the solutions and set to zero. Consequently, the first law remains $dM = TdS$. The intriguing property of our new solutions is that for given mass M, although the horizon radius is smaller than the Schwarzschild black hole, the entropy and hence temperature appear to be the same as those of the Schwarzschild black hole, as shown in Fig. 4. In other words, although the new hairy black hole is very different geometrically from the Schwarzschild black hole, the thermodynamical properties are the same.

We have analyzed a wide range of parameter space of the couplings (μ, β) and the properties described above hold in general. The allowed mass ranges for the new hairy black holes depend on the couplings (μ, β). For $\mu = 1/10$ and $1/100$, we plot the mass range dependence on β in Fig. 5. Data fitting up to the cubic order gives (for $\beta \geq 5$,)

$$\begin{aligned} \mu = \frac{1}{100}: M_{\max} &\approx 0.985 + 0.174\beta - 0.00278\beta^2 \\ &\quad + 0.0000223\beta^3, \\ M_{\min} &\approx 0.848 + 0.146\beta - 0.00226\beta^2 \\ &\quad + 0.0000175\beta^3, \\ \mu = \frac{1}{10}: M_{\max} &\approx 0.984 + 0.145\beta - 0.00257\beta^2 \\ &\quad + 0.0000207\beta^3, \\ M_{\min} &\approx 0.850 + 0.127\beta - 0.00219\beta^2 \\ &\quad + 0.0000173\beta^3. \end{aligned} \quad (15)$$

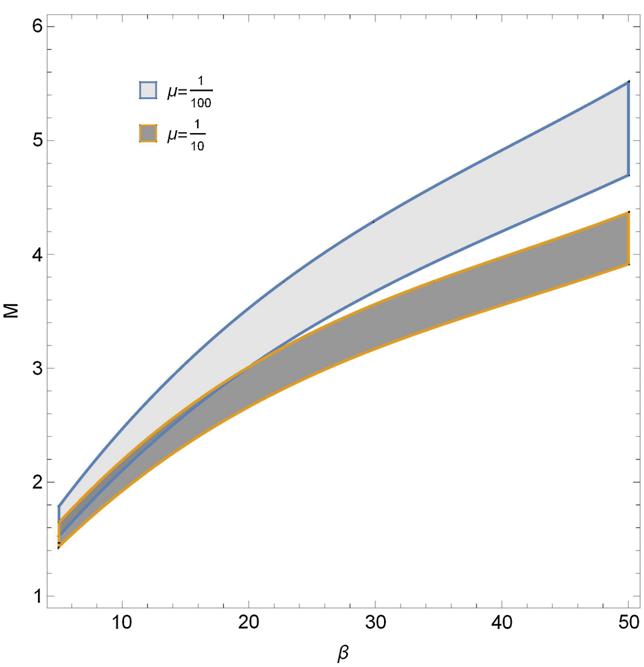

FIG. 5. The panel shows the ranges of M and β where black hole scalarization occurs for certain given μ's.

It shows that for the given μ, the bigger β gives a bigger mass range and for the given β, the smaller μ yields the bigger range. We verify the degeneracy of the black hole thermodynamics of the new black hole and the Schwarzschild black hole. Our numerical technique allows us to obtain reliably the solutions with $\beta \geq 5$. For smaller β, the equations become too unstable to get trustworthy data. The reason may be related to the fact that the solutions cease to be black holes in the $\beta \to 0$ limit.

IV. CONCLUSIONS

In this paper, we propose a class of higher-derivative extensions to Einstein gravity, constructed from the Ricci scalar and the Gauss-Bonnet combination. The theories are ghost-free and the linear spectrum contains a massive scalar mode, in addition to the usual graviton. We focus on the simplest such examples, namely, the Gauss-Bonnet extensions of the Starobinsky model. The extensions allow us to overcome the black hole no-hair theorem in the Starobinsky model and construct new hairy black holes for some restricted range of black hole mass. Our result indicates that black hole scalarization should be a common phenomenon in general higher-derivative gravities involving massive scalar modes. Owing to the massiveness, these black hole scalarizations are invisible in the long range, but should play important roles in quantum gravity and early cosmology. We find numerically that the difference of the entropy between the scalar hairy black hole and the Schwarzschild one of the same mass is within 0.3%, strongly indicating the degeneracy of these black hole thermodynamics. It is of great interest to investigate analytically whether the entropy difference indeed vanishes.

It is worth noting that for the Gauss-Bonnet extended Starobinsky model (8) that we focused on, in addition to the Minkowski vacuum, there are two de Sitter vacua for positive β and two anti–de Sitter vacua for negative β. It is worth investigating the black hole scalarization in these backgrounds as well. Our construction can also be easily generalized to general $f(R)$ gravity, by considering

$$\mathcal{L} = \sqrt{-g}(\Phi R - V(\Phi) + U(\Phi)E^{\mathrm{GB}}). \quad (16)$$

Integrating out the algebraic Φ gives rise a class of $f(R, E^{\mathrm{GB}})$ gravities and the Gauss-Bonnet extended Starobinsky model is the simplest example. It is of great interest to study the general aspects of black hole scalarization and also implications in cosmology.

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