Perturbations and quasinormal modes of black holes with time-dependent scalar hair in shift-symmetric scalar-tensor theories

Keitaro Tomikawa* and Tsutomu Kobayashi†

Department of Physics, Rikkyo University, Toshima, Tokyo 171-8501, Japan

(Received 16 January 2021; accepted 31 March 2021; published 22 April 2021)

We study odd parity perturbations of spherically symmetric black holes with time-dependent scalar hair in shift-symmetric higher-order scalar-tensor theories. The analysis is performed in a general way without assuming the degeneracy conditions. Nevertheless, we end up with second-order equations for a single master variable, similarly to cosmological tensor modes. We thus identify the general form of the quadratic Lagrangian for the odd parity perturbations, leading to a generalization of the Regge-Wheeler equation. We also investigate the structure of the effective metric for the master variable and refine the stability conditions. As an application of our generalized Regge-Wheeler equation, we compute the quasinormal modes of a certain nontrivial black-hole solution. Finally, our result is extended to include the matter energy-momentum tensor as a source term.

DOI: 10.1103/PhysRevD.103.084041

I. INTRODUCTION

The remarkable first direct detection of gravitational waves from a binary black-hole merger [1] has opened a new era of astrophysics and gravitational physics. Gravitational waves are becoming more and more important as a probe of strong gravitational fields and as a tool for testing gravity in the strong field regime. To establish general relativity on a firmer basis, it is necessary to study the predictions from alternative theories such as scalar-tensor theories and test them against gravitational-wave observations. In modified theories gravity, black holes may have hair, i.e., nontrivial configurations of scalar or other extra degrees of freedom around themselves, and the perturbation dynamics may also differ from that in general relativity. Therefore, identifying the general action for perturbations around a hairy black hole will help to achieve the above purpose. The results will be useful for instance for the computation of quasinormal modes (QNMs) in modified gravity.

In this paper, we determine the general action governing odd parity perturbations around a spherically symmetric black hole dressed with a linearly time-dependent scalar field. To do so, we start from a covariant action for shift-symmetric higher-order scalar-tensor theories admitting such time-dependent scalar hair and second-order field equations at least in the odd parity sector. In contrast to Refs. [2–4], we do not take the effective-field-theory (EFT) approach, because our background scalar-field configuration depends not only on the radial coordinate but also on time, which breaks the usual assumption of the EFT on the symmetry. In such a case, it is probably more convenient to start from a covariant action.

To make the results as general as possible, we work with general shift-symmetric higher-order scalar-tensor theories whose action depends on the curvature tensors and first and second derivatives of the scalar field in such a way that yields healthy second-order field equations for gravitational-wave degrees of freedom. Probably the most well-known example of such theories is the Horndeski theory [5] or, equivalently, the generalized Galileon theory [6,7]. The perturbation theory for spherically symmetric black holes with static hair has been developed in Refs. [8,9]. When restricted to the shift-symmetric subclass, the Horndeski theory admits linearly time-dependent scalar hair, as was first demonstrated in Ref. [10] and later generalized in Ref. [11], thus evading the no-hair theorem in shift-symmetric scalar-tensor theories [12]. The black-hole perturbation theory can be extended to the case with linearly time-dependent scalar hair [13,14] (see Ref. [15] for a recent update). Though the Horndeski theory is the most general scalar-tensor theory having second-order field equations both for the metric and scalar field, later it was noticed that it can further be generalized while maintaining one scalar and two tensorial degrees of freedom [16–19] (see also Ref. [20] for an earlier work seeking theories beyond Horndeski by means of a disformal transformation of the metric). The basic idea behind this generalization is that if some of the field equations are degenerate then the number of dynamical degrees of freedom is reduced, and thus the system can retain one scalar and two tensorial degrees of freedom even if the field equations are apparently of higher order. Theories with such a structure are

*k.tomikawa@rikkyo.ac.jp
†tsutomu@rikkyo.ac.jp

called degenerate higher-order scalar-tensor (DHOST) theories. See Refs. [21–23] for a review. In DHOST theories in which second derivatives of the scalar field appear quadratically in the Lagrangian (i.e., quadratic DHOST theories), odd parity perturbations of spherically symmetric black holes with linearly time-dependent scalar hair have been studied in Ref. [24], with the results showing that the master variable for the odd parity perturbations obeys a second-order equation having essentially the same structure as that in the case of the Horndeski theory [13,14]. More recently, it was argued that the dangerous ghost degrees of freedom remain to be absent even if one relaxes the degeneracy conditions so that the system is degenerate only in the unitary gauge, giving rise to the notion of "U-degenerate" theories [25]. If a theory is treated as a low-energy effective theory rather than a complete one, it is sufficient to require that no ghost degrees of freedom emerge within the regime of validity of the effective theory. This viewpoint allows us to consider the theories in which the degeneracy conditions are detuned slightly [26]. Detuning the degeneracy conditions help to resolve the problem of infinite strong coupling in the even parity sector [27,28].

In light of these developments, we will study black-hole perturbations in higher-order scalar-tensor theories that are most closely related to cubic DHOST theories [19], but without imposing the degeneracy conditions. Still, at least in the odd parity sector, we will have a second-order equation for a single master variable and can thus determine the general form of the action for the master variable, generalizing the previous results [13,14,24].

This paper is organized as follows. In the next section, we present the covariant action for scalar-tensor theories which we will work with. In Sec. III, we give an example of spherically symmetric background solutions with time-dependent scalar hair. Then, in Sec. IV, we determine the general action for odd parity perturbations and derive the generalized Regge-Wheeler equation. We also refine the previous notion of the stability conditions by investigating the structure of the effective metric for gravitons. In Sec. V, we calculate the QNMs of the black-hole solution we present in Sec. III. Section VI is devoted to a summary of conclusions. In the Appendixes, we argue the generality of our action for odd parity perturbations. We also generalize the Regge-Wheeler equation derived in the main text to include the matter energy-momentum tensor as a source term.

II. HIGHER-ORDER SCALAR-TENSOR THEORIES

We consider a system composed of the metric $g_{\mu\nu}$ and the scalar field ϕ described by the action [17–19]

$$S_{\rm grav} = \int d^4x\sqrt{-g}\bigg[F_0(X) + F_1(X)\Box\phi + F_2(X)R + \sum_{I=1}^{5} A_I(X)L_I^{(2)} + F_3(X)G_{\mu\nu}\phi^{\mu\nu} + \sum_{I=1}^{10} B_I(X)L_I^{(3)}\bigg], \tag{1}$$

where $X \coloneqq -\phi_\mu\phi^\mu/2$, $\phi_\mu \coloneqq \nabla_\mu\phi$, $\phi_{\mu\nu} = \nabla_\nu\nabla_\mu\phi$, R is the Ricci scalar, and $G_{\mu\nu}$ is the Einstein tensor. Here, $L_I^{(2)}$ are quadratic in the second derivatives of the scalar field and are written explicitly as

$$L_1^{(2)} = \phi_{\mu\nu}\phi^{\mu\nu}, \quad L_2^{(2)} = (\Box\phi)^2, \quad L_3^{(2)} = (\Box\phi)\phi^\mu\phi_{\mu\nu}\phi^\nu, \\ L_4^{(2)} = \phi^\mu\phi_{\mu\rho}\phi^{\rho\nu}\phi_\nu, \quad L_5^{(2)} = (\phi^\mu\phi_{\mu\nu}\phi^\nu)^2. \tag{2}$$

Similarly, $L_I^{(3)}$ are cubic in the second derivatives of the scalar field and are given by

$$L_1^{(3)} = (\Box\phi)^3, \quad L_2^{(3)} = (\Box\phi)\phi_{\mu\nu}\phi^{\mu\nu}, \\ L_3^{(3)} = \phi_{\mu\nu}\phi^{\nu\rho}\phi^\mu_\rho, \quad L_4^{(3)} = (\Box\phi)^2\phi_\mu\phi^{\mu\nu}\phi_\nu, \\ L_5^{(3)} = \Box\phi\phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^\rho, \quad L_6^{(3)} = \phi_{\mu\nu}\phi^{\mu\nu}\phi_\rho\phi^{\rho\sigma}\phi_\sigma, \\ L_7^{(3)} = \phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^{\rho\sigma}\phi_\sigma, \quad L_8^{(3)} = \phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^\rho\phi_\sigma\phi^{\sigma\lambda}\phi_\lambda, \\ L_9^{(3)} = \Box\phi(\phi_\mu\phi^{\mu\nu}\phi_\nu)^2, \quad L_{10}^{(3)} = (\phi_\mu\phi^{\mu\nu}\phi_\nu)^3. \tag{3}$$

These exhaust possible terms built from ϕ_μ and $\phi_{\mu\nu}$ and quadratic/cubic in $\phi_{\mu\nu}$. The functions F_0, F_1, F_2, F_3, A_I, and B_I depend only on X, so that the theory has shift symmetry, $\phi \to \phi + c$.

In general, the action (1) yields higher-order equations of motion for the metric and the scalar field, resulting in the dangerous Ostrogradsky ghost. One can circumvent this by imposing the degeneracy conditions among the functions F_2, F_3, A_I, and B_I [17–19]. In such degenerate theories, one arrives in the end at a set of second-order equations by combining the different components of field equations, and thus can remove the unstable ghost degrees of freedom. One may relax the degeneracy conditions so that the theory is degenerate at least in the unitary gauge, which can still provide a healthy class of theories called U-degenerate theories [25]. If the action (1) is regarded as a low-energy truncation of some complete theory, detuning the degeneracy conditions is acceptable because a ghost degree of freedom itself is not problematic from the effective-field-theory viewpoint [26].

In this paper, we do not assume any particular relations among the functions in the action. Nevertheless, we can handle the relevant equations and derive the universal form of the quadratic Lagrangian for odd mode perturbations around a spherically symmetric background with time-dependent scalar hair.

III. SPHERICALLY SYMMETRIC BACKGROUND

Let us start with a background solution. We consider static and spherically symmetric spacetime whose metric is of the form

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{B(r)} + r^2 C(r) d\sigma^2, \tag{4}$$

where $d\sigma^2 \coloneqq d\theta^2 + \sin^2\theta d\varphi^2$. At this point, we introduce $C(r)$ to reproduce all the relevant field equations from the action principle. After deriving the field equations, one may put $C(r) = 1$ by redefining the radial coordinate (see, e.g., Ref. [29]).

The scalar field is assumed to be dependent linearly on the time coordinate,

$$\phi(t, r) = \mu t + \psi(r), \tag{5}$$

where μ is a constant. Without loss of generality, we assume that $\mu > 0$. This configuration is consistent with the static metric (4) because the action (1) depends on ϕ only through its derivatives.

The crucial points of the ansatz (5) are the following. First, by assuming a linearly time-dependent scalar field, one can avoid the postulate in the no-hair theorem of [12], which makes it easier to obtain hairy solutions. Indeed, spherically symmetric black-hole solutions with such a scalar field configuration have been found in the context of the Horndeski theory [10,11,30–32] and beyond-Horndeski/DHOST theories [33–37]. Second, it has been assumed in the formulation of the EFT of black-hole perturbations [3,4] that the scalar field depends only on the radial coordinate. Therefore, for a time-dependent scalar field configuration, the previous result from the effective field theory approach cannot be used straightforwardly, and it is interesting to explore a general form of the effective action for black-hole perturbations in the presence of time-dependent hair.

Substituting the metric (4) and the scalar field ansatz (5) to the action (1) and varying it with respect to A, B, C, and ψ, one is able to derive the background field equations. We write the resultant field equations as $\mathcal{E}_A = 0$, $\mathcal{E}_B = 0$, $\mathcal{E}_C = 0$, and $\mathcal{E}_\psi = 0$, whose explicit expressions are not important in the present paper. These equations do not reduce to second-order differential equations in general because we do not impose any degeneracy conditions. However, as far as the odd mode perturbations are concerned, we do not need to care about the higher-order nature of the background equations. We will just use (some of) these background equations in their original form to simplify the quadratic action for the odd mode perturbations, whether they are of second order or higher.

Before proceeding to the analysis of perturbations, let us present a simple explicit example of background solutions. An interesting class of solutions often studied in the literature is a stealth Schwarzschild black hole with $X = X_0 = \text{const}$. One can see that our field equations admit the solution

$$A = B = 1 - \frac{r_h}{r}, \qquad X = X_0 = \frac{\mu^2}{2}, \tag{6}$$

provided that the functions in the action (1) satisfy the following equations (cf. Ref. [36]):

$$\begin{aligned} &F_0(X_0) = 0, \qquad F_{0X}(X_0) = 0, \qquad F_{1X}(X_0) = 0 \\ &A_1(X_0) + A_2(X_0) = 0, \qquad A_{1X}(X_0) + A_{2X}(X_0) = 0, \\ &B_2(X_0) = -\frac{1}{2}B_3(X_0) = 9B_1(X_0), \\ &B_4(X_0) + B_6(X_0) - B_{1X}(X_0) - B_{2X}(X_0) - \frac{5}{9}B_{3X}(X_0) \\ &\quad = \frac{6}{X_0}B_1(X_0). \end{aligned} \tag{7}$$

Note that these relations are compatible with the degeneracy conditions in the class ^{2}N-I + ^{3}M-I degenerate theories in the terminology of Ref. [19], and therefore the above solution is admitted even if one concentrates on a degenerate theory. From $2X = \mu^2 = \mu^2/A - B(d\psi/dr)^2$, we have

$$\psi = \pm\mu\left[2\sqrt{r_h r} + r_h \ln\left(\frac{\sqrt{r} - \sqrt{r_h}}{\sqrt{r} + \sqrt{r_h}}\right)\right]. \tag{8}$$

We choose the "+" branch because we have $\phi \simeq \mu[t \pm r_h \ln(r/r_h - 1)] + \text{const}$ near the horizon and it is regular at the horizon only in the "+" branch, as is clear by expressing ϕ in terms of the ingoing null coordinate $v = t + r + r_h \ln(r/r_h - 1)$ [10].

IV. ODD PARITY PERTURBATIONS

A. Derivation of the quadratic lagrangian and the effective metric

Let us consider the odd mode metric perturbations,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \tag{9}$$

where $\bar{g}_{\mu\nu}$ is the background metric (4) with $C(r) = 1$. The scalar field does not have an odd mode perturbation. Among the ten components, h_{ta}, h_{ra}, and h_{ab} are concerned with odd parity modes, where $a = \theta, \varphi$. Using the spherical harmonics $Y_{\ell m}(\theta, \varphi)$, we follow the standard procedure and expand the odd mode perturbations as

$$h_{t\theta} = -\frac{1}{\sin\theta}\partial_\varphi \sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell} h_0^{(\ell m)}(t,r)Y_{\ell m}(\theta,\varphi), \tag{10}$$

$$h_{t\varphi} = \sin\theta\partial_\theta \sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell} h_0^{(\ell m)}(t,r)Y_{\ell m}(\theta,\varphi), \tag{11}$$

$$h_{r\theta} = -\frac{1}{\sin\theta}\partial_\varphi \sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell} h_1^{(\ell m)}(t,r)Y_{\ell m}(\theta,\varphi), \tag{12}$$

$$h_{r\varphi} = \sin\theta\partial_\theta \sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell} h_1^{(\ell m)}(t,r)Y_{\ell m}(\theta,\varphi). \tag{13}$$

The odd parity part of h_{ab} can also be expressed using a single pseudoscalar function, say h_2, but we adopt the Regge-Wheeler gauge in which $h_2 = 0$ and accordingly $h_{ab} = 0$.

We substitute Eqs. (10)–(13) to the action (1) and expand it to second order in perturbations. In doing so, one can remove many terms by using the background equations. Performing the angular integrations, we arrive in the end at the general action

$$S_{\rm grav} = \sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell}\int {\rm d}t{\rm d}r\mathcal{L}^{(2)}_{\ell m}, \tag{14}$$

where, omitting the labels (ℓm) from h_0 and h_1,

$$\mathcal{L}^{(2)}_{\ell m} = \frac{1}{2}\left\{\left[\frac{2}{r^2}(ra_3)' + a_1\right]|h_0|^2 + a_2|h_1|^2 + a_3\left(|\dot{h}_1|^2 - 2\dot{h}_1^* h_0' + |h_0'|^2 + \frac{4}{r}\dot{h}_1^* h_0\right) + a_4 h_1^* h_0\right\} + {\rm c.c.}. \tag{15}$$

The coefficients are given by

$$a_1 = \frac{c_\ell}{2r^2\sqrt{AB}}\left\{F_2 + \frac{\mu^2}{A}A_1 + \frac{B\psi' X'}{2}F_{3X} + \frac{\mu^2}{A\psi'}\left[\frac{2B(\psi')^2}{r} - \frac{(AX)'}{A}\right]B_2 + \frac{3\mu^2 B\psi'}{rA}B_3 - \frac{\mu^2 B\psi' X'}{A}B_6\right\}, \tag{16}$$

$$a_2 = -\frac{c_\ell}{2}\frac{\sqrt{AB}}{r^2}\left\{F_2 - B(\psi')^2 A_1 - \frac{B\psi' X'}{2}F_{3X} - B\psi'\left[\frac{2B(\psi')^2}{r} - \frac{(AX)'}{A}\right]B_2 - \frac{3B^2(\psi')^3}{r}B_3 + B^2(\psi')^3 X' B_6\right\}, \tag{17}$$

$$a_3 = \frac{\ell(\ell+1)}{2}\sqrt{\frac{B}{A}}\left\{F_2 + 2XA_1 - \frac{B\psi' X'}{2}F_{3X} + \frac{2X}{\psi'}\left[\frac{2B(\psi')^2}{r} - \frac{(AX)'}{A}\right]B_2 + \frac{3X}{\psi'}\left[\frac{B(\psi')^2}{r} - X\frac{A'}{A} - \frac{\mu^2 X'}{2AX}\right]B_3 - 2B\psi' XX' B_6\right\}, \tag{18}$$

$$a_4 = -\frac{c_\ell}{r^2}\sqrt{\frac{B}{A}}\mu\left\{\psi' A_1 + \frac{X'}{2}F_{3X} + \left[\frac{2B(\psi')^2}{r} - \frac{(AX)'}{A}\right]B_2 + \frac{3B(\psi')^2}{r}B_3 - B(\psi')^2 X' B_6\right\}, \tag{19}$$

with $c_\ell = (\ell-1)\ell(\ell+1)(\ell+2)$. Here, a dot and a prime denote differentiation with respect to t and r, respectively. Following Ref. [24], it is convenient to write these coefficients as

$$a_1 = \frac{c_\ell}{4r^2\sqrt{AB}}\mathcal{F}(r), \qquad a_2 = -\frac{c_\ell}{4}\frac{\sqrt{AB}}{r^2}\mathcal{G}(r),$$
$$a_3 = \frac{\ell(\ell+1)}{4}\sqrt{\frac{B}{A}}\mathcal{H}(r), \qquad a_4 = \frac{c_\ell}{2r^2}\sqrt{\frac{B}{A}}\mathcal{J}(r), \tag{20}$$

where $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$, and $\mathcal{J}$ have a dimension of $({\rm mass})^2$. For the Schwarzschild solution in general relativity, we simply have $\mathcal{F} = \mathcal{G} = \mathcal{H} = M_{\rm Pl}^2 = (8\pi G)^{-1}$ and $\mathcal{J} = 0$.

Before proceeding to further reduction of the Lagrangian (15), let us point out two things, both of which come essentially from the fact that the odd parity modes constitute a part of tensorial metric perturbations. First, note that only F_2, F_3, A_1, B_2, B_3, and B_6 participate in the above result. Contributions from the other terms are dropped from the action upon using the background equations. This is as expected because it is known that only these terms contribute to the tensor modes on a cosmological background [38,39]. More specifically, gravitational waves are transverse and traceless metric perturbations, and hence they cannot arise from the terms such as $\Box\phi$ and $\phi^\nu\phi_{\mu\nu}$. Only a few functions thus appear in the quadratic Lagrangian for metric perturbations that correspond to gravitational waves. Second, it should be emphasized that the quadratic Lagrangian (15) is derived without using any degeneracy conditions. This is also not surprising because tensorial metric perturbations in the theory (1) obey second-order equations without regard to the degeneracy conditions. Therefore, our result can be used, for example, to U-degenerate theories [25] and detuned ("scordatura") DHOST theories [26].

Now, let us rewrite the Lagrangian (15) in terms of a single master variable. This can be done straightforwardly, following closely Refs. [13,14,24]. First, we introduce an auxiliary field $\chi = \chi^{(\ell m)}(t, r)$ and rewrite the Lagrangian (15) in an equivalent way as

$$\mathcal{L}^{(2)}_{\ell m} = \frac{1}{2}\left[a_1|h_0|^2 + a_2|h_1|^2 + a_4 h_1^* h_0 + 2a_3\chi^*\left(-\frac{1}{2}\chi + \dot{h}_1 - h_0' + \frac{2}{r}h_0\right)\right] + \text{c.c.} \quad (21)$$

Variation with respect to h_0^* and h_1^* leads, respectively, to

$$a_1 h_0 + (a_3\chi)' + \frac{2a_3}{r}\chi + \frac{1}{2}a_4 h_1 = 0, \quad (22)$$

$$a_2 h_1 - a_3\dot{\chi} + \frac{1}{2}a_4 h_0 = 0, \quad (23)$$

which can be solved for h_0 and h_1 to express them in terms of χ, $\dot{\chi}$, and χ':

$$h_0 = -\frac{8a_2a_3(\chi/r) + 4a_2(a_3\chi)' + 2a_3a_4\dot{\chi}}{4a_1a_2 - a_4^2}, \quad (24)$$

$$h_1 = \frac{4a_3a_4(\chi/r) + 2a_4(a_3\chi)' + 2a_1a_3\dot{\chi}}{4a_1a_2 - a_4^2}. \quad (25)$$

[Here, we assumed that $\mathcal{F}\mathcal{G} + (B/A)\mathcal{J}^2 \neq 0$.] Substituting Eqs. (24) and (25) back to Eq. (21), we obtain

$$\mathcal{L}^{(2)}_{\ell m} = \frac{\ell(\ell+1)r^2}{4(\ell-1)(\ell+2)}\sqrt{\frac{B}{A}}\left\{b_1|\dot{\chi}|^2 - b_2|\chi'|^2 + b_3\dot{\chi}^*\chi' - \left[\frac{\ell(\ell+1)}{2}\frac{\mathcal{H}}{r^2} + \frac{V}{2}\right]|\chi|^2\right\} + \text{c.c.}, \quad (26)$$

where

$$b_1 = \frac{\mathcal{F}}{2A}\cdot\frac{A\mathcal{H}^2}{A\mathcal{F}\mathcal{G} + B\mathcal{J}^2}, \qquad b_2 = \frac{\mathcal{G}B}{2}\cdot\frac{A\mathcal{H}^2}{A\mathcal{F}\mathcal{G} + B\mathcal{J}^2}, \qquad b_3 = \frac{B\mathcal{J}}{A}\cdot\frac{A\mathcal{H}^2}{A\mathcal{F}\mathcal{G} + B\mathcal{J}^2}, \quad (27)$$

and

$$V = 2\mathcal{H}\left[r^2 b_2\sqrt{\frac{B}{A}}\left(\frac{\sqrt{A/B}}{r^2\mathcal{H}}\right)'\right]' - \frac{2\mathcal{H}}{r^2}. \quad (28)$$

The equation of motion that follows from this Lagrangian is given by

$$b_1\ddot{\chi} - \frac{\sqrt{A/B}}{r^2}\left(r^2\sqrt{\frac{B}{A}}b_2\chi'\right)' + \frac{b_3}{2}\dot{\chi}' + \frac{\sqrt{A/B}}{2r^2}\left(r^2\sqrt{\frac{B}{A}}b_3\dot{\chi}\right)' + \left[\frac{\ell(\ell+1)}{2}\frac{\mathcal{H}}{r^2} + \frac{V}{2}\right]\chi = 0. \quad (29)$$

At this stage, it can be seen from the Lagrangian (26) that we need to impose

$$\mathcal{H} > 0, \quad (30)$$

as otherwise modes with large ℓ would have large negative energy and make the system unstable quickly.

One notices that Eq. (29) can be written in the form

$$\mathcal{H}\Omega^2 Z^{\mu\nu}D_\mu D_\nu\chi - V\chi = 0, \quad (31)$$

where $Z^{\mu\nu}$ is the inverse of the effective metric $Z_{\mu\nu}$ [40],

$$Z_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = \Omega^2\left(-\frac{\mathcal{G}}{\mathcal{H}}A\mathrm{d}t^2 - \frac{2\mathcal{J}}{\mathcal{H}}\mathrm{d}t\mathrm{d}r + \frac{\mathcal{F}}{\mathcal{H}}\frac{\mathrm{d}r^2}{B} + r^2\mathrm{d}\sigma^2\right), \quad (32)$$

with

$$\Omega^2 := \frac{B}{A}\frac{\mathcal{H}^2}{\sqrt{\mathcal{F}\mathcal{G} + (B/A)\mathcal{J}^2}}, \quad (33)$$

and D_μ is the covariant derivative operator defined in terms of a connection compatible with $Z_{\mu\nu}$. Note here that the metric perturbations have already been expanded in terms of the spherical harmonics and hence the spherical Laplacian in $Z^{\mu\nu}D_\mu D_\nu$ must be replaced with its eigenvalue $-\ell(\ell+1)$. Note also that

$$\zeta^2(r) := \mathcal{F}\mathcal{G} + \frac{B}{A}\mathcal{J}^2 > 0 \quad (34)$$

must be imposed in order for the effective metric to be well defined. It is easy to see that one has $Z_{\mu\nu} = M_{\rm Pl}^2\bar{g}_{\mu\nu}$ in general relativity, where $\mathcal{F} = \mathcal{G} = \mathcal{H} = M_{\rm Pl}^2$ and $\mathcal{J} = 0$. However, $Z_{\mu\nu}$ may not be proportional to $\bar{g}_{\mu\nu}$ in modified gravity. This fact itself has already been known in the context of the Horndeski theory [41–43].

We introduce a new time coordinate τ defined by

$$\mathrm{d}\tau = \mathrm{d}t + \frac{\mathcal{J}}{A\mathcal{G}}\mathrm{d}r. \quad (35)$$

Using τ, the effective metric (32) can be written in a diagonal form as

$$Z_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = \Omega^2\left(-\frac{\mathcal{G}}{\mathcal{H}}A\mathrm{d}\tau^2 + \frac{\zeta^2}{\mathcal{G}H}\frac{\mathrm{d}r^2}{B} + r^2\mathrm{d}\sigma^2\right). \quad (36)$$

It is sometimes more convenient to work in the conformally related effective metric $\tilde{Z}_{\mu\nu}$ defined as

$$\tilde{Z}_{\mu\nu} = \Omega^{-2}Z_{\mu\nu}. \quad (37)$$

In the tilded frame, Eq. (31) is written as

$$\tilde{Z}^{\mu\nu}\tilde{D}_\mu\tilde{D}_\nu\left(\frac{\tilde{\chi}}{r}\right) - \left[\frac{V}{\mathcal{H}} + \frac{\tilde{Z}^{\mu\nu}\tilde{D}_\mu\tilde{D}_\nu\Omega}{\Omega}\right]\frac{\tilde{\chi}}{r} = 0, \quad (38)$$

where $\tilde{\chi} \coloneqq \Omega r\chi$ and $\tilde{D}_\mu$ is the covariant derivative operator defined in terms of a connection compatible with $\tilde{Z}_{\mu\nu}$.

Defining the generalized tortoise coordinate by

$$\mathrm{d}r_* = \frac{\zeta}{\mathcal{G}\sqrt{AB}}\mathrm{d}r, \quad (39)$$

Eq. (38) can further be rewritten in a more familiar form as

$$(-\partial_\tau^2 + \partial_{r_*}^2 - \tilde{V})\tilde{\chi} = 0, \quad (40)$$

where

$$\tilde{V} = \frac{\mathcal{G}A}{\mathcal{H}}\left\{\frac{(\ell+2)(\ell-1)}{r^2} + \Omega r\left[\frac{\mathcal{G}\sqrt{AB}}{\zeta}\left(\frac{1}{r\zeta^{1/2}}\right)'\right]'\right\}. \quad (41)$$

This generalizes the Regge-Wheeler equation known in general relativity [44] to higher-order scalar-tensor theories. In Appendix B, we extend the main result of this section to include the energy-momentum tensor of matter and derive the generalized Regge-Wheeler equation with a matter source term.

So far, we have focused on the modes with $\ell \geq 2$. The dipole ($\ell = 1$) mode must be treated separately, but here we only comment that the dipole perturbation corresponds to adding a slow rotation, as has been already discussed in detail in the previous literature [8,13,14,24].

B. Propagation speed

In theories described by the action (1), the propagation speed of gravitational waves differs in general from the speed of light. In light of the constraint from GW170817 [45–47], let us identify the subclass of scalar-tensor theories that admits a luminal speed of gravitational waves at least at large r. This weak requirement was also employed in Ref. [4] (see, however, Refs. [48,49]).

We assume that the background is given by

$$A = 1 + \mathcal{O}(r^{-1}), \qquad B = 1 + \mathcal{O}(r^{-1}),$$
$$\psi' = \psi'_\infty + \mathcal{O}(r^{-1/2}), \quad (42)$$

for large r, where ψ'_∞ is a constant. We then find

$$\mathcal{F} = 2[F_2(X_\infty) + \mu^2 A_1(X_\infty)] + \mathcal{O}(r^{-1/2}), \quad (43)$$

$$\mathcal{G} = 2[F_2(X_\infty) - (\psi'_\infty)^2 A_1(X_\infty)] + \mathcal{O}(r^{-1/2}), \quad (44)$$

$$\mathcal{H} = 2[F_2(X_\infty) + 2X_\infty A_1(X_\infty)] + \mathcal{O}(r^{-1/2}), \quad (45)$$

$$\mathcal{J} = -2\mu\psi'_\infty A_1(X_\infty) + \mathcal{O}(r^{-1/2}), \quad (46)$$

where $X_\infty \coloneqq [\mu^2 - (\psi'_\infty)^2]/2$. Thus, if one has

$$A_1(X_\infty) = 0, \quad (47)$$

Eq. (40) reduces to $[-\partial_t^2 + \partial_r^2 - \ell(\ell+1)/r^2]\tilde{\chi} \simeq 0$ for large r, rendering luminal propagation of gravitational waves sufficiently away from a black hole. Note that F_3 and B_I appear only in the $\mathcal{O}(r^{-1})$ or higher-order terms in $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$, and $\mathcal{J}$.

A comment is now in order. In the even parity sector, there must be a mode that can be identified as gravitational waves. It is expected that in general the propagation speed of that mode coincides with that of the odd parity mode in the absence of gravitational parity violating interactions. This is indeed the case in the Horndeski theory [9]. Note in passing that we also have a mode in the even parity sector that can be identified as fluctuations of the scalar field, and its propagation speed differs in general from the speed of gravitational waves.

C. Horizons for photons and gravitons

Suppose that r_h is the location of the horizon in the metric $\bar{g}_{\mu\nu}$ and the metric components are expanded as

$$A(r) = \sum_{n=1}\alpha_n\epsilon^n, \qquad B(r) = \sum_{n=1}\beta_n\epsilon^n, \quad (48)$$

where $\epsilon \coloneqq r/r_h - 1 > 0$. We assume that X is regular at the horizon, so that X is of the form

$$X = X_h + \sum_{n=1}X_n\epsilon^n. \quad (49)$$

Accordingly, one has

$$\psi' = \frac{\mu}{\sqrt{\alpha_1\beta_1}}\frac{1}{\epsilon} + \sum_{n=0}\gamma_n\epsilon^n. \quad (50)$$

Note that ψ' diverges as $r \to r_h$, but this is not problematic. See the comment below Eq. (8). Substituting Eqs. (48)–(50) into Eqs. (16)–(19), we find, in the vicinity of the horizon,

$$\mathcal{F}=-\frac{d_0}{\epsilon}-d_1+\mathcal{O}(\epsilon),\quad \mathcal{G}=\frac{d_0}{\epsilon}+d_2+\mathcal{O}(\epsilon),$$
$$\mathcal{H}=d_3+\mathcal{O}(\epsilon),\quad \sqrt{\frac{B}{A}}\mathcal{J}=\frac{d_0}{\epsilon}+\frac{d_1+d_2}{2}+\mathcal{O}(\epsilon),\tag{51}$$

and hence $\zeta=\text{const}+\mathcal{O}(\epsilon)$, where

$$d_0=-\frac{2\mu^2}{\alpha_1}A_1(X_h)+\frac{2\mu}{r_h}\sqrt{\frac{\beta_1}{\alpha_1^3}}[(\alpha_1X_h-2\mu^2)B_2(X_h)$$
$$-3\mu^2B_3(X_h)+\mu^2X_1B_6(X_h)],\tag{52}$$

while the explicit expressions for d_1, d_2, and d_3 are more involved. Hereafter, we will consider the case where d_0 is nonvanishing. Thus, at $r\simeq r_h$,

$$\Omega\simeq\text{const},\qquad \tilde{Z}_{\tau\tau}\simeq\text{const},\qquad \tilde{Z}_{rr}\simeq\text{const},\tag{53}$$

which shows that nothing special happens in the effective metric at the horizon of the metric $\bar{g}_{\mu\nu}$. In particular, this fact implies that $r=r_h$ is not an appropriate place to impose the inner boundary conditions when solving the Regge-Wheeler equation (40). Rather, the form of the effective metric implies that a possible appropriate boundary will be $r=r_g$, where $\mathcal{G}(r_g)=0$. To see this more explicitly, let us study some concrete examples.

The first example is given by the special case of the solution in Sec. III, with $A_1(X_0)\neq 0$ and $B_1(X_0)=0$. Essentially the same solution is also studied in Ref. [24]. This does not satisfy Eq. (47) but is a good illustrative example. We have

$$\mathcal{G}=2F_2(X_0)\cdot\frac{1-r_g/r}{1-r_h/r},\qquad \mathcal{H}=2F_2(X_0)(1+\mathcal{A}),\tag{54}$$

where

$$r_g\coloneqq(1+\mathcal{A})r_h,\qquad \mathcal{A}\coloneqq\frac{2X_0A_1(X_0)}{F_2(X_0)},\tag{55}$$

and we assume that $F_2(X_0)>0$ and $1+\mathcal{A}>0$. The conformal factor is a nonvanishing constant, $\Omega^2=2F_2(X_0)(1+\mathcal{A})^{3/2}$, and the components of the (tilded) effective metric are given by

$$\tilde{Z}_{\tau\tau}=-\frac{1-r_g/r}{1+\mathcal{A}},\qquad \tilde{Z}_{rr}=\frac{1}{1-r_g/r},\tag{56}$$

which shows that the horizon of the effective metric is at $r=r_g(\neq r_h)$. In this case, the generalized tortoise coordinate is given by $r_*=(1+\mathcal{A})^{1/2}[r+r_g\ln(r/r_g-1)]$ and the potential in Eq. (40) reads

$$\tilde{V}=\frac{1-r_g/r}{1+\mathcal{A}}\left[\frac{\ell(\ell+1)}{r^2}-\frac{3r_g}{r^3}\right].\tag{57}$$

Aside from the constant factor of $(1+\mathcal{A})^{-1}$, this coincides with the well-known potential in the Regge-Wheeler equation in general relativity with the horizon at $r=r_g$.

In this example, $\mathcal{G}$ is singular at $r=r_h$. One also notices that $\mathcal{G}<0$ for $r_g<r<r_h$ if $\mathcal{A}<0$. However, the effective metric and the potential do not depend on r_h explicitly and are free from any pathologies. In particular, the sign of $\mathcal{G}$ does not directly related to the stability of the solution. Indeed, it is now clear that the above solution is stable provided that $F_2(X_0)>0$ and $1+\mathcal{A}>0$ are satisfied.

The second example is again the special case of the solution in Sec. III, but now with $A_1(X_0)=0$ and $B_1(X_0)\neq 0$. In this case, we have

$$\mathcal{G}=2F_2(X_0)\cdot\frac{f(r)}{1-r_h/r},\quad \mathcal{H}=2F_2(X_0),\tag{58}$$

where

$$f(r)=1-\frac{r_h}{r}+\mathcal{B}\left(\frac{r_h}{r}\right)^{5/2},\qquad \mathcal{B}\coloneqq\frac{81}{2}\frac{\mu^3}{r_h}\frac{B_1(X_0)}{F_2(X_0)}.\tag{59}$$

The conformal factor is given by

$$\Omega^2=\frac{2F_2(X_0)}{g^{1/2}(r)},\tag{60}$$

and the (tilded) effective metric reduces to

$$\tilde{Z}_{\tau\tau}=-f(r),\qquad \tilde{Z}_{rr}=\frac{g(r)}{f(r)},\tag{61}$$

where

$$g(r)=1-\mathcal{B}\left(\frac{r_h}{r}\right)^{3/2}.\tag{62}$$

We see that the horizon of the effective metric is at $r=r_g\neq r_h$, where $f(r_g)=0$.

Let us investigate the structure of the effective metric (61) in more detail. For $\mathcal{B}>6/25(\sqrt{3/5})(\simeq 0.186)$, f has no zeros, while $g=0$ at $r=\mathcal{B}^{2/3}r_h$. We are not interested in this case. For $0<\mathcal{B}\leq 6/25(\sqrt{3/5})$, we have $f=0$ at $r=r_g<r_h$. In this case, g remains positive outside the horizon of the effective metric, but $g=0$ occurs at $r=\mathcal{B}^{2/3}r_h<r_g$. Finally, for $\mathcal{B}<0$, we have $f=0$ at $r=r_g>r_h$, and g is always positive for $r>0$. Therefore, in the latter two cases, the solution has an outer horizon of the effective metric at $r=r_g$. It is straightforward to write the potential $\tilde{V}$, but the expression is messy. The shape of

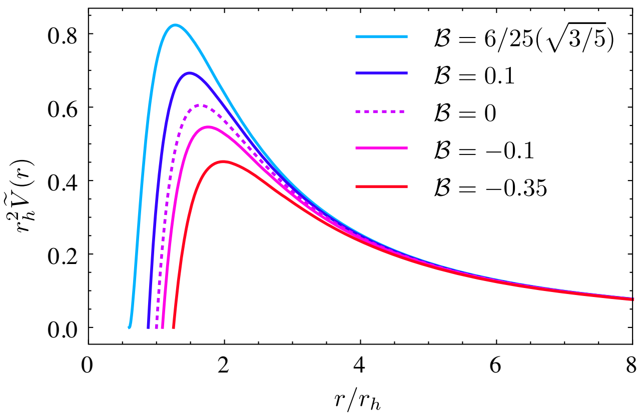

FIG. 1. Potential $\tilde{V}$ with $\ell = 2$ as a function of r/r_h.

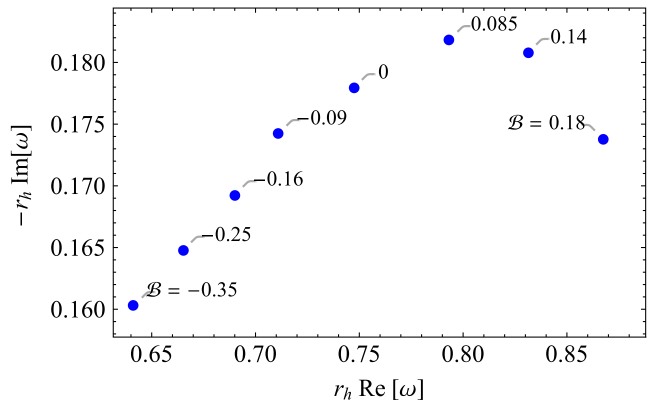

FIG. 2. Lowest overtone quasinormal frequencies for $\ell = 2$ and some representative values of $\mathcal{B}$.

the potential is shown for different values of $\mathcal{B}$ in Fig. 1. One can check that $r_* \to -\infty$ as $r \to r_g$.

V. QUASINORMAL MODES

In this section, we compute the QNMs of the second example of the previous section. Quasinormal modes in the Horndeski theory have been studied in the case of the Schwarzschild background with a constant scalar field [50] and a nearly Schwarzschild background with a nearly constant scalar field [51].

We assume the time dependence of the master variable as $\tilde{\chi} = Q(r)e^{-i\omega\tau}$ and solve

$$\frac{\mathrm{d}^2 Q}{\mathrm{d}r_*^2} + [\omega^2 - \tilde{V}(r)]Q = 0, \tag{63}$$

where $\tilde{V}$ is the potential obtained from the second example of the background solutions in the previous section, which is characterized by the dimensionless parameter $\mathcal{B}$ (Fig. 1). The boundary conditions for Q are given by

$$Q \propto \begin{cases} e^{+i\omega r_*} & r_* \to \infty (r \to \infty) \\ e^{-i\omega r_*} & r_* \to -\infty (r \to r_g) \end{cases}. \tag{64}$$

Note again that the inner boundary is located at $r = r_g$ rather than at $r = r_h$. In order to obtain the QNMs, we employ direct numerical integration.[1] The lowest overtone quasinormal frequencies for $\ell = 2$ are given in Fig. 2, showing how the frequencies depend on the modified gravity parameter $\mathcal{B}$.

[1]Taking $\mathcal{B}$ as a small expansion parameter, one may write the potential as $\tilde{V} = V_{\mathrm{GR}} + \delta V$, where V_{GR} is the Regge-Wheeler potential in general relativity and δV is a small correction. In the present case, δV contains fractional powers of r, which hinders us from using the convenient formalism of Ref. [52].

VI. CONCLUSIONS

In this paper, we have studied odd parity perturbations of black holes with linearly time-dependent scalar hair in shift-symmetric scalar-tensor theories. Due to the time dependence of the scalar field background, the EFT approach [2–4] cannot be applied straightforwardly to the present case. Therefore, we have started from a general covariant action that is most similar to the action of cubic degenerate higher-order scalar-tensor theories [19] and derived the general quadratic action for odd parity perturbations without imposing the degeneracy conditions. The degeneracy conditions are not essential for retaining the healthy odd parity perturbations that are not mixed with the perturbation of the scalar field. We have thus derived a second-order equation for a single master variable as a generalization of the Regge-Wheeler equation in general relativity. Starting from the more general action, we have arrived at qualitatively the same results as the previous ones [13,14,24], showing that no new terms appear in the quadratic action for odd parity perturbations. Our generalized Regge-Wheeler equation can be used in a wide class of scalar-tensor theories such as U-degenerate theories [25] and scordatura theories [26].

We have also refined the stability conditions explored in the previous literature [24]. The previous conditions were actually sufficient conditions, and we have argued that one of the conditions is not directly related to the stability.

As another application of our results, we have computed the quasinormal modes of a certain nontrivial black-hole solution. In doing so, we have demonstrated that it is important to identify the correct location of the inner boundary by inspecting the effective metric for gravitons.

It would be interesting to extend the present analysis to the even parity sector, which would be much more involved due to its higher derivative nature. It would also be interesting to perform a complementary analysis based on the EFT approach along the lines of Refs. [3,4].

ACKNOWLEDGMENTS

We are grateful to Masashi Kimura for sharing his *Mathematica* code for calculating quasinormal modes. We also thank Hayato Motohashi and Kazufumi Takahashi for their comments on the manuscript. The work of K. T. was supported by the Rikkyo University Special Fund for Research (SFR). The work of T. K. was supported by JSPS KAKENHI Grants No. JP20H04745 and No. JP20K03936.

APPENDIX A: GENERALITY OF THE QUADRATIC LAGRANGIAN

Starting from the action (1), we have shown in the main text that the quadratic Lagrangian for the odd parity modes is given by Eq. (15). Actually, one can show that more general scalar-tensor theories lead to the quadratic Lagrangian for the odd parity modes having the same structure as Eq. (15) as long as the equation of motion for gravitational-wave degrees of freedom remains of second order.

For example, one may add to the action (1)

$$\tilde{F}_3(X)R\Box\phi, \tag{A1}$$

to consider a general derivative coupling of the form $F_3G_{\mu\nu}\phi^{\mu\nu}+\tilde{F}_3R\Box\phi=F_3R_{\mu\nu}\phi^{\mu\nu}+(\tilde{F}_3-F_3/2)R\Box\phi$. This only shifts the coefficients as

$$\mathcal{F},G,H\to\mathcal{F},G,H+\left[\frac{B\psi'}{r}-\frac{(AX)'}{A\psi'}\right]\tilde{F}_3, \tag{A2}$$

$$\mathcal{J}\to\mathcal{J} \tag{A3}$$

and does not give rise to any new terms in Eq. (15).

Similarly, one may also add terms quartic in second derivatives of ϕ such as

$$C_1(X)\phi_{\mu\nu}\phi^{\nu\rho}\phi_{\rho\lambda}\phi^{\lambda\mu},\qquad C_2(X)(\Box\phi)^4,\cdots. \tag{A4}$$

One can verify by direct computation that such quartic terms merely shift the coefficients without altering the structure of the Lagrangian (15) or have no contribution to the odd parity sector.

We thus conclude that the form of the Lagrangian (15) is generic to scalar-tensor theories in which gravitational-wave degrees of freedom obey a second-order equation of motion.

APPENDIX B: SOURCED REGGE-WHEELER EQUATION

In this Appendix, we generalize our main result to include the source term, which has not been considered in the previous similar studies [8,13,14,24]. Assuming that matter is minimally coupled to gravity, the source term can be obtained from

$$S_{\rm source}=\frac{1}{2}\int{\rm d}^4x\sqrt{-\bar{g}}h^{\mu\nu}T_{\mu\nu}, \tag{B1}$$

where $T_{\mu\nu}$ is the matter energy-momentum tensor. Similarly to the metric perturbations, the odd parity part of the energy momentum tensor can also be expanded as

$$T_{t\theta}=-\frac{1}{\sin\theta}\partial_\varphi\sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell}S_0^{(\ell m)}(t,r)Y_{\ell m}, \tag{B2}$$

$$T_{t\varphi}=\sin\theta\partial_\theta\sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell}S_0^{(\ell m)}(t,r)Y_{\ell m}, \tag{B3}$$

$$T_{r\theta}=-\frac{1}{\sin\theta}\partial_\varphi\sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell}S_1^{(\ell m)}(t,r)Y_{\ell m}, \tag{B4}$$

$$T_{r\varphi}=\sin\theta\partial_\theta\sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell}S_1^{(\ell m)}(t,r)Y_{\ell m}, \tag{B5}$$

$$T_{\theta\theta}=\frac{2}{\sin\theta}(\partial_\theta\partial_\varphi-\cot\theta\partial_\varphi)\sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell}S_2^{(\ell m)}(t,r)Y_{\ell m}, \tag{B6}$$

$$T_{\theta\varphi}=\left(\frac{1}{\sin\theta}\partial_\varphi^2+\cos\theta\partial_\theta-\sin\theta\partial_\theta^2\right)\times\sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell}S_2^{(\ell m)}(t,r)Y_{\ell m}, \tag{B7}$$

$$T_{\varphi\varphi}=-2\sin\theta(\partial_\theta\partial_\varphi-\cot\theta\partial_\varphi)\sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell}S_2^{(\ell m)}(t,r)Y_{\ell m}. \tag{B8}$$

The conservation of the matter energy-momentum tensor, $\nabla_\nu T^{\mu\nu}=0$, yields

$$-\frac{\dot{S}_0^{(\ell m)}}{A}+\frac{\sqrt{B/A}}{r^2}(r^2\sqrt{AB}S_1^{(\ell m)})'+\frac{(\ell-1)(\ell+2)}{r^2}S_2^{(\ell m)}=0. \tag{B9}$$

It is straightforward to perform the angular integrations in Eq. (B1) to obtain

$$S_{\rm source}=-\sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell}\frac{\ell(\ell+1)}{2}\times\int{\rm d}t{\rm d}r\left(\frac{h_0^*S_0}{\sqrt{AB}}-\sqrt{AB}h_1^*S_1+{\rm c.c.}\right), \tag{B10}$$

where we omitted the labels (ℓm) from S_0 and S_1. This is the source action for the odd mode perturbations (see also

Ref. [4]). We add the above source action to the gravitational part of the action (14). Then, Eqs. (22) and (23) are generalized to

$$a_1 h_0 + (a_3\chi)' + \frac{2a_3}{r}\chi + \frac{1}{2}a_4 h_1 = \frac{\ell(\ell+1)}{2\sqrt{AB}}S_0, \tag{B11}$$

$$a_2 h_1 - a_3\dot{\chi} + \frac{1}{2}a_4 h_0 = -\frac{\ell(\ell+1)}{2}\sqrt{AB}S_1. \tag{B12}$$

Solving these equations for h_0 and h_1 and removing h_0 and h_1 from the quadratic Lagrangian, we see that the Lagrangian (26) is generalized to include the source as

$$\mathcal{L}^{(2)}_{\ell m,\mathrm{total}} = \mathcal{L}^{(2)}_{\ell m} - \frac{\ell(\ell+1)r^2}{4(\ell-1)(\ell+2)}\sqrt{\frac{B}{A}}(\chi^* S_{\mathrm{odd}} + \mathrm{c.c.}), \tag{B13}$$

where $\mathcal{L}^{(2)}_{\ell m}$ in the right-hand side is the same Lagrangian as the one defined as Eq. (26) and

$$S^{(\ell m)}_{\mathrm{odd}}(t,r) := 2\mathcal{H}\left(\frac{\mathcal{G}}{\zeta^2}S^{(\ell m)}_0\right)' - \frac{2\mathcal{F}H}{\zeta^2}\dot{S}^{(\ell m)}_1 - \frac{2\mathcal{H}J}{A\zeta^2}\dot{S}^{(\ell m)}_0 - 2\mathcal{H}\left(\frac{B\mathcal{J}}{\zeta^2}S^{(\ell m)}_1\right)'. \tag{B14}$$

Now, Eq. (31) with the source term reads

$$\mathcal{H}\Omega^2 Z^{\mu\nu}D_\mu D_\nu\chi - V\chi = S_{\mathrm{odd}}, \tag{B15}$$

and, accordingly, Eq. (40) with the source term is given by

$$(-\partial_\tau^2 + \partial_{r_*}^2 - \tilde{V})\tilde{\chi} = \frac{\mathcal{G}r\sqrt{AB}}{\mathcal{H}\zeta^{1/2}}S_{\mathrm{odd}}. \tag{B16}$$

This is the generalization of the sourced Regge-Wheeler equation.

[1] B. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), Observation of Gravitational Waves from a Binary Black Hole Merger, Phys. Rev. Lett. **116**, 061102 (2016).

[2] R. Kase, L. A. Gergely, and S. Tsujikawa, Effective field theory of modified gravity on the spherically symmetric background: Leading order dynamics and the odd-type perturbations, Phys. Rev. D **90**, 124019 (2014).

[3] G. Franciolini, L. Hui, R. Penco, L. Santoni, and E. Trincherini, Effective field theory of black hole quasinormal modes in scalar-tensor theories, J. High Energy Phys. 02 (2019) 127.

[4] A. Kuntz, R. Penco, and F. Piazza, Extreme mass ratio inspirals with scalar hair, J. Cosmol. Astropart. Phys. 08 (2020) 023.

[5] G. W. Horndeski, Second-order scalar-tensor field equations in a four-dimensional space, Int. J. Theor. Phys. **10**, 363 (1974).

[6] C. Deffayet, X. Gao, D. Steer, and G. Zahariade, From k-essence to generalised Galileons, Phys. Rev. D **84**, 064039 (2011).

[7] T. Kobayashi, M. Yamaguchi, and J. Yokoyama, Generalized G-inflation: Inflation with the most general second-order field equations, Prog. Theor. Phys. **126**, 511 (2011).

[8] T. Kobayashi, H. Motohashi, and T. Suyama, Black hole perturbation in the most general scalar-tensor theory with second-order field equations I: The odd-parity sector, Phys. Rev. D **85**, 084025 (2012).

[9] T. Kobayashi, H. Motohashi, and T. Suyama, Black hole perturbation in the most general scalar-tensor theory with second-order field equations II: The even-parity sector, Phys. Rev. D **89**, 084042 (2014).

[10] E. Babichev and C. Charmousis, Dressing a black hole with a time-dependent Galileon, J. High Energy Phys. 08 (2014) 106.

[11] T. Kobayashi and N. Tanahashi, Exact black hole solutions in shift symmetric scalar–tensor theories, Prog. Theor. Exp. Phys. **2014**, 073E02 (2014).

[12] L. Hui and A. Nicolis, No-Hair Theorem for the Galileon, Phys. Rev. Lett. **110**, 241104 (2013).

[13] H. Ogawa, T. Kobayashi, and T. Suyama, Instability of hairy black holes in shift-symmetric Horndeski theories, Phys. Rev. D **93**, 064078 (2016).

[14] K. Takahashi and T. Suyama, Linear perturbation analysis of hairy black holes in shift-symmetric Horndeski theories: Odd-parity perturbations, Phys. Rev. D **95**, 024034 (2017).

[15] J. Khoury, M. Trodden, and S. S. Wong, Existence and instability of novel hairy black holes in shift-symmetric Horndeski theories, J. Cosmol. Astropart. Phys. 11 (2020) 044.

[16] J. Gleyzes, D. Langlois, F. Piazza, and F. Vernizzi, Healthy Theories Beyond Horndeski, Phys. Rev. Lett. **114**, 211101 (2015).

[17] D. Langlois and K. Noui, Degenerate higher derivative theories beyond Horndeski: Evading the Ostrogradski instability, J. Cosmol. Astropart. Phys. 02 (2016) 034.

[18] M. Crisostomi, K. Koyama, and G. Tasinato, Extended scalar-tensor theories of gravity, J. Cosmol. Astropart. Phys. 04 (2016) 044.

[19] J. B. Achour, M. Crisostomi, K. Koyama, D. Langlois, K. Noui, and G. Tasinato, Degenerate higher order scalar-tensor theories beyond Horndeski up to cubic order, J. High Energy Phys. 12 (2016) 100.

[20] M. Zumalacárregui and J. García-Bellido, Transforming gravity: From derivative couplings to matter to second-order

scalar-tensor theories beyond the Horndeski Lagrangian, Phys. Rev. D **89**, 064046 (2014).
[21] D. Langlois, Degenerate higher-order scalar-tensor (DHOST) theories, in *52nd Rencontres de Moriond on Gravitation* (2017), pp. 221–228 [arXiv:1707.03625].
[22] D. Langlois, Dark energy and modified gravity in degenerate higher-order scalar–tensor (DHOST) theories: A review, Int. J. Mod. Phys. D **28**, 1942006 (2019).
[23] T. Kobayashi, Horndeski theory and beyond: A review, Rep. Prog. Phys. **82**, 086901 (2019).
[24] K. Takahashi, H. Motohashi, and M. Minamitsuji, Linear stability analysis of hairy black holes in quadratic degenerate higher-order scalar-tensor theories: Odd-parity perturbations, Phys. Rev. D **100**, 024041 (2019).
[25] A. De Felice, D. Langlois, S. Mukohyama, K. Noui, and A. Wang, Generalized instantaneous modes in higher-order scalar-tensor theories, Phys. Rev. D **98**, 084024 (2018).
[26] H. Motohashi and S. Mukohyama, Weakly-coupled stealth solution in Scordatura degenerate theory, J. Cosmol. Astropart. Phys. 01 (2020) 030.
[27] M. Minamitsuji and H. Motohashi, Stealth Schwarzschild solution in shift symmetry breaking theories, Phys. Rev. D **98**, 084027 (2018).
[28] C. de Rham and J. Zhang, Perturbations of stealth black holes in degenerate higher-order scalar-tensor theories, Phys. Rev. D **100**, 124023 (2019).
[29] H. Motohashi, T. Suyama, and K. Takahashi, Fundamental theorem on gauge fixing at the action level, Phys. Rev. D **94**, 124021 (2016).
[30] E. Babichev, C. Charmousis, and A. Lehébel, Black holes and stars in Horndeski theory, Classical Quantum Gravity **33**, 154002 (2016).
[31] E. Babichev, C. Charmousis, A. Lehébel, and T. Moskalets, Black holes in a cubic Galileon universe, J. Cosmol. Astropart. Phys. 09 (2016) 011.
[32] W. T. Emond, A. Lehébel, and P. M. Saffin, Black holes in self-tuning cubic Horndeski cosmology, Phys. Rev. D **101**, 084008 (2020).
[33] E. Babichev and G. Esposito-Farese, Cosmological self-tuning and local solutions in generalized Horndeski theories, Phys. Rev. D **95**, 024020 (2017).
[34] E. Babichev, C. Charmousis, and A. Lehébel, Asymptotically flat black holes in Horndeski theory and beyond, J. Cosmol. Astropart. Phys. 04 (**2017**) 027.
[35] H. Motohashi and M. Minamitsuji, Exact black hole solutions in shift-symmetric quadratic degenerate higher-order scalar-tensor theories, Phys. Rev. D **99**, 064040 (2019).
[36] M. Minamitsuji and J. Edholm, Black hole solutions in shift-symmetric degenerate higher-order scalar-tensor theories, Phys. Rev. D **100**, 044053 (2019).
[37] K. Takahashi and H. Motohashi, General Relativity solutions with stealth scalar hair in quadratic higher-order scalar-tensor theories, J. Cosmol. Astropart. Phys. 06 (2020) 034.
[38] C. de Rham and A. Matas, Ostrogradsky in theories with multiple fields, J. Cosmol. Astropart. Phys. 06 (2016) 041.
[39] D. Langlois, M. Mancarella, K. Noui, and F. Vernizzi, Effective description of higher-order scalar-tensor theories, J. Cosmol. Astropart. Phys. 05 (2017) 033.
[40] E. Babichev, C. Charmousis, G. Esposito-Farèse, and A. Lehébel, Hamiltonian unboundedness vs stability with an application to Horndeski theory, Phys. Rev. D **98**, 104050 (2018).
[41] M. Minamitsuji, Causal structure in the scalar–tensor theory with field derivative coupling to the Einstein tensor, Phys. Lett. B **743**, 272 (2015).
[42] N. Tanahashi and S. Ohashi, Wave propagation and shock formation in the most general scalar-tensor theories, Classical Quantum Gravity **34**, 215003 (2017).
[43] R. Benkel, N. Franchini, M. Saravani, and T. P. Sotiriou, Causal structure of black holes in shift-symmetric Horndeski theories, Phys. Rev. D **98**, 064006 (2018).
[44] T. Regge and J. A. Wheeler, Stability of a Schwarzschild singularity, Phys. Rev. **108**, 1063 (1957).
[45] B. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral, Phys. Rev. Lett. **119**, 161101 (2017).
[46] B. Abbott *et al.* (LIGO Scientific, Virgo, Fermi-GBM, and INTEGRAL Collaborations), Gravitational waves and gamma-rays from a binary neutron star merger: GW170817 and GRB 170817A, Astrophys. J. Lett. **848**, L13 (2017).
[47] B. Abbott *et al.* (LIGO Scientific, Virgo, Fermi GBM, INTEGRAL, IceCube, AstroSat Cadmium Zinc Telluride Imager Team, IPN, Insight-Hxmt, ANTARES, Swift, AGILE Team, 1M2H Team, Dark Energy Camera GW-EM, DES, DLT40, GRAWITA, Fermi-LAT, ATCA, ASKAP, Las Cumbres Observatory Group, OzGrav, DWF [Deeper Wider Faster Program], AST3, CAASTRO, VINROUGE, MASTER, J-GEM, GROWTH, JAGWAR, CaltechNRAO, TTU-NRAO, NuSTAR, Pan-STARRS, MAXI Team, TZAC Consortium, KU, Nordic Optical Telescope, ePESSTO, GROND, Texas Tech University, SALT Group, TOROS, BOOTES, MWA, CALET, IKI-GW Follow-up, H.E.S.S., LOFAR, LWA, HAWC, Pierre Auger, ALMA, Euro VLBI Team, Pi of Sky, Chandra Team at McGill University, DFN, ATLAS Telescopes, High Time Resolution Universe Survey, RIMAS, RATIR, and SKA South Africa/MeerKAT Collaborations), Multi-messenger Observations of a Binary Neutron Star Merger, Astrophys. J. Lett. **848**, L12 (2017).
[48] E. Babichev, C. Charmousis, G. Esposito-Farèse, and A. Lehébel, Stability of Black Holes and the Speed of Gravitational Waves within Self-Tuning Cosmological Models, Phys. Rev. Lett. **120**, 241101 (2018).
[49] O. J. Tattersall, P. G. Ferreira, and M. Lagos, Speed of gravitational waves and black hole hair, Phys. Rev. D **97**, 084005 (2018).
[50] O. J. Tattersall and P. G. Ferreira, Quasinormal modes of black holes in Horndeski gravity, Phys. Rev. D **97**, 104047 (2018).
[51] O. J. Tattersall, Quasi-normal modes of hairy scalar tensor black holes: Odd parity, Classical Quantum Gravity **37**, 115007 (2020).
[52] V. Cardoso, M. Kimura, A. Maselli, E. Berti, C. F. Macedo, and R. McManus, Parametrized black hole quasinormal ringdown: Decoupled equations for nonrotating black holes, Phys. Rev. D **99**, 104077 (2019).