Approximate analytical description of apparent horizons for initial data with momentum and spin

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We construct analytical initial data for a slowly moving and rotating black hole for generic orientations of the linear momentum and the spin. We solve the Hamiltonian constraint approximately and work out the properties of the apparent horizon and show the dependence of its shape on the angle between the spin and the linear momentum. In particular, a dimple, whose location depends on the mentioned angle, arises on the two-sphere geometry of the apparent horizon. We exclusively work in the case of conformally flat initial metrics.

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I. INTRODUCTION

Since the first observation of black hole merger [1], there have been many observations of the merger of compact objects via gravitational waves. The gravitational waves produced by these mergers are consistent with the numerical solutions of the field equations of general relativity. Besides the highly accurate numerical results, it always pays to have approximate solutions of relativistic gravitating systems. Here, we give an approximate analytical description of a self-gravitating system that has a conserved total energy, total spin, and a linear momentum in an asymptotically flat spacetime. The initial configuration is expected to evolve and settle to a single rotating black hole after emitting some gravitational radiation.

The problem was studied in [2] in the case of vanishing linear momentum but with a nonzero spin and in [3] in the case of vanishing spin with a nonzero linear momentum. See a remarkable exposition in [4]. Here, we assume both of these quantities to be nonzero and pointing arbitrarily in three-dimensional space. It will turn out that the shape of the apparent horizon depends on the angle between the linear momentum and the spin; even though at the next to leading order, the magnitude of the spin does not appear in the shape of the apparent horizon, its direction does. On the other hand, the shape of the apparent horizon depends on the magnitude of the linear momentum at the first order. The area of the apparent horizon does not depend on the angle between the spin and linear momentum. We also observe that a dimple arises on the two-sphere geometry of the apparent horizon.

The layout of the paper is as follows. In the next section, we discuss briefly the constraint equations in general relativity and present the Bowen-York method [5] in finding solutions to the initial value problem. In Sec. III, we give the approximate solution of the Hamiltonian constraint for a slowly rotating and moving black hole. In Sec. IV, we compute the position of the apparent horizon as a function of the angle between the spin and the linear momentum.

II. INITIAL DATA FOR A BLACK HOLE WITH MOMENTUM AND SPIN

Assuming the usual Arnowitt-Deser-Misner (ADM) split of the metric [6],

$$ds^2 = (N_i N^i - N^2)dt^2 + 2N_i dt dx^i + \gamma_{ij} dx^i dx^j, \quad i, j \in (1, 2, 3), \tag{1}$$

the Einstein equations in vacuum without a cosmological constant split into constraints and the evolution equations. The constraint equations are given as

$$-{}^{\Sigma}R - K^2 + K_{ij}K^{ij} = 0, \qquad -2D_k K^k_i + 2D_i K = 0, \tag{2}$$

where Σ is the Cauchy surface; $K_{ij} = K_{ij}(t, x^k)$ is its extrinsic curvature defined as

$$K_{ij} = \frac{1}{2N}(\dot{\gamma}_{ij} - D_i N_j - D_j N_i), \quad \dot{\gamma}_{ij} = \frac{\partial}{\partial t}\gamma_{ij}, \tag{3}$$

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with the trace $K := \gamma^{ij}K_{ij}$; and $D_i\gamma_{kl} = 0$. For further details of the construction, including the evolution equations that we do not depict here, see the Appendix of [7]. Following Bowen-York [5], let us assume that Σ is conformally flat with the metric,

$$\gamma_{ij} = \psi^4 f_{ij}, \quad \psi > 0, \tag{4}$$

with f_{ij} denoting the flat metric in some generic coordinates. One also sets the extrinsic curvature of the hypersurface to be given as $K_{ij} = \psi^{-2}\hat{K}_{ij}$. Furthermore, we assume that Σ is a maximally embedded hypersurface in the spacetime such that the trace of the extrinsic curvature vanishes,[1]

$$K = 0. \tag{5}$$

Under these conditions, the Hamiltonian constraint reduces to a nonlinear elliptic equation,

$$\hat{D}_i\hat{D}^i\psi = -\frac{1}{8}\psi^{-7}\hat{K}_{ij}^2, \tag{6}$$

and the momentum constraint reduces to

$$\hat{D}^i\hat{K}_{ij} = 0, \tag{7}$$

with $\hat{D}_i f_{jk} = 0$. The momentum constraint equations can be solved easily, following [5]. Let us choose the six-parameter solution,

$$\hat{K}_{ij} = \frac{3}{2r^2}(p_i n_j + p_j n_i + (n_i n_j - f_{ij})p\cdot n) + \frac{3}{r^3}\mathcal{J}^l n^k(\varepsilon_{kil}n_j + \varepsilon_{kjl}n_i), \tag{8}$$

where n^i is the unit normal on a sphere of radius r. For other solutions, see [8]. Assuming the following asymptotic behavior for the conformal factor,

$$\psi(r) = 1 + \frac{E}{2r} + \mathcal{O}(1/r^2), \tag{9}$$

one can easily show that (see [9]) the p^i in the solution (8) corresponds to the total conserved linear momentum via

$$P_i = \frac{1}{8\pi}\int_{S^2_\infty} dSn^j K_{ij} = \frac{1}{8\pi}\int_{S^2_\infty} dSn^j\hat{K}_{ij}. \tag{10}$$

Similarly, one can show that J_i corresponds to the total conserved angular momentum expressed in terms of the coordinates and the extrinsic curvature as

$$J_i = \frac{1}{16\pi}\varepsilon_{ijk}\int_{S^2_\infty} dSn_l(x^j K^{kl} - x^k K^{jl}) = \frac{1}{16\pi}\varepsilon_{ijk}\int_{S^2_\infty} dSn_l(x^j\hat{K}^{kl} - x^k\hat{K}^{jl}). \tag{11}$$

Finally, the ADM energy,

$$E_{ADM} = \frac{1}{16\pi}\int_{S^2_\infty} dS\, n_i(\partial_j h^{ij} - \partial_i h^j_j), \tag{12}$$

becomes

$$E_{ADM} = -\frac{1}{2\pi}\int_{S^2_\infty} dSn^i\partial_i\psi, \tag{13}$$

and so using the asymptotic form (9), one finds $E_{ADM} = E$. This has been a brief description of the solution of the momentum constraints. Now, the important task is to solve the Hamiltonian constraint, which, as we noted, is a nonlinear elliptic equation, and thus, generically, it can only be solved numerically. However, in the next section, we shall give an approximate solution for small momentum and small rotation.

III. INITIAL DATA WITH SMALL MOMENTUM AND SMALL SPIN

Computation of $\hat{K}_{ij}\hat{K}^{ij}$ [from Eq. (8)] yields

$$\hat{K}_{ij}\hat{K}^{ij} = \frac{9}{2r^4}(p^2 + 2(\vec{p}\cdot\vec{n})^2) + \frac{18}{r^5}(\vec{J}\times\vec{n})\cdot\vec{p} + \frac{18}{r^6}(\vec{J}\times\vec{n})\cdot(\vec{J}\times\vec{n}). \tag{14}$$

Without loss of generality, let us assume that the direction of the spin is the $\hat{k}$ direction, namely

$$\vec{J} = J\hat{k}, \tag{15}$$

and $\vec{p}$ is lying in the xz plane and given as

$$\vec{p} = p\sin\theta_0\hat{i} + p\cos\theta_0\hat{k}, \tag{16}$$

with θ_0 a fixed angle. To simplify the notation of the following discussion, let us denote

$$c_1 := \sin\theta_0, \qquad c_2 := \cos\theta_0. \tag{17}$$

The Hamiltonian constraint, after these conventions, becomes

[1]For physically relevant decay conditions in the case of asymptotically flat initial data, we refer the reader to Sec. III C of [7] where a slightly extended discussion is compiled.

$$\hat{D}_i\hat{D}^i\psi = \psi^{-7}\left(\frac{9Jp}{4r^5}c_1\sin\theta\sin\phi - \frac{9J^2}{4r^6}\sin^2\theta - \frac{9p^2}{16r^4}(1+2(c_1\sin\theta\cos\phi + c_2\cos\theta)^2)\right). \tag{18}$$

As it clear from the right-hand side, the correct perturbative expansion in terms of the momentum and spin reads,

$$\psi(r,\theta,\phi) \coloneqq \psi^{(0)} + J^2\psi^{(J)} + p^2\psi^{(p)} + Jp\psi^{(Jp)} + \mathcal{O}(p^4, J^4, p^2J^2), \tag{19}$$

where the functions on the right-hand side depend on all coordinates (r,θ,ϕ). At the lowest order, one has

$$\hat{D}_i\hat{D}^i\psi^{(0)} = 0. \tag{20}$$

To proceed, let us discuss the boundary conditions that we shall employ. Following [10,3], we chose the following boundary conditions:

$$\lim_{r\to\infty}\psi(r) = 1, \quad \psi(r) > 0, \tag{21}$$

and

$$\lim_{r\to 0}\psi(r) = \psi^{(0)}. \tag{22}$$

At the lowest order, the solution satisfying these boundary conditions reads,

$$\psi^{(0)} = 1 + \frac{a}{r}. \tag{23}$$

Inserting (19) into (18), one arrives at three linear partial differential equations to be solved:

$$\hat{D}_i\hat{D}^i\psi^{(J)} = -\frac{9}{4}\sin^2\theta\frac{r}{(r+a)^7},$$
$$\hat{D}_i\hat{D}^i\psi^{(Jp)} = \frac{9}{4}c_1\sin\theta\sin\phi\frac{r^2}{(r+a)^7}, \tag{24}$$

and also

$$\hat{D}_i\hat{D}^i\psi^{(p)} = -\frac{9}{16}(1+2(c_1\sin\theta\cos\phi + c_2\cos\theta)^2)\frac{r^3}{(r+a)^7}. \tag{25}$$

In finding the solutions to these equations, we will need the following spherical harmonics:

$$Y_0^0(\theta,\phi) = \frac{1}{\sqrt{4\pi}}, \qquad Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta,$$
$$Y_2^0(\theta,\phi) = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1),$$
$$Y_1^{-1}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\sin\theta\sin\phi,$$
$$Y_2^1(\theta,\phi) = \sqrt{\frac{15}{4\pi}}\sin\theta\cos\theta\cos\phi,$$
$$Y_1^1(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\sin\theta\cos\phi. \tag{26}$$

Then, ansatz for $\psi^{(J)}$ can be taken as

$$\psi^{(J)}(r,\theta,\phi) = \psi_0^{(J)}(r)Y_0^0(\theta,\phi) + \psi_1^{(J)}(r)Y_2^0(\theta,\phi), \tag{27}$$

which, upon insertion to the first equation of (24), yields two ordinary differential equations,

$$\frac{d}{dr}\left(r^2\frac{d\psi_0^{(J)}(r)}{dr}\right) = -3\sqrt{\pi}\frac{r^3}{(r+a)^7},$$
$$\frac{d}{dr}\left(r^2\frac{d\psi_1^{(J)}(r)}{dr}\right) - 6\psi_1^{(J)}(r) = 3\sqrt{\frac{\pi}{5}}\frac{r^3}{(r+a)^7}. \tag{28}$$

The solution obeying the boundary conditions (21) reads,

$$\psi^{(J)}(r,\theta,\phi) = \frac{(a^4 + 5a^3r + 10a^2r^2 + 5ar^3 + r^4)}{40a^3(a+r)^5} - \frac{r^2}{40a(a+r)^5}(3\cos^2\theta - 1). \tag{29}$$

Similarly, setting

$$\psi^{(Jp)}(r,\theta,\phi) = \psi_0^{(Jp)}(r)Y_0^0(\theta,\phi) + \psi_1^{(Jp)}(r)Y_1^{-1}(\theta,\phi) \tag{30}$$

in the second equation of (24), one finds that $\psi_0^{(Jp)}(r) = 0$ satisfies the boundary conditions, and the $\psi_1^{(Jp)}(r)$ piece satisfies

$$\frac{d}{dr}\left(r^2\frac{d\psi_1^{(Jp)}(r)}{dr}\right) - 2\psi_1^{(Jp)}(r) = \frac{3\sqrt{3\pi}}{2}c_1\frac{r^4}{(r+a)^7}, \tag{31}$$

of which, the solution can be found, and one has

$$\psi^{(Jp)}(r,\theta,\phi) = -\frac{c_1r(a^2 + 5ar + 10r^2)}{80a(a+r)^5}\sin\theta\sin\phi. \tag{32}$$

Finally, let us do the $\psi^{(p)}(r,\theta,\phi)$ part, which is slightly more complicated. One sets

$$\psi^{(p)} = \psi_0^{(p)}(r)Y_0^0(\theta,\phi) + \psi_1^{(p)}(r)Y_1^1(\theta,\phi)^2 + \psi_2^{(p)}(r)Y_2^1(\theta,\phi) + \psi_3^{(p)}(r)Y_1^0(\theta,\phi)^2 \quad (33)$$

to arrive at four equations, two of which are

$$\frac{d}{dr}\left(r^2\frac{d\psi_0^{(p)}}{dr}\right) + \frac{3}{\sqrt{\pi}}(\psi_1^{(p)} + \psi_3^{(p)}) = -\frac{9}{8}\sqrt{\pi}\frac{r^5}{(r+a)^7}, \quad (34)$$

and

$$\frac{d}{dr}\left(r^2\frac{d\psi_1^{(p)}}{dr}\right) - 6\psi_1^{(p)} = -\frac{3}{2}\pi c_1^2\frac{r^5}{(r+a)^7}. \quad (35)$$

The $\psi_2^{(p)}(r)$ equation can be obtained from (35) with the replacement $c_1^2 \to \sqrt{\frac{3}{5\pi}}c_1c_2$, and the $\psi_3^{(p)}(r)$ equation can be obtained from (35) via $c_1^2 \to c_2^2$. The solutions read, respectively, as follows:

$$\psi_0^{(p)}(r) = -\frac{\sqrt{\pi}(84a^6 + 378a^5r + 653a^4r^2 + 514a^3r^3 + 142a^2r^4 - 35ar^5 - 25r^6)}{80ar^2(a+r)^5} - \frac{21\sqrt{\pi}a}{20r^3}\log\frac{a}{a+r}, \quad (36)$$

and

$$\psi_1^{(p)}(r) = \frac{\pi c_1^2(84a^5 + 378a^4r + 658a^3r^2 + 539a^2r^3 + 192ar^4 + 15r^5)}{40r^2(a+r)^5} + \frac{21\pi ac_1^2}{10r^3}\log\frac{a}{r+a}, \quad (37)$$

from which, one can find $\psi^{(p)}$, but we do not depict it here since it is a little long.

Recall that, for the ADM energy computation, we need the dominant terms up to and including $\mathcal{O}(\frac{1}{r})$ in $\psi(r,\theta,\phi)$. Collecting these parts in the above solutions, one gets

$$\psi(r) = 1 + \frac{a}{r} + \frac{J^2}{40a^3r} + \frac{5p^2}{32ar} + \mathcal{O}\left(\frac{1}{r^2}\right). \quad (38)$$

Therefore, from (9), the ADM energy of the solution reads,

$$E_{\mathrm{ADM}} = 2a + \frac{J^2}{20a^3} + \frac{5p^2}{16a}. \quad (39)$$

Observe that the Jp term does not contribute to the energy since it is of $\mathcal{O}(\frac{1}{r^2})$.

Next, as in [3], let us express the ADM energy in terms of the irreducible mass M_{irr}, which is defined [11] as

$$M_{\mathrm{irr}} \coloneqq \sqrt{\frac{A}{16\pi}}, \quad (40)$$

with A being the area of a *section* of the event horizon. However, as the event horizon is a four-dimensional concept, which cannot be derived from the three-dimensional initial data, we will approximate this with the area of the apparent horizon, A_{AH}, following [3].

IV. COMPUTATION OF THE APPARENT HORIZON FOR THE BOOSTED, ROTATING SOLUTIONS

Let S be a two-dimensional subspace of Σ and s^i be the normalized unit vector of S, *i.e.,* $s^is_i = 1$. Then, the metric on S is the pullback metric from Σ given as

$$m_{ij} \coloneqq \gamma_{ij} - s_is_j. \quad (41)$$

The expansion of the null geodesic congruence vanishes at the apparent horizon by definition; *i.e.,* it is a marginally trapped surface, and the defining equation becomes

$$(\gamma^{ij} - s^is^j)(D_is_j - K_{ij}) = 0. \quad (42)$$

Assuming the surface to be defined as a level set of a function,

$$\Phi \coloneqq r - h(\theta,\phi) = 0, \quad (43)$$

then the normal one-form reads,

$$s_i \coloneqq \lambda m_i = \lambda\partial_i\Phi, \quad (44)$$

which explicitly becomes

$$s_i = \lambda(1, -\partial_\theta h, -\partial_\phi h). \quad (45)$$

Recall that the metric on Σ is

$$\gamma_{ij} = \psi^4\begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}, \quad (46)$$

and then one has

$$s^i = \lambda(\gamma^{rr}, -\gamma^{\theta\theta}\partial_\theta h, -\gamma^{\phi\phi}\partial_\phi h), \quad (47)$$

and

$$\lambda = (\gamma^{rr} + \gamma^{\theta\theta}(\partial_\theta h)^2 + \gamma^{\phi\phi}(\partial_\phi h)^2)^{-1/2}. \quad (48)$$

Equation (42) reads more explicitly as

$$\gamma^{ij}\partial_i m_j - \gamma^{ij}\Gamma^k_{ij} m_k - \lambda^2 m^i m^j \partial_i m_j + \lambda^2 m^i m^j m_k \Gamma^k_{ij} + \lambda m^i m^j K_{ij} = 0, \tag{49}$$

where we have used $\gamma^{ij}K_{ij} = K = 0$. After working out each piece, one arrives at

$$\begin{aligned}
&-\gamma^{\theta\theta}\partial_\theta^2 h - \gamma^{\phi\phi}\partial_\phi^2 h - \frac{1}{2}((\gamma^{rr})^2\partial_r\gamma_{rr} - \gamma^{\theta\theta}\gamma^{rr}\partial_r\gamma_{\theta\theta} - \gamma^{\phi\phi}\gamma^{rr}\partial_r\gamma_{\phi\phi} + \partial_\theta h\gamma^{\phi\phi}\gamma^{\theta\theta}\partial_\theta\gamma_{\phi\phi})\\
&\quad + \lambda^2((\gamma^{\theta\theta})^2(\partial_\theta h)^2\partial_\theta^2 h + (\gamma^{\phi\phi})^2(\partial_\phi h)^2\partial_\phi^2 h + 2\gamma^{\phi\phi}\gamma^{\theta\theta}\partial_\phi h\partial_\theta h\partial_\theta\partial_\phi h)\\
&\quad + \frac{\lambda^2}{2}((\gamma^{rr})^3\partial_r\gamma_{rr} + (\gamma^{\theta\theta})^2\gamma^{rr}(\partial_\theta h)^2\partial_r\gamma_{\theta\theta} + (\gamma^{\phi\phi})^2\gamma^{rr}(\partial_\phi h)^2\partial_r\gamma_{\phi\phi}\\
&\quad - (\partial_\phi h)^2\partial_\theta h(\gamma^{\phi\phi})^2\gamma^{\theta\theta}\partial_\theta\gamma_{\phi\phi})\\
&\quad + \lambda((\gamma^{rr})^2K_{rr} + (\gamma^{\theta\theta})^2(\partial_\theta h)^2K_{\theta\theta} + (\gamma^{\phi\phi})^2(\partial_\phi h)^2K_{\phi\phi} - 2\gamma^{rr}\gamma^{\theta\theta}\partial_\theta hK_{r\theta}\\
&\quad - 2\gamma^{rr}\gamma^{\phi\phi}\partial_\phi hK_{r\phi} + 2\gamma^{\theta\theta}\gamma^{\phi\phi}\partial_\theta h\partial_\phi hK_{\theta\phi}) = 0.
\end{aligned} \tag{50}$$

An exact solution to this equation is beyond reach, and we do not really need it. All we need is an approximate solution of the form,

$$h(\theta,\phi) = h^0 + ph^p + Jh^J + \mathcal{O}(p^2, J^2, Jp), \tag{51}$$

where

$$\partial_r h = 0, \qquad \partial_r h^0 = 0 = \partial_\theta h^0 = \partial_\phi h^0. \tag{52}$$

Note that to compute the area of the apparent horizon and the irreducible mass up to and including the $\mathcal{O}(p^2, J^2, Jp)$ terms, one only needs the shape of the horizon up to and including the $\mathcal{O}(p, J)$ terms, which becomes clear when one studies the area integral. (See also [3].) Ignoring the higher order terms such as $(\partial_\theta h)^2$, $(\partial_\phi h)^2$, and $\partial_\theta h\partial_\phi h$, the apparent horizon equation becomes

$$\begin{aligned}
&-\gamma^{\theta\theta}\partial_\theta^2 h - \gamma^{\phi\phi}\partial_\phi^2 h - \frac{1}{2}((\gamma^{rr})^2\partial_r\gamma_{rr} - \gamma^{\theta\theta}\gamma^{rr}\partial_r\gamma_{\theta\theta}\\
&\quad - \gamma^{\phi\phi}\gamma^{rr}\partial_r\gamma_{\phi\phi} + \partial_\theta h\gamma^{\phi\phi}\gamma^{\theta\theta}\partial_\theta\gamma_{\phi\phi}) + \frac{\lambda^2}{2}(\gamma^{rr})^3\partial_r\gamma_{rr}\\
&\quad + \lambda\gamma^{rr}(\gamma^{rr}K_{rr} - 2\gamma^{\theta\theta}\partial_\theta hK_{r\theta} - 2\gamma^{\phi\phi}\partial_\phi hK_{r\phi}) = 0.
\end{aligned} \tag{53}$$

To proceed, we need the components of the extrinsic curvature in the (r,θ,ϕ) coordinates. After coordinate transformations, one finds

$$\begin{aligned}
\hat{K}_{rr} &= \frac{3p}{r^2}(c_1\sin\theta\cos\phi + c_2\cos\theta),\\
\hat{K}_{r\theta} &= \frac{3p}{2r}(c_1\cos\theta\cos\phi - c_2\sin\theta),
\end{aligned} \tag{54}$$

and

$$\hat{K}_{r\phi} = -\frac{3p}{2r}c_1\sin\theta\sin\phi + \frac{3J}{r^2}\sin^2\theta. \tag{55}$$

Therefore, the resulting equation is

$$\begin{aligned}
&\partial_\theta^2 h + \frac{1}{\sin^2\theta}\partial_\phi^2 h + \cot\theta\partial_\theta h - 2r - 4r^2\frac{\partial_r\psi}{\psi} + \frac{6J}{\psi^4 r^2}\partial_\phi h\\
&\quad - \frac{3p}{\psi^4}(c_1\sin\theta\cos\phi + c_2\cos\theta) = 0.
\end{aligned} \tag{56}$$

At order $\mathcal{O}(p^0, J^0)$, this equation yields

$$1 + 2r\frac{\partial_r\psi}{\psi} = 0, \tag{57}$$

where $\psi = 1 + \frac{a}{r}$. And, setting $r = h$, one finds

$$h^0 = a. \tag{58}$$

This explains the physical meaning of the parameter a; it is the location of the apparent horizon at the lowest order. The next order contribution, which we shall find below, will be perturbations to this location. At $\mathcal{O}(p)$ and $\mathcal{O}(J)$, we have the following equations, respectively:

$$\begin{aligned}
&\partial_\theta^2 h^p + \frac{1}{\sin^2\theta}\partial_\phi^2 h^p + \cot\theta\partial_\theta h^p - h^p\\
&\quad - \frac{3}{16}(c_1\sin\theta\cos\phi + c_2\cos\theta) = 0,
\end{aligned} \tag{59}$$

and

$$\partial_\theta^2 h^J + \frac{1}{\sin^2\theta}\partial_\phi^2 h^J + \cot\theta\partial_\theta h^J - h^J = 0. \tag{60}$$

These are linear partial differential equations, and a close scrutiny shows that the h^J equation is the homogenous Helmholtz equation on a sphere (S^2), while the h^P equation is the inhomogeneous Helmholtz equation with a nontrivial source. So, the next task is to find everywhere finite solutions of the following equation:

$$(\vec{\nabla}^2_{S^2} + k) f(\theta,\phi) = g(\theta,\phi), \tag{61}$$

where $\vec{\nabla}^2_{S^2}$ is the Laplacian on S^2:

$$\vec{\nabla}^2_{S^2} = \partial^2_\theta + \cot\theta\partial_\theta + \frac{1}{\sin^2\theta}\partial^2_\phi. \tag{62}$$

It is clear that the Green's function technique is the most suitable approach to this problem. For the Helmholtz operator on the sphere, the Green function $G(\hat{x},\hat{x}')$ is defined as

$$(\vec{\nabla}^2_{S^2} + \lambda(\lambda+1)) G(\hat{x},\hat{x}') = \delta^{(2)}(\hat{x}-\hat{x}'), \tag{63}$$

which can be found to be (for example, see [12])

$$G(\hat{x},\hat{x}') = \frac{1}{4\sin\pi\lambda}\sum_{n=0}^{\infty}\frac{1}{(n!)^2}\frac{\Gamma(n-\lambda)}{\Gamma(-\lambda)} \times \frac{\Gamma(n+\lambda+1)}{\Gamma(\lambda+1)}\left(\frac{1+\hat{x}\cdot\hat{x}'}{2}\right)^n, \tag{64}$$

where $\hat{x} = \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k}$, and $\hat{x}'$ is a similar expression with some other θ and ϕ. Employing this Green's function with $\lambda = \frac{-1+i\sqrt{3}}{2}$, one finds

$$h^p = -\frac{1}{16}(c_1\sin\theta\cos\phi + c_2\cos\theta), \tag{65}$$

and $h^J = 0$. Therefore, the apparent horizon is located at

$$r = h(\theta,\phi) = a - \frac{1}{16}(\vec{p}\cdot\hat{J}\cos\theta + |\vec{p}\wedge\hat{J}|\sin\theta\cos\phi), \tag{66}$$

where $\hat{J} = \frac{\vec{J}}{J}$. In the limit $\theta_0 = 0$, h reduces to the form given in [3], that is, $h(\theta) = a - \frac{p}{16}\cos\theta$, and the apparent horizon in this axially symmetric case is a squashed sphere from the North Pole. Note that the shape of the apparent horizon (66) at this order does not depend on the magnitude of the spin, but it does depend on its orientation with respect to the linear momentum. In Fig. 1, we plot the apparent horizon. To be able to see the dimple clearly in the whole figure, we have chosen a high momentum value.

Let us now evaluate the area of the apparent horizon from the formula,

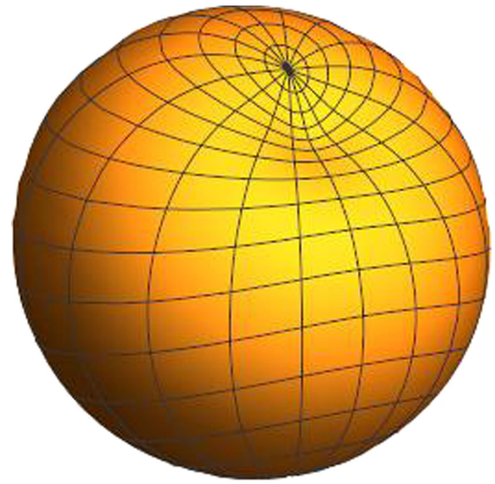

FIG. 1. The shape of the apparent horizon when the angle between $\vec{p}$ and $\vec{J}$ is 45 degrees; to be able to see the dimple, we have chosen $p/a = 8\sqrt{2}$, which is outside the validity of the approximation we have worked with, but the dimple exists for even small p.

$$A_{\mathrm{AH}} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sqrt{\det m}, \tag{67}$$

which, at the order that we are working, yields

$$A_{\mathrm{AH}} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \sin\theta\psi^4 h^2 \times \left(1 + \frac{1}{h^2}(\partial_\theta h)^2 + \frac{1}{h^2\sin^2\theta}(\partial_\phi h)^2\right)^{1/2}. \tag{68}$$

This is a pretty long computation since the conformal factor is quite complicated. However, at the end, one finds

$$A_{\mathrm{AH}} = 64\pi a^2 + 4\pi p^2 + \frac{11\pi J^2}{5a^2}. \tag{69}$$

Note that the angle between the spin and the linear momentum does not appear in the area. Then, the irreducible mass M_{irr} reads,

$$M_{\mathrm{irr}} = 2a + \frac{p^2}{16a} + \frac{11J^2}{320a^3}. \tag{70}$$

Comparing with E_{ADM}, we have

$$E_{\mathrm{ADM}} = M_{\mathrm{irr}} + \frac{p^2}{2M_{\mathrm{irr}}} + \frac{J^2}{8M^3_{\mathrm{irr}}}, \tag{71}$$

which matches the slow momentum and spin limit of the result in [11].

V. CONCLUSIONS

Momentum constraints in general relativity are easily solved with the method of Bowen-York, while the Hamiltonian constraint is a nontrivial elliptic equation. Here, extending earlier works, [2–3], we gave an approximate analytical solution that describes a spinning and moving system with a conserved spin and linear momentum pointing in arbitrary directions. We computed the properties of the apparent horizon, such as its shape and surface area, and showed the dependence of the shape on the angle between the spin and the linear momentum. We calculated the relation between the conserved quantities, such as the ADM mass, the spin, the linear momentum, and the irreducible mass. The area of the apparent horizon does not depend on the angle between the spin and the linear momentum, but a dimple arises in the apparent horizon whose location depends on this angle.

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