Stationary scalar clouds supported by rapidly-rotating acoustic black holes in a photon-fluid model

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It has recently been proved that, in the presence of vortex flows, the fluctuation dynamics of a rotating photon-fluid model is governed by the Klein-Gordon equation of an effective massive scalar field in a $(2+1)$-dimensional acoustic black-hole spacetime. Interestingly, it has been demonstrated numerically that the rotating acoustic black hole, like the familiar Kerr black-hole spacetime, may support spatially regular stationary density fluctuations (linearized acoustic scalar "clouds") in its exterior regions. In particular, it has been shown that the composed rotating acoustic black-hole-stationary-scalar-field configurations of the photon-fluid model exist in the narrow dimensionless regime $\alpha \equiv \Omega_0/m\Omega_{\rm H} \in (1, \alpha_{\rm max})$ with $\alpha_{\rm max} \simeq 1.08$ (here $\Omega_{\rm H}$ is the angular velocity of the black-hole horizon and $\{\Omega_0, m\}$ are the effective proper mass and the azimuthal harmonic index of the acoustic scalar field, respectively). In the present paper, we use analytical techniques in order to explore the physical and mathematical properties of the acoustic scalar clouds of the photon-fluid model in the regime $\Omega_{\rm H} r_{\rm H} \gg 1$ of rapidly spinning central supporting acoustic black holes. In particular, we derive a remarkably compact analytical formula for the discrete resonance spectrum $\{\Omega_0(\Omega_{\rm H}, m; n)\}$ which characterizes the stationary bound-state acoustic scalar clouds of the photon-fluid model. Interestingly, it is proved that the critical (maximal) mass parameter $\alpha_{\rm max}$, which determines the regime of existence of the composed acoustic black-hole-stationary-bound-state-massive-scalar-field configurations, is given by the exact dimensionless relation $\alpha_{\rm max} = \sqrt{\frac{32}{27}}$.

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I. INTRODUCTION

Kerr black-hole spacetimes [1] are characterized by the presence of an ergoregion [2], a region outside the black-hole horizon in which matter fields are bound to corotate with the central spinning black holes. Interestingly, it has been demonstrated, both analytically [3] and numerically [4], that this physically intriguing feature of spinning black-hole spacetimes allows them to support spatially regular stationary configurations of bosonic (integer-spin) fields that corotate with the central black hole.

The stationary hairy scalar-field configurations, which in the linearized regime have received the nickname "scalar clouds" in the physics literature [3,4], are characterized by proper frequencies that are in resonance with the angular velocity $\Omega_{\rm H}$ of the central supporting spinning black hole [3,4]. In particular, the characteristic proper frequency of a stationary scalar cloud with an azimuthal harmonic index m coincides with the critical (marginal) frequency for the superradiant scattering phenomenon [5,6] in the rotating black-hole spacetime [3,4,7]:

$$\omega = m\Omega_{\rm H}. \tag{1}$$

In addition, spatially regular (bounded) bosonic clouds are characterized by the simple upper bound [3,4]

$$\omega^2 < \mu^2, \tag{2}$$

where μ is the proper mass [8] of the supported stationary scalar field. The relation (2) implies that the corotating massive scalar-field configurations are spatially bounded to the central black hole and cannot radiate their energy and angular momentum to infinity.

Intriguingly, an analogous physical phenomenon in a rotating photon-fluid system has recently been revealed in the highly important work of Ref. [9]. Photon fluids are nonlinear optical systems whose physical and mathematical properties can be described by the hydrodynamic equations of an interacting Bose gas [9–12]. In particular, it has been shown [9,13] that photon-fluid systems are characterized by the presence of long-wavelength elementary excitations (phonons) that behave as massive scalar fields in an effective acoustic curved spacetime.

Fluid systems are known to provide physically interesting platforms and mathematically elegant toy models for analogue gravity investigations [14–17]. For example, the intriguing physical phenomenon of superradiant scattering, which characterizes the spinning Kerr black-hole spacetime, has also been analyzed in rotating photon-fluids models [18–21]. In particular, a rotating black-hole spacetime has been studied in the physically interesting work of Ref. [22]. In addition, an acoustic black hole which is enclosed in a cylindrical cavity has been analyzed in Ref. [23].

The dynamics of massive phonons over a draining vortex flow in the photon-fluid model has been investigated recently in Ref. [9] as the acoustic analogue of the (more familiar) dynamics of massive scalar fields in rotating curved black-hole spacetimes. In particular, it has been explicitly proved [9] that the dynamics of linearized acoustic excitations in the photon-fluid model with a draining vortex flow are governed by the familiar Klein-Gordon equation of an effective scalar field of proper mass Ω_0 that propagates in an acoustic $(2+1)$-dimensional spinning black-hole spacetime which, like the familiar Kerr black-hole spacetime, possesses an ergoregion.

Intriguingly, using direct numerical techniques, it has been explicitly demonstrated in Ref. [9] that the acoustic spinning black-hole spacetime may support stationary linearized density fluctuations (acoustic scalar "clouds") in its exterior regions. In particular, it has been revealed [9] that the composed acoustic black-hole-stationary-massive-scalar-field configurations of the photon-fluid model are characterized by the narrow regime of existence [9,24]

$$\alpha \equiv \frac{\Omega_0}{m\Omega_{\rm H}} \in (1, \alpha_{\rm max}) \quad \text{with} \quad \alpha_{\rm max} \simeq 1.08, \tag{3}$$

where $\Omega_{\rm H}$ is the angular velocity that characterizes the acoustic horizon of the central supporting spinning black hole.

The main goal of the present paper is to explore, using analytical techniques, the physical and mathematical properties of the composed acoustic spinning black-hole-stationary-linearized-scalar-field configurations of the photon-fluid model. In particular, we shall derive a remarkably compact analytical formula for the discrete resonance spectrum $\{\Omega_0(\Omega_{\rm H}, m; n)\}$ [25] that characterizes the spatially regular stationary acoustic scalar clouds in the dimensionless regime $\Omega_{\rm H} r_{\rm H} \gg 1$ [26] of rapidly spinning central supporting black holes. In addition, we shall provide a simple analytical explanation for the existence of the numerically observed [9] interesting upper bound $\alpha < \alpha_{\rm max} \simeq 1.08$ [see Eq. (3)] on the regime of existence of the composed acoustic black-hole-stationary-bound-state-massive-scalar-field configurations.

II. DESCRIPTION OF THE SYSTEM

We study the physical and mathematical properties of density fluctuations in a rotating photon-fluid model. Intriguingly, a formal equivalence has recently been established in the physically important work of Ref. [9] between the dynamics of linearized acoustic phonons that propagate on top of an inhomogeneous photon fluid and the dynamics of linearized massive scalar fields in a spinning curved spacetime. In particular, it has been explicitly proved in Ref. [9] that, in the presence of vortex flows, the dynamics of acoustic density fluctuations in the long-wavelength regime of the photon-fluid model are governed by the Klein-Gordon equation of a massive scalar field in an effective $(2+1)$-dimensional curved spacetime.

The effective acoustic spacetime of the $(2+1)$-dimensional rotating photon-fluid model can be described, using polar coordinates, by the nontrivial curved line element [9,27]

$$ds^2 = -\left(1 - \frac{r_{\rm H}}{r} - \frac{\Omega_{\rm H}^2 r_{\rm H}^4}{r^2}\right)dt^2 + \left(1 - \frac{r_{\rm H}}{r}\right)^{-1} dr^2 - 2\Omega_{\rm H} r_{\rm H}^2 d\theta dt + r^2 d\theta^2. \tag{4}$$

Here $r_{\rm H}$ is the radius of the acoustic black-hole horizon, which is determined as the circular ring at which the inward radial velocity $v_{\rm r}$ of the fluid flow equals the speed of sound $c_{\rm s}$ [9,28,29]. The physical parameter $\Omega_{\rm H}$ in the curved line element (4) is the angular velocity of the effective acoustic horizon.

Interestingly, like the familiar Kerr black-hole solution of the Einstein field equations, the rotating acoustic spacetime (4) of the photon-fluid model is characterized by the presence of an effective ergoregion, whose outer boundary [9]

$$r_{\rm E} = \frac{1}{2} r_{\rm H}\left(1 + \sqrt{1 + 4\Omega_{\rm H}^2 r_{\rm H}^2}\right) \tag{5}$$

is determined by the condition $g_{tt} = 0$.

As explicitly proved in Ref. [9], the spatial behavior of density fluctuations of the form

$$\rho(t, r, \theta) = \frac{\psi(r)}{\sqrt{r}} e^{im\theta - i\Omega t} \tag{6}$$

in the effective acoustic spacetime (4) of the rotating photon-fluid model are governed by the radial differential equation

$$\left[\Delta \frac{d}{dr}\left(\Delta \frac{d}{dr}\right) - V(r;\Omega)\right]\psi(r) = 0, \tag{7}$$

where

$$\Delta \equiv 1 - \frac{r_{\rm H}}{r}. \tag{8}$$

The effective radial potential of the photon-fluid system is given by the functional expression

$$V(r;\Omega) = -\left(\Omega - \frac{m\Omega_H r_H^2}{r^2}\right)^2 + \Delta\left(\Omega_0^2 + \frac{m^2}{r^2} + \frac{r_H}{2r^3} - \frac{\Delta}{4r^2}\right). \tag{9}$$

The physical parameter Ω_0, which plays the role of an effective scalar mass, is the rest energy of the collective excitations (phonons) [9]. The θ-periodicity of the angular function $e^{im\theta}$ in the field decomposition (6) implies that the azimuthal harmonic index $|m|$ of the scalar perturbation modes is an integer [30].

In the next section, we shall use analytical techniques in order to derive the discrete resonance spectrum $\{\Omega_0(\Omega_H, m; n)\}_{n=0}^{n=\infty}$ of the composed acoustic black-hole-stationary-bound-state-linearized-massive-scalar-field configurations. The radial eigenfunctions that characterize the stationary scalar clouds of the acoustic curved spacetime (4) are determined by the ordinary differential equation (7) with the physically motivated boundary conditions of spatially regular (bounded) scalar eigenfunctions at the acoustic black-hole horizon and at spatial infinity [9]:

$$\psi(r = r_H) < \infty \tag{10}$$

and

$$\psi(r \to \infty) \sim e^{-\sqrt{\Omega_0^2 - \Omega^2}\, r} \quad \text{for } \Omega^2 < \Omega_0^2. \tag{11}$$

III. THE DISCRETE RESONANCE SPECTRUM OF THE COMPOSED ACOUSTIC BLACK-HOLE SCALAR-CLOUDS CONFIGURATIONS OF THE PHOTON-FLUID MODEL

In the present section, we shall analyze the discrete resonance spectrum $\{\Omega_0(\Omega_H, m; n)\}$ that characterizes the composed acoustic black-hole-stationary-bound-state-linearized-massive-scalar-field configurations of the photon-fluid model [9]. The stationary bound-state scalar clouds of the effective rotating black-hole spacetime (4) are characterized by the resonance condition [9]

$$\Omega = m\Omega_H < \Omega_0. \tag{12}$$

Interestingly, we shall now prove that the resonance spectrum $\{\Omega_0(\Omega_H, m; n)\}$ of the acoustic scalar clouds can be studied *analytically* in the eikonal large-frequency regime [31]

$$\Omega_H r_H \gg m \tag{13}$$

of the central supporting spinning black hole.

To this end, it is convenient to write the radial differential equation (7), which determines the spatial behavior of the scalar eigenfunctions in the acoustic black-hole spacetime (4), in the form of the mathematically compact Schrödinger-like ordinary differential equation

$$\frac{d^2\psi}{dy^2} - V(y)\psi = 0, \tag{14}$$

where the tortoise radial coordinate y is defined by the differential relation [32]

$$dy = \Delta^{-1} dr. \tag{15}$$

Substituting into Eq. (9) the resonant frequency $\Omega = m\Omega_H$ [see Eq. (1)], which characterizes the stationary acoustic scalar clouds of the photon-fluid model, one obtains the functional expression

$$V(r) = -(m\Omega_H)^2\left(1 - \frac{r_H^2}{r^2}\right)^2 + \Delta\left(\Omega_0^2 + \frac{m^2}{r^2} + \frac{r_H}{2r^3} - \frac{\Delta}{4r^2}\right) \tag{16}$$

for the effective radial potential $V[r(y)]$ of the composed acoustic black-hole-stationary-bound-state-massive-scalar-field configurations.

We shall now show explicitly that the Schrödinger-like ordinary differential equation (14) for the spatially regular stationary scalar clouds in the rotating acoustic black-hole spacetime (4) is amenable to a standard WKB analysis [33–37] in the dimensionless regime $\Omega_H r_H \gg 1$ of rapidly spinning supporting black holes.

Interestingly, the potential (16) of the composed acoustic black-hole-stationary-scalar-clouds configurations has an effective binding well in the vicinity of the acoustic horizon. As shown in Refs. [33,34,37], in the eikonal large-frequency regime (13), one can express the WKB resonance condition for the bound-state field configurations of the one-dimensional Schrödinger-like ordinary differential equation (14) in the remarkably compact form

$$\frac{V_{\min}}{\sqrt{2V''_{\min}}} = -\left(n + \frac{1}{2}\right); \qquad n = 0, 1, 2, \ldots, \tag{17}$$

where $V'' \equiv d^2V/dy^2$. The effective binding potential $V_{\min}$ and its second spatial derivative $V''_{\min}$ in the WKB resonance condition (17) are evaluated at the minimum point $r = r_{\min}$ of the potential (16), where

$$V' \equiv \frac{dV}{dy} = 0 \quad \text{for } r(y) = r_{\min}. \tag{18}$$

Substituting the effective binding potential (16) of the composed spinning black-hole-acoustic-scalar-clouds configurations into Eq. (18), one finds the relation

$$\Omega_0^2 = \frac{4(m\Omega_H)^2(r_{min}^2 - r_H^2)r_H}{r_{min}^3}\{1 + O[(\Omega_H r_H)^{-2}]\} \quad \text{for } \Omega_H r_H \gg 1 \tag{19}$$

in the eikonal large-frequency regime (13) of the central supporting acoustic black hole.

Substituting the relation (19) into the WKB equation (17), one finds the resonance equation

$$4r_H(r_{min}^2 - r_H^2) - (r_{min} - r_H)(r_{min} + r_H)^2 = -\frac{2r_H\sqrt{2r_{min}^2 - 6r_H^2}}{m\Omega_H}\left(n + \frac{1}{2}\right) \tag{20}$$

for the radial location of the minimum $r = r_{min}$ of the effective binding potential (16) that characterizes the composed acoustic black-hole-stationary-scalar-clouds configurations.

As we shall now show, the (rather cumbersome) resonance equation (20) can be solved analytically using an iteration scheme. The zeroth-order solution $r_{min}^{(0)} \equiv r_{min}(\Omega_H r_H \to \infty)$ of the resonance equation (20) is given by the simple asymptotic value

$$r_{min}^{(0)} = 3r_H. \tag{21}$$

Next, substituting

$$r_{min} = 3r_H[1 + \alpha(\Omega_H r_H)^{-1}] \tag{22}$$

into the resonance equation (20), one finds

$$\alpha = \frac{n + \frac{1}{2}}{\sqrt{12m}}\{1 + O[(\Omega_H r_H)^{-1}]\}, \tag{23}$$

which yields the functional expression [see Eq. (22)]

$$r_{min} = 3r_H\left\{1 + \frac{1}{\sqrt{12m}}\left(n + \frac{1}{2}\right)(\Omega_H r_H)^{-1} + O[(\Omega_H r_H)^{-2}]\right\} \tag{24}$$

for the radial location of the minimum of the effective binding potential (16).

Finally, substituting Eq. (24) into the relation (19), one obtains the discrete resonance spectrum

$$\Omega_0 = m\Omega_H\sqrt{\frac{32}{27}}\left\{1 - \frac{\sqrt{3}}{16m}\left(n + \frac{1}{2}\right)(\Omega_H r_H)^{-1} + O[(\Omega_H r_H)^{-2}]\right\}, \tag{25}$$

which characterizes the composed acoustic black-hole-stationary-bound-state-linearized-massive-scalar-field configurations of the photon-fluid model.

IV. NUMERICAL CONFIRMATION

It is of physical interest to test the accuracy of the analytically derived resonance formula (25) which characterizes the composed acoustic black-hole stationary scalar-field configurations of the photon-fluid model. The corresponding effective field masses $\{\Omega_0(\Omega_H, m; n)\}$ of the acoustic scalar clouds were recently computed numerically in the interesting Ref. [9].

In Table I, we display the dimensionless ratios $\alpha_{numerical}$ and α_{wkb} for the fundamental ($n = 0$) resonant mode of the stationary bound-state acoustic scalar clouds with $m = 1$ and for various values of the dimensionless angular velocity $\Omega_H r_H$ of the central supporting spinning black hole. Here $\{\alpha_{numerical}(\Omega_H r_H)\}$ are the exact (*numerically* computed [9]) values of the dimensionless ratio $\Omega_0/m\Omega_H$, which characterizes the composed acoustic black-hole stationary bound-state linearized massive scalar-field configurations, and $\{\alpha_{wkb}(\Omega_H r_H)\}$ are the corresponding *analytically* derived values of this dimensionless physical parameter as given by the WKB resonance formula (25).

Interestingly, the data presented in Table I reveal an excellent agreement between the numerical data of Ref. [9] and the analytically derived WKB resonance formula (25) of the composed acoustic black-hole-stationary-massive-scalar-field configurations. It is worth noting that the agreement between the numerical data of Ref. [9] and the analytically derived WKB resonance formula (25) of the composed acoustic black-hole-stationary-bound-state-linearized-massive-scalar-field configurations is generally better than 0.1% in the $\Omega_H r_H \gtrsim 1$ regime. This observation is quite remarkable, since the analytically derived WKB

TABLE I. Stationary bound-state massive scalar clouds linearly coupled to acoustic spinning black holes of the photon-fluid model. We present, for various values of the black-hole dimensionless angular velocity $\Omega_H r_H$, the exact (numerically computed [9]) values of the dimensionless ratio $\alpha \equiv \Omega_0/m\Omega_H$ for the fundamental ($n = 0$) resonant mode of the stationary spatially regular massive scalar clouds with $m = 1$. We also present the corresponding analytically derived values of the dimensionless ratio $\alpha \equiv \Omega_0/m\Omega_H$ as calculated directly from the WKB resonance formula (25). One finds a remarkably good agreement between the analytically derived formula (25) and the numerically computed values [9] of the dimensionless ratio $\Omega_0/m\Omega_H$, which characterizes the composed acoustic black-hole stationary bound-state massive scalar-field configurations. Note that the agreement between the numerical data of Ref. [9] and the analytically derived WKB resonance formula (25) is generally better than 0.1% in the $\Omega_H r_H \gtrsim 1$ regime.

$\Omega_H r_H$	$\alpha_{numerical}$	α_{wkb}
2	1.058	1.059
4	1.073	1.074
6	1.078	1.079
8	1.081	1.081

resonance spectrum (25) is formally valid in the eikonal large-$\Omega_H r_H$ regime.

V. SUMMARY

The recently published highly important work of Ref. [9] has revealed the physically interesting fact that, in the presence of vortex flows, the dynamics of fluctuations in a rotating photon-fluid model is governed by the Klein-Gordon equation of an effective massive scalar field in a spinning acoustic black-hole spacetime. In particular, it has been demonstrated numerically [9] that corotating acoustic scalar clouds, spatially regular bound-state configurations which are made of stationary linearized massive scalar fields, can be supported by the central spinning $(2+1)$-dimensional acoustic black holes.

The important numerical results presented in Ref. [9] have nicely demonstrated the fact that, for a given value of the horizon angular velocity Ω_H of the central supporting black hole, the stationary bound-state acoustic clouds of the photon-fluid model are characterized by a discrete resonance spectrum $\{\Omega_0(\Omega_H, m; n)\}_{n=0}^{n=\infty}$ for the effective mass parameter of the supported scalar fields. In particular, it has been revealed that the composed acoustic black-hole-stationary-bound-state-massive-scalar-field configurations of the photon-fluid model [9] exist in the narrow dimensionless regime $\alpha \equiv \Omega_0/m\Omega_H \in (1, \alpha_{max})$ with $\alpha_{max} \simeq 1.08$.

In the present paper we have used analytical techniques in order to explore the physical and mathematical properties of the composed bound-state acoustic black-hole-stationary-linearized-massive-scalar-field configurations. In particular, we have derived the remarkably compact WKB analytical formula [see Eq. (25)]

$$\frac{\Omega_0}{m\Omega_H} = \sqrt{\frac{32}{27}}\Big\{1 - \frac{\sqrt{3}}{16m}\Big(n+\frac{1}{2}\Big)(\Omega_H r_H)^{-1} + O[(\Omega_H r_H)^{-2}]\Big\} \tag{26}$$

for the discrete resonant spectrum that characterizes the acoustic scalar clouds in the dimensionless regime $\Omega_H r_H \gg m$ of rapidly spinning central supporting black holes. The *analytically* derived formula (26) for the discrete resonant spectrum of the composed acoustic spinning black-hole-massive-scalar-field configurations was shown to agree remarkably well with direct *numerical* computations [9] of the corresponding resonant modes of the photon-fluid model.

Interestingly, from the resonance formula (26), one finds the asymptotic upper bound

$$\Big(\frac{\Omega_0}{m\Omega_H}\Big)_{max} = \sqrt{\frac{32}{27}} \tag{27}$$

on the regime of existence of the composed acoustic black-hole-stationary-bound-state-massive-scalar-field configurations of the photon-fluid model. Our results therefore provide a simple analytical explanation for the intriguing upper bound (3) on the regime of existence of the cloudy acoustic black-hole spacetimes that has recently been observed numerically in the physically interesting work of Ref. [9].

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[24] Here m is the azimuthal harmonic index of the stationary acoustic scalar mode [see Eq. (6) below].
[25] The parameter n, which characterizes the discrete resonance spectrum of the composed acoustic black-hole stationary scalar-clouds configurations, is an integer [see Eq. (17) below].
[26] Here, r_{H} is the horizon radius of the central supporting acoustic black hole.
[27] Here, the coordinates $\{r, \theta\}$ are, respectively, the radial and azimuthal polar coordinates in the xy plane.
[28] Note that $g_{rr} \to \infty$ at the horizon $r = r_{\mathrm{H}}$ of the effective acoustic spacetime (4).
[29] We shall use natural units in which $c_{\mathrm{s}} \equiv 1$.
[30] We shall henceforth assume $m > 0$ for the stationary spatially regular acoustic scalar modes.
[31] It is interesting to note that, unlike spinning Kerr black holes, whose horizon angular velocity is bounded from above by the simple dimensionless relation $\Omega_{\mathrm{H}} r_{\mathrm{H}} \leq 1/2$, the acoustic black-hole spacetimes of the photon-fluid model have no strict upper bound on their angular velocities [9].
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