Double covering of the modular A_5 group and lepton flavor mixing in the minimal seesaw model

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In this paper, we investigate the double covering of the modular $\Gamma_5 \simeq A_5$ group and derive all the modular forms of weight one for the first time. The modular forms of higher weights are also explicitly given by decomposing the direct products of weight-one forms. For the double-covering group $\Gamma'_5 \simeq A'_5$, there exist two inequivalent two-dimensional irreducible representations, into which we can assign two right-handed neutrino singlets in the minimal seesaw model. Two concrete models with such a salient feature have been constructed to successfully explain lepton mass spectra and flavor mixing pattern. The allowed parameter space for these two minimal scenarios has been numerically explored and analytically studied with some reasonable assumptions.

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I. INTRODUCTION

The discovery of neutrino oscillations calls for new physics beyond the standard model (SM) to dynamically generate tiny neutrino masses and significant lepton flavor mixing [1–3]. In order to account for tiny neutrino masses, one can extend the SM with three right-handed neutrino singlets $N_{i\mathrm{R}}$ (for $i = 1, 2, 3$) such that the gauge-invariant Lagrangian for lepton masses and flavor mixing is given by

$$-\mathcal{L}_{\mathrm{lepton}} = \overline{\ell_{\mathrm{L}}} Y_l H E_{\mathrm{R}} + \overline{\ell_{\mathrm{L}}} Y_\nu \tilde{H} N_{\mathrm{R}} + \frac{1}{2}\overline{N^{\mathrm{C}}_{\mathrm{R}}} M_{\mathrm{R}} N_{\mathrm{R}} + \mathrm{H.c.}, \tag{1.1}$$

where $\ell_{\mathrm{L}} \equiv (\nu_{\mathrm{L}}, E_{\mathrm{L}})^{\mathrm{T}}$ and $H \equiv (H^+, H^0)^{\mathrm{T}}$ stand, respectively, for the left-handed lepton doublet and the Higgs doublet, $\tilde{H} \equiv \mathrm{i}\sigma_2 H^*$ and $N^{\mathrm{C}}_{\mathrm{R}} \equiv \mathcal{C}\overline{N_{\mathrm{R}}}^{\mathrm{T}}$ with $\mathcal{C} \equiv \mathrm{i}\gamma^2\gamma^0$ being the charge-conjugation matrix defined, Y_l and Y_ν are the charged-lepton and Dirac neutrino Yukawa coupling matrices, respectively, and M_{R} is the Majorana mass matrix of right-handed neutrinos. After the spontaneous symmetry breaking, the charged-lepton and Dirac neutrino mass matrices are given by $M_l \equiv Y_l v/\sqrt{2}$ and $M_{\mathrm{D}} \equiv Y_\nu v/\sqrt{2}$, respectively, with $v \approx 246$ GeV being the vacuum expectation value (VEV) of the SM Higgs field. Given $\mathcal{O}(M_{\mathrm{D}}) \approx 100$ GeV and $\mathcal{O}(M_{\mathrm{R}}) \approx 10^{14}$ GeV, the effective Majorana mass matrix of three light neutrinos is determined by the seesaw formula $M_\nu \approx -M_{\mathrm{D}} M_{\mathrm{R}}^{-1} M_{\mathrm{D}}^{\mathrm{T}}$ [4–8] and turns out to be on the right order, i.e., $\mathcal{O}(M_\nu) \approx 0.1$ eV.

Although the smallness of light Majorana neutrino masses can naturally be ascribed to the heaviness of right-handed neutrino singlets in the canonical seesaw model [4–8], the observed pattern of lepton flavor mixing remains completely unexplained [2]. According to the latest global-fit analysis of all neutrino oscillation data [3], from which the current knowledge on neutrino mass-squared differences $\Delta m^2_{ij} \equiv m^2_i - m^2_j$ (for $ij = 21, 31, 32$), three mixing angles $\{\theta_{12}, \theta_{13}, \theta_{23}\}$, and the Dirac-type CP-violating phase δ has been summarized in Table I, we can observe that the flavor mixing angles $\theta_{12} \approx 33°$, $\theta_{13} \approx 8.6°$, and $\theta_{23} \approx 49°$ in the lepton sector are dramatically different from those in the quark sector [9]. An attractive way to understand the lepton flavor mixing is to impose discrete flavor symmetries on the seesaw model that is further extended with a number of scalar fields, which may transform nontrivially under the flavor symmetry group. See, e.g., Refs. [10–13] for recent reviews on discrete flavor symmetries and their applications to lepton flavor mixing and CP violation.

Apart from the discrete flavor symmetries, the finite modular symmetries have been recently implemented to account for lepton flavor mixing [14–16]. In the frameworks of string theories and supersymmetric field theories, the modular invariance and its connection to discrete flavor symmetries have been excellently elaborated in Ref. [16]. In addition, the practical applications of finite modular

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TABLE I. The best-fit values and the 1 σ and 3 σ intervals, together with the values of σ_i being the symmetrized 1σ uncertainties, for three neutrino mixing angles $\{\theta_{12}, \theta_{13}, \theta_{23}\}$, two neutrino mass-squared differences $\{\Delta m^2_{21}, \Delta m^2_{31}$ or $\Delta m^2_{32}\}$, and the Dirac *CP*-violating phase δ from a global-fit analysis of current experimental data [3].

Parameter	Best fit	1σ range	3σ range	σ_i
Normal neutrino mass ordering $(m_1 < m_2 < m_3)$				
$\sin^2\theta_{12}$	0.304	0.292–0.317	0.269–0.343	0.0125
$\sin^2\theta_{13}$	0.02221	0.02159–0.02289	0.02034–0.02430	0.00065
$\sin^2\theta_{23}$	0.570	0.546–0.588	0.407–0.618	0.021
$\delta/°$	195	170–246	107–403	38
$\Delta m^2_{21}/(10^{-5}\ \mathrm{eV}^2)$	7.42	7.22–7.63	6.82–8.04	0.205
$\Delta m^2_{31}/(10^{-3}\ \mathrm{eV}^2)$	+2.514	+2.487– + 2.542	+2.431– + 2.598	0.0275
Inverted neutrino mass ordering $(m_3 < m_1 < m_2)$				
$\sin^2\theta_{12}$	0.304	0.292–0.317	0.269–0.343	0.0125
$\sin^2\theta_{13}$	0.02240	0.02178–0.02302	0.02053–0.02436	0.00062
$\sin^2\theta_{23}$	0.575	0.554–0.592	0.411–0.621	0.019
$\delta/°$	286	254–313	192–360	29.5
$\Delta m^2_{21}/(10^{-5}\ \mathrm{eV}^2)$	7.42	7.22–7.63	6.82–8.04	0.205
$\Delta m^2_{32}/(10^{-3}\ \mathrm{eV}^2)$	−2.497	−2.525– − 2.469	−2.583– − 2.412	0.028

groups Γ_N (for $N = 2, 3, 4, 5, \ldots$) to the model building of neutrino masses and flavor mixing can be found in the vast literature [17–74]. Meanwhile, the double covering of Γ_N has also been discussed in Refs. [75–79]. In the present paper, we investigate the double covering of the modular A_5 symmetry group, i.e., $\Gamma'_5 \simeq A'_5$, which has not yet been explored in the previous works. For the finite modular group Γ'_5, the modular forms with both even and odd weights are present. The explicit expressions of the modular forms of weights up to six in the nontrivial representations of Γ'_5 are derived for the first time. Interestingly, there are two-dimensional irreducible representations $\hat{\mathbf{2}}$ and $\hat{\mathbf{2}}'$ for the double-covering group $\Gamma'_5 \simeq A'_5$, which are, however, absent for the $\Gamma_5 \simeq A_5$ group. Motivated by this observation, we further apply the Γ'_5 group to the minimal seesaw model (MSM) [80–86] and assign two right-handed neutrino singlets into the two-dimensional representation of Γ'_5. Two concrete examples in the MSM have been given to explain the observed pattern of lepton flavor mixing.

The remaining part of this paper is structured as follows. In Sec. II, the double-covering group $\Gamma'_5 \simeq A'_5$ of the modular $\Gamma_5 \simeq A_5$ group is examined and the modular forms of weights up to six are explicitly given. The applications of the Γ'_5 group to the MSM are explored in Sec. III, and two concrete models are built to account for lepton flavor mixing as observed in neutrino oscillation experiments. The exact numerical results of the allowed parameter space are presented, while the approximate analytical results are also derived in order to understand the flavor structures of the charged-lepton and neutrino mass matrices. In Sec. IV, we summarize our main results. Finally, the basic properties of the finite group A'_5 are presented in Appendixes A and B.

II. DOUBLE-COVERING GROUP

A. Double covering of the modular group

In this subsection, we introduce the double covering of the modular group and explain why the modular group $\Gamma \simeq \mathrm{SL}(2, \mathbb{Z})$ and its principal congruence subgroups $\Gamma(N)$ (for $N > 2$ being positive integers) can accommodate the modular forms with odd weights. Although the basic properties of the modular group can be found in the existing literature [16] and mathematical monographs [87–90], a concise introduction will be helpful in establishing our notations and conventions for the subsequent discussions. First, let us recall the definition of the modular group $\Gamma \simeq \mathrm{SL}(2, \mathbb{Z})$, namely,

$$\Gamma \equiv \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}, \tag{2.1}$$

which is generated by S, T, and R satisfying $S^2 = R$, $(ST)^3 = \mathbb{I}$, $R^2 = \mathbb{I}$, and $RT = TR$, respectively. More explicitly, the matrix representations of these three generators are given by

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.2}$$

Suppose that γ is an element of the modular group Γ. Then the modular transformations of the complex parameter τ in the upper half of the complex plane $\mathcal{H} = \{\tau \in \mathbb{C} | \mathrm{Im}\tau > 0\}$ and the chiral supermultiplet $\chi^{(I)}$ can be defined as

$$\gamma : \tau \to \frac{a\tau + b}{c\tau + d}, \qquad \chi^{(I)} \to (c\tau + d)^{-k_I} \rho^{(I)}(\gamma) \chi^{(I)}, \qquad (2.3)$$

where k_I is the weight of the chiral supermultiplet and $\rho^{(I)}(\gamma)$ denotes the representation matrix of γ. Obviously, the transformations of τ induced by γ and $-\gamma$ are actually identical; i.e., the modulus τ does not transform under R. Hence, one can define the so-called inhomogeneous modular group as $\bar{\Gamma} \simeq \mathrm{PSL}(2, \mathbb{Z}) \equiv \mathrm{SL}(2, \mathbb{Z})/\{\mathbb{I}, -\mathbb{I}\}$ with $\mathbb{I}$ being the identity element, for which two generators are found to be $S: \tau \to -1/\tau$ and $T: \tau \to \tau + 1$. On the other hand, the modular group has an important class of subgroups, i.e., the principal congruence subgroups, whose exact definition is as follows:

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}), \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}, \qquad (2.4)$$

for a given positive integer N. In a similar way, one can introduce $\bar{\Gamma}(N) \equiv \Gamma(N)/\{\mathbb{I}, -\mathbb{I}\}$ and the quotient group $\Gamma_N \equiv \bar{\Gamma}/\bar{\Gamma}(N)$, which are generated by S and T satisfying the identities $S^2 = (ST)^3 = T^N = \mathbb{I}$. However, as has been pointed out in Refs. [53,77], matter fields in modular-invariant theories are generally allowed to transform under R. Therefore, we should consider Γ rather than $\bar{\Gamma}$ as the symmetry group in such theories. As a consequence, the finite modular group Γ_N will be extended to its double-covering group, defined as $\Gamma'_N \equiv \Gamma/\Gamma(N)$, which is generated by S, T, and R that obey the following identities:

$$S^2 = R, \quad (ST)^3 = \mathbb{I}, \quad T^N = \mathbb{I}, \quad R^2 = \mathbb{I}, \quad RT = TR. \qquad (2.5)$$

Then, we consider the modular forms. By definition, the modular form $f(\tau)$ of level N and weight k is a holomorphic function of τ, and it transforms under $\Gamma(N)$ as

$$f(\gamma\tau) = (c\tau + d)^k f(\tau), \qquad \gamma \in \Gamma(N), \qquad (2.6)$$

where $k \geq 0$ is an integer. Note that $-\mathbb{I}$ belongs to $\Gamma(N)$ for $N = 1, 2$, leading to $f(\tau) = (-1)^k f(\tau)$ if we substitute $\gamma = -\mathbb{I}$ into Eq. (2.6). Therefore, the nonzero modular forms with odd weights can exist only in $\Gamma(N)$ with $N > 2$. As has been proved in Ref. [75], for a given modular space $\mathcal{M}_k[\Gamma(N)]$, the modular forms can always be decomposed into several multiplets that transform as irreducible unitary representations of the finite modular group Γ'_N. To be more precise, we can always find a proper basis for the modular space $\mathcal{M}_k[\Gamma(N)]$ such that a modular multiplet $Y^{(k)}_{\mathbf{r}} = (f_1(\tau), f_2(\tau), \ldots)^{\mathrm{T}}$ in the representation $\mathbf{r}$ satisfies the following equation:

$$Y^{(k)}_{\mathbf{r}}(\gamma\tau) = (c\tau + d)^k \rho_{\mathbf{r}}(\gamma) Y^{(k)}_{\mathbf{r}}(\tau), \qquad \gamma \in \Gamma, \qquad (2.7)$$

where $\rho_{\mathbf{r}}(\gamma)$ denotes the representation matrix of γ and $\rho_{\mathbf{r}}(\gamma) = 1$ for $\gamma \in \Gamma(N)$. In particular, for the generators $\gamma = S$ and T, we get

$$Y^{(k)}_{\mathbf{r}}(S\tau) = (-\tau)^k \rho_{\mathbf{r}}(S) Y^{(k)}_{\mathbf{r}}(\tau), \quad Y^{(k)}_{\mathbf{r}}(T\tau) = \rho_{\mathbf{r}}(T) Y^{(k)}_{\mathbf{r}}(\tau). \qquad (2.8)$$

Now that two elements γ and $S^2\gamma$ correspond to the same fractional linear transformation of τ in Eq. (2.3), we can substitute them into Eq. (2.7) and arrive at

$$\begin{aligned} Y^{(k)}_{\mathbf{r}}(\gamma\tau) &= (c\tau + d)^k \rho_{\mathbf{r}}(\gamma) Y^{(k)}_{\mathbf{r}}(\tau), \\ Y^{(k)}_{\mathbf{r}}(S^2\gamma\tau) &= (-1)^k (c\tau + d)^k \rho_{\mathbf{r}}(S^2\gamma) Y^{(k)}_{\mathbf{r}}(\tau), \end{aligned} \qquad (2.9)$$

which should be identical, leading to the relation

$$\rho_{\mathbf{r}}(S^2\gamma) = (-1)^k \rho_{\mathbf{r}}(\gamma). \qquad (2.10)$$

Since $S^2 = \mathbb{I}$ in the finite modular group Γ_N, Eq. (2.10) implies $\rho_{\mathbf{r}}(\gamma) = (-1)^k \rho_{\mathbf{r}}(\gamma)$, which is expected to hold only for k being an even integer. However, in the double-covering group Γ'_N, we can see that the generator $R = S^2$ fulfills the relations $\rho_{\mathbf{r}}(R) = \rho_{\mathbf{r}}(\mathbb{I}) = \mathbb{I}$ for an even k and $\rho_{\mathbf{r}}(R) = -\rho_{\mathbf{r}}(\mathbb{I}) = -\mathbb{I}$ for an odd k. Therefore, we demonstrate that the modular group Γ and its principal congruence subgroup $\Gamma(N)$ with $N > 2$ have to be considered for the modular forms of both even and odd weights.

B. The group $\Gamma'_5 \simeq A'_5$

In this paper, we focus on the finite modular group $\Gamma_5 \simeq A_5$ and its double-covering group $\Gamma'_5 \simeq A'_5$, corresponding to the specific case of $N = 5$. The basic properties of the finite group A'_5 have already been studied in the existing literature [91–94], so we just briefly summarize the key points relevant for our later discussions. The A'_5 group has 120 elements, which can be produced by three generators S, T, and R satisfying the identities in Eq. (2.5) for $N = 5$. All 120 elements can be divided into nine conjugacy classes, indicating that A'_5 has nine distinct irreducible representations, which are normally denoted as $\mathbf{1}, \hat{\mathbf{2}}, \hat{\mathbf{2}}', \mathbf{3}, \mathbf{3}', \mathbf{4}, \hat{\mathbf{4}}, \mathbf{5}$, and $\hat{\mathbf{6}}$ by their dimensions. The conjugacy classes and character table of A'_5, together with the representation matrices of all three generators S, T, and R in the irreducible representations, are explicitly given in Appendix A. Notice that the representations $\mathbf{1}$, $\mathbf{3}$, $\mathbf{3}'$, $\mathbf{4}$, and $\mathbf{5}$ with $R = \mathbb{I}$ coincide with those for A_5, whereas $\hat{\mathbf{2}}$, $\hat{\mathbf{2}}'$, $\hat{\mathbf{4}}$, and $\hat{\mathbf{6}}$ are unique for A'_5 with $R = -\mathbb{I}$. In addition, the decomposition rules of the Kronecker products of any two nontrivial irreducible representations can be found in Appendix B and will be frequently used in the subsequent discussions.

C. The modular space of $\Gamma(5)$

To construct the modular forms that transform nontrivially under Γ'_5, which is isomorphic to A'_5, we need first to find out the modular space of $\Gamma(5)$. For a given non-negative integer k, the modular space $\mathcal{M}_k[\Gamma(5)]$ of weight k for $\Gamma(5)$ contains $5k+1$ linearly independent modular forms, which can be regarded as the basis vectors of the modular space. According to Ref. [90], we have

$$\mathcal{M}_k[\Gamma(5)] = \bigoplus_{\substack{a+b=5k\\ a,b\geq 0}} \mathbb{C}\frac{\eta(5\tau)^{15k}}{\eta(\tau)^{3k}}\mathfrak{k}^a_{\frac{1}{5},\frac{0}{5}}(5\tau)\mathfrak{k}^b_{\frac{2}{5},\frac{0}{5}}(5\tau), \qquad (2.11)$$

where $\eta(\tau)$ is the Dedekind eta function

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \qquad (2.12)$$

with $q \equiv e^{2\mathrm{i}\pi\tau}$, and $\mathfrak{k}_{r_1,r_2}(\tau)$ is the Klein form

$$\mathfrak{k}_{r_1,r_2}(\tau) = q_z^{(r_1-1)/2}(1-q_z) \times \prod_{n=1}^{\infty}(1-q^n q_z)(1-q^n q_z^{-1})(1-q^n)^{-2}, \qquad (2.13)$$

with (r_1, r_2) being a pair of rational numbers in the domain of $\mathbb{Q}^2 - \mathbb{Z}^2$, $z \equiv \tau r_1 + r_2$, and $q_z \equiv e^{2\mathrm{i}\pi z}$. Under the transformations of S and T, the eta function and the Klein form change as follows:

$$S:\ \eta(\tau) \to \sqrt{-\mathrm{i}\tau}\,\eta(\tau), \qquad \mathfrak{k}_{r_1,r_2}(\tau) \to -\frac{1}{\tau}\mathfrak{k}_{-r_2,r_1}(\tau),$$
$$T:\ \eta(\tau) \to e^{\mathrm{i}\pi/12}\eta(\tau), \qquad \mathfrak{k}_{r_1,r_2}(\tau) \to \mathfrak{k}_{r_1,r_1+r_2}(\tau). \qquad (2.14)$$

More information about the properties of the Klein form $\mathfrak{k}_{r_1,r_2}(\tau)$ can be found in Refs. [36,90] and, thus, will not be further discussed here.

Then we take $k=1$ and find out the modular forms of the lowest weight. With the help of Eq. (2.11), it is straightforward to obtain the basis vectors of the modular space $\mathcal{M}_1[\Gamma(5)]$, i.e.,

$$\hat{e}_1(\tau) = \frac{\eta^{15}(5\tau)}{\eta^3(\tau)}\mathfrak{k}^5_{\frac{2}{5},\frac{0}{5}}(5\tau), \qquad \hat{e}_2(\tau) = \frac{\eta^{15}(5\tau)}{\eta^3(\tau)}\mathfrak{k}_{\frac{1}{5},\frac{0}{5}}(5\tau)\mathfrak{k}^4_{\frac{2}{5},\frac{0}{5}}(5\tau),$$
$$\hat{e}_3(\tau) = \frac{\eta^{15}(5\tau)}{\eta^3(\tau)}\mathfrak{k}^2_{\frac{1}{5},\frac{0}{5}}(5\tau)\mathfrak{k}^3_{\frac{2}{5},\frac{0}{5}}(5\tau), \qquad \hat{e}_4(\tau) = \frac{\eta^{15}(5\tau)}{\eta^3(\tau)}\mathfrak{k}^3_{\frac{1}{5},\frac{0}{5}}(5\tau)\mathfrak{k}^2_{\frac{2}{5},\frac{0}{5}}(5\tau),$$
$$\hat{e}_5(\tau) = \frac{\eta^{15}(5\tau)}{\eta^3(\tau)}\mathfrak{k}^4_{\frac{1}{5},\frac{0}{5}}(5\tau)\mathfrak{k}_{\frac{2}{5},\frac{0}{5}}(5\tau), \qquad \hat{e}_6(\tau) = \frac{\eta^{15}(5\tau)}{\eta^3(\tau)}\mathfrak{k}^5_{\frac{1}{5},\frac{0}{5}}(5\tau). \qquad (2.15)$$

Furthermore, making use of Eqs. (2.12) and (2.13), we can derive the q expansions of the above six basis vectors:

$$\hat{e}_1 = 1 + 3q + 4q^2 + 2q^3 + q^4 + 3q^5 + 6q^6 + 4q^7 - q^9 + \cdots,$$
$$\hat{e}_2 = q^{1/5}(1 + 2q + 2q^2 + q^3 + 2q^4 + 2q^5 + 2q^6 + q^7 + 2q^8 + 2q^9 + \cdots),$$
$$\hat{e}_3 = q^{2/5}(1 + q + q^2 + q^3 + 2q^4 + q^6 + q^7 + 2q^8 + q^9 + \cdots),$$
$$\hat{e}_4 = q^{3/5}(1 + q^2 + q^3 + q^4 - q^5 + 2q^6 + 2q^8 + q^9 + \cdots),$$
$$\hat{e}_5 = q^{4/5}(1 - q + 2q^2 + 2q^6 - 2q^7 + 2q^8 + q^9 + \cdots),$$
$$\hat{e}_6 = q(1 - 2q + 4q^2 - 3q^3 + q^4 + 2q^5 - 2q^6 + 3q^8 - 2q^9 + \cdots). \qquad (2.16)$$

From Eq. (2.15), one can immediately observe that $\hat{e}_i\hat{e}_j = \hat{e}_m\hat{e}_n$ exactly holds for $i+j = m+n$. These relations are very useful for the calculations of higher-weight modular forms, as we shall see in the next subsection. Under the transformation of T, we have

$$\hat{e}_1 \to \hat{e}_1, \qquad \hat{e}_2 \to \omega\hat{e}_2, \qquad \hat{e}_3 \to \omega^2\hat{e}_3, \qquad \hat{e}_4 \to \omega^3\hat{e}_4, \qquad \hat{e}_5 \to \omega^4\hat{e}_5, \qquad \hat{e}_6 \to \hat{e}_6, \qquad (2.17)$$

with $\omega \equiv e^{2\mathrm{i}\pi/5}$, while under the transformation of S we obtain

$$\hat{e}_1 \to (-\mathrm{i}\tau)\frac{\sqrt{\phi}}{\sqrt[4]{5}}\left[\frac{\phi^2}{5}\hat{e}_1 + \phi\hat{e}_2 + 2\hat{e}_3 + 2(\phi-1)\hat{e}_4 + (\phi-1)^2\hat{e}_5 + \frac{2\phi-3}{5}\hat{e}_6\right],$$
$$\hat{e}_2 \to (-\mathrm{i}\tau)\frac{\sqrt{\phi-1}}{\sqrt[4]{5}}\left[\frac{\phi^2}{5}\hat{e}_1 + \frac{\sqrt{5}}{5}\hat{e}_2 - \frac{2\sqrt{5}}{5}\hat{e}_3 - \frac{2\sqrt{5}\phi}{5}\hat{e}_4 - \frac{\sqrt{5}\phi}{5}\hat{e}_5 - \frac{(\phi-1)}{5}\hat{e}_6\right],$$

$$\hat{e}_3 \to (-\mathrm{i}\tau)\frac{\sqrt{\phi}}{\sqrt[4]{5}}\left[\frac{1}{5}\hat{e}_1 - \frac{\sqrt{5}(\phi-1)}{5}\hat{e}_2 - \frac{\sqrt{5}}{5}\hat{e}_3 + \frac{\sqrt{5}(\phi-1)}{5}\hat{e}_4 + \frac{\sqrt{5}}{5}\hat{e}_5 + \frac{(\phi-1)}{5}\hat{e}_6\right],$$
$$\hat{e}_4 \to (-\mathrm{i}\tau)\frac{\sqrt{\phi-1}}{\sqrt[4]{5}}\left[\frac{1}{5}\hat{e}_1 - \frac{\sqrt{5}\phi}{5}\hat{e}_2 + \frac{\sqrt{5}}{5}\hat{e}_3 + \frac{\sqrt{5}\phi}{5}\hat{e}_4 - \frac{\sqrt{5}}{5}\hat{e}_5 - \frac{\phi}{5}\hat{e}_6\right],$$
$$\hat{e}_5 \to (-\mathrm{i}\tau)\frac{\sqrt{\phi}}{\sqrt[4]{5}}\left[\frac{(\phi-1)^2}{5}\hat{e}_1 - \frac{\sqrt{5}}{5}\hat{e}_2 + \frac{2\sqrt{5}}{5}\hat{e}_3 - \frac{2\sqrt{5}(\phi-1)}{5}\hat{e}_4 - \frac{\sqrt{5}(\phi-1)}{5}\hat{e}_5 + \frac{\phi}{5}\hat{e}_6\right],$$
$$\hat{e}_6 \to (-\mathrm{i}\tau)\frac{\sqrt{\phi-1}}{\sqrt[4]{5}}\left[\frac{(\phi-1)^2}{5}\hat{e}_1 - (\phi-1)\hat{e}_2 + 2\hat{e}_3 - 2\phi\hat{e}_4 + \phi^2\hat{e}_5 - \frac{2\phi+1}{5}\hat{e}_6\right], \tag{2.18}$$

with $\phi \equiv (\sqrt{5}+1)/2$. The derivation of Eq. (2.17) is quite straightforward, since one can simply use the T transformation properties of $\eta(\tau)$ and $\mathfrak{k}_{r_1,r_2}(\tau)$ shown in Eq. (2.14). However, the derivation of Eq. (2.18) would be very tedious if one strictly followed the S transformation properties of $\eta(\tau)$ and $\mathfrak{k}_{r_1,r_2}(\tau)$. Our strategy for such a derivation is as follows. First of all, we know that each function $\hat{e}_i$ will be transformed under S into a linear combination of all six basis vectors with coefficients to be determined. In each linear combination, the q expansions of $\hat{e}_i$ given in Eq. (2.16) will be performed. On the other hand, given the transformation rules for $\eta(\tau)$ and $\mathfrak{k}_{r_1,r_2}(\tau)$ under S in Eq. (2.14), one can calculate $\hat{e}_i$ after the S transformation by using Eq. (2.15) and perform the q expansions as well. Since the expressions derived in those two different ways should be equivalent to each other, we can extract the coefficients in the linear combinations by comparing the first few terms in the q expansions. With those coefficients, we can obtain the final results in Eq. (2.18).

Now that the transformations of $\hat{e}_i$ under S and T are known, we are ready to determine the transformation rule for the modular form $Y^{(1)}_{\hat{\mathbf{6}}}(\tau)$ of weight one in the irreducible representation $\hat{\mathbf{6}}$, which can be expressed in terms of the basis vectors of the modular space $\mathcal{M}_1[\Gamma(5)]$. More explicitly, the components of $Y^{(1)}_{\hat{\mathbf{6}}}(\tau)$ can be written as

$$(Y^{(1)}_{\hat{\mathbf{6}}})_i = \sum_{j=1}^{6} a_{ij}\hat{e}_j, \tag{2.19}$$

for $i = 1, 2, \ldots, 6$, where the coefficients a_{ij} (for $i, j = 1, 2, \ldots, 6$) need to be calculated. Then, after applying Eq. (2.8) in the case of $k = 1$ and $\mathbf{r} = \hat{\mathbf{6}}$ and taking the transformation properties of $\hat{e}_i$ in Eqs. (2.17) and (2.18) into consideration, we can find all the nonzero coefficients:

$$a_{11} = 1, \qquad a_{16} = a_{61} = -3, \qquad a_{22} = a_{55} = 5\sqrt{2},$$
$$a_{33} = a_{44} = 10, \qquad a_{66} = -1. \tag{2.20}$$

As a consequence, the explicit expressions of all the components of $Y^{(1)}_{\hat{\mathbf{6}}}$ can be obtained. For later convenience, we denote six components of $Y^{(1)}_{\hat{\mathbf{6}}}(\tau)$ as $Y_i(\tau)$ (for $i = 1, 2, \ldots, 6$) and then have

$$Y^{(1)}_{\hat{\mathbf{6}}} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \end{pmatrix} = \begin{pmatrix} \hat{e}_1 - 3\hat{e}_6 \\ 5\sqrt{2}\hat{e}_2 \\ 10\hat{e}_3 \\ 10\hat{e}_4 \\ 5\sqrt{2}\hat{e}_5 \\ -3\hat{e}_1 - \hat{e}_6 \end{pmatrix}, \tag{2.21}$$

where the argument τ has been suppressed for all the relevant functions, and the exact formulas of $\hat{e}_i$ and their q expansions are given in Eqs. (2.15) and (2.16), respectively.

D. Modular forms of higher weights

As usual, the modular forms of higher weights can be constructed through the tensor products of those of lower weights. We start with the modular forms of weight two (i.e., $k = 2$), which can be generated by the tensor product of two modular forms of weight one, namely, $Y^{(1)}_{\hat{\mathbf{6}}} \otimes Y^{(1)}_{\hat{\mathbf{6}}}$. With the help of the decomposition rules of tensor products in Appendix B, we get all the nonzero modular forms with weight two as

$$
\begin{aligned}
Y^{(2)}_{\mathbf{3},\mathrm{i}} &= [Y^{(1)}_{\hat{\mathbf{6}}} \otimes Y^{(1)}_{\hat{\mathbf{6}}}]_{\mathbf{3}_{\mathrm{s},1}} = -3\begin{pmatrix} \hat{e}_1^2 - 36\hat{e}_1\hat{e}_6 - \hat{e}_6^2 \\ 5\sqrt{2}\hat{e}_2(\hat{e}_1 - 3\hat{e}_6) \\ 5\sqrt{2}\hat{e}_5(3\hat{e}_1 + \hat{e}_6) \end{pmatrix} = -3\begin{pmatrix} Y_1^2 - 3Y_1Y_6 - Y_6^2 \\ Y_1Y_2 \\ -Y_5Y_6 \end{pmatrix}, \\
Y^{(2)}_{\mathbf{3},\mathrm{ii}} &= [Y^{(1)}_{\hat{\mathbf{6}}} \otimes Y^{(1)}_{\hat{\mathbf{6}}}]_{\mathbf{3}_{\mathrm{s},2}} = -\frac{\sqrt{6}}{9} Y^{(2)}_{\mathbf{3},\mathrm{i}}, \\
Y^{(2)}_{\mathbf{3}',\mathrm{i}} &= [Y^{(1)}_{\hat{\mathbf{6}}} \otimes Y^{(1)}_{\hat{\mathbf{6}}}]_{\mathbf{3}'_{\mathrm{s},1}} = \sqrt{6}\begin{pmatrix} -(\hat{e}_1^2 + 14\hat{e}_1\hat{e}_6 - \hat{e}_6^2) \\ 5\sqrt{2}\hat{e}_3(\hat{e}_1 + 2\hat{e}_6) \\ 5\sqrt{2}\hat{e}_4(2\hat{e}_1 - \hat{e}_6) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sqrt{6}(Y_1^2 - 2Y_1Y_6 - Y_6^2) \\ -\sqrt{3}Y_3(Y_1 + Y_6) \\ \sqrt{3}Y_4(Y_1 - Y_6) \end{pmatrix}, \\
Y^{(2)}_{\mathbf{3}',\mathrm{ii}} &= [Y^{(1)}_{\hat{\mathbf{6}}} \otimes Y^{(1)}_{\hat{\mathbf{6}}}]_{\mathbf{3}'_{\mathrm{s},2}} = -\frac{4\sqrt{6}}{3} Y^{(2)}_{\mathbf{3}',\mathrm{i}}, \\
Y^{(2)}_{\mathbf{5}} &= [Y^{(1)}_{\hat{\mathbf{6}}} \otimes Y^{(1)}_{\hat{\mathbf{6}}}]_{\mathbf{5}_{\mathrm{s}}} = 5\begin{pmatrix} \sqrt{2}(\hat{e}_1^2 + \hat{e}_6^2) \\ -2\sqrt{3}\hat{e}_2(\hat{e}_1 + 7\hat{e}_6) \\ 2\sqrt{3}\hat{e}_3(4\hat{e}_6 - 3\hat{e}_1) \\ -2\sqrt{3}\hat{e}_4(4\hat{e}_1 + 3\hat{e}_6) \\ 2\sqrt{3}\hat{e}_5(\hat{e}_6 - 7\hat{e}_1) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sqrt{2}(Y_1^2 + Y_6^2) \\ 2\sqrt{6}Y_2(2Y_1 + Y_6) \\ \sqrt{3}Y_3(Y_6 - 3Y_1) \\ \sqrt{3}Y_4(Y_1 + 3Y_6) \\ 2\sqrt{6}Y_5(2Y_6 - Y_1) \end{pmatrix}.
\end{aligned} \tag{2.22}
$$

Some comments on the above modular forms are in order. First, the dimensionality of the modular space $\mathcal{M}_k[\Gamma(5)]$ is $5k+1$, implying 11 independent modular forms of $k=2$, which we take as $Y^{(2)}_{\mathbf{3}} \equiv Y^{(2)}_{\mathbf{3},\mathrm{i}}$, $Y^{(2)}_{\mathbf{3}'} \equiv Y^{(2)}_{\mathbf{3}',\mathrm{i}}$ and $Y^{(2)}_{\mathbf{5}}$. Second, substituting the q expansions of $\hat{e}_i$ in Eq. (2.16) into the expressions of those three modular forms, we find that they are consistent up to some overall factors with the results obtained in Ref. [36] for the modular A_5 group. This should be the case, as the modular forms of even weights for the modular A_5 group coincide with those for the double-covering group A'_5.

Following a similar procedure, we can derive the modular forms of weights three, four, five, and six. For weight three, there exist 16 independent modular forms, transforming as the irreducible representations $\hat{\mathbf{4}}$, $\hat{\mathbf{6}}_1$, and $\hat{\mathbf{6}}_2$ of the modular A'_5 group, which can be expressed as

$$
\begin{aligned}
Y^{(3)}_{\hat{\mathbf{4}}} &= [Y^{(1)}_{\hat{\mathbf{6}}} \otimes Y^{(2)}_{\mathbf{3}}]_{\hat{\mathbf{4}}} = -\frac{3\sqrt{30}}{10}(Y_1^2 - 4Y_1Y_6 - Y_6^2)\begin{pmatrix} -\sqrt{2}Y_2 \\ \sqrt{3}Y_3 \\ \sqrt{3}Y_4 \\ \sqrt{2}Y_5 \end{pmatrix}, \\
Y^{(3)}_{\hat{\mathbf{6}},1} &= [Y^{(1)}_{\hat{\mathbf{6}}} \otimes Y^{(2)}_{\mathbf{3}}]_{\hat{\mathbf{6}},2} = -\frac{\sqrt{3}}{2}\begin{pmatrix} 5Y_1^3 - 12Y_1^2Y_6 - 11Y_1Y_6^2 - 2Y_6^3 \\ -2Y_2(Y_1^2 - 5Y_1Y_6 - 2Y_6^2) \\ Y_3(Y_1 + Y_6)(Y_1 + 2Y_6) \\ Y_4(Y_1 - Y_6)(2Y_1 - Y_6) \\ 2Y_5(2Y_1^2 - 5Y_1Y_6 - Y_6^2) \\ 2Y_1^3 - 11Y_1^2Y_6 + 12Y_1Y_6^2 + 5Y_6^3 \end{pmatrix}, \\
Y^{(3)}_{\hat{\mathbf{6}},2} &= [Y^{(1)}_{\hat{\mathbf{6}}} \otimes Y^{(2)}_{\mathbf{3}'}]_{\hat{\mathbf{6}},2} = -\frac{\sqrt{3}}{2}\begin{pmatrix} 3Y_1^3 - 9Y_1^2Y_6 - Y_1Y_6^2 + Y_6^3 \\ -2Y_2(2Y_1^2 - 2Y_1Y_6 - Y_6^2) \\ 2Y_1^2Y_3 \\ 2Y_4Y_6^2 \\ 2Y_5(Y_1^2 - 2Y_1Y_6 - 2Y_6^2) \\ -Y_1^3 - Y_1^2Y_6 + 9Y_1Y_6^2 + 3Y_6^3 \end{pmatrix}.
\end{aligned} \tag{2.23}
$$

For weight four, we have

$$
\begin{aligned}
Y_{\mathbf{1}}^{(4)} &= [Y_{\hat{\mathbf{6}}}^{(1)} \otimes Y_{\hat{\mathbf{6}},2}^{(3)}]_{\mathbf{1}} = -\sqrt{2}(Y_1^4 - 3Y_1^3Y_6 - Y_1^2Y_6^2 + 3Y_1Y_6^3 + Y_6^4), \\
Y_{\mathbf{3}}^{(4)} &= [Y_{\hat{\mathbf{6}}}^{(1)} \otimes Y_{\hat{\mathbf{6}},2}^{(3)}]_{\mathbf{3}_{\mathrm{s},1}} = \frac{\sqrt{3}}{4}\begin{pmatrix} (Y_1^2+Y_6^2)(7Y_1^2-18Y_1Y_6-7Y_6^2) \\ Y_2(13Y_1^3-3Y_1^2Y_6-29Y_1Y_6^2-9Y_6^3) \\ -Y_5(9Y_1^3-29Y_1^2Y_6+3Y_1Y_6^2+13Y_6^3) \end{pmatrix}, \\
Y_{\mathbf{3}'}^{(4)} &= [Y_{\hat{\mathbf{6}}}^{(1)} \otimes Y_{\hat{\mathbf{6}},2}^{(3)}]_{\mathbf{3}'_{\mathrm{s},1}} = -\frac{1}{2}\begin{pmatrix} \sqrt{2}(Y_1^2+Y_6^2)(4Y_1^2-11Y_1Y_6-4Y_6^2) \\ -Y_3(Y_1-2Y_6)(7Y_1^2-3Y_1Y_6-2Y_6^2) \\ Y_4(2Y_1+Y_6)(2Y_1^2-3Y_1Y_6-7Y_6^2) \end{pmatrix}, \\
Y_{\mathbf{4}}^{(4)} &= [Y_{\hat{\mathbf{6}}}^{(1)} \otimes Y_{\hat{\mathbf{6}},2}^{(3)}]_{\mathbf{4}_{\mathrm{a}}} = (Y_1^2-4Y_1Y_6-Y_6^2)\begin{pmatrix} \sqrt{2}Y_2(2Y_1+Y_6) \\ Y_3(2Y_1+Y_6) \\ -Y_4(Y_1-2Y_6) \\ \sqrt{2}Y_5(Y_1-2Y_6) \end{pmatrix}, \\
Y_{\mathbf{5},1}^{(4)} &= [Y_{\hat{\mathbf{6}}}^{(1)} \otimes Y_{\hat{\mathbf{4}}}^{(3)}]_{\mathbf{5},2} = -\frac{3\sqrt{30}}{10}(Y_1^2-4Y_1Y_6-Y_6^2)\begin{pmatrix} \sqrt{3}(Y_1-3Y_6)(3Y_1+Y_6) \\ -Y_2(5Y_1+Y_6) \\ \sqrt{2}Y_3(Y_1-Y_6) \\ \sqrt{2}Y_4(Y_1+Y_6) \\ -Y_5(Y_1-5Y_6) \end{pmatrix}, \\
Y_{\mathbf{5},2}^{(4)} &= [Y_{\hat{\mathbf{6}}}^{(1)} \otimes Y_{\hat{\mathbf{6}},2}^{(3)}]_{\mathbf{5}_{\mathrm{a},1}} = \frac{1}{4}\begin{pmatrix} 11Y_1^4-60Y_1^3Y_6+58Y_1^2Y_6^2+60Y_1Y_6^3+11Y_6^4 \\ \sqrt{3}Y_2(Y_1+Y_6)(Y_1^2+8Y_1Y_6+3Y_6^2) \\ -\sqrt{6}Y_3(Y_1^3+3Y_1^2Y_6-9Y_1Y_6^2-3Y_6^3) \\ \sqrt{6}Y_4(3Y_1^3-9Y_1^2Y_6-3Y_1Y_6^2+Y_6^3) \\ -\sqrt{3}Y_5(Y_1-Y_6)(3Y_1^2-8Y_1Y_6+Y_6^2) \end{pmatrix}.
\end{aligned} \tag{2.24}
$$

For weight five, we have

$$
\begin{aligned}
Y_{\hat{\mathbf{2}}}^{(5)} &= [Y_{\hat{\mathbf{6}}}^{(1)} \otimes Y_{\mathbf{3}'}^{(4)}]_{\hat{\mathbf{2}}} = \frac{\sqrt{6}}{4}(Y_1^2-4Y_1Y_6-Y_6^2)^2\begin{pmatrix} Y_3 \\ Y_4 \end{pmatrix}, \\
Y_{\hat{\mathbf{2}}'}^{(5)} &= [Y_{\hat{\mathbf{6}}}^{(1)} \otimes Y_{\mathbf{3}}^{(4)}]_{\hat{\mathbf{2}}'} = \frac{3}{4}(Y_1^2-4Y_1Y_6-Y_6^2)\begin{pmatrix} Y_2(7Y_1^2-3Y_1Y_6-2Y_6^2) \\ Y_5(2Y_1^2-3Y_1Y_6-7Y_6^2) \end{pmatrix}, \\
Y_{\hat{\mathbf{4}}}^{(5)} &= [Y_{\hat{\mathbf{6}}}^{(1)} \otimes Y_{\mathbf{3}'}^{(4)}]_{\hat{\mathbf{4}}} = \frac{\sqrt{10}}{4}(Y_1^2-4Y_1Y_6-Y_6^2)\begin{pmatrix} \sqrt{2}Y_2(Y_1^2-6Y_1Y_6-2Y_6^2) \\ \sqrt{3}Y_3Y_6(2Y_1+Y_6) \\ \sqrt{3}Y_1Y_4(Y_1-2Y_6) \\ \sqrt{2}Y_5(2Y_1^2-6Y_1Y_6-Y_6^2) \end{pmatrix}, \\
Y_{\hat{\mathbf{6}},1}^{(5)} &= [Y_{\hat{\mathbf{6}}}^{(1)} \otimes Y_{\mathbf{1}}^{(4)}]_{\hat{\mathbf{6}}} = -\sqrt{2}(Y_1^4-3Y_1^3Y_6-Y_1^2Y_6^2+3Y_1Y_6^3+Y_6^4)\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \end{pmatrix},
\end{aligned}
$$

$$Y^{(5)}_{\hat{\mathbf{6}},2} = [Y^{(1)}_{\hat{\mathbf{6}}} \otimes Y^{(4)}_{\mathbf{3}}]_{\hat{\mathbf{6}},1} = \frac{\sqrt{6}}{16}\begin{pmatrix} 41Y_1^5 - 195Y_1^4Y_6 + 214Y_1^3Y_6^2 + 66Y_1^2Y_6^3 - 127Y_1Y_6^4 - 39Y_6^5 \\ 2Y_2(40Y_1^4 - 81Y_1^3Y_6 - 49Y_1^2Y_6^2 + 33Y_1Y_6^3 + 13Y_6^4) \\ -Y_3(Y_1^4 + 6Y_1^3Y_6 + 20Y_1^2Y_6^2 - 114Y_1Y_6^3 - 41Y_6^4) \\ Y_4(41Y_1^4 - 114Y_1^3Y_6 - 20Y_1^2Y_6^2 + 6Y_1Y_6^3 - Y_6^4) \\ 2Y_5(13Y_1^4 - 33Y_1^3Y_6 - 49Y_1^2Y_6^2 + 81Y_1Y_6^3 + 40Y_6^4) \\ 39Y_1^5 - 127Y_1^4Y_6 - 66Y_1^3Y_6^2 + 214Y_1^2Y_6^3 + 195Y_1Y_6^4 + 41Y_6^5 \end{pmatrix},$$

$$Y^{(5)}_{\hat{\mathbf{6}},3} = [Y^{(1)}_{\hat{\mathbf{6}}} \otimes Y^{(4)}_{\mathbf{4}}]_{\hat{\mathbf{6}},2} = \frac{21\sqrt{5}}{40}(Y_1^2 - 4Y_1Y_6 - Y_6^2)\begin{pmatrix} (3Y_1 - Y_6)(3Y_1 + Y_6)(3Y_6 - Y_1) \\ 2Y_2(Y_1 + Y_6)(2Y_1 + Y_6) \\ Y_3(5Y_1^2 - 6Y_1Y_6 - 3Y_6^2) \\ Y_4(3Y_1^2 - 6Y_1Y_6 - 5Y_6^2) \\ -2Y_5(Y_1 - 2Y_6)(Y_1 - Y_6) \\ (3Y_1 + Y_6)(Y_1 + 3Y_6)(3Y_6 - Y_1) \end{pmatrix}.$$

For weight six, we have

$$Y^{(6)}_{\mathbf{1}} = [Y^{(1)}_{\hat{\mathbf{6}}} \otimes Y^{(5)}_{\hat{\mathbf{6}},2}]_{\mathbf{1}} = -\frac{3}{8}(Y_1^2 + Y_6^2)(41Y_1^4 - 198Y_1^3Y_6 + 154Y_1^2Y_6^2 + 198Y_1Y_6^3 + 41Y_6^4),$$

$$Y^{(6)}_{\mathbf{3},1} = [Y^{(1)}_{\hat{\mathbf{6}}} \otimes Y^{(5)}_{\hat{\mathbf{2}}'}]_{\mathbf{3}} = \frac{9\sqrt{2}}{16}(Y_1^2 - 4Y_1Y_6 - Y_6^2)\begin{pmatrix} (Y_1 - 3Y_6)(3Y_1 + Y_6)(3Y_1^2 - 2Y_1Y_6 - 3Y_6^2) \\ 2Y_2(2Y_1^3 - 9Y_1Y_6^2 - 3Y_6^3) \\ 2Y_5(3Y_1^3 - 9Y_1^2Y_6 + 2Y_6^3) \end{pmatrix},$$

$$Y^{(6)}_{\mathbf{3},2} = [Y^{(1)}_{\hat{\mathbf{6}}} \otimes Y^{(5)}_{\hat{\mathbf{6}},1}]_{\mathbf{3}_{s,1}} = 3\sqrt{2}(Y_1^4 - 3Y_1^3Y_6 - Y_1^2Y_6^2 + 3Y_1Y_6^3 + Y_6^4)\begin{pmatrix} Y_1^2 - 3Y_1Y_6 - Y_6^2 \\ Y_1Y_2 \\ -Y_5Y_6 \end{pmatrix},$$

$$Y^{(6)}_{\mathbf{3}',1} = [Y^{(1)}_{\hat{\mathbf{6}}} \otimes Y^{(5)}_{\hat{\mathbf{2}}}]_{\mathbf{3}'} = -\frac{\sqrt{3}}{2}(Y_1^2 - 4Y_1Y_6 - Y_6^2)^2\begin{pmatrix} (3Y_1 + Y_6)(Y_1 - 3Y_6) \\ \sqrt{2}Y_1Y_3 \\ \sqrt{2}Y_4Y_6 \end{pmatrix},$$

$$Y^{(6)}_{\mathbf{3}',2} = [Y^{(1)}_{\hat{\mathbf{6}}} \otimes Y^{(5)}_{\hat{\mathbf{6}},1}]_{\mathbf{3}'_{s,1}} = -\frac{\sqrt{6}}{2}(Y_1^4 - 3Y_1^3Y_6 - Y_1^2Y_6^2 + 3Y_1Y_6^3 + Y_6^4)\begin{pmatrix} \sqrt{2}(Y_1^2 - 2Y_1Y_6 - Y_6^2) \\ -Y_3(Y_1 + Y_6) \\ Y_4(Y_1 - Y_6) \end{pmatrix},$$

$$Y^{(6)}_{\mathbf{4},1} = [Y^{(1)}_{\hat{\mathbf{6}}} \otimes Y^{(5)}_{\hat{\mathbf{2}}}]_{\mathbf{4}} = -\frac{3}{4}(Y_1^2 - 4Y_1Y_6 - Y_6^2)^2\begin{pmatrix} -\sqrt{2}Y_2(3Y_1 + Y_6) \\ Y_3(Y_1 + Y_6) \\ Y_4(Y_1 - Y_6) \\ \sqrt{2}Y_5(Y_1 - 3Y_6) \end{pmatrix},$$

$$Y^{(6)}_{\mathbf{4},2} = [Y^{(1)}_{\hat{\mathbf{6}}} \otimes Y^{(5)}_{\hat{\mathbf{2}}'}]_{\mathbf{4}} = -\frac{\sqrt{6}}{8}(Y_1^2 - 4Y_1Y_6 - Y_6^2)\begin{pmatrix} \sqrt{2}Y_2(Y_1^3 + 11Y_1^2Y_6 + 19Y_1Y_6^2 + 5Y_6^3) \\ Y_3(13Y_1^3 - 31Y_1^2Y_6 - 17Y_1Y_6^2 - Y_6^3) \\ Y_4(Y_1^3 - 17Y_1^2Y_6 + 31Y_1Y_6^2 + 13Y_6^3) \\ \sqrt{2}Y_5(5Y_1^3 - 19Y_1^2Y_6 + 11Y_1Y_6^2 - Y_6^3) \end{pmatrix},$$

$$Y^{(6)}_{\mathbf{5},1} = [Y^{(1)}_{\hat{\mathbf{6}}} \otimes Y^{(5)}_{\hat{\mathbf{4}}}]_{\mathbf{5},2} = \frac{\sqrt{10}}{8}(Y_1^2 - 4Y_1Y_6 - Y_6^2)\begin{pmatrix} \sqrt{3}(Y_1 - 3Y_6)(3Y_1 + Y_6)(Y_1^2 + Y_6^2) \\ -2Y_2(2Y_1 + Y_6)(2Y_1^2 - 3Y_1Y_6 - Y_6^2) \\ \sqrt{2}Y_3(Y_1^3 + 2Y_1^2Y_6 - 11Y_1Y_6^2 - 4Y_6^3) \\ \sqrt{2}Y_4(4Y_1^3 - 11Y_1^2Y_6 - 2Y_1Y_6^2 + Y_6^3) \\ 2Y_5(Y_1 - 2Y_6)(Y_1^2 - 3Y_1Y_6 - 2Y_6^2) \end{pmatrix},$$

$$Y^{(6)}_{\mathbf{5},2} = [Y^{(1)}_{\hat{\mathbf{6}}} \otimes Y^{(5)}_{\hat{\mathbf{6}},1}]_{\mathbf{5}_{\rm s}} = -\frac{\sqrt{2}}{2}(Y_1^4 - 3Y_1^3Y_6 - Y_1^2Y_6^2 + 3Y_1Y_6^3 + Y_6^4)\begin{pmatrix} \sqrt{2}(Y_1^2 + Y_6^2) \\ 2\sqrt{6}Y_2(2Y_1 + Y_6) \\ -\sqrt{3}Y_3(3Y_1 - Y_6) \\ \sqrt{3}Y_4(Y_1 + 3Y_6) \\ -2\sqrt{6}Y_5(Y_1 - 2Y_6) \end{pmatrix}.$$

Although only part of the above modular forms will be implemented to build the concrete models of neutrino masses and flavor mixing in the present paper, a complete list of them up to the weight $k = 6$ should be useful for future works.

III. MINIMAL SEESAW MODEL

A. Simple viable scenarios

As a practical application of the modular A_5' group explored in the previous section, we consider the MSM of neutrino masses and lepton flavor mixing, in which two right-handed neutrino singlets can be just assigned into the two-dimensional irreducible representation of A_5'. As we have mentioned, two-dimensional representations do not exist for the original A_5 group. Therefore, the MSM with two right-handed neutrinos is a well-motivated and economical scenario for the model building with the modular A_5' group.

Together with two right-handed neutrino singlets assigned into $\hat{\mathbf{2}}'$ of the modular A_5' group, the charge assignments of other chiral superfields under the $\mathrm{SU}(2)_{\rm L}$ gauge symmetry and the flavor A_5' symmetry in our model have been summarized in Table II. After the weights and representations of all the superfields under A_5' are fixed, it is easy to determine the Yukawa couplings that have to be the modular forms of weights $k_Y = k_{I_1} + k_{I_2} + \cdots + k_{I_n}$, where k_{I_i} (for $i = 1, 2, \ldots, n$) are the weights of the superfields involved. As for lepton masses and flavor mixing, the gauge- and modular-invariant superpotentials read

$$\begin{aligned} \mathcal{W}_l &= +\gamma_1[(\hat{L}\hat{E}_1^{\rm C})_{\mathbf{3}}Y^{(2)}_{\mathbf{3}}]_{\mathbf{1}}\hat{H}_{\rm d} + \gamma_2[(\hat{L}\hat{E}_2^{\rm C})_{\mathbf{3}}Y^{(4)}_{\mathbf{3}}]_{\mathbf{1}}\hat{H}_{\rm d} \\ &\quad + \gamma_3[(\hat{L}\hat{E}_3^{\rm C})_{\mathbf{3}}Y^{(6)}_{\mathbf{3},1}]_{\mathbf{1}}\hat{H}_{\rm d} + \gamma_4[(\hat{L}\hat{E}_3^{\rm C})_{\mathbf{3}}Y^{(6)}_{\mathbf{3},2}]_{\mathbf{1}}\hat{H}_{\rm d}, \\ \mathcal{W}_{\rm D} &= g[(\hat{L}\hat{N}^{\rm C})_{\hat{\mathbf{6}}}Y^{(1)}_{\hat{\mathbf{6}}}]_{\mathbf{1}}\hat{H}_{\rm u}, \\ \mathcal{W}_{\rm R} &= \frac{1}{2}\Lambda_1[(\hat{N}^{\rm C}\hat{N}^{\rm C})_{\mathbf{3}'}Y^{(6)}_{\mathbf{3}',1}]_{\mathbf{1}} + \frac{1}{2}\Lambda_2[(\hat{N}^{\rm C}\hat{N}^{\rm C})_{\mathbf{3}'}Y^{(6)}_{\mathbf{3}',2}]_{\mathbf{1}}. \end{aligned} \tag{3.1}$$

Without loss of generality, it is always possible to render γ_i (for $i = 1, 2, 3$), g, and Λ_1 to be real by redefining the unphysical phases of lepton fields. Hence, we are left with two complex parameters $\gamma_4/\gamma_3 \equiv \tilde{\gamma} = \gamma e^{{\rm i}\varphi_\gamma}$ and $\Lambda_2/\Lambda_1 \equiv \tilde{\Lambda} = \Lambda e^{{\rm i}\varphi_\Lambda}$. After implementing the Kronecker product rules in Appendix B, we obtain the charged-lepton mass matrix M_l, the Dirac neutrino mass matrix $M_{\rm D}$, and Majorana neutrino mass matrix $M_{\rm R}$, i.e.,

$$\begin{aligned} M_l &= \frac{v_{\rm d}}{\sqrt{2}}\begin{pmatrix} (Y^{(2)}_{\mathbf{3}})_1 & (Y^{(4)}_{\mathbf{3}})_1 & (Y^{(6)}_{\mathbf{3},1})_1 + \tilde{\gamma}(Y^{(6)}_{\mathbf{3},2})_1 \\ (Y^{(2)}_{\mathbf{3}})_3 & (Y^{(4)}_{\mathbf{3}})_3 & (Y^{(6)}_{\mathbf{3},1})_3 + \tilde{\gamma}(Y^{(6)}_{\mathbf{3},2})_3 \\ (Y^{(2)}_{\mathbf{3}})_2 & (Y^{(4)}_{\mathbf{3}})_2 & (Y^{(6)}_{\mathbf{3},1})_2 + \tilde{\gamma}(Y^{(6)}_{\mathbf{3},2})_2 \end{pmatrix}^* \cdot \begin{pmatrix} \gamma_1 & 0 & 0 \\ 0 & \gamma_2 & 0 \\ 0 & 0 & \gamma_3 \end{pmatrix}, \\ M_{\rm D} &= \frac{g v_{\rm u}}{\sqrt{2}}\begin{pmatrix} (Y^{(1)}_{\hat{\mathbf{6}}})_5 & (Y^{(1)}_{\hat{\mathbf{6}}})_2 \\ -(Y^{(1)}_{\hat{\mathbf{6}}})_4 & -(Y^{(1)}_{\hat{\mathbf{6}}})_1 \\ -(Y^{(1)}_{\hat{\mathbf{6}}})_6 & (Y^{(1)}_{\hat{\mathbf{6}}})_3 \end{pmatrix}^*, \\ M_{\rm R} &= \Lambda_1\begin{pmatrix} -\sqrt{2}[(Y^{(6)}_{\mathbf{3}',1})_3 + \tilde{\Lambda}(Y^{(6)}_{\mathbf{3}',2})_3] & (Y^{(6)}_{\mathbf{3}',1})_1 + \tilde{\Lambda}(Y^{(6)}_{\mathbf{3}',2})_1 \\ (Y^{(6)}_{\mathbf{3}',1})_1 + \tilde{\Lambda}(Y^{(6)}_{\mathbf{3}',2})_1 & \sqrt{2}[(Y^{(6)}_{\mathbf{3}',1})_2 + \tilde{\Lambda}(Y^{(6)}_{\mathbf{3}',2})_2] \end{pmatrix}^*, \end{aligned} \tag{3.2}$$

TABLE II. Summary of the charge assignments of the chiral superfields under the $\mathrm{SU}(2)_\mathrm{L}$ gauge symmetry and the modular A_5' symmetry in our model, where the corresponding weights have been listed in the last row.

	$\hat{L}$	$\hat{E}_1^\mathrm{C}$	$\hat{E}_2^\mathrm{C}$	$\hat{E}_3^\mathrm{C}$	$\hat{N}^\mathrm{C}$	$\hat{H}_\mathrm{u}$, $\hat{H}_\mathrm{d}$
$\mathrm{SU}(2)_L$	2	1	1	1	1	2
A_5'	3	1	1	1	$\hat{\mathbf{2}}'$	1
k_I	−2	4	6	8	3	0

where the asterisk "*" indicates the complex conjugation, $(Y_\mathbf{r}^{(k)})_i$ denotes the ith component of $Y_\mathbf{r}^{(k)}$, and v_u and v_d stand for the VEVs of the up- and down-type Higgs fields, respectively, in the minimal supersymmetric standard model. Once M_D and M_R are known, we shall be able to get the neutrino mass matrix M_ν via the seesaw formula $M_\nu = -M_\mathrm{D} M_\mathrm{R}^{-1} M_\mathrm{D}^\mathrm{T}$. With both M_l and M_ν, one can diagonalize them and get lepton masses and flavor mixing matrix.

First, we carry out a detailed numerical analysis of our model, for which a parameter counting should be helpful. In addition to the real and imaginary parts $\{\mathrm{Re}\tau, \mathrm{Im}\tau\}$ of the modulus τ, there are five parameters in the charged-lepton sector (i.e., $\mu_l \equiv v_\mathrm{d}\gamma_3/\sqrt{2}$, $b_1 \equiv \gamma_1/\gamma_3$, $b_2 \equiv \gamma_2/\gamma_3$, and $\tilde{\gamma} \equiv \gamma e^{\mathrm{i}\varphi_\gamma}$) and three parameters in the neutrino sector [i.e., $\mu_\nu \equiv g^2 v_\mathrm{u}^2/(2\Lambda_1)$ and $\tilde{\Lambda} \equiv \Lambda e^{\mathrm{i}\varphi_\Lambda}$]. Totally, we have ten real model parameters. The most general case, where all the ten parameters are taken into account, will be numerically studied. Then, we assume two of the free parameters to be zero and investigate two such special scenarios. The strategy for our numerical analysis is quite simple. Upon scanning over the parameter space of the model parameters, we can compute the low-energy observables, namely, two neutrino mass-squared differences $\{\Delta m_{21}^2, \Delta m_{31}^2 \,\mathrm{or}\, \Delta m_{32}^2\}$ and three flavor mixing angles $\{\theta_{12}, \theta_{13}, \theta_{23}\}$, which are then confronted with their allowed ranges at the 1σ or 3σ level according to the global-fit results from NuFIT 5.0 [3,95] without including the atmospheric neutrino data from SuperKamiokande, as summarized in Table I. Then the CP-violating phases and effective neutrino masses for beta decays and neutrinoless double-beta decays are calculated as theoretical predictions. More details of our numerical calculations and comments on the final results can be found below.

(i) The values of τ (i.e., both real and imaginary parts) are randomly generated from the region

$$\mathcal{G}_\mathrm{R} = \{\tau \in \mathbb{C} \colon \mathrm{Im}\tau > 0, 0 \leq \mathrm{Re}\tau \leq 0.5, |\tau| \geq 1\}, \tag{3.3}$$

which is the right-half part of the fundamental domain $\mathcal{G}$. As for the conjugate part with $-0.5 \leq \mathrm{Re}\tau \leq 0$, the corresponding results can be obtained by reversing the signs of all the phases. Moreover, γ and φ_γ are chosen from the regions $\gamma \in [10^{-4}, 10^4]$ and $\varphi_\gamma \in [0, 2\pi)$, respectively. Then three real parameters μ_l, b_1, and b_2 in the charged-lepton sector can be numerically determined via the following identities:

$$\mathrm{Tr}(M_l M_l^\dagger) = m_e^2 + m_\mu^2 + m_\tau^2, \tag{3.4}$$

$$\mathrm{Det}(M_l M_l^\dagger) = m_e^2 m_\mu^2 m_\tau^2, \tag{3.5}$$

$$\frac{1}{2}[\mathrm{Tr}(M_l M_l^\dagger)]^2 - \frac{1}{2}\mathrm{Tr}[(M_l M_l^\dagger)^2] = m_e^2 m_\mu^2 + m_\mu^2 m_\tau^2 + m_\tau^2 m_e^2, \tag{3.6}$$

where the running charged-lepton masses $m_e = 0.48307$ MeV, $m_\mu = 0.101766$ GeV, and $m_\tau = 1.72856$ GeV are evaluated at the electroweak scale characterized by $m_Z = 91.2$ GeV [96].[1] After so doing, the charged-lepton mass matrix M_l is also fixed, and the unitary matrix U_l used to diagonalize it via $U_l^\dagger M_l M_l^\dagger U_l = \mathrm{Diag}\{m_e^2, m_\mu^2, m_\tau^2\}$ can be obtained.

(ii) In the neutrino sector, we randomly generate the values of Λ and φ_Λ in the regions $\Lambda \in [10^{-4}, 10^4]$ and $\varphi_\Lambda \in [0, 2\pi)$, respectively. Given the modulus parameter τ in the previous step, the neutrino mass matrix M_ν can be determined up to the overall factor μ_ν. Therefore, we can calculate the ratio $r \equiv \sqrt{\Delta m_{21}^2/\Delta m_{31}^2} = m_2/m_3$ in the case of normal mass ordering (NO) with $m_1 = 0 < m_2 < m_3$ or $r \equiv \sqrt{\Delta m_{21}^2/|\Delta m_{32}^2|} = \sqrt{1 - m_1^2/m_2^2}$ in the case of inverted mass ordering (IO) with $m_3 = 0 < m_1 < m_2$. This ratio is independent of the overall neutrino mass scale μ_ν, and so is the unitary matrix U_ν diagonalizing the neutrino mass matrix M_ν via $U_\nu^\dagger M_\nu M_\nu^\dagger U_\nu = \mathrm{Diag}\{m_1^2, m_2^2, m_3^2\}$. The lepton flavor mixing matrix U, or the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix [97,98], is thus given by $U = U_l^\dagger U_\nu$. In the standard parametrization of the PMNS matrix [9], we have

[1]Although the modular symmetry is usually supposed to work at some high-energy scale, the renormalization-group running effects can be safely neglected in our model for two reasons. First, neutrino masses cannot be nearly degenerate in the MSM, where the lightest neutrino turns out to be massless. Second, a sufficiently small value of $\tan\beta = v_\mathrm{u}/v_\mathrm{d}$ is assumed such that the charged-lepton Yukawa couplings are highly suppressed.

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-\mathrm{i}\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{\mathrm{i}\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{\mathrm{i}\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{\mathrm{i}\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{\mathrm{i}\delta} & c_{23}c_{13} \end{pmatrix} \cdot \begin{pmatrix} 1 & & \\ & e^{\mathrm{i}\sigma} & \\ & & 1 \end{pmatrix}, \tag{3.7}$$

where $c_{ij} \equiv \cos\theta_{ij}$ and $s_{ij} \equiv \sin\theta_{ij}$ (for $ij = 12$, 13, 23) have been defined. Note that, in the MSM, the lightest neutrino is massless, and, thus, only one Majorana CP-violating phase σ is physical. Comparing the obtained values of r and $\{\sin^2\theta_{12}, \sin^2\theta_{13}, \sin^2\theta_{23}\}$ with their allowed 3σ (or 1σ) ranges from global-fit results, we find out the parameter space of our model that is compatible with experimental data at the 3σ (or 1σ) level. The overall factor μ_ν can be pinned down by further reproducing the correct values of Δm^2_{21} or $|\Delta m^2_{31}|$, but it is not relevant for our discussions about lepton flavor mixing and CP violation.

(iii) In order to determine the best-fit values of free parameters in our model, we construct the χ^2 function as the sum of one-dimensional functions χ^2_j, namely,

$$\chi^2(p_i) = \sum_j \chi^2_j(p_i), \tag{3.8}$$

where $p_i \in \{\mathrm{Re}\tau, \mathrm{Im}\tau, \gamma, \varphi_\gamma, \Lambda, \varphi_\Lambda\}$ stand for the model parameters and j is summed over the observables $\{\sin^2\theta_{12}, \sin^2\theta_{13}, \sin^2\theta_{23}, r\}$. For $\sin^2\theta_{12}$, $\sin^2\theta_{13}$, and r, we simply take the Gaussian approximations and assume

$$\chi^2_j(p_i) = \left(\frac{q_j(p_i) - q_j^{\mathrm{bf}}}{\sigma_j}\right)^2, \tag{3.9}$$

where $q_j(p_i)$ denote the model predictions for these observables and q_j^{bf} are their best-fit values from the global analysis in Ref. [3]. The associated uncertainties σ_j are derived by symmetrizing 1σ uncertainties from the global-fit analysis and have already been given in Table I. For $\sin^2\theta_{23}$, we use the one-dimensional projection of the χ^2 function provided by Refs. [3,95]. By minimizing the overall χ^2 function in Eq. (3.8), we can determine the best-fit values of our model parameters.

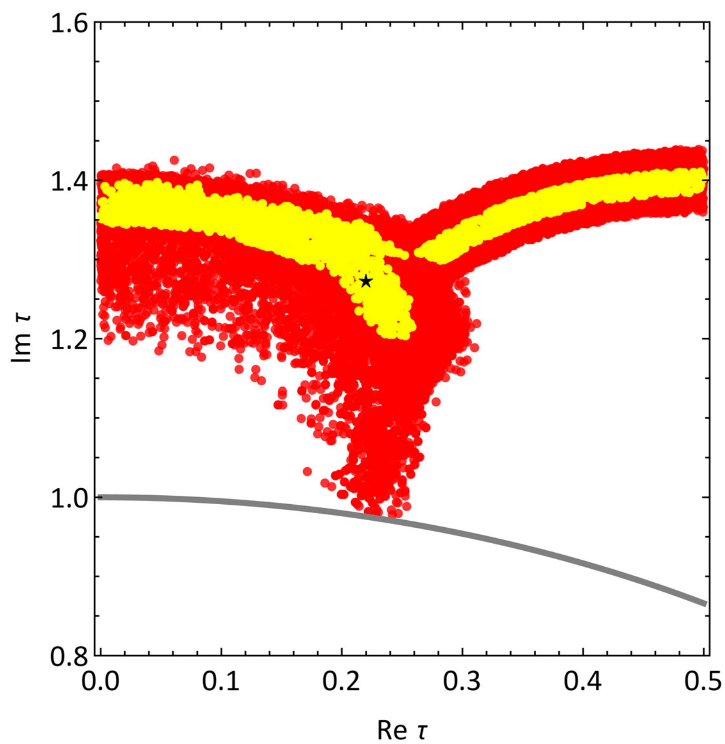

FIG. 1. The allowed parameter space of the model parameters $\{\mathrm{Re}\tau, \mathrm{Im}\tau\}$ in the most general case with ten real parameters, where the 1σ (yellow dots) and 3σ (red dots) ranges of flavor mixing angles and neutrino mass-squared differences from the global-fit analysis of neutrino oscillation data have been input [3]. The best-fit values of $\{\mathrm{Re}\tau, \mathrm{Im}\tau\}$ from the χ^2-fit analysis are denoted by the black star. In addition, the lower boundary of $\mathcal{G}_{\mathrm{R}}$ is plotted as the gray curve.

In the most general case, where all ten real parameters are included, our model is compatible with the global-fit results of oscillation parameters at the 1σ level in the NO case, while the IO case is excluded at the 3σ level. The allowed parameter space of $\{\mathrm{Re}\tau, \mathrm{Im}\tau\}$ in the NO case is shown in Fig. 1, where one can observe that the whole range [0, 0.5] of $\mathrm{Re}\tau$ is allowed, while $\mathrm{Im}\tau$ can vary from 0.98 to 1.44 at the 3σ level. In Table III, we summarize the best-fit values, as well as the 1σ and 3σ ranges, of the free parameters and the allowed ranges of low-energy observables. Notice that the parameter γ can vary in a broad range from 0 to 1293. It is interesting to see that γ_3 could be much smaller than γ_1 and γ_2, so the contribution to the flavor mixing from the third column of M_l can be negligible when compared to those from the first two columns. On the other hand, Table III reveals that $\tilde{\gamma}$ or $\tilde{\Lambda}$ could even be zero, providing the possibility to reduce the number of free model parameters from ten to eight.

Next, motivated by the above observations, we now consider two special cases, namely, case A with $\tilde{\Lambda} = 0$ and case B with $\tilde{\gamma} = 0$.

(i) In case A with $\tilde{\Lambda} = 0$, only $Y^{(6)}_{\mathbf{3}',1}$ is retained in the right-handed neutrino mass matrix $M_{\rm R}$. We find that this scenario is consistent with the global-fit analysis of neutrino oscillation data at the 3σ level in the NO case. The allowed parameter space and the constraint on low-energy observables are shown in Fig. 2. From Fig. 2, we observe that the allowed parameter space of τ is restricted to the region of $0.12 \lesssim \mathrm{Re}\tau \lesssim 0.27$ and $1.326 \lesssim \mathrm{Im}\tau \lesssim 1.352$. In addition, $1/\gamma$ varies from 0.045 to 0.33, or equivalently $3 \lesssim \gamma \lesssim 22$, and $115° \lesssim \phi_\gamma \lesssim 235°$ is obtained. There exist strong correlations among the low-energy observables. As can be seen from the top-right panel, the allowed range of δ decreases as the value of θ_{13} becomes smaller. In particular, when θ_{13} is as small as 8.2°, the value of δ is tightly restricted to be around 257°. From the bottom-left panel, we see the correlation between θ_{12} and θ_{23}, indicating that relatively large values of θ_{12} and small values of θ_{23} are predicted in case A. Next-generation neutrino oscillation experiments, e.g., JUNO [99], Hyper-Kamiokande [100], and DUNE [101], will unambiguously pin down the neutrino mass ordering and determine the octant of θ_{23}, so case A will be hopefully confirmed or ruled out in the near future. Notice that $m_1 = 0$ in the NO case, so the absolute neutrino masses can be immediately determined from neutrino mass-squared differences, namely, $m_2 = \sqrt{\Delta m^2_{21}}$ and $m_3 = \sqrt{\Delta m^2_{31}}$. The value of m_2 cannot reach its upper bound of the 3σ allowed range from neutrino oscillation data. For the maximum $m_2 = 8.81$ meV, the effective mass for beta decays $m_\beta \equiv \sqrt{m_1^2|U_{e1}|^2 + m_2^2|U_{e2}|^2 + m_3^2|U_{e3}|^2}$ is constrained to be 9.45 meV. Meanwhile, the effective mass for neutrinoless double-beta decays $m_{\beta\beta} \equiv |m_1 U^2_{e1} + m_2 U^2_{e2} + m_3 U^2_{e3}|$ is lying in the range $(1.35\ldots3.86)$ meV, which will be a great challenge for future neutrinoless double-beta decay experiments [102,103].

TABLE III. Summary of the best-fit values with $\chi^2_{\rm min} = 0.0153$, together with the 1 σ and 3 σ ranges, of all the free model parameters in the NO case in our model with ten real parameters. The allowed ranges of the low-energy observables are also given.

		Best fit	1σ range	3σ range
Free model parameters	$\mathrm{Re}\tau$	0.2201	0–0.5	0–0.5
	$\mathrm{Im}\tau$	1.272	1.204–1.410	0.978–1.439
	γ	13.51	0–570	0–1293
	$\varphi_\gamma/°$	346.8	0–360	0–360
	Λ	5.606	1.061–17.17	0–19.90
	$\varphi_\Lambda/°$	209.3	82.47–317.0	0–360
	μ_l/MeV	0.1369	0.1158–150.5	5.163×10^{-2}–1.789×10^2
	b_1	2413	9.243×10^{-3}–2.771×10^3	2.174×10^{-3}–5.301×10^3
	b_2	85.60	1.235×10^{-3}–1.361×10^2	5.210×10^{-4}–5.713×10^2
	μ_ν/meV	26.60	24.11–48.93	23.68–87.16
Observables	m_2/meV	8.614	8.497–8.735	8.258–8.967
	m_3/meV	50.19	49.87–50.42	49.30–50.98
	$\theta_{12}/°$	33.51	32.70–34.21	31.27–35.86
	$\theta_{13}/°$	8.582	8.45–8.70	8.20–8.97
	$\theta_{23}/°$	49.02	47.6–50.1	39.6–51.8
	$\delta/°$	359.6	0–360	0–360
	$\sigma/°$	128.2	0–180	0–180
	m_β/meV	8.816	8.711–8.943	8.259–9.445
	$m_{\beta\beta}$/meV	2.404	1.428–3.676	1.043–4.167
	$\chi^2_{\rm min}$	0.0153	...	...

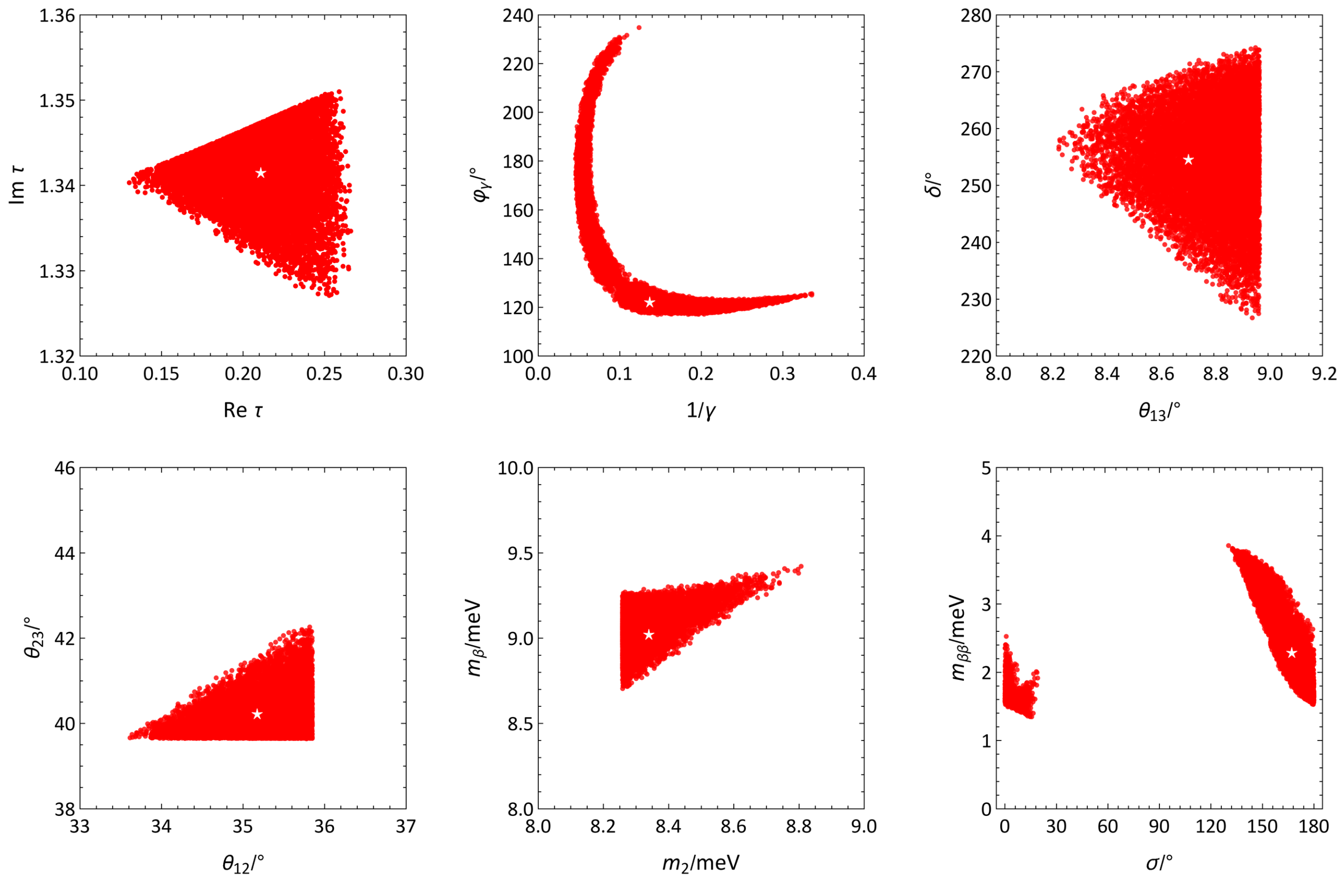

FIG. 2. The allowed parameter space of model parameters, as well as the constraints on the low-energy observables in case A with $\tilde{\Lambda} = 0$, where the 3σ ranges of flavor mixing angles and neutrino mass-squared differences from the global-fit analysis of neutrino oscillation data have been input [3]. The best-fit values from the χ^2-fit analysis are denoted by the white stars.

Based on the χ^2-fit analysis, we find that the minimum $\chi^2_{\rm min} = 15.99$ is obtained in the NO case with the following best-fit values of the model parameters:

$$\begin{aligned} {\rm Re}\tau &= 0.2110, \qquad {\rm Im}\tau = 1.341, \\ \gamma &= 7.294, \qquad \varphi_\gamma = 121.9°, \end{aligned} \tag{3.10}$$

which together with the charged-lepton masses lead to $\mu_l = 0.03639$ GeV, $b_1 = 0.04809$, and $b_2 = 0.3528$. The overall factor μ_ν in the neutrino sector turns out to be 27.11 meV. Given the best-fit values of model parameters, we get the neutrino mass spectrum $m_1 = 0$, $m_2 = 8.340$ meV, and $m_3 = 50.65$ meV, three mixing angles $\theta_{12} = 35.17°$, $\theta_{13} = 8.707°$, and $\theta_{23} = 40.21°$, and two *CP*-violating phases $\delta = 254.5°$ and $\sigma = 167.2°$. Meanwhile, the model predictions for the effective neutrino masses m_β and $m_{\beta\beta}$ are found to be 9.020 and 2.282 meV, respectively.

Furthermore, a brief illustration on why the IO case is excluded in case A is helpful. In fact, the observed values of Δm^2_{21} and Δm^2_{32} impose strong constraints on the parameter space of τ in the IO case. To be specific, if both Δm^2_{21} and Δm^2_{32} are within their individual 3σ ranges from global-fit results, the allowed range of $\{{\rm Re}\tau, {\rm Im}\tau\}$ is restricted to be in a small ring centered on $\tau = 1/2 + {\rm i}\sqrt{3}/2$, where the predicted values of θ_{13} are found to be around either 0 or 90°. Therefore, the IO case is not compatible with the neutrino oscillation data.

(ii) In case B with $\tilde{\gamma} = 0$, the total number of model parameters is also reduced to be eight. Unlike case A, we find that case B is compatible with neutrino oscillation data even at the 1σ level in the NO case. The allowed ranges of model parameters as well as the constrained ranges of low-energy observables are shown in Fig. 3. As one can see from the top-left panel, there are two separated parts in the 3σ allowed parameter space of $\{{\rm Re}\tau, {\rm Im}\tau\}$, corresponding to two distinct allowed hierarchies with $b_1 > b_2 > 1$ and $b_1 > 1 > b_2$ in the charged-lepton sector, respectively. While at the 1σ level, only one narrow region remains in the allowed parameter

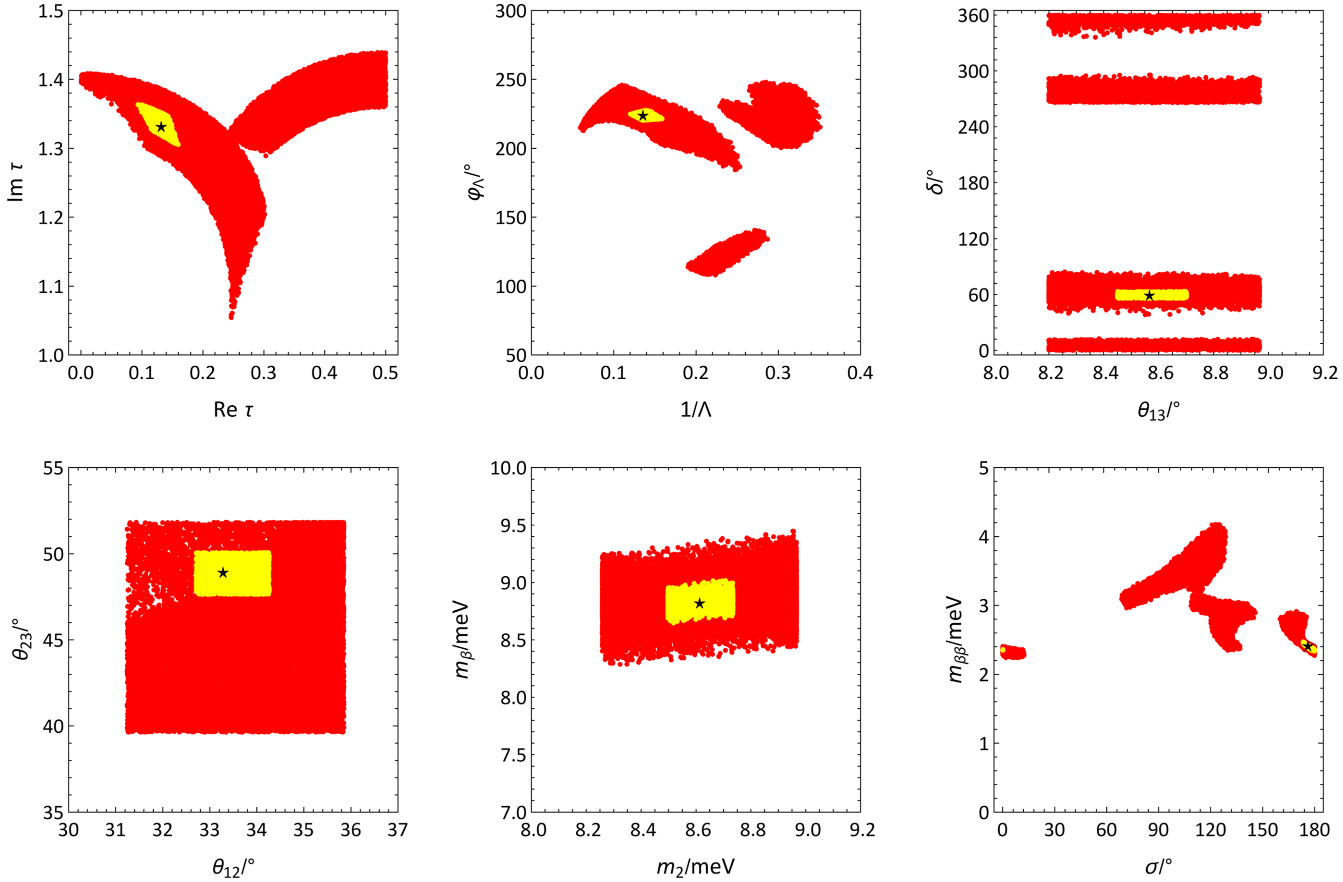

FIG. 3. The allowed parameter space of model parameters, as well as the constraints on the low-energy observables in case B with $\tilde{\gamma} = 0$, where the 1σ (yellow dots) and 3σ (red dots) ranges of flavor mixing angles and neutrino mass-squared differences from the global-fit analysis of neutrino oscillation data have been input [3]. The best-fit values from the χ^2-fit analysis are denoted by the black stars.

space of $\{\mathrm{Re}\tau, \mathrm{Im}\tau\}$, where $0.1 \lesssim \mathrm{Re}\tau \lesssim 0.16$ and $1.31 \lesssim \mathrm{Im}\tau \lesssim 1.36$. Similarly, there appear three separated regions in the 3σ allowed parameter space of $\{\Lambda^{-1}, \phi_\Lambda\}$ in the top-middle panel in Fig. 3. However, as has been mentioned before, under the transformations $\mathrm{Re}\tau \to -\mathrm{Re}\tau$ and $\phi_\Lambda \to 2\pi - \phi_\Lambda$, the predictions for low-energy observables remain unchanged, except that the signs of all CP-violating phases are reversed. Therefore, the rightmost two parts in the top-middle panel seem to be connected to each other by the transformation $\phi_\Lambda \to 2\pi - \phi_\Lambda$. In the top-right panel, the allowed values of δ are lying in several separated regions, probably originating from different regions in the allowed parameter space of $\{\Lambda^{-1}, \varphi_\Lambda\}$. In case B, the predicted values of three mixing angles can essentially saturate their individual 3σ ranges allowed by neutrino oscillation data, and there is no significant correlation among them. The 3σ allowed ranges of m_β and $m_{\beta\beta}$ are larger than those in case A, which is mainly due to the fact that the allowed range of θ_{12} becomes larger in case B. Performing the χ^2-fit analysis in case B, we find that the minimum $\chi^2_{\rm min} = 0.0741$ is achieved at

$$\mathrm{Re}\tau = 0.1320, \qquad \mathrm{Im}\tau = 1.331,$$
$$\Lambda = 7.362, \qquad \varphi_\Lambda = 223.4°, \tag{3.11}$$

which together with the charged-lepton masses lead to $\mu_l = 1.853 \times 10^{-4}$ GeV, $b_1 = 1.872 \times 10^3$, $b_2 = 64.30$, and $\mu_\nu = 29.47$ meV. Furthermore, we get the neutrino mass spectrum $m_1 = 0$, $m_2 = 8.625$ meV, and $m_3 = 50.18$ meV, three mixing angles $\theta_{12} = 33.28°$, $\theta_{13} = 8.567°$, and $\theta_{23} = 48.88°$, and two CP-violating phases $\delta = 58.67°$ and $\sigma = 176.1°$. Meanwhile, the predictions for two effective neutrino masses m_β and $m_{\beta\beta}$ are 8.827 and 2.404 meV, respectively. The remarkable difference between case A and case B is their predictions for the octant of θ_{23}. Case B shows no clear preference for the octant of θ_{23}, while case A prefers the first octant. If the present hint for the second octant of θ_{23} is confirmed by future neutrino oscillation data, case A will be definitely ruled out. In addition, the precision measurement of θ_{12} and the determination of the CP-violating phase δ will also shed some light on the discrimination between these two simple but viable scenarios.

In summary, by a detailed numerical analysis, we have demonstrated that our model with ten real parameters is well compatible with current neutrino oscillation data. Even more, if the complex parameter $\tilde{\Lambda}$ in case A or $\tilde{\gamma}$ in case B is set to zero, the model is still allowed by experimental observations.

B. Analytical results

It is worthwhile to notice that there are no additional free parameters other than the complex modulus τ in the neutrino sector in case A with $\tilde{\Lambda} = 0$. As a result, the effective Majorana neutrino mass matrix M_ν in this case turns out to be simple enough, rendering analytical calculations under some reasonable approximations to be possible. Analytical calculations, though approximate, will be helpful for understanding the flavor structures of lepton mass matrices.

First, as we have seen from the numerical calculations in the previous section, the allowed values of $\mathrm{Im}\tau$ are located in a very narrow region $1.326 \lesssim \mathrm{Im}\tau \lesssim 1.352$, for which $|q| = e^{-2\pi \mathrm{Im}\tau}$ is small enough, and, thus, it is safe to retain only the leading-order terms in the q expansions of all the modular forms. For this purpose, we introduce two auxiliary real parameters

$$x \equiv \exp\left(-\frac{2\pi}{5}\mathrm{Im}\tau\right), \qquad y \equiv \frac{2\pi}{5}\mathrm{Re}\tau, \tag{3.12}$$

which are actually the modulus and argument of $q^{1/5}$ (i.e., $q^{1/5} \equiv xe^{\mathrm{i}y}$), and six basis vectors given in Eq. (2.16) approximate to

$$\hat{e}_1 \approx 1, \qquad \hat{e}_2 \approx xe^{\mathrm{i}y}, \qquad \hat{e}_3 \approx x^2 e^{2\mathrm{i}y}, \qquad \hat{e}_4 \approx x^3 e^{3\mathrm{i}y}, \qquad \hat{e}_5 \approx x^4 e^{4\mathrm{i}y}, \qquad \hat{e}_6 \approx x^5 e^{5\mathrm{i}y}, \tag{3.13}$$

where x serves as the expansion parameter. To have a ballpark feeling about the size of x, we take the best-fit value of $\mathrm{Im}\tau = 1.341$, as found in Eq. (3.10), and then obtain $x \approx 0.185$ from Eq. (3.12), which is not perfect but reasonably good for perturbation calculations.

Then, applying the approximate expressions in Eq. (3.13) to M_D and M_R in Eq. (3.2), one can get the analytical form of the effective neutrino mass matrix via $M_\nu = -M_\mathrm{D} M_\mathrm{R}^{-1} M_\mathrm{D}^\mathrm{T}$, i.e.,

$$M_\nu \approx \mu_\nu \left[\frac{5\sqrt{3}}{96}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} + \frac{\sqrt{3}e^{3\mathrm{i}y}}{480x^3}\begin{pmatrix} 0 & -5\sqrt{2}x^2 e^{-2\mathrm{i}y} & 80\sqrt{2}x^4 e^{-4\mathrm{i}y} \\ -5\sqrt{2}x^2 e^{-2\mathrm{i}y} & xe^{-\mathrm{i}y} & 0 \\ 80\sqrt{2}x^4 e^{-4\mathrm{i}y} & 0 & 3 \end{pmatrix}\right], \tag{3.14}$$

where μ_ν denotes the absolute scale of neutrino masses and only the terms up to $\mathcal{O}(x^4)$ in the second matrix in the square brackets are kept. Some comments on the general structure of M_ν are helpful. In Eq. (3.14), we have intentionally divided the neutrino mass matrix into two parts, the first of which is constant. Considering the absolute value of the ratio of the coefficient in front of the first matrix on the right-hand side of Eq. (3.14) to that of the second matrix, namely, $25x^3 \approx 0.158$, we can see that both of these two parts make remarkable contributions to M_ν. However, one can observe from Eq. (3.14) that only the (3,3) element of M_ν survives at the leading order, whose absolute value is approximately the largest mass eigenvalue m_3 in the NO case. On the other hand, the lightest neutrino must be massless in the MSM, i.e., $m_1 = 0$. Therefore, it is straightforward to derive three neutrino mass eigenvalues:

$$m_1 = 0, \quad m_2 \approx (1 + 50x^2)\frac{\sqrt{3}}{480x^2}\mu_\nu, \quad m_3 \approx \frac{\sqrt{3}}{160x^3}\mu_\nu, \tag{3.15}$$

from which one can determine the neutrino mass ratio

$$r \equiv \frac{m_2}{m_3} \approx \frac{x}{3}(1 + 50x^2). \tag{3.16}$$

When setting $x \approx 0.185$ from the best-fit value of $\mathrm{Im}\tau = 1.341$, we find that the prediction from Eq. (3.16) is $r \approx 0.167$, which is in excellent agreement with the best-fit value $r \approx 0.165$.

As the symmetric and complex neutrino mass matrix is diagonalized by the unitary matrix U_ν via $U_\nu^\dagger M_\nu U_\nu^* = \mathrm{Diag}\{m_1, m_2, m_3\}$, we can also obtain

$$U_\nu \approx \begin{pmatrix} e^{-3iy/2} & 0 & 0 \\ 0 & e^{-iy/2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta_\nu & -\sin\theta_\nu & 0 \\ +\sin\theta_\nu & \cos\theta_\nu & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} e^{-5iy/2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.17}$$

where the rotation angle θ_ν is determined by $\tan\theta_\nu = 5\sqrt{2}x$. Given $x \approx 0.185$, one can estimate $\theta_\nu = \arctan(5\sqrt{2}x) \approx 52.60°$, which is quite large. Furthermore, with the help of Eq. (3.16), we can immediately establish a rather simple relation between the rotation angle and the neutrino mass ratio $m_2/m_3 \approx \tan\theta_\nu \sec^2\theta_\nu/(15\sqrt{2})$. We have numerically checked that the approximate analytical result of U_ν in Eq. (3.17) agrees very well with the exact result, and the difference arises from the omission of higher-order terms.

Next, we proceed with the charged-lepton sector. Up to the order of $\mathcal{O}(x^3)$, the charged-lepton mass matrix M_l in Eq. (3.2) can be expressed as

$$M_l \approx \mu_l \begin{pmatrix} -3 & -5\sqrt{3} & 3\sqrt{2}\gamma e^{-i\varphi_\gamma} \\ 0 & 0 & 0 \\ -15\sqrt{2}xe^{-iy} & 5\sqrt{6}xe^{-iy} & 30(3+\gamma e^{-i\varphi_\gamma})xe^{-iy} \end{pmatrix} \cdot \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.18}$$

while the corresponding Hermitian matrix $H_l \equiv M_l M_l^\dagger$ reads

$$H_l \approx \mu_l^2 \begin{pmatrix} 75b_2^2 + 18\gamma^2 & 0 & 15\sqrt{2}[6\gamma(\gamma + 3e^{-i\varphi_\gamma}) - 5b_2^2]xe^{iy} \\ 0 & 0 & 0 \\ 15\sqrt{2}[6\gamma(\gamma + 3e^{i\varphi_\gamma}) - 5b_2^2]xe^{-iy} & 0 & 150x^2[b_2^2 + 6(\gamma^2 + 6\gamma\cos\varphi_\gamma + 9)] \end{pmatrix}, \tag{3.19}$$

where b_1^2 has been set to be zero because of the strong hierarchy $b_1^2 \ll b_2^2 \ll 1$ as indicated by numerical calculations in the previous section. If $b_2^2 = 0$ is further assumed, then one can verify that the masses of the first two generations of charged leptons are vanishing, i.e., $m_e = m_\mu = 0$. Therefore, the terms associated with b_2^2 will be retained. It is worth stressing that, even if b_1^2 is set to be nonzero, the electron mass is still vanishing due to the special structure of M_l in Eq. (3.18). To generate a nonzero electron mass, we have to keep higher-order terms of x in M_l. For illustration, we work with the accuracy of $\mathcal{O}(x^3)$ and, thus, focus on the charged-lepton mass matrix M_l in Eq. (3.18) and accordingly H_l in Eq. (3.19), implying $m_e = 0$. As the Hermitian matrix H_l is diagonalized by the unitary matrix U_l via $U_l^\dagger H_l U_l = \mathrm{Diag}\{m_e^2, m_\mu^2, m_\tau^2\}$, we can obtain

$$U_l = \begin{pmatrix} 0 & -\sin\theta_l e^{i(y-\xi)} & \cos\theta_l \\ 1 & 0 & 0 \\ 0 & \cos\theta_l & \sin\theta_l e^{-i(y-\xi)} \end{pmatrix}, \tag{3.20}$$

where the rotation angle θ_l is determined by $\tan^2\theta_l = 50x^2(\gamma^2 + 6\gamma\cos\varphi_\gamma + 9)/\gamma^2$ and the phase by $\tan\xi = 3\sin\varphi_\gamma/(3\cos\varphi_\gamma + \gamma)$. Given the best-fit values of model parameters, we have $\theta_l \approx 48.17°$ and $\xi \approx 23.16°$. Meanwhile, three charged-lepton mass eigenvalues are found to be $m_e = 0$ and

$$m_\mu^2 \approx 75b_2^2x^2\gamma^{-4}(25 + 20\gamma\cos\varphi_\gamma + 4\gamma^2)m_\tau^2|\cos 2\theta_l|, \qquad m_\tau^2 \approx 18\gamma^2\mu_l^2\sec^2\theta_l, \tag{3.21}$$

where the rotation angle θ_l has been defined below Eq. (3.20). Substituting the best-fit values of relevant parameters in Eq. (3.21), one obtains $m_e = 0$, $m_\mu/\mu_l = 2.044$, and $m_\tau/\mu_l = 47.862$, while the exact numerical results are $(m_e, m_\mu, m_\tau)/\mu_l = (0.013, 2.764, 48.924)$. Bearing in mind that only the leading-order terms are kept in the charged-lepton mass matrix, we can see a reasonably good agreement between analytical and numerical results.

Finally, with the unitary matrices U_ν in Eq. (3.17) and U_l in Eq. (3.20), the PMNS matrix is thus given by

$$U = U_l^\dagger U_\nu \approx \begin{pmatrix} \sin\theta_\nu & \cos\theta_\nu & 0 \\ -\sin\theta_l\cos\theta_\nu & \sin\theta_l\sin\theta_\nu & \cos\theta_l \\ \cos\theta_l\cos\theta_\nu & -\cos\theta_l\sin\theta_\nu & \sin\theta_l \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-i(5y/2-\xi)} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.22}$$

where the unphysical phases have been eliminated by redefining the charged-lepton fields and the neutrino field with mass $m_1 = 0$. From the PMNS matrix in Eq. (3.22), we can extract three flavor mixing angles as below:

$$\sin\theta_{13} \approx 0, \qquad \sin\theta_{12} \approx \cos\theta_\nu, \qquad \sin\theta_{23} \approx \cos\theta_l, \tag{3.23}$$

and the Majorana CP phase $\sigma \approx \xi - 5y/2$. Interestingly, the rotation angle θ_ν from the neutrino sector and θ_l from the charged-lepton sector are directly related to the mixing angle θ_{12} and θ_{23} via the simple relation $\theta_{12} \approx \pi/2 - \theta_\nu$ and $\theta_{23} \approx \pi/2 - \theta_l$, respectively. Substituting the best-fit values of model parameters shown in Eq. (3.10), we find that our approximate analytical results lead to

$$\theta_{13} \approx 0, \qquad \theta_{12} \approx 37.40°, \qquad \theta_{23} \approx 41.83°, \qquad \sigma \approx 165.8°, \tag{3.24}$$

where one can see that the values of θ_{12}, θ_{23}, and σ are in good agreement with their individual best-fit values shown in the paragraph below Eq. (3.10). In addition, recalling the approximate analytical expressions of r and m_τ in Eqs. (3.16) and (3.21), we can see that the free model parameters x and $\gamma\mu_l$ are directly related to the observables by, respectively,

$$x = 3\sqrt{\frac{\Delta m^2_{21}}{\Delta m^2_{31}}\sin^2\theta_{12}}, \qquad \gamma\mu_l = \frac{m_\tau \sin\theta_{23}}{3\sqrt{2}}. \tag{3.25}$$

Before closing this subsection, let us briefly discuss how to generate a nonzero θ_{13}. Note that the unitary matrix U_ν in the neutrino sector in Eq. (3.17) is kept only to $\mathcal{O}(x)$ and, thus, parametrized by a pure (1,2) rotation. If we improve our calculations with the accuracy of $\mathcal{O}(x^3)$, U_ν will be modified with an extra (2,3) rotation, which can be expressed as

$$U_\nu \approx \begin{pmatrix} e^{-4iy} & 0 & 0 \\ 0 & e^{-3iy} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta'_\nu & -\sin\theta'_\nu \\ 0 & +\sin\theta'_\nu & \cos\theta'_\nu \end{pmatrix} \cdot \begin{pmatrix} \cos\theta_\nu & -\sin\theta_\nu & 0 \\ +\sin\theta_\nu & \cos\theta_\nu & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{5iy/2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.26}$$

where the rotation angle θ'_ν is determined by $\sin\theta'_\nu = 25x^3/3$. Adopting the best-fit value of $x \approx 0.185$, we obtain $\theta'_\nu = \arcsin(25x^3/3) \approx 3.023°$, indicating θ'_ν is quite a small angle. Then the modified version of the PMNS matrix reads

$$U = U_l^\dagger U_\nu \approx \begin{pmatrix} \sin\theta_\nu & \cos\theta_\nu & \sin\theta'_\nu e^{-i(5y-\xi+\pi)} \\ -\sin\theta_l\cos\theta_\nu & \sin\theta_l\sin\theta_\nu & \cos\theta_l \\ \cos\theta_l\cos\theta_\nu & -\cos\theta_l\sin\theta_\nu & \sin\theta_l \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-i(5y/2-\xi)} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.27}$$

where one can observe that the (1,3) element of U acquires a small nonzero value proportional to $\sin\theta'_\nu$ while all the other elements remain approximately unchanged. Thanks to this small correction, we can obtain the approximate expressions of θ_{13} and δ as, respectively,

$$\theta_{13} \approx \theta'_\nu, \qquad \delta \approx 5y - \xi + \pi. \tag{3.28}$$

Substituting the best-fit values of model parameters shown in Eq. (3.10) into Eq. (3.28), one can arrive at

$$\theta_{13} \approx 3.023°, \qquad \delta \approx 231.6°, \tag{3.29}$$

where the value of θ_{13} is still much smaller than its best-fit value. However, if we retain the terms of $\mathcal{O}(x^4)$ in M_l in Eq. (3.18) from the very beginning, then a nonzero value of m_e will be obtained and θ_{13} will receive an additional correction from the charged-lepton sector. In fact, we have checked that $\theta_{13} \approx 9.012°$ can indeed be generated at this order of approximations if the best-fit values of model parameters are taken.

IV. SUMMARY

The double covering of modular groups can accommodate the modular forms with odd weights and, thus, provide us with new possibilities to account for tiny neutrino masses, lepton flavor mixing, and CP violation. In this paper, we investigate the basic properties of the double covering of the modular $\Gamma_5 \simeq A_5$ group, i.e., the modular A'_5 group, which has not been studied in the previous literature. As a practical application, we have considered the minimal seesaw model with a modular A'_5 symmetry, in which we numerically explore the allowed parameter space and analytically study the mass spectrum and flavor mixing in the lepton sector with some reasonable assumptions.

The main results are summarized as follows. First of all, we begin with the modular forms of weight one. With the help of the Dedekind eta function and Klein form, we obtain the basis vectors of the modular space $\mathcal{M}_1[\Gamma(5)]$, and the modular forms $Y^{(1)}_{\hat{\mathbf{6}}}$ with weight one turn out to be the linear combination of these basis vectors. Next, we derive the modular forms of weights up to six using the Kronecker product rules of A'_5 and present their explicit expressions. After the modular forms are determined, we proceed to apply the double-covering group A'_5 to the concrete models. There exists a two-dimensional irreducible representation $\hat{\mathbf{2}}'$ in the group A'_5, into which we assign two right-handed neutrino singlets $\hat{N}^{\rm C}$ in the minimal seesaw model, and the charged-lepton doublets $\hat{L}$ and three charged-lepton singlets $\{\hat{E}^{\rm C}_1, \hat{E}^{\rm C}_2, \hat{E}^{\rm C}_3\}$ are assumed to transform as **3** and three one-dimensional representations of A'_5, respectively. In the most general case, the model contains ten real parameters, which are the modulus parameter $\tau = {\rm Re}\tau + {\rm iIm}\tau$ together with the parameters $\mu_l \equiv v_{\rm d}\gamma_3/\sqrt{2}$, $b_1 \equiv \gamma_1/\gamma_3$, $b_2 \equiv \gamma_2/\gamma_3$, and $\tilde{\gamma} \equiv \gamma e^{{\rm i}\varphi_\gamma}$ in the charged-lepton sector and $\mu_\nu \equiv g^2 v_{\rm u}^2/(2\Lambda_1)$ and $\tilde{\Lambda} \equiv \Lambda e^{{\rm i}\varphi_\Lambda}$ in the neutrino sector. We find that our model is consistent with the global-fit results of neutrino oscillation data at the 1σ level only in the NO case. The best-fit values, together with 1σ and 3σ allowed ranges of model parameters and low-energy observables, are also given.

In addition, we have investigated two simple but viable cases, which are case A with $\tilde{\Lambda} = 0$ and case B with $\tilde{\gamma} = 0$. In these two cases, only eight real model parameters are involved. Numerically, we find that case A is compatible with the oscillation data at the 3σ level in the NO case, while case B can be consistent with the global-fit results within the 1σ level in the NO case. In particular, the effective Majorana neutrino mass matrix M_ν in case A turns out to be phenomenologically appealing, since no additional parameters other than the modulus τ are present. This allows us to perform analytical calculations under some reasonable approximations. Expanding lepton mass matrices in terms of the parameter $x \equiv \exp[-(2\pi{\rm Im}\tau)/5]$, which is about 0.185 given the best-fit value of ${\rm Im}\tau \approx 1.341$, we show that the PMNS matrix up to the order of $\mathcal{O}(x^2)$ can be described by the combination of two rotations coming from the neutrino sector with the rotation angle $\theta_\nu = \arctan(5\sqrt{2}x) \approx 52.60°$ and the charged-lepton sector with the angle $\theta_l = \arctan\left[5\sqrt{2}x\sqrt{\gamma^2 + 6\gamma\cos\varphi_\gamma + 9}/\gamma\right] \approx 48.17°$, respectively. As a consequence, we obtain simple expressions of the mixing angles θ_{12} and θ_{23}, namely, $\theta_{12} \approx \pi/2 - \theta_\nu$ and $\theta_{23} \approx \pi/2 - \theta_l$, which agree well with their individual numerical results. A nonzero θ_{13} can be generated only if the higher-order corrections are taken into account. At the order of $\mathcal{O}(x^3)$, an additional rotation with the rotation angle $\theta'_\nu = \arcsin(25x^3/3)$ in the neutrino sector will contribute to the PMNS matrix, leading to $\theta_{13} \approx \theta'_\nu \approx 3.023°$. Furthermore, the approximate expressions of two CP-violating phases δ and σ have also been gained.

For further exploration along this direction, it will be interesting to bring the modular A'_5 group into the model building of both lepton and quark masses and give a unified description of both quark and lepton masses, flavor mixing patterns, and CP violation. We hope to come back to this possibility in the near future.

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APPENDIX A: CONJUGACY CLASSES AND REPRESENTATION MATRICES

As has been mentioned in Sec. II, the group A'_5 has 120 elements, which can be divided into the following nine conjugacy classes [93]:

TABLE IV. The character table of the group A'_5.

A'_5	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{3}'$	$\mathbf{4}$	$\mathbf{5}$	$\hat{\mathbf{2}}$	$\hat{\mathbf{2}}'$	$\hat{\mathbf{4}}$	$\hat{\mathbf{6}}$
$1C_1$	1	3	3	4	5	2	2	4	6
$12C_5$	1	ϕ	$1-\phi$	-1	0	$-\phi$	$\phi-1$	-1	1
$12C'_5$	1	$1-\phi$	ϕ	-1	0	$\phi-1$	$-\phi$	-1	1
$20C_3$	1	0	0	1	-1	-1	-1	1	0
$30C_4$	1	-1	-1	0	1	0	0	0	0
$1C_2$	1	3	3	4	5	-2	-2	-4	-6
$12C_{10}$	1	ϕ	$1-\phi$	-1	0	ϕ	$1-\phi$	1	-1
$12C'_{10}$	1	$1-\phi$	ϕ	-1	0	$1-\phi$	ϕ	1	-1
$20C_6$	1	0	0	1	-1	1	1	-1	0

$$
\begin{aligned}
1C_1:&\ 1;\\
12C_5:&\ T, T^4, ST^2R, T^2SR, ST^3, T^3S, STSR, TSTR, TST^2, T^2ST, T^3ST^4R, T^4ST^3R;\\
12C_5':&\ T^2, T^3, ST^2SR, ST^3SR, (ST^2)^2, (T^2S)^2, (ST^3)^2, (T^3S)^2, (T^2S)^2T^3R, T^3(ST^2)^2R,\\
&\ T^3ST^2ST^4, T^4ST^2ST^3;\\
20C_3:&\ ST, TS, ST^4R, T^4SR, TST^3R, T^2ST^2R, T^2ST^4, T^3STR, T^3ST^3, T^4ST^2, TST^3SR,\\
&\ T^2ST^3S, T^3ST^2S, ST^2ST^3, ST^3STR, ST^3ST^2, (T^2S)^2T^2R, T^2(T^2S)^2R, (ST^2)^2S,\\
&\ (ST^2)^2T^2R;\\
30C_4:&\ ST^2ST^3S, TST^4, T^4(ST^2)^2, T^2ST^3, (T^2S)^2T^3S, ST^2ST, S, T^3ST^2ST^3,\\
&\ T^3ST^2ST^3S, T^3ST^2, T^4ST^2ST^3S, TST^2S, ST^3ST^2S, T^4ST, (T^2S)^2T^4,\\
&\ ST^2ST^3SR, TST^4R, T^4(ST^2)^2R, T^2ST^3R, (T^2S)^2T^3SR, ST^2STR, SR,\\
&\ T^3ST^2ST^3R, T^3ST^2ST^3SR, T^3ST^2R, T^4ST^2ST^3SR, TST^2SR, ST^3ST^2SR,\\
&\ T^4STR, (T^2S)^2T^4R;\\
1C_2:&\ R;\\
12C_{10}:&\ TR, T^4R, ST^2, T^2S, ST^3R, T^3SR, STS, TST, TST^2R, T^2STR, T^3ST^4, T^4ST^3;\\
12C_{10}':&\ T^2R, T^3R, ST^2S, ST^3S, (ST^2)^2R, (T^2S)^2R, (ST^3)^2R, (T^3S)^2R, (T^2S)^2T^3,\\
&\ T^3(ST^2)^2, T^3ST^2ST^4R, T^4ST^2ST^3R;\\
20C_6:&\ STR, TSR, ST^4, T^4S, TST^3, T^2ST^2, T^2ST^4R, T^3ST, T^3ST^3R, T^4ST^2R,\\
&\ TST^3S, T^2ST^3SR, T^3ST^2SR, ST^2ST^3R, ST^3ST, ST^3ST^2R, (T^2S)^2T^2,\\
&\ T^2(T^2S)^2, (ST^2)^2SR, (ST^2)^2T^2.
\end{aligned}
\tag{A1}
$$

The character table of A_5' has been shown in Table IV, where $\phi \equiv (\sqrt{5}+1)/2$ has been defined. The irreducible representation matrices of three generators S, T, and R are summarized as below:

$$
\begin{aligned}
\mathbf{1}:&\ \rho(S) = +1, \qquad \rho(T) = +1, \qquad \rho(R) = +1,\\
\hat{\mathbf{2}}:&\ \rho(S) = \frac{\mathrm{i}}{\sqrt[4]{5}}\begin{pmatrix} \sqrt{\phi} & \sqrt{\phi-1} \\ \sqrt{\phi-1} & -\sqrt{\phi} \end{pmatrix}, \qquad \rho(T) = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^3 \end{pmatrix}, \qquad \rho(R) = -\mathbb{I}_{2\times 2},\\
\hat{\mathbf{2}}':&\ \rho(S) = \frac{\mathrm{i}}{\sqrt[4]{5}}\begin{pmatrix} \sqrt{\phi-1} & \sqrt{\phi} \\ \sqrt{\phi} & -\sqrt{\phi-1} \end{pmatrix}, \qquad \rho(T) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^4 \end{pmatrix}, \qquad \rho(R) = -\mathbb{I}_{2\times 2},\\
\mathbf{3}:&\ \rho(S) = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & -\phi & \phi-1 \\ -\sqrt{2} & \phi-1 & -\phi \end{pmatrix}, \qquad \rho(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^4 \end{pmatrix}, \qquad \rho(R) = +\mathbb{I}_{3\times 3},\\
\mathbf{3}':&\ \rho(S) = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 1-\phi & \phi \\ \sqrt{2} & \phi & 1-\phi \end{pmatrix}, \qquad \rho(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega^3 \end{pmatrix}, \qquad \rho(R) = +\mathbb{I}_{3\times 3},\\
\mathbf{4}:&\ \rho(S) = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & \phi-1 & \phi & -1 \\ \phi-1 & -1 & 1 & \phi \\ \phi & 1 & -1 & \phi-1 \\ -1 & \phi & \phi-1 & 1 \end{pmatrix}, \qquad \rho(T) = \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & \omega^2 & 0 & 0 \\ 0 & 0 & \omega^3 & 0 \\ 0 & 0 & 0 & \omega^4 \end{pmatrix}, \qquad \rho(R) = +\mathbb{I}_{4\times 4},
\end{aligned}
$$

$$\hat{\mathbf{4}}\colon\ \rho(S)=\frac{\mathrm{i}}{5^{\frac{3}{4}}}\begin{pmatrix}-\sqrt{2\phi+1} & \sqrt{3\phi} & \sqrt{3(\phi-1)} & \sqrt{2\phi-3}\\ \sqrt{3\phi} & \sqrt{2\phi-3} & \sqrt{2\phi+1} & \sqrt{3(\phi-1)}\\ \sqrt{3(\phi-1)} & \sqrt{2\phi+1} & -\sqrt{2\phi-3} & -\sqrt{3\phi}\\ \sqrt{2\phi-3} & \sqrt{3(\phi-1)} & -\sqrt{3\phi} & \sqrt{2\phi+1}\end{pmatrix},\qquad \rho(T)=\begin{pmatrix}\omega & 0 & 0 & 0\\ 0 & \omega^2 & 0 & 0\\ 0 & 0 & \omega^3 & 0\\ 0 & 0 & 0 & \omega^4\end{pmatrix},$$

$$\rho(R)=-\mathbb{I}_{4\times4},$$

$$\mathbf{5}\colon\ \rho(S)=\frac{1}{5}\begin{pmatrix}-1 & \sqrt{6} & \sqrt{6} & \sqrt{6} & \sqrt{6}\\ \sqrt{6} & (\phi-1)^2 & -2\phi & 2(\phi-1) & \phi^2\\ \sqrt{6} & -2\phi & \phi^2 & (\phi-1)^2 & 2(\phi-1)\\ \sqrt{6} & 2(\phi-1) & (\phi-1)^2 & \phi^2 & -2\phi\\ \sqrt{6} & \phi^2 & 2(\phi-1) & -2\phi & (\phi-1)^2\end{pmatrix},\qquad \rho(T)=\begin{pmatrix}1 & 0 & 0 & 0 & 0\\ 0 & \omega & 0 & 0 & 0\\ 0 & 0 & \omega^2 & 0 & 0\\ 0 & 0 & 0 & \omega^3 & 0\\ 0 & 0 & 0 & 0 & \omega^4\end{pmatrix},$$

$$\rho(R)=+\mathbb{I}_{5\times5},$$

$$\hat{\mathbf{6}}\colon\ \rho(S)=\frac{-\mathrm{i}}{5^{\frac{3}{4}}}\begin{pmatrix}\sqrt{\phi} & -\sqrt{2(\phi-1)} & \sqrt{2\phi-3} & -\sqrt{2\phi+1} & \sqrt{2\phi} & \sqrt{\phi-1}\\ -\sqrt{2(\phi-1)} & -\sqrt{\phi-1} & \sqrt{2(\phi-1)} & \sqrt{2\phi} & \sqrt{\phi} & \sqrt{2\phi}\\ \sqrt{2\phi-3} & \sqrt{2(\phi-1)} & \sqrt{\phi} & -\sqrt{\phi-1} & -\sqrt{2\phi} & \sqrt{2\phi+1}\\ -\sqrt{2\phi+1} & \sqrt{2\phi} & -\sqrt{\phi-1} & -\sqrt{\phi} & \sqrt{2(\phi-1)} & \sqrt{2\phi-3}\\ \sqrt{2\phi} & \sqrt{\phi} & -\sqrt{2\phi} & \sqrt{2(\phi-1)} & \sqrt{\phi-1} & \sqrt{2(\phi-1)}\\ \sqrt{\phi-1} & \sqrt{2\phi} & \sqrt{2\phi+1} & \sqrt{2\phi-3} & \sqrt{2(\phi-1)} & -\sqrt{\phi}\end{pmatrix},$$

$$\rho(T)=\begin{pmatrix}1 & 0 & 0 & 0 & 0 & 0\\ 0 & \omega & 0 & 0 & 0 & 0\\ 0 & 0 & \omega^2 & 0 & 0 & 0\\ 0 & 0 & 0 & \omega^3 & 0 & 0\\ 0 & 0 & 0 & 0 & \omega^4 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix},\qquad \rho(R)=-\mathbb{I}_{6\times6}, \tag{A2}$$

where $\omega\equiv e^{2\pi\mathrm{i}/5}$ and $\mathbb{I}_{n\times n}$ denotes the n-dimensional identity matrix. Notice that the above representation matrices are equivalent to those in Ref. [93] via unitary transformations.

APPENDIX B: THE KRONECKER PRODUCT RULES OF A_5'

In this Appendix, we summarize the decomposition rules of the Kronecker products of any two nontrivial irreducible representations of A_5', namely,

$$\hat{\mathbf{2}}\otimes\hat{\mathbf{2}}=\mathbf{1}_{\mathrm{a}}\oplus\mathbf{3}_{\mathrm{s}}\qquad \hat{\mathbf{2}}'\otimes\hat{\mathbf{2}}'=\mathbf{1}_{\mathrm{a}}\oplus\mathbf{3}'_{\mathrm{s}}\qquad \hat{\mathbf{2}}\otimes\hat{\mathbf{2}}'=\mathbf{4}$$

$$\begin{cases}\mathbf{1}_{\mathrm{a}}\colon\ -\frac{\sqrt{2}}{2}(\alpha_1\beta_2-\alpha_2\beta_1)\\ \mathbf{3}_{\mathrm{s}}\colon\ \frac{\sqrt{2}}{2}\begin{pmatrix}\alpha_1\beta_2+\alpha_2\beta_1\\ -\sqrt{2}\alpha_2\beta_2\\ \sqrt{2}\alpha_1\beta_1\end{pmatrix}\end{cases}\qquad \begin{cases}\mathbf{1}_{\mathrm{a}}\colon\ \frac{\sqrt{2}}{2}(\alpha_1\beta_2-\alpha_2\beta_1)\\ \mathbf{3}'_{\mathrm{s}}\colon\ \frac{\sqrt{2}}{2}\begin{pmatrix}\alpha_1\beta_2+\alpha_2\beta_1\\ -\sqrt{2}\alpha_1\beta_1\\ \sqrt{2}\alpha_2\beta_2\end{pmatrix}\end{cases}\qquad \mathbf{4}\colon\begin{pmatrix}\alpha_1\beta_2\\ \alpha_2\beta_2\\ -\alpha_1\beta_1\\ \alpha_2\beta_1\end{pmatrix}$$

$$\hat{\mathbf{2}}\otimes\mathbf{3}=\hat{\mathbf{2}}\oplus\hat{\mathbf{4}}\qquad \begin{cases}\hat{\mathbf{2}}:\ \frac{\sqrt{3}}{3}\begin{pmatrix}-\alpha_1\beta_1+\sqrt{2}\alpha_2\beta_3\\ \sqrt{2}\alpha_1\beta_2+\alpha_2\beta_1\end{pmatrix}\\ \hat{\mathbf{4}}:\ \frac{\sqrt{3}}{3}\begin{pmatrix}-\sqrt{3}\alpha_1\beta_3\\ \sqrt{2}\alpha_1\beta_1+\alpha_2\beta_3\\ -\alpha_1\beta_2+\sqrt{2}\alpha_2\beta_1\\ -\sqrt{3}\alpha_2\beta_2\end{pmatrix}\end{cases}$$

$$\hat{\mathbf{2}}'\otimes\mathbf{3}'=\hat{\mathbf{2}}'\oplus\hat{\mathbf{4}}\qquad \begin{cases}\hat{\mathbf{2}}':\ \frac{\sqrt{3}}{3}\begin{pmatrix}\alpha_1\beta_1+\sqrt{2}\alpha_2\beta_2\\ \sqrt{2}\alpha_1\beta_3-\alpha_2\beta_1\end{pmatrix}\\ \hat{\mathbf{4}}:\ \frac{\sqrt{3}}{3}\begin{pmatrix}-\sqrt{2}\alpha_1\beta_1+\alpha_2\beta_2\\ -\sqrt{3}\alpha_2\beta_3\\ -\sqrt{3}\alpha_1\beta_2\\ \alpha_1\beta_3+\sqrt{2}\alpha_2\beta_1\end{pmatrix}\end{cases}$$

$$\hat{\mathbf{2}}\otimes\mathbf{3}'=\hat{\mathbf{6}}\qquad \hat{\mathbf{6}}:\ -\frac{\sqrt{2}}{2}\begin{pmatrix}\alpha_1\beta_3-\alpha_2\beta_2\\ \sqrt{2}\alpha_2\beta_3\\ -\sqrt{2}\alpha_1\beta_1\\ \sqrt{2}\alpha_2\beta_1\\ -\sqrt{2}\alpha_1\beta_2\\ \alpha_1\beta_3+\alpha_2\beta_2\end{pmatrix}$$

$$\hat{\mathbf{2}}'\otimes\mathbf{3}=\hat{\mathbf{6}}\qquad \hat{\mathbf{6}}:\ \begin{pmatrix}\alpha_1\beta_3\\ -\alpha_1\beta_1\\ -\alpha_1\beta_2\\ -\alpha_2\beta_3\\ \alpha_2\beta_1\\ -\alpha_2\beta_2\end{pmatrix}$$

$$\hat{\mathbf{2}}\otimes\mathbf{4}=\hat{\mathbf{2}}'\oplus\hat{\mathbf{6}}\qquad \begin{cases}\hat{\mathbf{2}}':\ -\frac{\sqrt{2}}{2}\begin{pmatrix}\alpha_1\beta_4+\alpha_2\beta_3\\ \alpha_1\beta_2-\alpha_2\beta_1\end{pmatrix}\\ \hat{\mathbf{6}}:\ \frac{\sqrt{2}}{2}\begin{pmatrix}-\sqrt{2}\alpha_1\beta_3\\ -\alpha_1\beta_4+\alpha_2\beta_3\\ \sqrt{2}\alpha_2\beta_4\\ -\sqrt{2}\alpha_1\beta_1\\ \alpha_1\beta_2+\alpha_2\beta_1\\ \sqrt{2}\alpha_2\beta_2\end{pmatrix}\end{cases}$$

$$\hat{\mathbf{2}}'\otimes\mathbf{4}=\hat{\mathbf{2}}\oplus\hat{\mathbf{6}}\qquad \begin{cases}\hat{\mathbf{2}}:\ \frac{\sqrt{2}}{2}\begin{pmatrix}\alpha_1\beta_1+\alpha_2\beta_3\\ \alpha_1\beta_2-\alpha_2\beta_4\end{pmatrix}\\ \hat{\mathbf{6}}:\ \frac{\sqrt{2}}{2}\begin{pmatrix}-\alpha_1\beta_4-\alpha_2\beta_1\\ -\sqrt{2}\alpha_2\beta_2\\ \alpha_1\beta_1-\alpha_2\beta_3\\ -\alpha_1\beta_2-\alpha_2\beta_4\\ \sqrt{2}\alpha_1\beta_3\\ \alpha_1\beta_4-\alpha_2\beta_1\end{pmatrix}\end{cases}$$

$$\hat{\mathbf{2}}\otimes\hat{\mathbf{4}}=\mathbf{3}\oplus\mathbf{5}\qquad \begin{cases}\mathbf{3}:\ -\frac{1}{2}\begin{pmatrix}\sqrt{2}(\alpha_1\beta_3-\alpha_2\beta_2)\\ -\sqrt{3}\alpha_1\beta_4+\alpha_2\beta_3\\ \alpha_1\beta_2+\sqrt{3}\alpha_2\beta_1\end{pmatrix}\\ \mathbf{5}:\ \frac{1}{2}\begin{pmatrix}\sqrt{2}(\alpha_1\beta_3+\alpha_2\beta_2)\\ \alpha_1\beta_4+\sqrt{3}\alpha_2\beta_3\\ 2\alpha_2\beta_4\\ -2\alpha_1\beta_1\\ -\sqrt{3}\alpha_1\beta_2+\alpha_2\beta_1\end{pmatrix}\end{cases}$$

$$\hat{\mathbf{2}}'\otimes\hat{\mathbf{4}}=\mathbf{3}'\oplus\mathbf{5}\qquad \begin{cases}\mathbf{3}':\ \frac{1}{2}\begin{pmatrix}\sqrt{2}(\alpha_1\beta_4+\alpha_2\beta_1)\\ \alpha_1\beta_1+\sqrt{3}\alpha_2\beta_3\\ -\sqrt{3}\alpha_1\beta_2-\alpha_2\beta_4\end{pmatrix}\\ \mathbf{5}:\ \frac{1}{2}\begin{pmatrix}\sqrt{2}(\alpha_1\beta_4-\alpha_2\beta_1)\\ -2\alpha_2\beta_2\\ -\sqrt{3}\alpha_1\beta_1+\alpha_2\beta_3\\ \alpha_1\beta_2-\sqrt{3}\alpha_2\beta_4\\ 2\alpha_1\beta_3\end{pmatrix}\end{cases}$$

$$\hat{\mathbf{2}}\otimes\mathbf{5}=\hat{\mathbf{4}}\oplus\hat{\mathbf{6}}$$

$$\begin{cases}\hat{\mathbf{4}}:\ -\frac{\sqrt{5}}{5}\begin{pmatrix}\alpha_1\beta_5+2\alpha_2\beta_4\\ \sqrt{2}\alpha_1\beta_1+\sqrt{3}\alpha_2\beta_5\\ \sqrt{3}\alpha_1\beta_2-\sqrt{2}\alpha_2\beta_1\\ 2\alpha_1\beta_3-\alpha_2\beta_2\end{pmatrix}\\ \hat{\mathbf{6}}:\ \frac{\sqrt{10}}{10}\begin{pmatrix}-\alpha_1\beta_4+3\alpha_2\beta_3\\ \sqrt{2}(2\alpha_1\beta_5-\alpha_2\beta_4)\\ -\sqrt{6}\alpha_1\beta_1+2\alpha_2\beta_5\\ -2\alpha_1\beta_2-\sqrt{6}\alpha_2\beta_1\\ -\sqrt{2}(\alpha_1\beta_3+2\alpha_2\beta_2)\\ 3\alpha_1\beta_4+\alpha_2\beta_3\end{pmatrix}\end{cases}$$

$$\hat{\mathbf{2}}'\otimes\mathbf{5}=\hat{\mathbf{4}}\oplus\hat{\mathbf{6}}$$

$$\begin{cases}\hat{\mathbf{4}}:\ \frac{\sqrt{5}}{5}\begin{pmatrix}\sqrt{2}\alpha_1\beta_1-\sqrt{3}\alpha_2\beta_3\\ 2\alpha_1\beta_2+\alpha_2\beta_4\\ -\alpha_1\beta_3+2\alpha_2\beta_5\\ \sqrt{3}\alpha_1\beta_4+\sqrt{2}\alpha_2\beta_1\end{pmatrix}\\ \hat{\mathbf{6}}:\ \frac{\sqrt{5}}{5}\begin{pmatrix}-\alpha_1\beta_5+2\alpha_2\beta_2\\ \sqrt{3}\alpha_1\beta_1+\sqrt{2}\alpha_2\beta_3\\ \alpha_1\beta_2-2\alpha_2\beta_4\\ -2\alpha_1\beta_3-\alpha_2\beta_5\\ -\sqrt{2}\alpha_1\beta_4+\sqrt{3}\alpha_2\beta_1\\ -2\alpha_1\beta_5-\alpha_2\beta_2\end{pmatrix}\end{cases}$$

$$\hat{\mathbf{2}}\otimes\hat{\mathbf{6}}=\mathbf{3}'\oplus\mathbf{4}\oplus\mathbf{5}$$

$$\begin{cases}\mathbf{3}':\ \frac{1}{2}\begin{pmatrix}\sqrt{2}(\alpha_1\beta_4+\alpha_2\beta_3)\\ -\alpha_1\beta_1+\alpha_1\beta_6+\sqrt{2}\alpha_2\beta_5\\ \sqrt{2}\alpha_1\beta_2-\alpha_2\beta_1-\alpha_2\beta_6\end{pmatrix}\\ \mathbf{4}:\ \frac{\sqrt{3}}{3}\begin{pmatrix}-\alpha_1\beta_5-\sqrt{2}\alpha_2\beta_4\\ -\sqrt{2}\alpha_1\beta_6+\alpha_2\beta_5\\ -\alpha_1\beta_2-\sqrt{2}\alpha_2\beta_1\\ -\sqrt{2}\alpha_1\beta_3-\alpha_2\beta_2\end{pmatrix}\\ \mathbf{5}:\ \frac{\sqrt{3}}{6}\begin{pmatrix}\sqrt{6}(\alpha_1\beta_4-\alpha_2\beta_3)\\ 2(\sqrt{2}\alpha_1\beta_5-\alpha_2\beta_4)\\ -3\alpha_1\beta_1-\alpha_1\beta_6-\sqrt{2}\alpha_2\beta_5\\ \sqrt{2}\alpha_1\beta_2-\alpha_2\beta_1+3\alpha_2\beta_6\\ -2(\alpha_1\beta_3-\sqrt{2}\alpha_2\beta_2)\end{pmatrix}\end{cases}$$

$$\hat{\mathbf{2}}'\otimes\hat{\mathbf{6}}=\mathbf{3}\oplus\mathbf{4}\oplus\mathbf{5}$$

$$\begin{cases}\mathbf{3}:\ \frac{\sqrt{2}}{2}\begin{pmatrix}-\alpha_1\beta_5-\alpha_2\beta_2\\ \alpha_1\beta_6-\alpha_2\beta_3\\ \alpha_1\beta_4+\alpha_2\beta_1\end{pmatrix}\\ \mathbf{4}:\ \frac{\sqrt{3}}{3}\begin{pmatrix}-\alpha_1\beta_1-\alpha_1\beta_6-\alpha_2\beta_3\\ -\sqrt{2}\alpha_1\beta_2+\alpha_2\beta_4\\ -\alpha_1\beta_3-\sqrt{2}\alpha_2\beta_5\\ -\alpha_1\beta_4+\alpha_2\beta_1-\alpha_2\beta_6\end{pmatrix}\\ \mathbf{5}:\ \frac{\sqrt{6}}{6}\begin{pmatrix}-\sqrt{3}(\alpha_1\beta_5-\alpha_2\beta_2)\\ -2\alpha_1\beta_1+\alpha_1\beta_6+\alpha_2\beta_3\\ -\sqrt{2}(\alpha_1\beta_2+\sqrt{2}\alpha_2\beta_4)\\ \sqrt{2}(\sqrt{2}\alpha_1\beta_3-\alpha_2\beta_5)\\ \alpha_1\beta_4-\alpha_2\beta_1-2\alpha_2\beta_6\end{pmatrix}\end{cases}$$

$$\mathbf{3}\otimes\mathbf{3}=\mathbf{1}_{\rm s}\oplus\mathbf{3}_{\rm a}\oplus\mathbf{5}_{\rm s}$$

$$\begin{cases}\mathbf{1}_{\rm s}:\ \frac{\sqrt{3}}{3}(\alpha_1\beta_1+\alpha_2\beta_3+\alpha_3\beta_2)\\ \mathbf{3}_{\rm a}:\ \frac{\sqrt{2}}{2}\begin{pmatrix}\alpha_2\beta_3-\alpha_3\beta_2\\ \alpha_1\beta_2-\alpha_2\beta_1\\ \alpha_3\beta_1-\alpha_1\beta_3\end{pmatrix}\\ \mathbf{5}_{\rm s}:\ \frac{\sqrt{6}}{6}\begin{pmatrix}2\alpha_1\beta_1-\alpha_2\beta_3-\alpha_3\beta_2\\ -\sqrt{3}\alpha_1\beta_2-\sqrt{3}\alpha_2\beta_1\\ \sqrt{6}\alpha_2\beta_2\\ \sqrt{6}\alpha_3\beta_3\\ -\sqrt{3}(\alpha_1\beta_3+\alpha_3\beta_1)\end{pmatrix}\end{cases}$$

$$\mathbf{3}'\otimes\mathbf{3}'=\mathbf{1}_{\rm s}\oplus\mathbf{3}'_{\rm a}\oplus\mathbf{5}_{\rm s}$$

$$\begin{cases}\mathbf{1}_{\rm s}:\ \frac{\sqrt{3}}{3}(\alpha_1\beta_1+\alpha_2\beta_3+\alpha_3\beta_2)\\ \mathbf{3}'_{\rm a}:\ \frac{\sqrt{2}}{2}\begin{pmatrix}\alpha_2\beta_3-\alpha_3\beta_2\\ \alpha_1\beta_2-\alpha_2\beta_1\\ \alpha_3\beta_1-\alpha_1\beta_3\end{pmatrix}\\ \mathbf{5}_{\rm s}:\ \frac{\sqrt{6}}{6}\begin{pmatrix}2\alpha_1\beta_1-\alpha_2\beta_3-\alpha_3\beta_2\\ \sqrt{6}\alpha_3\beta_3\\ -\sqrt{3}(\alpha_1\beta_2+\alpha_2\beta_1)\\ -\sqrt{3}(\alpha_1\beta_3+\alpha_3\beta_1)\\ \sqrt{6}\alpha_2\beta_2\end{pmatrix}\end{cases}$$

$$\mathbf{3}\otimes\mathbf{3}'=\mathbf{4}\oplus\mathbf{5}$$

$$\begin{cases}\mathbf{4}:\ \frac{\sqrt{3}}{3}\begin{pmatrix}\sqrt{2}\alpha_2\beta_1+\alpha_3\beta_2\\ -\sqrt{2}\alpha_1\beta_2-\alpha_3\beta_3\\ -\sqrt{2}\alpha_1\beta_3-\alpha_2\beta_2\\ \sqrt{2}\alpha_3\beta_1+\alpha_2\beta_3\end{pmatrix}\\ \mathbf{5}:\ \frac{\sqrt{3}}{3}\begin{pmatrix}\sqrt{3}\alpha_1\beta_1\\ \alpha_2\beta_1-\sqrt{2}\alpha_3\beta_2\\ \alpha_1\beta_2-\sqrt{2}\alpha_3\beta_3\\ \alpha_1\beta_3-\sqrt{2}\alpha_2\beta_2\\ \alpha_3\beta_1-\sqrt{2}\alpha_2\beta_3\end{pmatrix}\end{cases}$$

$$\mathbf{3}\otimes\mathbf{4}=\mathbf{3}'\oplus\mathbf{4}\oplus\mathbf{5}$$

$$\begin{cases}\mathbf{3}': \frac{1}{2}\begin{pmatrix}-\sqrt{2}(\alpha_2\beta_4+\alpha_3\beta_1)\\ \sqrt{2}\alpha_1\beta_2-\alpha_2\beta_1+\alpha_3\beta_3\\ \sqrt{2}\alpha_1\beta_3+\alpha_2\beta_2-\alpha_3\beta_4\end{pmatrix}\\ \mathbf{4}: \frac{\sqrt{3}}{3}\begin{pmatrix}\alpha_1\beta_1-\sqrt{2}\alpha_3\beta_2\\ -\alpha_1\beta_2-\sqrt{2}\alpha_2\beta_1\\ \alpha_1\beta_3+\sqrt{2}\alpha_3\beta_4\\ -\alpha_1\beta_4+\sqrt{2}\alpha_2\beta_3\end{pmatrix}\\ \mathbf{5}: \frac{\sqrt{3}}{6}\begin{pmatrix}\sqrt{6}(\alpha_2\beta_4-\alpha_3\beta_1)\\ 2(\sqrt{2}\alpha_1\beta_1+\alpha_3\beta_2)\\ -\sqrt{2}\alpha_1\beta_2+\alpha_2\beta_1+3\alpha_3\beta_3\\ \sqrt{2}\alpha_1\beta_3-3\alpha_2\beta_2-\alpha_3\beta_4\\ -2(\sqrt{2}\alpha_1\beta_4+\alpha_2\beta_3)\end{pmatrix}\end{cases}$$

$$\mathbf{3}'\otimes\mathbf{4}=\mathbf{3}\oplus\mathbf{4}\oplus\mathbf{5}$$

$$\begin{cases}\mathbf{3}: \frac{1}{2}\begin{pmatrix}-\sqrt{2}(\alpha_2\beta_3+\alpha_3\beta_2)\\ \sqrt{2}\alpha_1\beta_1+\alpha_2\beta_4-\alpha_3\beta_3\\ \sqrt{2}\alpha_1\beta_4-\alpha_2\beta_2+\alpha_3\beta_1\end{pmatrix}\\ \mathbf{4}: \frac{\sqrt{3}}{3}\begin{pmatrix}\alpha_1\beta_1+\sqrt{2}\alpha_3\beta_3\\ \alpha_1\beta_2-\sqrt{2}\alpha_3\beta_4\\ -\alpha_1\beta_3+\sqrt{2}\alpha_2\beta_1\\ -\alpha_1\beta_4-\sqrt{2}\alpha_2\beta_2\end{pmatrix}\\ \mathbf{5}: \frac{\sqrt{3}}{6}\begin{pmatrix}\sqrt{6}(\alpha_2\beta_3-\alpha_3\beta_2)\\ \sqrt{2}\alpha_1\beta_1-3\alpha_2\beta_4-\alpha_3\beta_3\\ 2(\sqrt{2}\alpha_1\beta_2+\alpha_3\beta_4)\\ -2(\sqrt{2}\alpha_1\beta_3+\alpha_2\beta_1)\\ -\sqrt{2}\alpha_1\beta_4+\alpha_2\beta_2+3\alpha_3\beta_1\end{pmatrix}\end{cases}$$

$$\mathbf{3}\otimes\hat{\mathbf{4}}=\hat{\mathbf{2}}\oplus\hat{\mathbf{4}}\oplus\hat{\mathbf{6}}$$

$$\begin{cases}\hat{\mathbf{2}}: \frac{\sqrt{6}}{6}\begin{pmatrix}-\sqrt{2}\alpha_1\beta_2+\sqrt{3}\alpha_2\beta_1+\alpha_3\beta_3\\ -\sqrt{2}\alpha_1\beta_3-\alpha_2\beta_2+\sqrt{3}\alpha_3\beta_4\end{pmatrix}\\ \hat{\mathbf{4}}: \frac{\sqrt{15}}{15}\begin{pmatrix}3\alpha_1\beta_1+\sqrt{6}\alpha_3\beta_2\\ \alpha_1\beta_2+\sqrt{6}\alpha_2\beta_1-2\sqrt{2}\alpha_3\beta_3\\ -\alpha_1\beta_3-2\sqrt{2}\alpha_2\beta_2-\sqrt{6}\alpha_3\beta_4\\ -3\alpha_1\beta_4-\sqrt{6}\alpha_2\beta_3\end{pmatrix}\\ \hat{\mathbf{6}}: \frac{\sqrt{10}}{10}\begin{pmatrix}-3\alpha_2\beta_4+\alpha_3\beta_1\\ 2\alpha_1\beta_1-\sqrt{6}\alpha_3\beta_2\\ -\sqrt{6}\alpha_1\beta_2-\alpha_2\beta_1-\sqrt{3}\alpha_3\beta_3\\ -\sqrt{6}\alpha_1\beta_3+\sqrt{3}\alpha_2\beta_2-\alpha_3\beta_4\\ -2\alpha_1\beta_4+\sqrt{6}\alpha_2\beta_3\\ -\alpha_2\beta_4-3\alpha_3\beta_1\end{pmatrix}\end{cases}$$

$$\mathbf{3}'\otimes\hat{\mathbf{4}}=\hat{\mathbf{2}}'\oplus\hat{\mathbf{4}}\oplus\hat{\mathbf{6}}$$

$$\begin{cases}\hat{\mathbf{2}}': \frac{\sqrt{6}}{6}\begin{pmatrix}\sqrt{2}\alpha_1\beta_1-\alpha_2\beta_4+\sqrt{3}\alpha_3\beta_3\\ -\sqrt{2}\alpha_1\beta_4+\sqrt{3}\alpha_2\beta_2-\alpha_3\beta_1\end{pmatrix}\\ \hat{\mathbf{4}}: \frac{\sqrt{15}}{15}\begin{pmatrix}\alpha_1\beta_1-2\sqrt{2}\alpha_2\beta_4-\sqrt{6}\alpha_3\beta_3\\ -3\alpha_1\beta_2-\sqrt{6}\alpha_3\beta_4\\ 3\alpha_1\beta_3-\sqrt{6}\alpha_2\beta_1\\ -\alpha_1\beta_4-\sqrt{6}\alpha_2\beta_2-2\sqrt{2}\alpha_3\beta_1\end{pmatrix}\\ \hat{\mathbf{6}}: \frac{\sqrt{10}}{10}\begin{pmatrix}-\sqrt{2}(\alpha_2\beta_3-2\alpha_3\beta_2)\\ \sqrt{6}\alpha_1\beta_1+\sqrt{3}\alpha_2\beta_4-\alpha_3\beta_3\\ 2\alpha_1\beta_2-\sqrt{6}\alpha_3\beta_4\\ 2\alpha_1\beta_3+\sqrt{6}\alpha_2\beta_1\\ -\sqrt{6}\alpha_1\beta_4-\alpha_2\beta_2+\sqrt{3}\alpha_3\beta_1\\ -\sqrt{2}(2\alpha_2\beta_3+\alpha_3\beta_2)\end{pmatrix}\end{cases}$$

$$\mathbf{3}\otimes\mathbf{5}=\mathbf{3}\oplus\mathbf{3}'\oplus\mathbf{4}\oplus\mathbf{5}$$

$$\begin{cases}\mathbf{3}:\ \frac{\sqrt{10}}{10}\begin{pmatrix}-2\alpha_1\beta_1+\sqrt{3}\alpha_2\beta_5+\sqrt{3}\alpha_3\beta_2\\ \sqrt{3}\alpha_1\beta_2+\alpha_2\beta_1-\sqrt{6}\alpha_3\beta_3\\ \sqrt{3}\alpha_1\beta_5-\sqrt{6}\alpha_2\beta_4+\alpha_3\beta_1\end{pmatrix}\\ \mathbf{3}':\ \frac{\sqrt{5}}{5}\begin{pmatrix}\sqrt{3}\alpha_1\beta_1+\alpha_2\beta_5+\alpha_3\beta_2\\ \alpha_1\beta_3-\sqrt{2}\alpha_2\beta_2-\sqrt{2}\alpha_3\beta_4\\ \alpha_1\beta_4-\sqrt{2}\alpha_2\beta_3-\sqrt{2}\alpha_3\beta_5\end{pmatrix}\\ \mathbf{4}:\ \frac{\sqrt{15}}{15}\begin{pmatrix}2\sqrt{2}\alpha_1\beta_2-\sqrt{6}\alpha_2\beta_1+\alpha_3\beta_3\\ -\sqrt{2}\alpha_1\beta_3+2\alpha_2\beta_2-3\alpha_3\beta_4\\ \sqrt{2}\alpha_1\beta_4+3\alpha_2\beta_3-2\alpha_3\beta_5\\ -2\sqrt{2}\alpha_1\beta_5-\alpha_2\beta_4+\sqrt{6}\alpha_3\beta_1\end{pmatrix}\\ \mathbf{5}:\ \frac{\sqrt{6}}{6}\begin{pmatrix}\sqrt{3}(\alpha_2\beta_5-\alpha_3\beta_2)\\ -\alpha_1\beta_2-\sqrt{3}\alpha_2\beta_1-\sqrt{2}\alpha_3\beta_3\\ -\sqrt{2}(\sqrt{2}\alpha_1\beta_3+\alpha_2\beta_2)\\ \sqrt{2}(\sqrt{2}\alpha_1\beta_4+\alpha_3\beta_5)\\ \alpha_1\beta_5+\sqrt{2}\alpha_2\beta_4+\sqrt{3}\alpha_3\beta_1\end{pmatrix}\end{cases}$$

$$\mathbf{3}'\otimes\mathbf{5}=\mathbf{3}\oplus\mathbf{3}'\oplus\mathbf{4}\oplus\mathbf{5}$$

$$\begin{cases}\mathbf{3}:\ \frac{\sqrt{5}}{5}\begin{pmatrix}\sqrt{3}\alpha_1\beta_1+\alpha_2\beta_4+\alpha_3\beta_3\\ \alpha_1\beta_2-\sqrt{2}\alpha_2\beta_5-\sqrt{2}\alpha_3\beta_4\\ \alpha_1\beta_5-\sqrt{2}\alpha_2\beta_3-\sqrt{2}\alpha_3\beta_2\end{pmatrix}\\ \mathbf{3}':\ \frac{\sqrt{10}}{10}\begin{pmatrix}-2\alpha_1\beta_1+\sqrt{3}\alpha_2\beta_4+\sqrt{3}\alpha_3\beta_3\\ \sqrt{3}\alpha_1\beta_3+\alpha_2\beta_1-\sqrt{6}\alpha_3\beta_5\\ \sqrt{3}\alpha_1\beta_4-\sqrt{6}\alpha_2\beta_2+\alpha_3\beta_1\end{pmatrix}\\ \mathbf{4}:\ \frac{\sqrt{15}}{15}\begin{pmatrix}\sqrt{2}\alpha_1\beta_2+3\alpha_2\beta_5-2\alpha_3\beta_4\\ 2\sqrt{2}\alpha_1\beta_3-\sqrt{6}\alpha_2\beta_1+\alpha_3\beta_5\\ -2\sqrt{2}\alpha_1\beta_4-\alpha_2\beta_2+\sqrt{6}\alpha_3\beta_1\\ -\sqrt{2}\alpha_1\beta_5+2\alpha_2\beta_3-3\alpha_3\beta_2\end{pmatrix}\\ \mathbf{5}:\ \frac{\sqrt{6}}{6}\begin{pmatrix}\sqrt{3}(\alpha_2\beta_4-\alpha_3\beta_3)\\ \sqrt{2}(\sqrt{2}\alpha_1\beta_2+\alpha_3\beta_4)\\ -\alpha_1\beta_3-\sqrt{3}\alpha_2\beta_1-\sqrt{2}\alpha_3\beta_5\\ \alpha_1\beta_4+\sqrt{2}\alpha_2\beta_2+\sqrt{3}\alpha_3\beta_1\\ -\sqrt{2}(\sqrt{2}\alpha_1\beta_5+\alpha_2\beta_3)\end{pmatrix}\end{cases}$$

$$\mathbf{3}\otimes\hat{\mathbf{6}}=\hat{\mathbf{2}}'\oplus\hat{\mathbf{4}}\oplus\hat{\mathbf{6}}_1\oplus\hat{\mathbf{6}}_2$$

$$\begin{cases}\hat{\mathbf{2}}':\ \frac{\sqrt{3}}{3}\begin{pmatrix}\alpha_1\beta_2-\alpha_2\beta_1+\alpha_3\beta_3\\ -\alpha_1\beta_5+\alpha_2\beta_4+\alpha_3\beta_6\end{pmatrix}\\ \hat{\mathbf{4}}:\ \frac{\sqrt{15}}{15}\begin{pmatrix}-2\alpha_1\beta_2-\alpha_2\beta_1+3\alpha_2\beta_6+\alpha_3\beta_3\\ \sqrt{6}\alpha_1\beta_3+\sqrt{6}\alpha_2\beta_2-\sqrt{3}\alpha_3\beta_4\\ \sqrt{6}\alpha_1\beta_4+\sqrt{3}\alpha_2\beta_3-\sqrt{6}\alpha_3\beta_5\\ 2\alpha_1\beta_5+\alpha_2\beta_4+3\alpha_3\beta_1+\alpha_3\beta_6\end{pmatrix}\\ \hat{\mathbf{6}}_1:\ \frac{\sqrt{2}}{2}\begin{pmatrix}\alpha_1\beta_1+\alpha_3\beta_2\\ \alpha_2\beta_1+\alpha_3\beta_3\\ -\alpha_1\beta_3+\alpha_2\beta_2\\ \alpha_1\beta_4+\alpha_3\beta_5\\ \alpha_2\beta_4-\alpha_3\beta_6\\ -\alpha_1\beta_6-\alpha_2\beta_5\end{pmatrix}\\ \hat{\mathbf{6}}_2:\ \frac{\sqrt{3}}{3}\begin{pmatrix}\alpha_1\beta_1+\alpha_1\beta_6-\alpha_2\beta_5\\ -\alpha_1\beta_2-\alpha_2\beta_6+\alpha_3\beta_3\\ \alpha_2\beta_2+\sqrt{2}\alpha_3\beta_4\\ \sqrt{2}\alpha_2\beta_3+\alpha_3\beta_5\\ \alpha_1\beta_5+\alpha_2\beta_4-\alpha_3\beta_1\\ \alpha_1\beta_1-\alpha_1\beta_6-\alpha_3\beta_2\end{pmatrix}\end{cases}$$

$$\mathbf{3}'\otimes\hat{\mathbf{6}}=\hat{\mathbf{2}}\oplus\hat{\mathbf{4}}\oplus\hat{\mathbf{6}}_1\oplus\hat{\mathbf{6}}_2$$

$$\begin{cases}\hat{\mathbf{2}}:\ \frac{\sqrt{6}}{6}\begin{pmatrix}-\sqrt{2}\alpha_1\beta_3+\alpha_2\beta_1+\alpha_2\beta_6-\sqrt{2}\alpha_3\beta_5\\ \sqrt{2}\alpha_1\beta_4+\sqrt{2}\alpha_2\beta_2-\alpha_3\beta_1+\alpha_3\beta_6\end{pmatrix}\\ \hat{\mathbf{4}}:\ \frac{\sqrt{15}}{15}\begin{pmatrix}-\sqrt{3}(\sqrt{2}\alpha_1\beta_2+\alpha_2\beta_5+\sqrt{2}\alpha_3\beta_4)\\ \sqrt{2}\alpha_2\beta_6-2\alpha_1\beta_3-2\sqrt{2}\alpha_2\beta_1+\alpha_3\beta_5\\ \alpha_2\beta_2-2\alpha_1\beta_4+\sqrt{2}\alpha_3\beta_1+2\sqrt{2}\alpha_3\beta_6\\ \sqrt{3}(\sqrt{2}\alpha_1\beta_5+\sqrt{2}\alpha_2\beta_3-\alpha_3\beta_2)\end{pmatrix}\\ \hat{\mathbf{6}}_1:\ \frac{\sqrt{3}}{3}\begin{pmatrix}-\alpha_1\beta_1+\sqrt{2}\alpha_2\beta_4\\ -\alpha_1\beta_2+\sqrt{2}\alpha_2\beta_5\\ -\alpha_1\beta_3-\sqrt{2}\alpha_2\beta_6\\ \alpha_1\beta_4+\sqrt{2}\alpha_3\beta_1\\ \alpha_1\beta_5+\sqrt{2}\alpha_3\beta_2\\ \alpha_1\beta_6-\sqrt{2}\alpha_3\beta_3\end{pmatrix}\\ \hat{\mathbf{6}}_2:\ \frac{1}{2}\begin{pmatrix}\sqrt{2}\alpha_1\beta_6-\alpha_2\beta_4+\alpha_3\beta_3\\ \sqrt{2}(\alpha_1\beta_2-\alpha_3\beta_4)\\ \alpha_2\beta_1+\alpha_2\beta_6+\sqrt{2}\alpha_3\beta_5\\ -\sqrt{2}\alpha_2\beta_2-\alpha_3\beta_1+\alpha_3\beta_6\\ -\sqrt{2}(\alpha_1\beta_5-\alpha_2\beta_3)\\ \sqrt{2}\alpha_1\beta_1+\alpha_2\beta_4+\alpha_3\beta_3\end{pmatrix}\end{cases}$$

$$\mathbf{4}\otimes\mathbf{4}=\mathbf{1}_{\rm s}\oplus\mathbf{3}_{\rm a}\oplus\mathbf{3}'_{\rm a}\oplus\mathbf{4}_{\rm s}\oplus\mathbf{5}_{\rm s}$$
$$\begin{cases}\mathbf{1}_{\rm s}: \ \frac{1}{2}(\alpha_1\beta_4+\alpha_2\beta_3+\alpha_3\beta_2+\alpha_4\beta_1)\\ \mathbf{3}_{\rm a}: \ \frac{1}{2}\begin{pmatrix}-\alpha_1\beta_4+\alpha_2\beta_3-\alpha_3\beta_2+\alpha_4\beta_1\\ \sqrt{2}(\alpha_2\beta_4-\alpha_4\beta_2)\\ \sqrt{2}(\alpha_1\beta_3-\alpha_3\beta_1)\end{pmatrix}\\ \mathbf{3}'_{\rm a}: \ \frac{1}{2}\begin{pmatrix}\alpha_1\beta_4+\alpha_2\beta_3-\alpha_3\beta_2-\alpha_4\beta_1\\ \sqrt{2}(\alpha_3\beta_4-\alpha_4\beta_3)\\ \sqrt{2}(\alpha_1\beta_2-\alpha_2\beta_1)\end{pmatrix}\\ \mathbf{4}_{\rm s}: \ \frac{\sqrt{3}}{3}\begin{pmatrix}\alpha_2\beta_4+\alpha_3\beta_3+\alpha_4\beta_2\\ \alpha_1\beta_1+\alpha_3\beta_4+\alpha_4\beta_3\\ \alpha_1\beta_2+\alpha_2\beta_1+\alpha_4\beta_4\\ \alpha_1\beta_3+\alpha_2\beta_2+\alpha_3\beta_1\end{pmatrix}\\ \mathbf{5}_{\rm s}: \ \frac{\sqrt{3}}{6}\begin{pmatrix}\sqrt{3}(\alpha_1\beta_4-\alpha_2\beta_3-\alpha_3\beta_2+\alpha_4\beta_1)\\ -\sqrt{2}(\alpha_2\beta_4-2\alpha_3\beta_3+\alpha_4\beta_2)\\ -\sqrt{2}(2\alpha_1\beta_1-\alpha_3\beta_4-\alpha_4\beta_3)\\ \sqrt{2}(\alpha_1\beta_2+\alpha_2\beta_1-2\alpha_4\beta_4)\\ -\sqrt{2}(\alpha_1\beta_3-2\alpha_2\beta_2+\alpha_3\beta_1)\end{pmatrix}\end{cases}$$

$$\hat{\mathbf{4}}\otimes\hat{\mathbf{4}}=\mathbf{1}_{\rm a}\oplus\mathbf{3}_{\rm s}\oplus\mathbf{3}'_{\rm s}\oplus\mathbf{4}_{\rm s}\oplus\mathbf{5}_{\rm s}$$
$$\begin{cases}\mathbf{1}_{\rm a}: \ \frac{1}{2}(\alpha_1\beta_4+\alpha_2\beta_3-\alpha_3\beta_2-\alpha_4\beta_1)\\ \mathbf{3}_{\rm s}: \ -\frac{\sqrt{5}}{10}\begin{pmatrix}3\alpha_1\beta_4+\alpha_2\beta_3+\alpha_3\beta_2+3\alpha_4\beta_1\\ \sqrt{2}(\sqrt{3}\alpha_2\beta_4-2\alpha_3\beta_3+\sqrt{3}\alpha_4\beta_2)\\ \sqrt{2}(\sqrt{3}\alpha_1\beta_3+2\alpha_2\beta_2+\sqrt{3}\alpha_3\beta_1)\end{pmatrix}\\ \mathbf{3}'_{\rm s}: \ -\frac{\sqrt{5}}{10}\begin{pmatrix}\alpha_1\beta_4-3\alpha_2\beta_3-3\alpha_3\beta_2+\alpha_4\beta_1\\ \sqrt{2}(2\alpha_1\beta_1-\sqrt{3}\alpha_3\beta_4-\sqrt{3}\alpha_4\beta_3)\\ \sqrt{2}(\sqrt{3}\alpha_1\beta_2+\sqrt{3}\alpha_2\beta_1-2\alpha_4\beta_4)\end{pmatrix}\\ \mathbf{4}_{\rm s}: \ \frac{\sqrt{5}}{5}\begin{pmatrix}\alpha_2\beta_4+\sqrt{3}\alpha_3\beta_3+\alpha_4\beta_2\\ -\sqrt{3}\alpha_1\beta_1-\alpha_3\beta_4-\alpha_4\beta_3\\ -\alpha_1\beta_2-\alpha_2\beta_1-\sqrt{3}\alpha_4\beta_4\\ -\alpha_1\beta_3+\sqrt{3}\alpha_2\beta_2-\alpha_3\beta_1\end{pmatrix}\\ \mathbf{5}_{\rm a}: \ \frac{1}{2}\begin{pmatrix}\alpha_1\beta_4-\alpha_2\beta_3+\alpha_3\beta_2-\alpha_4\beta_1\\ -\sqrt{2}(\alpha_2\beta_4-\alpha_4\beta_2)\\ -\sqrt{2}(\alpha_3\beta_4-\alpha_4\beta_3)\\ \sqrt{2}(\alpha_1\beta_2-\alpha_2\beta_1)\\ -\sqrt{2}(\alpha_1\beta_3-\alpha_3\beta_1)\end{pmatrix}\end{cases}$$

$$\mathbf{4}\otimes\hat{\mathbf{4}}=\hat{\mathbf{4}}\oplus\hat{\mathbf{6}}_1\oplus\hat{\mathbf{6}}_2$$
$$\begin{cases}\hat{\mathbf{4}}: \ \frac{\sqrt{5}}{5}\begin{pmatrix}-\sqrt{3}\alpha_2\beta_4-\alpha_3\beta_3+\alpha_4\beta_2\\ -\alpha_1\beta_1-\alpha_3\beta_4+\sqrt{3}\alpha_4\beta_3\\ -\sqrt{3}\alpha_1\beta_2+\alpha_2\beta_1-\alpha_4\beta_4\\ \alpha_1\beta_3+\alpha_2\beta_2+\sqrt{3}\alpha_3\beta_1\end{pmatrix}\\ \hat{\mathbf{6}}_1: \ \frac{1}{2}\begin{pmatrix}\alpha_3\beta_2-\sqrt{3}\alpha_4\beta_1\\ -\sqrt{2}(\alpha_3\beta_3+\alpha_4\beta_2)\\ \sqrt{3}\alpha_3\beta_4+\alpha_4\beta_3\\ \alpha_1\beta_2+\sqrt{3}\alpha_2\beta_1\\ -\sqrt{2}(\alpha_1\beta_3-\alpha_2\beta_2)\\ -\sqrt{3}\alpha_1\beta_4+\alpha_2\beta_3\end{pmatrix}\\ \hat{\mathbf{6}}_2: \ \frac{\sqrt{2}}{4}\begin{pmatrix}\alpha_1\beta_4+\sqrt{3}\alpha_2\beta_3-2\alpha_4\beta_1\\ \sqrt{2}(\alpha_2\beta_4-\sqrt{3}\alpha_3\beta_3)\\ 2\alpha_1\beta_1+\alpha_3\beta_4+\sqrt{3}\alpha_4\beta_3\\ \sqrt{3}\alpha_1\beta_2+\alpha_2\beta_1-2\alpha_4\beta_4\\ \sqrt{2}(\sqrt{3}\alpha_2\beta_2-\alpha_3\beta_1)\\ -2\alpha_1\beta_4-\sqrt{3}\alpha_3\beta_2-\alpha_4\beta_1\end{pmatrix}\end{cases}$$

$$\mathbf{4}\otimes\mathbf{5}=\mathbf{3}\oplus\mathbf{3}'\oplus\mathbf{4}\oplus\mathbf{5}_1\oplus\mathbf{5}_2$$

$$\begin{cases}
\mathbf{3}:\ \frac{\sqrt{5}}{10}\begin{pmatrix}\sqrt{2}(2\alpha_1\beta_5-\alpha_2\beta_4+\alpha_3\beta_3-2\alpha_4\beta_2)\\ -\sqrt{6}\alpha_1\beta_1+2\alpha_2\beta_5+3\alpha_3\beta_4-\alpha_4\beta_3\\ \alpha_1\beta_4-3\alpha_2\beta_3-2\alpha_3\beta_2+\sqrt{6}\alpha_4\beta_1\end{pmatrix}\\
\mathbf{3}':\ \frac{\sqrt{5}}{10}\begin{pmatrix}\sqrt{2}(\alpha_1\beta_5+2\alpha_2\beta_4-2\alpha_3\beta_3-\alpha_4\beta_2)\\ 3\alpha_1\beta_2-\sqrt{6}\alpha_2\beta_1-\alpha_3\beta_5+2\alpha_4\beta_4\\ -2\alpha_1\beta_3+\alpha_2\beta_2+\sqrt{6}\alpha_3\beta_1-3\alpha_4\beta_5\end{pmatrix}\\
\mathbf{4}:\ \frac{\sqrt{15}}{15}\begin{pmatrix}\sqrt{3}\alpha_1\beta_1-\sqrt{2}\alpha_2\beta_5+\sqrt{2}\alpha_3\beta_4-2\sqrt{2}\alpha_4\beta_3\\ -\sqrt{2}\alpha_1\beta_2-\sqrt{3}\alpha_2\beta_1+2\sqrt{2}\alpha_3\beta_5+\sqrt{2}\alpha_4\beta_4\\ \sqrt{2}\alpha_1\beta_3+2\sqrt{2}\alpha_2\beta_2-\sqrt{3}\alpha_3\beta_1-\sqrt{2}\alpha_4\beta_5\\ -2\sqrt{2}\alpha_1\beta_4+\sqrt{2}\alpha_2\beta_3-\sqrt{2}\alpha_3\beta_2+\sqrt{3}\alpha_4\beta_1\end{pmatrix}\\
\mathbf{5}_1:\ \frac{\sqrt{2}}{4}\begin{pmatrix}\sqrt{2}(\alpha_1\beta_5-\alpha_2\beta_4-\alpha_3\beta_3+\alpha_4\beta_2)\\ -\sqrt{2}\alpha_1\beta_1-\sqrt{3}\alpha_3\beta_4-\sqrt{3}\alpha_4\beta_3\\ \sqrt{3}\alpha_1\beta_2+\sqrt{2}\alpha_2\beta_1+\sqrt{3}\alpha_3\beta_5\\ \sqrt{3}\alpha_2\beta_2+\sqrt{2}\alpha_3\beta_1+\sqrt{3}\alpha_4\beta_5\\ -\sqrt{3}\alpha_1\beta_4-\sqrt{3}\alpha_2\beta_3-\sqrt{2}\alpha_4\beta_1\end{pmatrix}\\
\mathbf{5}_2:\ \frac{\sqrt{5}}{10}\begin{pmatrix}\sqrt{2}(\alpha_1\beta_5+2\alpha_2\beta_4+2\alpha_3\beta_3+\alpha_4\beta_2)\\ 2\sqrt{2}\alpha_1\beta_1+2\sqrt{3}\alpha_2\beta_5\\ -\sqrt{3}\alpha_1\beta_2+\sqrt{2}\alpha_2\beta_1-\sqrt{3}\alpha_3\beta_5+2\sqrt{3}\alpha_4\beta_4\\ 2\sqrt{3}\alpha_1\beta_3-\sqrt{3}\alpha_2\beta_2+\sqrt{2}\alpha_3\beta_1-\sqrt{3}\alpha_4\beta_5\\ 2\sqrt{3}\alpha_3\beta_2+2\sqrt{2}\alpha_4\beta_1\end{pmatrix}
\end{cases}$$

$$\hat{\mathbf{4}}\otimes\mathbf{5}=\hat{\mathbf{2}}\oplus\hat{\mathbf{2}}'\oplus\hat{\mathbf{4}}\oplus\hat{\mathbf{6}}_1\oplus\hat{\mathbf{6}}_2$$

$$\begin{cases}
\hat{\mathbf{2}}':\ \frac{\sqrt{10}}{10}\begin{pmatrix}\alpha_1\beta_2+\sqrt{2}\alpha_2\beta_1+\sqrt{3}\alpha_3\beta_5+2\alpha_4\beta_4\\ 2\alpha_1\beta_3+\sqrt{3}\alpha_2\beta_2-\sqrt{2}\alpha_3\beta_1-\alpha_4\beta_5\end{pmatrix}\quad \hat{\mathbf{2}}':\ \frac{\sqrt{10}}{10}\begin{pmatrix}\sqrt{2}\alpha_1\beta_1+2\alpha_2\beta_5-\alpha_3\beta_4+\sqrt{3}\alpha_4\beta_3\\ -\sqrt{3}\alpha_1\beta_4+\alpha_2\beta_3+2\alpha_3\beta_2+\sqrt{2}\alpha_4\beta_1\end{pmatrix}\\
\hat{\mathbf{4}}:\ \frac{\sqrt{5}}{5}\begin{pmatrix}\alpha_1\beta_1-\sqrt{2}\alpha_2\beta_5-\sqrt{2}\alpha_3\beta_4\\ -\sqrt{2}\alpha_1\beta_2-\alpha_2\beta_1+\sqrt{2}\alpha_4\beta_4\\ -\sqrt{2}\alpha_1\beta_3-\alpha_3\beta_1-\sqrt{2}\alpha_4\beta_5\\ \sqrt{2}\alpha_2\beta_3-\sqrt{2}\alpha_3\beta_2+\alpha_4\beta_1\end{pmatrix}\\
\hat{\mathbf{6}}_1:\ \frac{\sqrt{10}}{10}\begin{pmatrix}-\sqrt{3}\alpha_1\beta_5-2\alpha_2\beta_4+\sqrt{3}\alpha_4\beta_2\\ \sqrt{2}(\alpha_2\beta_5-\alpha_3\beta_4-\sqrt{3}\alpha_4\beta_3)\\ -\sqrt{3}\alpha_1\beta_2+\sqrt{6}\alpha_2\beta_1-\alpha_3\beta_5\\ -\alpha_2\beta_2-\sqrt{6}\alpha_3\beta_1+\sqrt{3}\alpha_4\beta_5\\ \sqrt{2}(\sqrt{3}\alpha_1\beta_4+\alpha_2\beta_3+\alpha_3\beta_2)\\ -\sqrt{3}\alpha_1\beta_5-2\alpha_3\beta_3-\sqrt{3}\alpha_4\beta_2\end{pmatrix}\quad \hat{\mathbf{6}}_2:\ \frac{\sqrt{10}}{10}\begin{pmatrix}\sqrt{2}(\alpha_1\beta_5-\sqrt{3}\alpha_3\beta_3+\alpha_4\beta_2)\\ -\sqrt{6}\alpha_1\beta_1-\sqrt{3}\alpha_3\beta_4+\alpha_4\beta_3\\ -\sqrt{2}(\alpha_1\beta_2-\sqrt{3}\alpha_3\beta_5+\alpha_4\beta_4)\\ -\sqrt{2}(\alpha_1\beta_3-\sqrt{3}\alpha_2\beta_2-\alpha_4\beta_5)\\ -\alpha_1\beta_4+\sqrt{3}\alpha_2\beta_3-\sqrt{6}\alpha_4\beta_1\\ -\sqrt{2}(\alpha_1\beta_5-\sqrt{3}\alpha_2\beta_4-\alpha_4\beta_2)\end{pmatrix}
\end{cases}$$

$$\mathbf{4}\otimes\hat{\mathbf{6}} = \hat{\mathbf{2}}\oplus\hat{\mathbf{2}}'\oplus\hat{\mathbf{4}}_1\oplus\hat{\mathbf{4}}_2\oplus\hat{\mathbf{6}}_1\oplus\hat{\mathbf{6}}_2$$

$$\left\{\begin{array}{l}
\hat{\mathbf{2}}:\ \frac{\sqrt{6}}{6}\begin{pmatrix}\alpha_1\beta_2+\sqrt{2}\alpha_2\beta_1-\alpha_3\beta_5+\sqrt{2}\alpha_4\beta_4\\ -\sqrt{2}\alpha_1\beta_3-\alpha_2\beta_2-\sqrt{2}\alpha_3\beta_6-\alpha_4\beta_5\end{pmatrix}\quad \hat{\mathbf{2}}':\ \frac{\sqrt{6}}{6}\begin{pmatrix}\alpha_1\beta_1-\alpha_1\beta_6-\sqrt{2}\alpha_2\beta_5+\alpha_3\beta_4-\alpha_4\beta_3\\ \alpha_1\beta_4+\alpha_2\beta_3+\sqrt{2}\alpha_3\beta_2+\alpha_4\beta_1+\alpha_4\beta_6\end{pmatrix}\\
\hat{\mathbf{4}}_1:\ \frac{\sqrt{6}}{6}\begin{pmatrix}\sqrt{3}(\alpha_1\beta_1-\alpha_3\beta_4)\\ \sqrt{2}\alpha_1\beta_2-\alpha_2\beta_1-\sqrt{2}\alpha_3\beta_5-\alpha_4\beta_4\\ -\alpha_1\beta_3+\sqrt{2}\alpha_3\beta_2-\alpha_3\beta_6+\sqrt{2}\alpha_4\beta_5\\ -\sqrt{3}(\alpha_2\beta_2-\alpha_4\beta_6)\end{pmatrix}\\
\hat{\mathbf{4}}_2:\ \frac{\sqrt{3}}{6}\begin{pmatrix}\alpha_1\beta_1-\alpha_1\beta_6-\sqrt{2}\alpha_2\beta_5-2\alpha_3\beta_4+2\alpha_4\beta_3\\ \sqrt{3}(\sqrt{2}\alpha_1\beta_2-\alpha_2\beta_1-\alpha_2\beta_6)\\ \sqrt{3}(\alpha_3\beta_1-\alpha_3\beta_6+\sqrt{2}\alpha_4\beta_5)\\ -2\alpha_1\beta_4-2\alpha_2\beta_3+\sqrt{2}\alpha_3\beta_2+\alpha_4\beta_1+\alpha_4\beta_6\end{pmatrix}\\
\hat{\mathbf{6}}_1:\ -\frac{\sqrt{6}}{6}\begin{pmatrix}\alpha_1\beta_5+\sqrt{2}\alpha_2\beta_4-\sqrt{2}\alpha_3\beta_3+\alpha_4\beta_2\\ \sqrt{2}(\sqrt{2}\alpha_1\beta_6-\alpha_2\beta_5)\\ \alpha_1\beta_2+\sqrt{2}\alpha_2\beta_1+\alpha_3\beta_5-\sqrt{2}\alpha_4\beta_4\\ \sqrt{2}\alpha_1\beta_3+\alpha_2\beta_2-\sqrt{2}\alpha_3\beta_6-\alpha_4\beta_5\\ \sqrt{2}(\alpha_3\beta_2-\sqrt{2}\alpha_4\beta_1)\\ \alpha_1\beta_5+\sqrt{2}\alpha_2\beta_4+\sqrt{2}\alpha_3\beta_3-\alpha_4\beta_2\end{pmatrix}\quad \hat{\mathbf{6}}_2:\ \frac{1}{2}\begin{pmatrix}\alpha_1\beta_5-\sqrt{2}\alpha_2\beta_4-\alpha_4\beta_2\\ -\alpha_1\beta_1-\alpha_1\beta_6-\alpha_3\beta_4-\alpha_4\beta_3\\ -\alpha_1\beta_2-\sqrt{2}\alpha_2\beta_6-\alpha_3\beta_5\\ -\alpha_2\beta_2-\sqrt{2}\alpha_3\beta_1+\alpha_4\beta_5\\ \alpha_1\beta_4-\alpha_2\beta_3+\alpha_4\beta_1-\alpha_4\beta_6\\ -\alpha_1\beta_5-\sqrt{2}\alpha_3\beta_3-\alpha_4\beta_2\end{pmatrix}
\end{array}\right.$$

$$\hat{\mathbf{4}}\otimes\hat{\mathbf{6}} = \mathbf{3}\oplus\mathbf{3}'\oplus\mathbf{4}_1\oplus\mathbf{4}_2\oplus\mathbf{5}_1\oplus\mathbf{5}_2$$

$$\left\{\begin{array}{l}
\mathbf{3}:\ -\frac{\sqrt{5}}{10}\begin{pmatrix}2\alpha_1\beta_5+\sqrt{6}\alpha_2\beta_4-\sqrt{6}\alpha_3\beta_3+2\alpha_4\beta_2\\ 3\alpha_1\beta_1+\alpha_1\beta_6-\sqrt{6}\alpha_2\beta_5+\sqrt{3}\alpha_3\beta_4-\alpha_4\beta_3\\ \alpha_1\beta_4+\sqrt{3}\alpha_2\beta_3-\sqrt{6}\alpha_3\beta_2+\alpha_4\beta_1-3\alpha_4\beta_6\end{pmatrix}\\
\mathbf{3}':\ \frac{\sqrt{5}}{10}\begin{pmatrix}-\sqrt{6}\alpha_1\beta_5+2\alpha_2\beta_4-2\alpha_3\beta_3-\sqrt{6}\alpha_4\beta_2\\ \sqrt{3}\alpha_1\beta_2-\sqrt{2}\alpha_2\beta_1-2\sqrt{2}\alpha_2\beta_6+\alpha_3\beta_5-\sqrt{6}\alpha_4\beta_4\\ -\sqrt{6}\alpha_1\beta_3-\alpha_2\beta_2-2\sqrt{2}\alpha_3\beta_1+\sqrt{2}\alpha_3\beta_6-\sqrt{3}\alpha_4\beta_5\end{pmatrix}\\
\mathbf{4}_1:\ \frac{\sqrt{6}}{6}\begin{pmatrix}\sqrt{3}\alpha_1\beta_6+\sqrt{2}\alpha_2\beta_5+\alpha_3\beta_4\\ -\alpha_2\beta_6+\sqrt{2}\alpha_3\beta_5+\sqrt{3}\alpha_4\beta_4\\ -\sqrt{3}\alpha_1\beta_3+\sqrt{2}\alpha_2\beta_2+\alpha_3\beta_1\\ -\alpha_2\beta_3-\sqrt{2}\alpha_3\beta_2-\sqrt{3}\alpha_4\beta_1\end{pmatrix}\quad \mathbf{4}_2:\ \frac{\sqrt{3}}{6}\begin{pmatrix}\alpha_1\beta_1-2\alpha_1\beta_6-\sqrt{3}\alpha_3\beta_4-2\alpha_4\beta_3\\ \sqrt{2}\alpha_1\beta_2+\sqrt{3}\alpha_2\beta_1-\sqrt{6}\alpha_3\beta_5-\alpha_4\beta_4\\ \alpha_1\beta_3-\sqrt{6}\alpha_2\beta_2+\sqrt{3}\alpha_3\beta_6+\sqrt{2}\alpha_4\beta_5\\ -2\alpha_1\beta_4+\sqrt{3}\alpha_2\beta_3+2\alpha_4\beta_1+\alpha_4\beta_6\end{pmatrix}\\
\mathbf{5}_1:\ \frac{\sqrt{3}}{6}\begin{pmatrix}\sqrt{6}(\alpha_2\beta_4+\alpha_3\beta_3)\\ -\sqrt{3}\alpha_1\beta_1+\sqrt{3}\alpha_1\beta_6-\sqrt{2}\alpha_2\beta_5-\alpha_3\beta_4-\sqrt{3}\alpha_4\beta_3\\ \sqrt{2}(\sqrt{3}\alpha_1\beta_2+\sqrt{2}\alpha_2\beta_6+\alpha_3\beta_5)\\ \sqrt{2}(\alpha_2\beta_2-\sqrt{2}\alpha_3\beta_1+\sqrt{3}\alpha_4\beta_5)\\ -\sqrt{3}\alpha_1\beta_4+\alpha_2\beta_3+\sqrt{2}\alpha_3\beta_2-\sqrt{3}\alpha_4\beta_1-\sqrt{3}\alpha_4\beta_6\end{pmatrix}\\
\mathbf{5}_2:\ \frac{\sqrt{3}}{6}\begin{pmatrix}\sqrt{6}(\alpha_1\beta_5-\alpha_4\beta_2)\\ -\sqrt{2}(\alpha_1\beta_1+\alpha_1\beta_6-\sqrt{3}\alpha_3\beta_4+\alpha_4\beta_3)\\ -\alpha_1\beta_2+\sqrt{6}\alpha_2\beta_1+\sqrt{3}\alpha_3\beta_5-\sqrt{2}\alpha_4\beta_4\\ \sqrt{2}\alpha_1\beta_3+\sqrt{3}\alpha_2\beta_2+\sqrt{6}\alpha_3\beta_6-\alpha_4\beta_5\\ -\sqrt{2}(\alpha_1\beta_4+\sqrt{3}\alpha_2\beta_3-\alpha_4\beta_1+\alpha_4\beta_6)\end{pmatrix}
\end{array}\right.$$

$$\mathbf{5}\otimes\mathbf{5}=\mathbf{1}_{\rm s}\oplus\mathbf{3}_{\rm a}\oplus\mathbf{3}'_{\rm a}\oplus\mathbf{4}_{\rm s}\oplus\mathbf{4}_{\rm a}\oplus\mathbf{5}_{\rm s,1}\oplus\mathbf{5}_{\rm s,2}$$

$$\begin{cases}
\mathbf{1}_{\rm s}:\ \frac{\sqrt{5}}{5}(\alpha_1\beta_1+\alpha_2\beta_5+\alpha_3\beta_4+\alpha_4\beta_3+\alpha_5\beta_2) \\
\mathbf{3}_{\rm a}:\ \frac{\sqrt{10}}{10}\begin{pmatrix}\alpha_2\beta_5+2\alpha_3\beta_4-2\alpha_4\beta_3-\alpha_5\beta_2\\ -\sqrt{3}\alpha_1\beta_2+\sqrt{3}\alpha_2\beta_1+\sqrt{2}\alpha_3\beta_5-\sqrt{2}\alpha_5\beta_3\\ \sqrt{3}\alpha_1\beta_5+\sqrt{2}\alpha_2\beta_4-\sqrt{2}\alpha_4\beta_2-\sqrt{3}\alpha_5\beta_1\end{pmatrix} \\
\mathbf{3}'_{\rm a}:\ \frac{\sqrt{10}}{10}\begin{pmatrix}2\alpha_2\beta_5-\alpha_3\beta_4+\alpha_4\beta_3-2\alpha_5\beta_2\\ \sqrt{3}\alpha_1\beta_3-\sqrt{3}\alpha_3\beta_1+\sqrt{2}\alpha_4\beta_5-\sqrt{2}\alpha_5\beta_4\\ -\sqrt{3}\alpha_1\beta_4+\sqrt{2}\alpha_2\beta_3-\sqrt{2}\alpha_3\beta_2+\sqrt{3}\alpha_4\beta_1\end{pmatrix} \\
\mathbf{4}_{\rm s}:\ \frac{\sqrt{30}}{30}\begin{pmatrix}\sqrt{6}\alpha_1\beta_2+\sqrt{6}\alpha_2\beta_1-\alpha_3\beta_5+4\alpha_4\beta_4-\alpha_5\beta_3\\ \sqrt{6}\alpha_1\beta_3+4\alpha_2\beta_2+\sqrt{6}\alpha_3\beta_1-\alpha_4\beta_5-\alpha_5\beta_4\\ \sqrt{6}\alpha_1\beta_4-\alpha_2\beta_3-\alpha_3\beta_2+\sqrt{6}\alpha_4\beta_1+4\alpha_5\beta_5\\ \sqrt{6}\alpha_1\beta_5-\alpha_2\beta_4+4\alpha_3\beta_3-\alpha_4\beta_2+\sqrt{6}\alpha_5\beta_1\end{pmatrix} \\
\mathbf{4}_{\rm a}:\ \frac{\sqrt{10}}{10}\begin{pmatrix}\sqrt{2}\alpha_1\beta_2-\sqrt{2}\alpha_2\beta_1+\sqrt{3}\alpha_3\beta_5-\sqrt{3}\alpha_5\beta_3\\ -\sqrt{2}\alpha_1\beta_3+\sqrt{2}\alpha_3\beta_1+\sqrt{3}\alpha_4\beta_5-\sqrt{3}\alpha_5\beta_4\\ -\sqrt{2}\alpha_1\beta_4-\sqrt{3}\alpha_2\beta_3+\sqrt{3}\alpha_3\beta_2+\sqrt{2}\alpha_4\beta_1\\ \sqrt{2}\alpha_1\beta_5-\sqrt{3}\alpha_2\beta_4+\sqrt{3}\alpha_4\beta_2-\sqrt{2}\alpha_5\beta_1\end{pmatrix} \\
\mathbf{5}_{\rm s,1}:\ \frac{\sqrt{14}}{14}\begin{pmatrix}2\alpha_1\beta_1+\alpha_2\beta_5-2\alpha_3\beta_4-2\alpha_4\beta_3+\alpha_5\beta_2\\ \alpha_1\beta_2+\alpha_2\beta_1+\sqrt{6}\alpha_3\beta_5+\sqrt{6}\alpha_5\beta_3\\ -2\alpha_1\beta_3+\sqrt{6}\alpha_2\beta_2-2\alpha_3\beta_1\\ -2\alpha_1\beta_4-2\alpha_4\beta_1+\sqrt{6}\alpha_5\beta_5\\ \alpha_1\beta_5+\sqrt{6}\alpha_2\beta_4+\sqrt{6}\alpha_4\beta_2+\alpha_5\beta_1\end{pmatrix} \\
\mathbf{5}_{\rm s,2}:\ \frac{\sqrt{14}}{14}\begin{pmatrix}2\alpha_1\beta_1-2\alpha_2\beta_5+\alpha_3\beta_4+\alpha_4\beta_3-2\alpha_5\beta_2\\ -2\alpha_1\beta_2-2\alpha_2\beta_1+\sqrt{6}\alpha_4\beta_4\\ \alpha_1\beta_3+\alpha_3\beta_1+\sqrt{6}\alpha_4\beta_5+\sqrt{6}\alpha_5\beta_4\\ \alpha_1\beta_4+\sqrt{6}\alpha_2\beta_3+\sqrt{6}\alpha_3\beta_2+\alpha_4\beta_1\\ -2\alpha_1\beta_5+\sqrt{6}\alpha_3\beta_3-2\alpha_5\beta_1\end{pmatrix}
\end{cases}$$

$$\mathbf{5}\otimes\hat{\mathbf{6}}=\hat{\mathbf{2}}\oplus\hat{\mathbf{2}}'\oplus\hat{\mathbf{4}}_1\oplus\hat{\mathbf{4}}_2\oplus\hat{\mathbf{6}}_1\oplus\hat{\mathbf{6}}_2\oplus\hat{\mathbf{6}}_3$$

$$\begin{cases}
\hat{\mathbf{2}}:\ -\frac{\sqrt{30}}{30}\begin{pmatrix}\sqrt{6}\alpha_1\beta_3-2\sqrt{2}\alpha_2\beta_2+\alpha_3\beta_1-3\alpha_3\beta_6+\sqrt{2}\alpha_4\beta_5+2\alpha_5\beta_4\\ \sqrt{6}\alpha_1\beta_4-2\alpha_2\beta_3+\sqrt{2}\alpha_3\beta_2-3\alpha_4\beta_1-\alpha_4\beta_6+2\sqrt{2}\alpha_5\beta_5\end{pmatrix}\\
\hat{\mathbf{2}}':\ \frac{\sqrt{15}}{15}\begin{pmatrix}\sqrt{3}\alpha_1\beta_2-\alpha_2\beta_1-2\alpha_2\beta_6-\sqrt{2}\alpha_3\beta_5-2\alpha_4\beta_4+\alpha_5\beta_3\\ \sqrt{3}\alpha_1\beta_5-\alpha_2\beta_4-2\alpha_3\beta_3+\sqrt{2}\alpha_4\beta_2+2\alpha_5\beta_1-\alpha_5\beta_6\end{pmatrix}\\
\hat{\mathbf{4}}_1:\ \frac{\sqrt{15}}{15}\begin{pmatrix}-\sqrt{6}\alpha_1\beta_2+\sqrt{2}\alpha_2\beta_1-\sqrt{2}\alpha_2\beta_6-\alpha_3\beta_5-\sqrt{2}\alpha_4\beta_4-\sqrt{2}\alpha_5\beta_3\\ \sqrt{2}(\sqrt{3}\alpha_3\beta_6+\alpha_4\beta_5+\sqrt{2}\alpha_5\beta_4)\\ \sqrt{2}(\sqrt{3}\alpha_2\beta_3-\alpha_3\beta_2-\sqrt{2}\alpha_4\beta_1)\\ -\sqrt{6}\alpha_1\beta_5+\sqrt{2}\alpha_2\beta_4-\sqrt{2}\alpha_3\beta_3+\alpha_4\beta_2+\sqrt{2}\alpha_5\beta_1+\sqrt{2}\alpha_5\beta_6\end{pmatrix}\\
\hat{\mathbf{4}}_2:\ \frac{\sqrt{15}}{15}\begin{pmatrix}-\sqrt{3}(\alpha_2\beta_1+\alpha_2\beta_6-\sqrt{2}\alpha_3\beta_5+\alpha_5\beta_3)\\ \sqrt{6}\alpha_1\beta_3+\sqrt{2}\alpha_2\beta_2-2\alpha_3\beta_1+\sqrt{2}\alpha_4\beta_5-\alpha_5\beta_4\\ -\sqrt{6}\alpha_1\beta_4-\alpha_2\beta_3-\sqrt{2}\alpha_3\beta_2-2\alpha_4\beta_6+\sqrt{2}\alpha_5\beta_5\\ \sqrt{3}(\alpha_2\beta_4-\sqrt{2}\alpha_4\beta_2+\alpha_5\beta_1-\alpha_5\beta_6)\end{pmatrix}\\
\hat{\mathbf{6}}_1:\ \frac{\sqrt{10}}{10}\begin{pmatrix}-\sqrt{6}\alpha_1\beta_6-\alpha_3\beta_4-\alpha_4\beta_3-\sqrt{2}\alpha_5\beta_2\\ \sqrt{2}(\alpha_2\beta_1+\sqrt{2}\alpha_3\beta_5-\alpha_4\beta_4+\alpha_5\beta_3)\\ -\sqrt{2}\alpha_2\beta_2+\alpha_3\beta_1+\alpha_3\beta_6+\sqrt{2}\alpha_4\beta_5-2\alpha_5\beta_4\\ 2\alpha_2\beta_3+\sqrt{2}\alpha_3\beta_2+\alpha_4\beta_1-\alpha_4\beta_6+\sqrt{2}\alpha_5\beta_5\\ -\sqrt{2}(\alpha_2\beta_4+\alpha_3\beta_3+\sqrt{2}\alpha_4\beta_2-\alpha_5\beta_6)\\ \sqrt{6}\alpha_1\beta_1-\sqrt{2}\alpha_2\beta_5+\alpha_3\beta_4-\alpha_4\beta_3\end{pmatrix}\\
\hat{\mathbf{6}}_2:\ \frac{\sqrt{10}}{10}\begin{pmatrix}\alpha_1\beta_1+\sqrt{6}\alpha_4\beta_3-\sqrt{3}\alpha_5\beta_2\\ -2\alpha_1\beta_2-\sqrt{3}\alpha_2\beta_1+\sqrt{3}\alpha_5\beta_3\\ \alpha_1\beta_3+\sqrt{3}\alpha_2\beta_2+\sqrt{6}\alpha_3\beta_1\\ \alpha_1\beta_4-\sqrt{6}\alpha_4\beta_6-\sqrt{3}\alpha_5\beta_5\\ -2\alpha_1\beta_5-\sqrt{3}\alpha_2\beta_4-\sqrt{3}\alpha_5\beta_6\\ \alpha_1\beta_6-\sqrt{3}\alpha_2\beta_5-\sqrt{6}\alpha_3\beta_4\end{pmatrix}\quad
\hat{\mathbf{6}}_3:\ \frac{\sqrt{10}}{10}\begin{pmatrix}\sqrt{3}\alpha_1\beta_1+2\alpha_2\beta_5-\sqrt{2}\alpha_3\beta_4+\alpha_5\beta_2\\ \alpha_2\beta_1-2\alpha_2\beta_6+2\alpha_4\beta_4+\alpha_5\beta_3\\ -\sqrt{3}\alpha_1\beta_3+\alpha_2\beta_2-\sqrt{2}\alpha_3\beta_6+2\alpha_4\beta_5\\ -\sqrt{3}\alpha_1\beta_4+2\alpha_3\beta_2-\sqrt{2}\alpha_4\beta_1-\alpha_5\beta_5\\ -\alpha_2\beta_4+2\alpha_3\beta_3+2\alpha_5\beta_1+\alpha_5\beta_6\\ \sqrt{3}\alpha_1\beta_6+\alpha_2\beta_5-\sqrt{2}\alpha_4\beta_3-2\alpha_5\beta_2\end{pmatrix}
\end{cases}$$

$$\hat{\mathbf{6}}\otimes\hat{\mathbf{6}}=\mathbf{1}_{\rm a}\oplus\mathbf{3}_{\rm s,1}\oplus\mathbf{3}_{\rm s,2}\oplus\mathbf{3}'_{\rm s,1}\oplus\mathbf{3}'_{\rm s,2}\oplus\mathbf{4}_{\rm s}\oplus\mathbf{4}_{\rm a}\oplus\mathbf{5}_{\rm s}\oplus\mathbf{5}_{\rm a,1}\oplus\mathbf{5}_{\rm a,2}$$

$$\begin{cases}
\mathbf{1}_{\rm a}:\ \frac{\sqrt{6}}{6}(\alpha_1\beta_6+\alpha_2\beta_5-\alpha_3\beta_4+\alpha_4\beta_3-\alpha_5\beta_2-\alpha_6\beta_1) \\
\mathbf{3}_{\rm s,1}:\ \frac{1}{2}\begin{pmatrix}\alpha_1\beta_6+\alpha_3\beta_4+\alpha_4\beta_3+\alpha_6\beta_1\\ \alpha_2\beta_6+\alpha_3\beta_5+\alpha_5\beta_3+\alpha_6\beta_2\\ \alpha_1\beta_5-\alpha_2\beta_4-\alpha_4\beta_2+\alpha_5\beta_1\end{pmatrix}\quad \mathbf{3}_{\rm s,2}:\ \frac{\sqrt{6}}{6}\begin{pmatrix}-\alpha_1\beta_1+\alpha_1\beta_6-\alpha_2\beta_5-\alpha_5\beta_2+\alpha_6\beta_1+\alpha_6\beta_6\\ \alpha_1\beta_2+\alpha_2\beta_1+\alpha_3\beta_5-\sqrt{2}\alpha_4\beta_4+\alpha_5\beta_3\\ -\alpha_2\beta_4+\sqrt{2}\alpha_3\beta_3-\alpha_4\beta_2-\alpha_5\beta_6-\alpha_6\beta_5\end{pmatrix} \\
\mathbf{3}'_{\rm s,1}:\ \frac{\sqrt{6}}{6}\begin{pmatrix}\alpha_1\beta_6+\alpha_2\beta_5-\alpha_3\beta_4-\alpha_4\beta_3+\alpha_5\beta_2+\alpha_6\beta_1\\ -\sqrt{2}(\alpha_1\beta_3-\alpha_2\beta_2+\alpha_3\beta_1)\\ -\sqrt{2}(\alpha_4\beta_6+\alpha_5\beta_5+\alpha_6\beta_4)\end{pmatrix} \\
\mathbf{3}'_{\rm s,2}:\ \frac{\sqrt{2}}{4}\begin{pmatrix}\sqrt{2}(\alpha_1\beta_1-\alpha_2\beta_5-\alpha_5\beta_2-\alpha_6\beta_6)\\ \alpha_1\beta_3+\alpha_3\beta_1-\alpha_3\beta_6+\sqrt{2}\alpha_4\beta_5+\sqrt{2}\alpha_5\beta_4-\alpha_6\beta_3\\ \alpha_1\beta_4+\sqrt{2}\alpha_2\beta_3+\sqrt{2}\alpha_3\beta_2+\alpha_4\beta_1+\alpha_4\beta_6+\alpha_6\beta_4\end{pmatrix} \\
\mathbf{4}_{\rm s}:\ \frac{\sqrt{30}}{30}\begin{pmatrix}-3\alpha_1\beta_2-3\alpha_2\beta_1-\alpha_2\beta_6+\alpha_3\beta_5-2\sqrt{2}\alpha_4\beta_4+\alpha_5\beta_3-\alpha_6\beta_2\\ \sqrt{2}\alpha_1\beta_3+2\sqrt{2}\alpha_2\beta_2+\sqrt{2}\alpha_3\beta_1+2\sqrt{2}\alpha_3\beta_6+\alpha_4\beta_5+\alpha_5\beta_4+2\sqrt{2}\alpha_6\beta_3\\ 2\sqrt{2}\alpha_1\beta_4-\alpha_2\beta_3-\alpha_3\beta_2+2\sqrt{2}\alpha_4\beta_1-\sqrt{2}\alpha_4\beta_6+2\sqrt{2}\alpha_5\beta_5-\sqrt{2}\alpha_6\beta_4\\ \alpha_1\beta_5+\alpha_2\beta_4-2\sqrt{2}\alpha_3\beta_3+\alpha_4\beta_2+\alpha_5\beta_1-3\alpha_5\beta_6-3\alpha_6\beta_5\end{pmatrix} \\
\mathbf{4}_{\rm a}:\ -\frac{\sqrt{6}}{6}\begin{pmatrix}\alpha_1\beta_2-\alpha_2\beta_1+\alpha_2\beta_6+\alpha_3\beta_5-\alpha_5\beta_3-\alpha_6\beta_2\\ \sqrt{2}\alpha_1\beta_3-\sqrt{2}\alpha_3\beta_1+\alpha_4\beta_5-\alpha_5\beta_4\\ \alpha_2\beta_3-\alpha_3\beta_2+\sqrt{2}\alpha_4\beta_6-\sqrt{2}\alpha_6\beta_4\\ \alpha_1\beta_5-\alpha_2\beta_4+\alpha_4\beta_2-\alpha_5\beta_1-\alpha_5\beta_6+\alpha_6\beta_5\end{pmatrix} \\
\mathbf{5}_{\rm s}:\ \frac{\sqrt{3}}{6}\begin{pmatrix}\sqrt{6}(\alpha_1\beta_1+\alpha_6\beta_6)\\ \sqrt{2}(\alpha_2\beta_6-\alpha_3\beta_5-\sqrt{2}\alpha_4\beta_4-\alpha_5\beta_3+\alpha_6\beta_2)\\ -\alpha_1\beta_3-2\alpha_2\beta_2-\alpha_3\beta_1+\alpha_3\beta_6+\sqrt{2}\alpha_4\beta_5+\sqrt{2}\alpha_5\beta_4+\alpha_6\beta_3\\ \alpha_1\beta_4-\sqrt{2}\alpha_2\beta_3-\sqrt{2}\alpha_3\beta_2+\alpha_4\beta_1+\alpha_4\beta_6-2\alpha_5\beta_5+\alpha_6\beta_4\\ -\sqrt{2}(\alpha_1\beta_5+\alpha_2\beta_4+\sqrt{2}\alpha_3\beta_3+\alpha_4\beta_2+\alpha_5\beta_1)\end{pmatrix} \\
\mathbf{5}_{\rm a,1}:\ \frac{\sqrt{3}}{6}\begin{pmatrix}\alpha_1\beta_6-2\alpha_2\beta_5-\alpha_3\beta_4+\alpha_4\beta_3+2\alpha_5\beta_2-\alpha_6\beta_1\\ -\sqrt{3}(\alpha_2\beta_6-\alpha_3\beta_5+\alpha_5\beta_3-\alpha_6\beta_2)\\ \sqrt{6}(\alpha_3\beta_6-\alpha_6\beta_3)\\ -\sqrt{6}(\alpha_1\beta_4-\alpha_4\beta_1)\\ -\sqrt{3}(\alpha_1\beta_5+\alpha_2\beta_4-\alpha_4\beta_2-\alpha_5\beta_1)\end{pmatrix} \\
\mathbf{5}_{\rm a,2}:\ \frac{\sqrt{3}}{6}\begin{pmatrix}\sqrt{3}(\alpha_1\beta_6+\alpha_3\beta_4-\alpha_4\beta_3-\alpha_6\beta_1)\\ -2\alpha_1\beta_2+2\alpha_2\beta_1+\alpha_2\beta_6+\alpha_3\beta_5-\alpha_5\beta_3-\alpha_6\beta_2\\ -\sqrt{2}(\alpha_1\beta_3-\alpha_3\beta_1-\sqrt{2}\alpha_4\beta_5+\sqrt{2}\alpha_5\beta_4)\\ \sqrt{2}(\sqrt{2}\alpha_2\beta_3-\sqrt{2}\alpha_3\beta_2-\alpha_4\beta_6+\alpha_6\beta_4)\\ \alpha_1\beta_5-\alpha_2\beta_4+\alpha_4\beta_2-\alpha_5\beta_1+2\alpha_5\beta_6-2\alpha_6\beta_5\end{pmatrix}.
\end{cases}$$

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