New solutions for rotating boson stars

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It has been shown that scalar fields can form gravitationally bound compact objects called boson stars. In this study, we analyze boson star configurations where the scalar fields contain a small amount of angular momentum and find two new classes of solutions. In the first case all particles are in the same slowly rotating state, and in the second case the majority of particles are in the nonrotating ground state and a small number of particles are in an excited rotating state. In both cases, we solve the underlying Gross-Pitaevskii-Poisson equations that describe the profile of these compact objects both numerically as well as analytically through series expansions.

I. INTRODUCTION

If light bosons, such as axions, form dark matter, it is potentially possible for them to collapse into bound compact objects, which are called boson stars [1–3] or axion stars [4–6]. Considerable work has been done in determining the wave functions for these compact objects, either numerically or semianalytically in both nonrelativistic and relativistic frameworks [7–22]. For a detailed comparison of the approximation methods and ansatz used in the literature see [23].

Rotating boson star configurations have also been studied, but all known solutions (that we have found in the literature) have the property that the total angular momentum increases proportionally to the mass of the star (e.g., [24–29]). In these solutions, the ratio of the angular momentum to the number of particles has a minimum value, and hence for a fixed number of particles, these solutions do not include configurations of rotating boson stars with an arbitrarily small angular momentum.

In this paper we remedy this gap by finding new solutions which carry an arbitrarily small angular momentum for a fixed number of particles. We in fact find *two* different classes of such solutions.

Our first approach is a generalization of the solutions which exist in the literature, where all the particles are in the same state. However, we impose that the total angular momentum in the bosons is constrained to be fixed at a small value. This produces a state dominated by a spherical component, with a small admixture of a higher harmonic, naturally leading to a star with a small rotation. Our second approach is to take a small number of particles in the star to be in a higher spherical harmonic, while most of the particles are in the nonrotating state. Note that it is clear that such a solution must exist; for instance, if a single particle is placed in a $\ell = 1$ harmonic, there is no lower energy state with this angular momentum. We shall call these two ansätze, respectively, the one-state and two-state solutions. We show that both these approaches successfully yield solutions for a rotating star with a small angular momentum.

This paper is organized as follows: In order to set our notation and to connect to previous work, we first review our previous results for the case of nonrotating boson stars in Sec. II. We then turn to the rotating star: we consider the one-state ansatz in Sec. III and the two-state ansatz in Sec. IV. In each case, we set up the perturbation expansion around the nonrotating star, and solve the equations both numerically and in a series expansion, thereby providing strong numerical evidence that these solutions exist. We conclude in Sec. V.

II. NONRELATIVISTIC BOSON STARS

A. Lagrangian and structure equations

Let us consider a real noninteracting scalar field $\phi(r, t)$ which is coupled to gravity. This scenario is described by the following Lagrangian:

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$$\mathcal{L} = \sqrt{g}\left[\frac{1}{16\pi G}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right]. \quad (1)$$

The scalar field can form gravitational bound states, or boson stars. In this study, we focus on the case of dilute boson stars, which can be described by the Newtonian nonrelativistic limit. For the case of QCD axions, it has been shown that only dilute axion stars are stable over astronomical timescales [30,31].

In the Newtonian limit, when the field ϕ couples only weakly to gravity, the metric can be written as $g_{\mu\nu} = \text{diag}(1 + 2\Phi, -1, -1, -1)$, where Φ is the Newtonian gravitational potential. We are interested in stationary solutions, in which case the gravitational potential is time independent. In this case the Ricci scalar takes the simple form $R = -2(\nabla\Phi)^2$. Also in the nonrelativistic limit, we can treat the energy as being close to the mass, and we have $(\partial_t\phi)^2\Phi = m^2\phi^2\Phi$. The Lagrangian in Eq. (1) then becomes

$$\mathcal{L} = \frac{1}{2}\frac{(\partial_t\phi)^2}{1 + 2\Phi} - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{(\nabla\Phi)^2}{8\pi G}$$
$$= \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2) - \frac{(\nabla\Phi)^2}{8\pi G} - m^2\phi^2\Phi. \quad (2)$$

Since the Lagrangian is quadratic in the scalar field, we can quantize the scalar in the usual way. We first find a set of wave functions satisfying

$$\nabla^2\phi_n - 2m^2\phi_n\Phi = -2mE_n\phi_n \quad (3)$$

and quantize by setting the scalar operator equal to

$$\phi = \sum a_n^\dagger\phi_n + a_n\phi_n^*. \quad (4)$$

The Hamiltonian is then

$$H = \sum\sqrt{2mE_n + m^2}a_n^\dagger a_n. \quad (5)$$

The eigenstates are of the form

$$\Psi = a_{n_1}^\dagger a_{n_2}^\dagger \cdots |0\rangle. \quad (6)$$

The gravitational potential interacts with the scalar through the term $m^2\phi^2\Phi$. This leads to the equation for the potential

$$\nabla^2\Phi = 4\pi Gm\langle\Psi|\phi^2|\Psi\rangle. \quad (7)$$

These field equations, often referred to as Gross-Pitaevskii-Poisson equations, are the structure equations for the boson star.

B. The nonrotating boson star

For the nonrotating star, we consider an ansatz where we have N particles in the ground state $\psi_{nr} \equiv \phi_0$, which has an energy eigenvalue $e_{nr} \equiv E_0$. The state is then

$$\Psi_{nr} = \frac{1}{N!}(a_0^\dagger)^N|0\rangle, \quad (8)$$

and the corresponding structure equations are given by a Schrödinger-type equation for the ground state wave function

$$\nabla^2\psi_{nr} - 2m^2\Phi_{nr}\psi_{nr} = -2me_{nr}\psi_{nr} \quad (9)$$

and a Poisson equation for the gravitational potential

$$\nabla^2\Phi_{nr} = 4\pi GNm|\psi_{nr}|^2. \quad (10)$$

To solve the structure equations for the boson stars, it is convenient to introduce dimensionless variables. Following Refs. [17,18], we define

$$s_0 = -\frac{\sqrt{2\pi GM}}{e_{nr}}\psi_{nr}, \qquad v_0 = -1 + \frac{m}{e_{nr}}\Phi_{nr},$$
$$z = \sqrt{-2me_{nr}}r, \qquad e_{nr} = -\frac{G^2M^2m^3}{2\beta^2}, \quad (11)$$

where $M = Nm$ is the star's mass. Using these variables, we can rewrite the Gross-Pitaevskii-Poisson equations as

$$\nabla_z^2 s_0 = -s_0 v_0 \quad \text{and} \quad \nabla_z^2 v_0 = -|s_0|^2, \quad (12)$$

where the derivatives ∇_z are now with respect to the dimensionless coordinate z. In dimensionless variables, we can write the normalization condition of the wave function, $\int|\psi|^2 dV = 1$, as $\int s_0^2 z^2 dz = 2\beta$ and associate 2β with the mass of the star. Note that up to scalings, there is only one ground state solution for noninteracting boson stars.

In [17,18] we have solved the Gross-Pitaevskii-Poisson equations and obtained a semianalytic solution for the ground state of the boson star. In this approach, the profiles at both small and large radii are separately described through a series expansion of the wave function and potential and matched at an intermediate point. At small radii, the profile can be described by an even polynomial around the center of the star ($z = 0$),

$$s_0^{near} = \sum_{n=0}^\infty s_n^0 z^n \quad \text{and} \quad v_0^{near} = \sum_{n=0}^\infty v_n^0 z^n. \quad (13)$$

At large radii, we take

$$s_0^{\text{far}} = \sum_{n,m=0,0}^{\infty,\infty} s_{n,m}^0 \left(\frac{e^{-z}}{z^\sigma}\right)^n z^{-m} \quad \text{and}$$

$$v_0^{\text{far}} = \sum_{n,m=0,0}^{\infty,\infty} v_{n,m}^0 \left(\frac{e^{-z}}{z^\sigma}\right)^n z^{-m}. \quad (14)$$

The potential and wave function are fully specified by knowing the parameters of the leading expansion

$$s_0^{\text{near}} \approx s_0^0 + \cdots, \qquad s_0^{\text{far}} \approx \alpha e^{-z} z^{\beta-1} + \cdots,$$

$$v_0^{\text{near}} \approx v_0^0 + \cdots, \qquad v_0^{\text{far}} \approx -1 + \frac{2\beta}{z} + \cdots. \quad (15)$$

The remaining coefficients can be obtained using recursion relations which can be derived from the Gross-Pitaevskii-Poisson equations and have been presented in [17,18]. We can determine the four expansion parameters either through a fit to the numerical solution or by matching the small and large radius wave functions and their derivatives at a matching point z^*. We have obtained the following solutions [17]:

$$s_0^0 = 1.02149303631 \pm 1.4 \times 10^{-10},$$
$$v_0^0 = 0.93832284019 \pm 1.3 \times 10^{-10},$$
$$\alpha = 3.4951309897 \pm 5.1 \times 10^{-9},$$
$$\beta = 1.7526648513 \pm 1.3 \times 10^{-9}. \quad (16)$$

C. Slowly rotating boson stars

We now turn to a study of rotating boson stars. In particular, we look for a slowly rotating boson stars solution, which can be treated as a perturbation around the nonrotating solution. That is, the nonrotating solution should admit a normalizable perturbation such that the perturbation carries angular momentum. The existence of such a perturbation would indicate that a slowly rotating boson star can be found at least at the linearized level, which is suggestive that the full solution should exist.

To look for these states, we impose a constraint on the total angular momentum of the boson star

$$N \int \phi \hat{L}^2 \phi dV = L_{\text{star}}^2 \neq 0, \quad (17)$$

where $\hat{L}^2$ is the usual total angular momentum operator $\hat{L}^2 = \partial_\theta^2 + (\sin\theta)^{-2}\partial_\phi^2$. On astrophysically relevant time-scales, the boson star's angular momentum L_{star} is a fixed quantity. We implement this constraint by introducing a Lagrange multiplier λ. The Lagrangian in Eq. (2) then becomes

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2) - \frac{(\nabla\Phi)^2}{8\pi G}$$
$$- m^2\phi^2\Phi + m\lambda(N\phi\hat{L}^2\phi - L_{\text{star}}^2\phi^2). \quad (18)$$

We can repeat the quantization procedure and find resulting equations of motion are the Poisson equation given in Eq. (10) and a modified Schrödinger-type equation

$$\frac{1}{2m}\nabla^2\phi_n = (m\Phi - E_n)\phi_n - \lambda\hat{L}^2\phi_n + \frac{\lambda L_{\text{star}}^2}{N}\phi_n. \quad (19)$$

In the following, we will present two possible solutions for the slowly rotating boson star and obtain the corresponding ground state wave function.

III. ROTATING BOSON STARS: ONE-STATE SOLUTION

A. The Ansatz

We first look for a solution where all the particles are in the ground state. The state is then

$$\Psi = \frac{1}{N!}(a_0^\dagger)^N|0\rangle. \quad (20)$$

This is formally similar to the nonrotating case, but because of the constraints, we must take the ground state in this sector to have nonzero angular momentum.

We take the ground state to be a perturbation around the nonrotating spherically symmetric solution $\psi_{nr}(r)$ obtained in Sec. II B. In particular, we choose an ansatz in which the wave function and potential perturbation are expanded in spherical harmonics $Y_{\ell m}$ with $\ell \geq 1$ and $m = 0$,

$$\phi_0(r,\theta,\phi) = \psi_{\text{nr}}(r) + \epsilon\psi_1(r)Y_{\ell 0}(\theta),$$
$$\Phi(r,\theta,\phi) = \Phi_{\text{nr}}(r) + \epsilon\Phi_1(r)Y_{\ell 0}(\theta), \quad (21)$$

as well as $-E_0 = e_{\text{nr}} + \epsilon e_1$. The expansion parameter ϵ is taken to be parametrically small, which allows us to work in linear order perturbation theory.

The angular momentum constraint in Eq. (17) relates the value of ϵ and the star's angular momentum, such that $\epsilon = L_{\text{star}} \times [N\ell(\ell+1) \cdot \int |\psi_1|^2 dV]^{-1/2}$. We then find that the last term in Eq. (19) is of order $L_{\text{star}}^2 \sim \epsilon^2$ and can therefore be ignored at linear order in perturbation theory.

We now insert this ansatz into the field equations Eqs. (19) and (10). Collecting terms at zeroth order in ϵ, we recover the equations of motion for a nonrotating boson star, whose solution we presented in Sec. II B. Matching the terms proportional to $\epsilon Y_{\ell 0}$ we find the structure equations for the perturbation

$$\frac{1}{2m}\nabla^2\psi_1 = m\Phi_1\psi_{nr} + (m\Phi_{nr} - e_{nr})\psi_1 - \lambda\ell(\ell+1)\psi_1,$$
$$\nabla^2\Phi_1 = 4\pi GNm(\psi_{nr}^*\psi_1 + \psi_1^*\psi_{nr}). \quad (22)$$

Finally, collecting the terms proportional to ϵY_{00} implies $e_1 = 0$, meaning that the rotation does not induce a shift in the binding energy at leading order in perturbation theory. Such a shift first appears at order ϵ^2.

We perform the change of variables in Eq. (11) and further define

$$s_1 = \left[\frac{2\pi GM}{e_{nr}^2}\right]^{\frac{1}{2}}\psi_1, \quad v_1 = \frac{m}{e_{nr}}\Phi_1, \quad \Gamma = \frac{\lambda\ell(\ell+1)}{e_{nr}}. \quad (23)$$

The resulting structure equations for the dimensionless field and potential perturbations s_1 and v_1 then read

$$\nabla_z^2 s_1 - \ell(\ell+1)/z^2 s_1 = -v_1 s_0 - v_0 s_1 + \Gamma s_1,$$
$$\nabla_z^2 v_1 - \ell(\ell+1)/z^2 v_1 = -2s_0 s_1. \quad (24)$$

B. Series expansion

We have seen in Sec. II B that we can describe the profile of the nonrotating boson star through an infinite series for the wave function and potential. We will follow the same approach to obtain a solution for Eq. (24).

At small radii, the profiles for s_1 and v_1 can be described via a polynomial around the center of the boson star $z = 0$,

$$s_1^{near} = \sum_{n=0}^{\infty} s_n^1 z^n \quad \text{and} \quad v_1^{near} = \sum_{n=0}^{\infty} v_n^1 z^n. \quad (25)$$

By matching the coefficients in Eq. (24) we obtain the recursion relations

$$[(n+2)(n+3) - \ell(\ell+1)]s_{n+2}^1 = \Gamma s_n^1$$
$$- \sum_{m=0}^{n}[s_m^0 v_{n-m}^1 + s_m^1 v_{n-m}^0],$$
$$\times [(n+2)(n+3) - \ell(\ell+1)]v_{n+2}^1 = -\sum_{m=0}^{n} s_m^0 s_{n-m}^1. \quad (26)$$

Requiring the left-hand side of Eq. (24) to be defined at $z = 0$ implies that the perturbation vanishes at the origin and hence $s_0^1 = v_0^1 = 0$. The profile at small radii can therefore be fully parametrized in terms of the derivative of the wave function and potential at the origin $\partial_z s_1 = s_1^1$ and $\partial_z v_1 = v_1^1$.

At large radii, we will once again use the series expansion ansatz

$$s_1^{far} = \sum_{n,m=0,0}^{\infty,\infty} s_{n,m}^1 \left(\frac{e^{-z}}{z^\sigma}\right)^n z^{-m} \quad \text{and}$$
$$v_1^{far} = \sum_{n,m=0,0}^{\infty,\infty} v_{n,m}^1 \left(\frac{e^{-z}}{z^\sigma}\right)^n z^{-m}, \quad (27)$$

and obtain the recursion relations

$$n^2 s_{n,m}^1 + 2n(n\sigma + m - 2)s_{n,m-1}^1$$
$$+ [(\sigma n + m - 2)(\sigma n + m - 3) - \ell(\ell+1)]s_{n,m-2}^1$$
$$= -\sum_{p,q=0,0}^{n,m} s_{p,q}^0 v_{n-p,m-q}^1 - \sum_{p,q=0,0}^{n,m} s_{p,q}^1 v_{n-p,m-q}^0 + \Gamma s_{n,m}^1 \quad (28)$$

and

$$n^2 v_{n,m}^1 + 2n(n\sigma + m - 2)v_{n,m-1}^1$$
$$+ [(n\sigma + m - 2)(n\sigma + m - 3) - \ell(\ell+1)]v_{n,m-2}^1$$
$$= -2\sum_{p,q=0,0}^{n,m} s_{p,q}^0 s_{n-p,m-q}^1. \quad (29)$$

Let us note the following properties of s_1^{far} and v_1^{far}: (i) Eq. (29) with $n = 0$ implies that $v_{0,m}^1 = 0$ unless $m = \ell + 1$. At large radius, the potential is then approximately described by $v_1^{far} = v_{0,\ell+1}^1 z^{-(\ell+1)}$, while all other terms in the expansion are at least exponentially suppressed. (ii) Normalizability of the wave function requires $s_{0,0}^1 = 0$. Equation (28) with $n = 0$ then implies that all coefficients $s_{0,m}^1$ vanish as well. This means that similar to the nonrotating wave function s_0^{far}, the wave function of the rotating perturbation s_1^{far} decays at least exponentially. (iii) Equations (28) and (29) further imply that the potential contains only nonvanishing components $v_{n,m}^1$ for even n while the wave function only has nonvanishing component $s_{n,m}^1$ for odd n.

The first nonvanishing terms for wave function s_1^{far} appear for $n = 1$. Using the known $n = 0$ solutions of the nonrotating case, we can simplify Eq. (28) and write

$$2(\beta + \sigma + m - 2)s_{1,m-1}^1 + s_{1,m-\ell-1}^0 v_{0,\ell+1}^1 - \Gamma s_{1,m}^1$$
$$= -[(\sigma + m - 2)(\sigma + m - 3) - \ell(\ell+1)]s_{1,m-2}^1. \quad (30)$$

Setting $m = 0$, Eq. (30) can be written as $s_{1,0}^1 \Gamma = 0$, which implies either $s_{1,0}^1 = 0$ or $\Gamma = 0$. Although both possibilities will lead to a solution, we will mainly focus on the $\Gamma = 0$ solution. For $m = 1$, we find that $1 - \sigma = \beta$, where the σ originates from the s_1^{far} expansion. This is the same relation we found for the nonrotating boson star in Ref. [17], justifying our ansatz to use the same σ for both the s_0^{far} expansion in Eq. (14) and the s_1^{far} expansion in

Eq. (27). Finally, setting $m = M + 1$ we can use Eq. (30) to obtain the recursion relation

$$2Ms_{1,M}^1 = [\ell(\ell + 1) - (\sigma + M - 1)(\sigma + M - 2)]s_{1,M-1}^1$$
$$- v_{0,\ell+1}^1 s_{1,M-\ell}^0. \quad (31)$$

This means that all coefficients can be determined recursively from $s_{1,0}^1$ and $v_{0,\ell}^1$. More generally, we can use Eqs. (28) and (29) to recursively calculate all coefficients $s_{n,m}^1$ and $v_{n,m}^1$ in the expansion of s_1^{far} and v_1^{far}.

We can now determine the expansion parameters by matching the near and far field wave function and potential and their derivatives at a matching point z^*. We have performed such a matching using the near field solution in Eq. (25) truncated with $n \leq 100$ and the far field solution in Eq. (27) truncated with $n \leq 5$, $m \leq 5$ and obtained

$$s_1^1 = 0.91848 \pm 0.00061,$$
$$s_{1,0}^1 = 10.125 \pm 0.052,$$
$$v_{0,2}^1 = 10.111 \pm 0.056, \quad (32)$$

where the perturbation is normalized such that $v_1^1 = 1$. To estimate the uncertainty associated with the matching procedure, we performed multiple matchings for $3 \leq z^* \leq 3.5$.

C. Leading order analytic far-field solution

Similar to the ground state, it is also possible to obtain an approximate analytical solution for the far field at leading order $n = 1$. Using that $v_0 \approx -1 + 2\beta z^{-1}$ and $v_1 \approx v_{0,\ell+1}^1 z^{-\ell+1}$, we can write the Schrödinger-like equation in Eq. (24) as

$$\nabla_z^2 s_1 - \frac{\ell(\ell+1)}{z^2} s_1 = \left(1 + \Gamma - \frac{2\beta}{z}\right)s_1 - \frac{v_{0,\ell+1}^1}{z^{\ell+1}} s_0. \quad (33)$$

After performing a change of variables to $w = 2zs_1$, $w_0 = 2zs_0$, and $y = 2z(1 + \Gamma)^{1/2}$, we can write

$$\frac{dw^2}{dy^2} + \left(-\frac{1}{4} + \frac{\kappa}{y} - \frac{\ell(\ell+1)}{y^2}\right)w = -\frac{\lambda}{y^{\ell-1}}\frac{v_{0,\ell+1}^1}{y^2}w_0 \quad (34)$$

with $\kappa = \beta \cdot (1 + \Gamma)^{-1/2}$ and $\lambda = [2(1 + \Gamma)^{1/2}]^{\ell-1}$. Looking at the homogeneous part on the left-hand side, we rediscover the *Whittaker equation*. Following the notation of Ref. [32], the solution to Eq. (34) is given by a linear combination of the *Whittaker functions* $W_{\kappa,\mu}(y)$ and $M_{\kappa,\mu}(y)$ as well as one solution $w_{\kappa,\mu}^{\text{in}}(y)$ to the inhomogeneous Whittaker equation

$$w(y) = c \cdot W_{\kappa,\mu}(y) + c' \cdot M_{\kappa,\mu}(y) + w_{\kappa,\mu}^{\text{in}}(y), \quad (35)$$

where $\mu^2 = \frac{1}{4} + \ell(\ell + 1)$ or $\mu = \ell + \frac{1}{2}$.

For $\Gamma = 0$, and hence $\kappa = \beta$, normalizability of the wave function requires $c' = 0$. For $\ell = 1$, we also see that $w = v_{02}^1[\ell(\ell+1)]^{-1}w_0$ is a solution of the inhomogeneous Whittaker equation in Eq. (34). This then implies that

$$s_1^{\text{far},\ell=1} = \frac{c}{2z}W_{\beta,\mu}(2z) + \frac{v_{0,2}^1}{2}\frac{\alpha}{2^\beta z}W_{\beta,\frac{1}{2}}(2z). \quad (36)$$

Expanding the Whittaker function, we obtain $s_1^{\text{far}} = (c2^{\beta-1} + \alpha v_{02}^1/2)e^{-z}z^{-\sigma} + \cdots$, which allows us to identify $c = (2s_{1,0}^1 - \alpha v_{02}^1)2^{-\beta}$.

For $\Gamma \neq 0$, additional normalizable solutions with $c' \neq 0$ could exist. The Whittaker function $M_{\kappa,\mu}(y)$ converges to zero for large values of y if κ is a natural number ≥ 2, fixing the corresponding values of $\Gamma = \beta^2/\kappa^2 - 1$.

D. Numerical analysis

In Sec. III B we have shown that the wave function and potential profile of the rotating boson star can be described by a series expansion, which is characterized by the expansion parameters given in Eq. (32). In the following, we will compare this result to the numerical solution of Eq. (24), focusing on the case $\ell = 1$.

As we have seen before, near $z = 0$ the solution takes the form $s_1 \sim s_1^1 z + \cdots$ and $v_1 \sim v_1^1 z + \cdots$. To obtain a numerical solution, it is convenient to normalize the field s_1 and the potential v_1 such that $v_1^1 = 1$, so that the solution is only parametrized by s_1^1. Using a Runge-Kutta four method, we then perform the numerical integration of Eq. (24). For most values of s_1^1, the wave function profile will diverge to positive or negative infinity at large radii $z \gg 1$. Using a shooting point method analogous to those used by the authors of Refs. [17,33], we adjust s_1^1 such that the wave function converges and becomes square integrable.

The numerical solution for $\Gamma = 0$ is shown in the left panel of Fig. 1 as a solid gray line. Fitting the solution by the far potential $v_1^{\text{far}} \approx v_{0,2}^1 z^{-2}$ and the far wave function given in Eq. (36), we can extract the expansion parameters of the series expansion

$$s_1^1 = 0.91835 \pm 0.00014,$$
$$s_{1,0}^1 = 10.123 \pm 0.018,$$
$$v_{0,2}^1 = 10.080089 \pm 0.000035, \quad (37)$$

where the uncertainties were obtained by varying the fit range. These results agree with our previous findings based on the matching between the near and far solutions obtained in Eq. (32).

The dashed curves show the wave function profile of the truncated near solution in Eq. (25) with $n \leq 15$ and $n \leq 100$ as well as the far solution of Eq. (27) with $n \leq 1$ and $m \leq 2$. Here the truncated solution takes the simple form
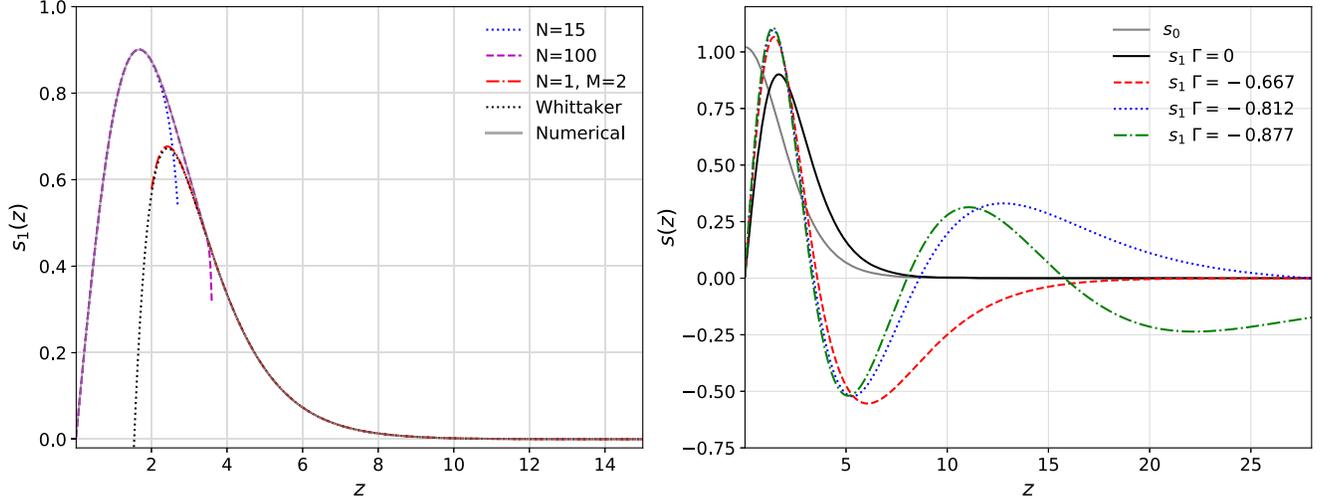
FIG. 1. One-state solution. Left: The numerical solution (solid gray curve) and truncated series expansion at small radius (dotted blue and dashed magenta curves) and large radii (dot-dashed red curve) as well as the analytic Whittaker approximation (dotted black curve) for the wave function of a rotating perturbation with $\ell = 1$ and $\Gamma = 0$. Right: Numerical solution for the wave function of a rotating perturbation with $\ell = 1$ and $\Gamma = 0$ (black curve), $\Gamma = -0.667$ (dashed red curve), $\Gamma = -0.812$ (dotted blue curve), and $\Gamma = -0.877$ (dot-dashed green curve). The nonrotating ground state wave function is shown for comparisons (gray solid curve).

$$
s_1 = \begin{cases} \begin{aligned} &0.918z - 0.188z^3 + 0.024z^5 \\ &-0.262 \times 10^{-2}z^7 + 0.251 \times 10^{-3}z^9 &\text{for} \\ &-2.239 \times 10^{-5}z^{11} + 1.896 \times 10^{-6}z^{13} &z < 3.45 \\ &-1.547 \times 10^{-7}z^{15} \\ \\ &10.12z^{0.752}e^{-z} - 14.16z^{-0.247}e^{-z} &\text{for} \\ &-1.93z^{-1.247}e^{-z} &z > 3.45 \end{aligned} \end{cases}
$$

$$
\tag{38}
$$

We can see that already such few terms in the series expansion are sufficient to describe the wave function well. The dotted black curve shows the Whittaker function solution of Eq. (36), which is already well described by the first few terms of the far field expansion.

The right panel of Fig. 1 shows the numerical solution for both $\Gamma = 0$ and $\Gamma \neq 0$, alongside the nonrotating ground-state solution s_0 discussed in Sec. II B. In particular, we found that solutions exist for $\Gamma = -0.667, -0.812, -0.887$. These values are consistent with the relation $\Gamma = \beta^2/\kappa^2 - 1$ found in Sec. III C for $\kappa = 3, 4, 5$. Notably, $\kappa - 2$ also characterizes at how many radii the wave function vanishes identically, $s_1 = 0$.

IV. ROTATING AXION STARS: TWO-STATE SOLUTIONS

A. The Ansatz

We now consider a second approach to find rotating boson star solutions. In this ansatz, we look for a state where N particles are in the ground state $a_0^\dagger|0\rangle$, and k particles are in the excited state $a_1^\dagger|0\rangle$. The state is then

$$
\Psi = \frac{1}{N!k!}(a_0^\dagger)^N(a_1^\dagger)^k|0\rangle. \tag{39}
$$

This leads to the Poisson-type equation for the potential

$$
\nabla^2\Phi = 4\pi Gm(N|\phi_0|^2 + k|\phi_1|^2), \tag{40}
$$

which should be solved along with the Schrödinger-type equations

$$
\nabla^2\phi_0 - 2m^2\phi_0\Phi = -2mE_0\phi_0,
$$
$$
\nabla^2\phi_1 - 2m^2\phi_1\Phi + 2m\lambda\hat{L}^2\phi_1 = -2mE_1\phi_1. \tag{41}
$$

We will assume $k \ll N$ and perturb in the small parameter $\epsilon = k/N$. For this reason, we dropped the L_{star}^2 term in Eq. (41) which only contributes at subleading order in ϵ. Now, to zeroth order in ϵ, Φ will just be equal to the potential for the nonrotating star Φ_{nr}, ϕ_0 is equal to the wave function for the nonrotating star ψ_{nr}, and $E_0 = e_{\text{nr}}$. As before, we will consider a single $Y_{\ell 0}$ mode, i.e.,

$$
\phi_1(r, \theta, \phi) = \psi_1(r)Y_{\ell 0}(\theta, \phi). \tag{42}
$$

We again perform the change of variables in Eq. (11) and further define

$$
s_1 = \left[\frac{2\pi GM}{e_{\text{nr}}^2}\right]^{\frac{1}{2}}\psi_1, \qquad \Gamma = \frac{E_1 + \lambda\ell(\ell+1)}{e_{\text{nr}}} - 1, \tag{43}
$$

and obtain the structure equation

$$\nabla_z^2 s_1 - \ell(\ell+1)/z^2 s_1 = -v_0 s_1 + \Gamma s_1. \quad (44)$$

The angular momentum of the boson star is equal to $L_{\text{star}} = [k \cdot \ell(\ell+1) \cdot \int |\psi_1|^2 dV]^{1/2}$. Note that unlike for the one-state case, in this case the rotation does induce a shift in the binding energy at leading order in perturbation theory.

B. Series expansion

As before, we will parametrize the wave function via an infinite series expansion. At small radii, the profile for s_1 can be described via a polynomial around the center of the boson star $z = 0$,

$$s_1^{\text{near}} = \sum_{n=0}^{\infty} s_n^1 z^n. \quad (45)$$

By matching the coefficients in Eq. (44) we obtain the recursion relation

$$[(n+2)(n+3) - \ell(\ell+1)]s_{n+2}^1 = \Gamma s_n^1 - \sum_{m=0}^{n} s_m^1 v_{n-m}^0. \quad (46)$$

As in the one-state case, requiring the left-hand side of Eq. (44) to be defined at $z = 0$ implies that the perturbation vanishes at the origin and hence $s_0^1 = 0$. The profile at small radii can therefore be fully parametrized in terms of the derivative of the wave function at the origin $\partial_z s_1 = s_1^1$.

At large radii, we will use the series expansion

$$s_1^{\text{far}} = \sum_{n,m=0,0}^{\infty,\infty} s_{n,m}^1 \left(\frac{e^{-\sqrt{1+\Gamma}z}}{(\sqrt{1+\Gamma}z)^{\sigma'}} \right)^n (\sqrt{1+\Gamma}z)^{-m}. \quad (47)$$

Note that the form of this ansatz is slightly different from the nonrotating boson star in Eq. (14) and the one-state solution in Eq. (27). As we will see later, two-state solutions only exist for $\Gamma \neq 0$, and the resulting far field solution would approximately follow the Whittaker function $W_{\kappa,\mu}(2z\sqrt{1+\Gamma})$. In order to match the asymptotic behavior of this Whittaker function solution, the additional $\sqrt{1+\Gamma}$ factor as well as a new parameter σ' have been included in the series expansion ansatz.

The coefficients of the expansion are related by the recursion relation

$$(1+\Gamma)(n^2 s_{n,m}^1 + 2n(n\sigma' + m - 2)s_{n,m-1}^1$$
$$+ [(\sigma'n + m - 2)(\sigma'n + m - 3) - \ell(\ell+1)]s_{n,m-2}^1)$$
$$= -s_{n,m}^1 v_{0,0}^0 - \sqrt{1+\Gamma}s_{n,m-1}^1 v_{0,1}^0 + \Gamma s_{n,m}^1. \quad (48)$$

Here we have used the approximate ground-state potential $v_0^{\text{far}} \approx -1 + 2\beta/z$, such that the Cauchy product $v_0 s_1$ is well defined.

As before, requiring the wave function to be normalizable implies that all coefficients $s_{0,m}^1$ vanish. The first nonvanishing terms appear for $n = 1$, in which case we can simplify Eq. (48) and write

$$2[\beta(1+\Gamma)^{-1/2} + \sigma' + m - 2]s_{1,m-1}^1$$
$$= [\ell(\ell+1) - (\sigma' + m - 2)(\sigma' + m - 3)]s_{1,m-2}^1. \quad (49)$$

By setting $m = 1$, we obtain $\sigma' = 1 - \beta(1+\Gamma)^{-1/2}$ and note that $\sigma' \neq \sigma$ for $\Gamma \neq 0$, where $\sigma = 1 - \beta$ appears in the nonrotating boson star expansion in Eq. (14). Setting $m = M + 1$ we find

$$s_{1,M}^1 = \frac{[\ell(\ell+1) - (\sigma' + M - 2)(\sigma' + M - 1)]s_{1,M-1}^1}{2M}. \quad (50)$$

Following the same procedure as in Sec. III B, we determine the expansion coefficient $s_{1,0}^1$ by matching the near field solution in Eq. (45) truncated with $n \leq 100$ and $s_1^1 = 1$ and the far field solution in Eq. (47) truncated with $n \leq 1$ and $m \leq 5$ at a matching point z^* and obtain

$$s_{1,0}^1 = 4.910 \pm 0.074. \quad (51)$$

The uncertainty was estimated by performing multiple matchings for $3.2 \leq z^* \leq 3.5$.

C. Leading order analytic far-field solution

Similar to the ground-state and the one-state solutions, we can obtain an approximate analytical solution for the far field at leading order $n = 1$. Using $v_0 \approx -1 + 2\beta z^{-1}$ and performing a change of variables to $w = 2zs_1$ and $y = 2z(1+\Gamma)^{1/2}$, we can rewrite Eq. (44) in the familiar Whittaker equation form

$$\frac{dw^2}{dy^2} + \left(-\frac{1}{4} + \frac{\kappa}{y} - \frac{\ell(\ell+1)}{y^2} \right)w = 0, \quad (52)$$

where $\kappa = \beta \cdot (1+\Gamma)^{-1/2}$. Note that Eq. (52) is homogeneous, while the Whittaker equation for the one-state ansatz in Eq. (34) contained an additional inhomogeneous component (arising from the product $s_0 v_1$). The general solution to Eq. (52) is given by a linear combination of $W_{\kappa,\mu}(y)$ and $M_{\kappa,\mu}(y)$,

$$w(y) = c \cdot W_{\kappa,\mu}(y) + c' \cdot M_{\kappa,\mu}(y), \quad (53)$$

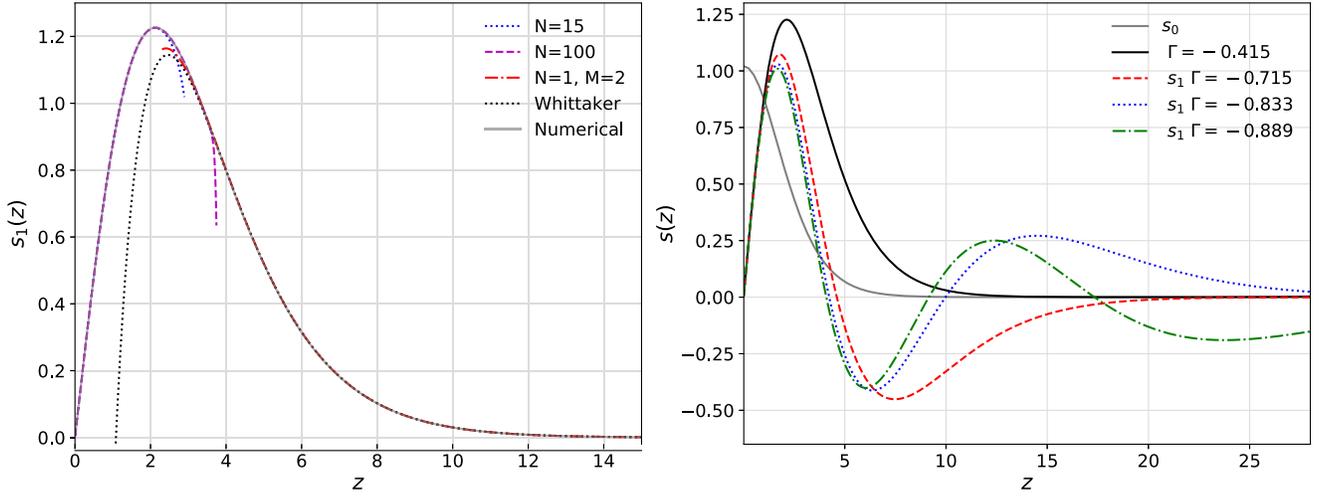where $\mu^2 = \frac{1}{4} + \ell(\ell+1)$.

FIG. 2. Two-state solution. Left: The numerical solution (solid gray curve) and truncated series expansion at small radius (dotted blue and dashed magenta curves) and large radii (dot-dashed red curve) as well as the analytic Whittaker approximation (dotted black curve) for the wave function of a rotating perturbation with $\ell = 1$ and $\Gamma = -0.415$ in the two-state ansatz. Right: Numerical solution for the wave function of a rotating perturbation with $\ell = 1$ and $\Gamma = -0.415$ (black curve), $\Gamma = -0.715$ (dashed red curve), $\Gamma = -0.833$ (dotted blue curve), and $\Gamma = -0.889$ (dot-dashed green curve). The nonrotating ground state wave function is shown for comparisons (gray solid curve).

The function $M_{\kappa,\mu}(y)$ diverges at large y unless $\Gamma = \beta^2/\kappa^2 - 1$ with κ being a natural number ≥ 2. We will see in the next section that the solutions and corresponding values of Γ do not fulfill this condition. Normalizability of the wave function then requires $c' = 0$. The solution must therefore be solely described by $W_{\kappa,\mu}(y)$ which allows us to write

$$s_1^{\text{far}} = \frac{c}{2z} W_{\kappa,\mu}(2z\sqrt{1+\Gamma}). \tag{54}$$

Expanding this function leads to the ansatz in Eq. (47) and matching the coefficients of the leading terms allows us to identify $c = s_{1,0}^1 [2^{-\sigma'}\sqrt{1+\Gamma}]^{-1}$ with $\sigma' = 1 - \kappa$.

D. Numerical results

As for the one state solution, we also obtain a numerical solution of Eq. (44), focusing on the case $\ell = 1$. The equation is linear in s_1 which allows is to choose $s_1^1 = 1$ without loss of generality. We then use a Runge-Kutta four method to perform the numerical integration of Eq. (44) and apply a shooting point method to find the values of Γ for which the wave function converges at large radii.

The lowest energy solution is obtained for $\Gamma = -0.415$, and the corresponding wave function is shown in the left panel of Fig. 1 as a solid gray line. As before, we can fit the numerical solution with the far wave function given in Eq. (54) and obtain

$$s_{1,0}^1 = 4.894 \pm 0.013, \tag{55}$$

which agrees with our previous finding in Eq. (51). We also show the wave function profile using the truncated near solution in Eq. (45) with $n \leq 15$ and $n \leq 100$, the far solution of Eq. (47) with $n \leq 1$ and $m \leq 2$, and the Whittaker solution of Eq. (54). The truncated solution takes the form

$$s_1 = \begin{cases} z - 0.135z^3 + 0.013z^5 \\ -0.106 \times 10^{-2}z^7 + 8.253 \times 10^{-5}z^9 \quad \text{for} \\ -6.207 \times 10^{-6}z^{11} + 4.551 \times 10^{-7}z^{13} \quad z < 3.45 \\ -3.275 \times 10^{-8}z^{15} \\ \\ (3.461 - 2.174z^{-1} - 1.153z^{-2}) \quad \text{for} \\ \times z^{1.292} e^{-0.765z} \quad z > 3.45 \end{cases}$$

$$\tag{56}$$

Again, with only a few terms in the expansion the wave function is described fairly well.

The right panel of Fig. 2 shows additional numerical solutions for $\Gamma = -0.415$, -0.715, -0.833, and -0.889. Note that these values of Γ do not coincide with $\Gamma = \beta^2/\kappa^2 - 1$ for $\kappa \geq 2$ similar to those of the one-state solutions. As argued before, this implies that the far field solution is solely described by the Whittaker W function.

V. CONCLUSIONS

Light scalar fields can form gravitationally bound compact objects, called boson stars. In the Newtonian

limit, the profiles of boson stars are described by the Gross-Pitaevskii-Poisson equations.

In previous works, we presented a semianalytic solution to these equations describing the profile of boson stars formed by scalar fields [17,18]. The solution was based on a series expansion which is parametrized by four expansion parameters that were obtained from numerical simulations at high accuracy. In this paper we have extended our methods to find new solutions which allow for slowly rotating boson stars; specifically, we have found solutions for boson stars where the ratio of the angular momentum to the number of particles can be made arbitrarily small.

We considered two possibilities: in one case, all the particles are in the same state, and in the second case the majority of the particles are in the zero angular momentum ground state and a small number of particles are in an excited state containing angular momentum. In each case, we obtained accurate numerical and semianalytic profiles (about 1% precision), thereby establishing the existence of these slowly rotating boson stars.

The results and methods presented in this paper allow for systematic studies of the properties of boson stars in an analytic way without further relying on numerical simulations. There are several directions for further research; in particular, it would be interesting to extend these solutions to interacting scalars and to relativistic stars. It would also be interesting to see how the profiles are modified in the presence of other astrophysical objects such as planets. We hope to return to these questions in future work.

ACKNOWLEDGMENTS

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