Symmetric orbifold theories from little string residues

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We study a class of little string theories (LSTs) of A type, described by N parallel M5-branes spread out on a circle and which in the low energy regime engineer supersymmetric gauge theories with $U(N)$ gauge group. The Bogomol'nyi-Prasad-Sommerfield (BPS) states in this setting correspond to M2-branes stretched between the M5-branes. Generalizing an observation made by Ahmed *et al.* [Bound states of little strings and symmetric orbifold conformal field theories, Phys. Rev. D **96**, 081901 (2017).], we provide evidence that the BPS counting functions of special subsectors of the latter exhibit a Hecke structure in the Nekrasov-Shatashvili (NS) limit; i.e., the different orders in an instanton expansion of the supersymmetric gauge theory are related through the action of Hecke operators. We extract N distinct such reduced BPS counting functions from the full free energy of the LST with the help of contour integrals with respect to the gauge parameters of the $U(N)$ gauge group. Physically, the states captured by these functions correspond to configurations where the same number of M2-branes is stretched between some of these neighboring M5-branes, while the remaining M5-branes are collapsed on top of each other and a particular singular contribution is extracted. The Hecke structures suggest that these BPS states form the spectra of symmetric orbifold conformal field theories. We show, furthermore, that to leading instanton order (in the NS limit) the reduced BPS counting functions factorize into simpler building blocks. These building blocks are the expansion coefficients of the free energy for $N = 1$ and the expansion of a particular function, which governs the counting of BPS states of a single M5-brane with single M2-branes ending on it on either side. To higher orders in the instanton expansion, we observe new elements appearing in this decomposition whose coefficients are related through a holomorphic anomaly equation.

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I. INTRODUCTION

Little string theories (LSTs) were first introduced in [1–7]. They are a type of quantum theory in six dimensions which behaves like an ordinary quantum field theory (with pointlike degrees of freedom) in the low energy regime but whose UV completion requires the inclusion of stringlike degrees of freedom. On the one hand, LSTs serve in many aspects as toy models of string theory, with the only difference being that the gravitational sector is absent. Indeed, in practice, many examples of LSTs are obtained from (type II) string theory or M theory through a particular decoupling limit which sends the string coupling to zero while leaving the string length finite. Thus studying properties of LSTs gives us an important window into string and M theories, which are intrinsically difficult to study by more direct means. On the other hand, conversely, a better understanding of LSTs also provides us with more information about the (supersymmetric) gauge theories that are engineered in the low energy sector: due to their stringy origins, LSTs inherit numerous symmetries and dualities from string and M theory, remnants of which are still visible in the low energy gauge theories engineered by the LSTs.

In the same spirit, there are many (geometric and computational) tools that have been developed in the framework of full-fledged string theory (or related applications), which allows us to perform many explicit computations for LSTs. For example, geometrical methods which have been used to classify conformal field theories in six dimensions or fewer [8–19] have recently also been deployed to attempt a classification of LSTs [15,20]. Indeed, while an ADE classification of LSTs has been known for some time [1–7], recent efforts have focused on sharpening the list of all possible such theories.

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Furthermore, a specific class of theories which have received a lot of attention recently are LSTs of type A. Such theories, compactified on a circle, have been studied using various dual approaches in string or M theory: on the one hand, they can be described by configurations of N parallel M5-branes that are separated along a circle S^1_ρ and compactified on a circle S^1_τ.[1] Bogomol'nyi-Prasad-Sommerfield (BPS) configurations in this setting correspond to M2-branes that are stretched between neighboring M5-branes and wrap S^1_τ. The partition function of LSTs compactified on S^1_τ, $\mathcal{Z}_{N,1}$, can be calculated by analyzing the theory on the intersection of the M2- and M5-branes [21,25,26] using a two-dimensional sigma-model description. In order to render $\mathcal{Z}_{N,1}$ well defined, the introduction of two regularization parameters $\epsilon_{1,2}$ is required. From the point of view of the low energy gauge theory description, the latter correspond to the parameters of the Ω background, which are needed to regularize the Nekrasov partition function. A dual approach is given by F theory compactified on a class of toric Calabi-Yau manifolds [27] called $X_{N,1}$. The web diagram of the latter can directly be read off from the brane-web diagram representing the system of (M2–M5)-branes mentioned above [21,25,26]. Furthermore, $\mathcal{Z}_{N,1}$ in this approach is captured by the topological string partition function on $X_{N,1}$, which in turn can be very efficiently calculated with the help of the topological vertex [28,29]. The regularization parameters $\epsilon_{1,2}$ in this approach are intrinsic to the refined topological string [30–32] (see also [33–36]).

Recent studies have exploited this efficient way to explicitly compute $\mathcal{Z}_{N,1}$ (or the corresponding free energy $\mathcal{F}_{N,1}$) to study symmetries and other properties of the corresponding LSTs. In the process, numerous very interesting and unexpected structures have been discovered including, among others, the following:

(i) *Dihedral and paramodular symmetries*.—The Calabi-Yau manifold $X_{N,1}$ engineers a supersymmetric gauge theory on $\mathbb{R}^4 \times S^1 \times S^1$ with a $U(N)$ gauge group and matter in the adjoint representation. The Kähler moduli space of $X_{N,1}$ can be understood as a subregion of a much larger so-called extended moduli space. Depending on the value of N, there are further regions in the latter which engineer supersymmetric gauge theories with different gauge structures and matter content. Many of these theories are dual to each other in the sense that they share the same partition function. The duality map, however, is intrinsically nonperturbative. More concretely, it was conjectured in [37] and proven in [38,39] that the $U(N)$ gauge theory above is dual to a circular quiver gauge theory with a gauge group made up of M' factors of $U(N')$ and bifundamental matter, for any pair (N', M') such that $M'N' = N$ and $\gcd(M', N') = 1$.[2]

It was shown in [42] that this web of dualities implies additional symmetries for the partition function $\mathcal{Z}_{N,1}$ (as well as the free energy $\mathcal{F}_{N,1}$). While it is clear (due to the structure of $X_{N,1}$ as a double-elliptic fibration) that the latter are symmetric with respect to two modular groups called $SL(2,\mathbb{Z})_\rho$ and $SL(2,\mathbb{Z})_\tau$,[3] it was shown in [42] that they also enjoy a dihedral symmetry which (from the perspective of the gauge theories) acts in an intrinsically nonperturbative fashion. Moreover, it was argued in [43] that a particular subsector of the BPS states [namely, the sector of states which carry the same $U(1)$ charges under all the generators of the Cartan subalgebra of the $U(N)$ gauge group], is invariant under the level N paramodular group $\Sigma_N \subset Sp(4,\mathbb{Q})$.

(ii) *Hecke structures*.—In [43] evidence was presented that in the Nekrasov-Shatashvili (NS) limit [44,45] (which in our notation essentially corresponds to the limit $\epsilon_2 \to 0$), the paramodular group Σ_N that is present in the above-mentioned subsector of BPS states, is further extended to Σ_N^*. The latter is obtained from Σ_N through the inclusion of a further generator that exchanges the modular parameters ρ and τ of the two modular groups mentioned above (see Appendix D for details). This result corroborates the observation made in [46] that the states of the subsector of BPS states mentioned above (in the NS limit) can be organized into a symmetric orbifold conformal field theory (CFT). The latter in particular implies that the expansion coefficients of the reduced free energy (which counts states only in this BPS subsector) are related through the action of Hecke operators. This relation was indeed observed in [46] in numerous examples.

(iii) *Factorization to leading instanton order and graph functions*.—In [47] nontrivial evidence was provided that the free energy $F_{N,1}$ in the so-called unrefined limit (i.e., for $\epsilon_2 = -\epsilon_1$) factorizes in a very intriguing fashion: for the examples $N = 2, 3, 4$, it was shown that the free energy to leading instanton order [from the perspective of the $U(N)$ gauge

[1]We refer the reader to [21–24] for more details on the brane setup.

[2]In [37], a much stronger conjecture was put forth: that the Calabi-Yau manifolds $X_{N,M}$ and $X_{N',M'}$ are dual to each other if $NM = N'M'$ and $\gcd(N,M) = \gcd(N',M')$. This implies a duality between gauge theories with gauge groups $U(N)^M$ and $U(N')^{M'}$. Numerous examples were successfully tested in [37,39,40]. Furthermore, the case $\gcd(N,M) = 1$ has been proven in [39] for arbitrary values of $\epsilon_{1,2}$, and a proof for generic N, M for $\epsilon_{1,2} \to 0$ was presented in [41].

[3]The notation follows the Kähler parameters which act us modular parameters for the two groups.

theory] can be decomposed into sums of products of the functions $H^{(r)}_{N=1}$ and $W^{(r)}_{\mathrm{NS}}$. The former are the expansion coefficients of the instanton expansion of the free energy $F_{N=1,1}$, while the latter are the expansion coefficients of a function that governs the counting of BPS states of an M5-brane with a single M2-brane ending on either side.[4] Furthermore, it was observed in [47] that the coefficients appearing in this expansion of $F_{N,1}$ resemble in many respects so-called modular graph functions, which have recently appeared in the study of scattering amplitudes in string theory [48–57]. Higher terms in the instanton expansion are more complicated: certain parts are still allowed to be factorized into simpler building blocks; however, on the one hand, the coefficient functions that appear in this way are more complicated, while, on the other hand, the inclusion of Hecke transformations of $H^{(r)}_{N=1}$ and $W^{(r)}_{\mathrm{NS}}$ is required. While primarily dealing with the unrefined free energy, preliminary results in [47] indicate similar decompositions (albeit more complicated ones) to be valid in the NS limit.

This paper is a continuation of the analysis of the symmetries and structures discovered in [43,46,47,58]: focusing on the NS limit of the free energy, we analyze the form of the free energy that was found in [43,47,58] and which is implied by property (i) above. We observe new subsectors of the BPS states that show a Hecke structure similar to that discussed under (ii) above. Based on the examples of $N=2$ and $N=3$ (as well as partial results for $N=4$), we observe, for a given N and at each order in an expansion of ϵ_1, N distinct subsectors of the NS limit of the BPS free energy $\mathcal{F}_{N,1}$ that exhibit such structures. We call the functions which count these BPS states at the rth instanton order $\mathcal{C}^{N,(r)}_{i,(2s,0)}$, where $i=1,\ldots,N$ and $s\in\mathbb{N}$ indicates the order in an expansion in powers of ϵ_1. The latter can abstractly be defined for general N through contour integrals of (an expansion in powers of ϵ_1 of the NS limit of) $\mathcal{F}_{N,1}$ with respect to the gauge parameters $\hat{a}_{1,\ldots,N}$ of the $\mathfrak{a}_{N-1}$ gauge algebra (or their exponentials $Q_{\hat{a}_i}=e^{2\pi i\hat{a}_i}$ for $i=1,\ldots,N$). These contours extract specific coefficients in a Fourier expansion of $\mathcal{F}_{N,1}$ in $Q_{\hat{a}_i}$ and/or particular poles in a limit where some of the $\hat{a}_i$ vanish [see Eq. (3.1) for the abstract definition of the $\mathcal{C}^{N,(r)}_{i,(2s,0)}$]. From a physical perspective the functions $\mathcal{C}^{N,(r)}_{i,(2s,0)}$ receive contributions only from M5-brane configurations where the same number of M2-branes is stretched between some of the adjacent M5-branes (see Fig. 4 for a schematic representation). From these configurations, in turn, specific poles are extracted in the limit where the remaining M5-branes coincide. The remaining functional dependence of $\mathcal{C}^{N,(r)}_{i,(2s,0)}$ is made up of two (remaining) Kähler moduli of $X_{N,1}$ (which we call ρ and S). Finally, we can resum the $\mathcal{C}^{N,(r)}_{i,(2s,0)}$ into a Laurent series expansion $\mathcal{C}^{N,(r)}_{i}(\rho,S,\epsilon_1)$ in powers of ϵ_1.

We observe that the functions $\mathcal{C}^{N,(r)}_{i,(2s,0)}$ obtained in this fashion show numerous interesting properties. First, they are quasi-Jacobi forms of index rN and weight $2s-2i$. Moreover, the functions for $r>1$ can be obtained through the action of the rth Hecke operator $\mathcal{H}_r$ [see (A10) in Appendix A for a definition] on $\mathcal{C}^{N,(r=1)}_{i,(2s,0)}(\rho,S)$:

$$\mathcal{C}^{N,(r)}_{i,(2s,0)}(\rho,S)=\mathcal{H}_r\left[\mathcal{C}^{N,(1)}_{i,(2s,0)}(\rho,S)\right],\qquad \begin{array}{l}\forall\, r\geq 1,\\ \forall\, i=1,\ldots,N.\end{array}\tag{1.1}$$

Following the logic of [46], this suggests that the corresponding BPS states can be organized into a symmetric torus orbifold CFT. However, since the seed function (i.e., the initial function $\mathcal{C}^{N,(r=1)}_{i,(2s,0)}$) is different in each case, the corresponding target spaces are different for all $i=1,\ldots,N$. Indeed, the functions $\mathcal{C}^{N,(r=1)}_{i}$ can be factorized in terms of $H^{(r=1)}_{N=1}$ and $W^{(r=1)}_{\mathrm{NS}}$ in a very simple fashion [see Eq. (3.9)]. For $r>1$, the $\mathcal{C}^{N,(r)}_{i}$ can still mostly be decomposed into $H^{(r=1)}_{N=1}(\rho,S)$ and $W^{(r=1)}_{\mathrm{NS}}(\rho,S)$ up to remainder functions [see Eq. (3.11)]. The latter, however, are not arbitrary but are connected by equations that strongly resemble holomorphic anomaly equations [59]. These results generalize the properties of the free energy discussed under (iii) above. Since the results in this paper are obtained by studying the examples of $N=2$ and $N=3$ (as well as partially $N=4$) their generalizations to higher N have to be considered conjectures. However, the large number of examples that all follow the same pattern provides rather strong evidence in their favor.

This paper is organized as follows: In Sec. II we review the LST partition function $\mathcal{Z}_{N,1}$ and the associated free energy $\mathcal{F}_{N,1}$, as well as some of their properties discovered in recent publications. Owing to the technical nature of some of the subsequent discussions, we provide a summary of the results of this paper in Sec. III. Sections IV–VI provide a detailed discussion of the LST free energies for $N=2$, $N=3$, and $N=4$, respectively. Finally, Sec. VII contains our conclusions. Additional details on modular objects, explicit discussions of properties of the free energy, the discussion (and explicit expressions for some of their expansion coefficients) of the fundamental building blocks $H^{(r)}_{N=1}$ and $W^{(r)}_{\mathrm{NS}}$, and the definition of paramodular groups, which have been deemed too long for the body of this paper, have been relegated to four appendixes.

[4]Explicit expressions for $H^{(r)}_{N=1}$ and $W^{(r)}_{\mathrm{NS}}$ as well as more information can be found in Appendix C.

II. LITTLE STRING FREE ENERGIES AND THEIR PROPERTIES

The LSTs of A type that we are interested in can be studied by exploiting various dual descriptions in string or M theory. On the one hand, they can be described through configurations of parallel M5-branes compactified on a circle of circumference τ and spread out on a circle with circumference ρ, where the distances between the neighboring M5-branes are denoted $(t_1, \ldots, t_N)$ such that

$$\rho = t_1 + t_2 + \cdots + t_N. \tag{2.1}$$

BPS states in this setting are given by M2-branes. The latter are stretched between the neighboring M5-branes and appear as strings in their worldvolumes, wrapping the circle S^1_τ on which the M5-branes are wrapped [21]. In this context, arbitrarily many M2-branes can be stretched between any of the neighbouring M5-branes (a schematic example is shown in Fig. 1). The space transverse to the M2-branes inside the M5-brane world volume is $\mathbb{R}^4_\parallel$ and the M2-branes appear as point particles in this space. The world-volume theory of M2-branes has $\mathcal{N} = (4,4)$ supersymmetry which is broken down to $\mathcal{N} = (0,2)$ by a $U(1)_{\epsilon_1} \times U(1)_{\epsilon_2} \times U(1)_m$ action on $R^4_\parallel \times \mathbb{R}^4_\perp$ [21],[5]

$$(z_1, z_2, w_1, w_2) \in \mathbb{C}^2_\parallel \times \mathbb{C}^2_\perp \mapsto (z_1 e^{i\epsilon_1}, z_2 e^{i\epsilon_2}, w_1 e^{i(m+\epsilon_+)}, w_2 e^{i(m-\epsilon_+)}). \tag{2.2}$$

The BPS degeneracies of the M2-branes is captured by the elliptic genus of the world-volume theory which depends on the parameters $(\tau, t_{1,\ldots,N}, m, \epsilon_{1,2})$. This can be calculated by studying the gauge and matter content of the $(0,2)$ world-volume theory and using the techniques developed in [60–62]. A different approach is to calculate the $(0,2)$ elliptic genus of the sigma model to which the world-volume theory flows in the infrared. The target space of the sigma model in this case is the product of Hilbert schemes of points on $\mathbb{C}^2_\parallel$, and the equivariant elliptic genus can be calculated using the details of the $U(1)_{\epsilon_1} \times U(1)_{\epsilon_2}$ action on the target space [21,63,64]. The theory on the world volume of the compactified M5-branes is the five-dimensional $\mathcal{N} = 1^\star$ quiver gauge theory. The partition function of this gauge theory captures the M2-brane BPS states as well and is given by the generating function of equivariant elliptic genera of the rank N charge K instanton moduli spaces $M(N,k)$ [21,26].

A dual approach to describing the same LST is through F theory compactified on a toric Calabi-Yau threefold [27] which in [22] was called $X_{N,1}$. The BPS string states are given by D3-branes wrapping various rational curves in the base of the Calabi-Yau threefold with Kähler parameters $t_1, \ldots, t_N$. The web diagram of the latter can be directly read off from the brane-web configuration discussed earlier and is shown in Fig. 2. This figure also includes a definition (shown in blue) of a basis of the Kähler parameters of $X_{N,1}$: besides $t_1, \ldots, t_N$, these include τ and m, which can be expressed in terms of the basis $(h_1, \ldots, h_N, m, v)$. From the perspective of F theory compactified on $X_{N,1}$, the little string partition function $\mathcal{Z}_{N,1}$ is captured by the topological string partition function on $X_{N,1}$ [20,22]. The latter can be computed in an efficient manner using the refined topological vertex formalism [28,29].

In [21,22,25,26,65,66] two different expansion of $\mathcal{Z}_{N,1}$ and their interpretations were studied,

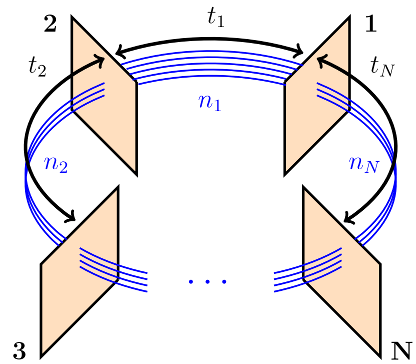

FIG. 1. N parallel M5-branes (orange) with $(n_1, \ldots, n_N)$ M2-branes (blue) stretched between them. The distances between the M5-branes are $t_{1,\ldots,N}$.

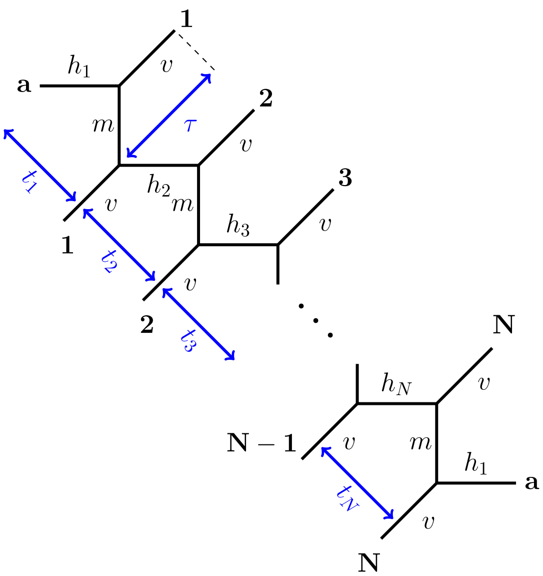

FIG. 2. Web diagram of $X_{N,1}$.

[5]We define $q = e^{i\epsilon_1}$ and $t = e^{-i\epsilon_2}$ so that the unrefined limit is $q = t$.

$$\mathcal{Z}_{N,1} = \sum_k Q_\tau^k Z_k(t_{1,\ldots,N}, m, \epsilon_{1,2}),$$
$$= \sum_{k_1,\ldots,k_N} Q_{t_1}^{k_1} \cdots Q_{t_N}^{k_N} Z_{k_1\cdots k_N}(\tau, m, \epsilon_{1,2}), \quad (2.3)$$

where $Q_\tau = e^{2\pi i \tau}$ and $Q_{t_i} = e^{2\pi i t_i}$.

As discussed in Appendix B, $Z_k(t_{1,\ldots,N}, m, \epsilon_{1,2})$ is the equivariant (2,2) elliptic genus of $M(N,k)$. This expansion of the partition function is natural when considering the theory on the M5-branes. The expansion on the second line of Eq. (2.3) gives the functions $Z_{k_1\cdots k_N}$, which capture the degeneracy of configurations of M2-branes in which k_i M2-branes are stretched between the ith and $(i+1)$th M5-brane. The $Z_{k_1\cdots k_N}$ is the equivariant (0,2) elliptic genus with target space $\otimes_{i=1}^N \mathrm{Hilb}^{k_i}[\mathbb{C}^2]$ with right moving fermions coupled to a bundle $V_{k_1\cdots k_N}$, the details of which are given in [21,26].

In [22,26] the following little string free energy $\mathcal{F}_{N,1}^{\mathrm{plet}}$ was discussed:

$$\mathcal{F}_{N,1}^{\mathrm{plet}}(t_{1,\ldots,N}, m, \tau, \epsilon_{1,2}) = \mathrm{Plog}\mathcal{Z}_{N,1}(t_{1,\ldots,N}, m, \tau, \epsilon_{1,2}), \quad (2.4)$$

where Plog denotes the plethystic logarithm[6] of $\mathcal{Z}_{N,1}$. The exact form of $\mathcal{Z}_{N,1}$ is given in Appendix B. From a physical perspective, $\mathcal{F}_{N,1}^{\mathrm{plet}}$ counts only single-particle BPS excitations of the LST projecting out multiparticle states. Similar to the two equivalent expansions of the partition function in Eq. (2.3), one can similarly consider two different ways of expanding $\mathcal{F}_{N,1}^{\mathrm{plet}}$,

$$\mathcal{F}_{N,1}^{\mathrm{plet}}(t_{1,\ldots,N}, m, \tau, \epsilon_{1,2})$$
$$= \sum_k Q_\tau^k F_k^{\mathrm{plet}}(t_{1,\ldots,N}, m, \epsilon_{1,2})$$
$$= \sum_{k_1\cdots k_N} Q_{t_1}^{k_1} \cdots Q_{t_N}^{k_N} F_{\mathrm{plet}}^{(k_1,\ldots,k_N)}(\tau, m, \epsilon_{1,2}). \quad (2.5)$$

In previous work numerous properties of the free energy $\mathcal{F}_{N,1}^{\mathrm{plet}}$ (or some of the coefficients appearing in these two expansions) have been discovered. In the following we shall discuss some of them which will turn out to be important for our current work:

(a) *Recursion relation.*—In [23,24] the counting functions of a particular class of single BPS states has been discussed: these states correspond to M-brane configurations of the type schematically shown in Fig. 1; however, they are special in the sense that they have one (or several neighboring) M5-brane(s) with only a single M2-brane ending on them on either side. In the notation of Fig. 1, these are characterized by the fact that several adjacent n_i are identical to 1, i.e.,

$$(n_1, \ldots, n_k, \underbrace{1, \ldots, 1}_{m\,\text{times}}, n_{k+m}, \ldots, n_N). \quad (2.6)$$

The BPS degeneracy of such states is captured by $F_{\mathrm{plet}}^{(k_1,\ldots,k_N)}$ [defined in Eq. (2.5)] with $(k_1,\ldots,k_N) = (n_1, \ldots, n_k, \underbrace{1, \ldots, 1}_{m\,\text{times}}, n_{k+m}, \ldots, n_N)$. It was observed that in this case

$$F_{\mathrm{plet}}^{(n_1,\ldots,n_k,1,\ldots,1,n_{k+m},\ldots,n_N)}$$
$$= F_{\mathrm{plet}}^{(n_1,\ldots,n_k,1,n_{k+m},\ldots,n_N)} W(\tau, m, \epsilon_{1,2})^{m-1}. \quad (2.7)$$

The relative factor W appearing in this relation is a quasimodular form and is given by

$$W(\tau, m, \epsilon_{1,2}) = \frac{\theta_1(\tau, m+\epsilon_+)\theta_1(\tau, m-\epsilon_+) - \theta_1(\tau, m+\epsilon_-)\theta_1(\tau, m-\epsilon_-)}{\theta_1(\tau, \epsilon_1)\theta_2(\tau, \epsilon_2)}, \quad (2.8)$$

with $\epsilon_\pm = \frac{\epsilon_1 \pm \epsilon_2}{2}$. Further information on this function and particular expansions that will be useful in the remainder of this paper can be found in Appendix C 2.

(b) *Self-similarity.*—In [24] it was observed that in the NS limit and in a certain region of the Kähler moduli space of $X_{N,1}$, the part of the free energy that counts only single-particle states, becomes directly related to the BPS counting function for the LST with $N = 1$ (and thus proportional to the free energy $\mathcal{F}_{N=1,1}^{\mathrm{plet}}$). With the notation introduced above, the particular region in the moduli space is defined as

$$t_1 = t_2 = \ldots = t_N = \frac{\rho}{N} \quad (2.9)$$

so that the M5-branes are all at an equal distance from each other on the circle. In this region of the

[6]The plethystic logarithm of a function $g(x_1, x_2, \ldots, x_K)$ is given by $\mathrm{Plog}g(x_1, x_2, \ldots, x_K) = \sum_{n=1}^\infty \frac{\mu(n)}{n} \log g(nx_1, nx_2, \ldots, nx_K)$ where $\mu(n)$ is the Möbius function.

moduli space the free energy is a function of (τ,ρ,m) only and

$$\mathcal{F}^{\mathrm{plet,NS}}_{N,1}\left(\frac{\rho}{N},\ldots,\frac{\rho}{N},\tau,m,\epsilon_1\right)=\mathcal{F}^{\mathrm{plet,NS}}_{N=1,1}\left(\frac{\rho}{N},\tau,m,\epsilon_1\right), \tag{2.10}$$

where the NS limit [44,45] is defined as

$$\mathcal{F}^{\mathrm{plet,NS}}_{N,1}(t_1,\ldots,t_N,\tau,m,\epsilon_1) = \lim_{\epsilon_2\mapsto 0}\frac{\epsilon_2}{\epsilon_1}\mathcal{F}^{\mathrm{plet}}_{N,1}(t_1,\cdots t_N,\tau,m,\epsilon_1). \tag{2.11}$$

(c) *Hecke structures and torus orbifold.*—In [46] contributions to the free energy coming from BPS states which carry the same charges under all the generators of the Cartan subalgebra of the gauge algebra $\mathfrak{a}_{N-1}$ were studied. In the language of the M-brane description, these correspond to configurations in which an equal number of M2-branes is stretched between any of the adjacent M5-branes. The degeneracy of such states is captured by

$$\mathcal{F}^{\mathrm{orb}}_{N,1}(\rho,\tau,m,\epsilon_{1,2})=\sum_k Q_\rho^k F^{(k,\ldots,k)}_{\mathrm{plet}}(\tau,m,\epsilon_{1,2}). \tag{2.12}$$

Based on the study of numerous examples (and supported by modular arguments), it was conjectured in [46] that[7] $\exp(\mathcal{F}^{\mathrm{orb,NS}}_{N,1}(\rho,\tau,m,\epsilon_1))$ is the partition function of a two-dimensional torus orbifold theory whose target space is the symmetric product of the moduli space $\mathcal{M}_{1\ldots1}$ of monopole strings with charge $(1,\ldots,1)$,

$$\begin{aligned}\exp(\mathcal{F}^{\mathrm{orb,NS}}_{N,1}(\rho,\tau,m,\epsilon_1)) &= \sum_k Q_\rho^k\chi_{\mathrm{ell}}(\mathrm{Sym}^k\mathcal{M}_{1\ldots1})\\ &=\prod_{k,n,\ell,r}(1-Q_\rho^kQ_\tau^nQ_m^\ell q^r)^{-c(kn,\ell,r,s)}.\end{aligned} \tag{2.13}$$

Here $c(k,\ell,r)$ are the coefficients in the Fourier expansion of $\chi_{\mathrm{ell}}(\mathcal{M}_{1\ldots1})$:

$$\chi_{\mathrm{ell}}(\mathcal{M}_{1\ldots1})=\sum_{k,\ell,r}c(k,\ell,r)Q_\tau^kQ_m^\ell q^r. \tag{2.14}$$

In this paper we shall discuss novel properties of the free energy, which (in a certain sense) generalize some of the points mentioned above. However, to render some of these properties more clearly visible, we shall choose to slightly modify two important points:

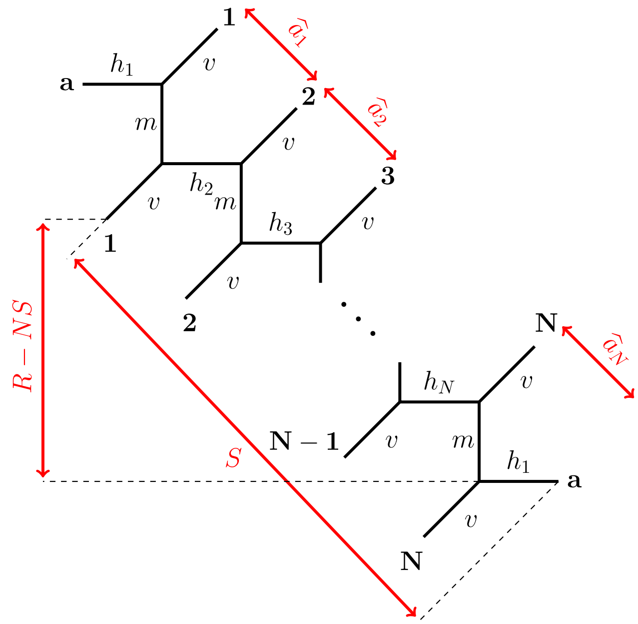

FIG. 3. Web diagram of $X_{N,1}$ labeled by the parameters $(\hat{a}_1,\ldots,\hat{a}_N,S,R)$.

(a) Instead of the basis $(t_1,\ldots,t_N,m,\tau)$, which is defined in Fig. 2 as certain Kähler parameters of $X_{N,1}$, we work in a different basis given by the parameters $(\hat{a}_1,\ldots,\hat{a}_N,S,R)$: this basis was first introduced in [38,40,42] and allows for a more streamlined definition of some of the symmetries of the free energy. With respect to the web diagram of $X_{N,1}$, this basis is shown in Fig. 3. Furthermore, as was discussed in [42,43], it can be obtained from the basis $(t_1,\ldots,t_N,m,\tau)$ through the following linear transformation[8] (with $v=\tau-m$):

$$\begin{aligned}&R=\tau-2Nm+N\rho, \qquad S=-m+\rho,\\ &\hat{a}_i=t_{i+1}, \qquad \forall\, i=1,\ldots,N.\end{aligned} \tag{2.15}$$

This transformation is part of a symmetry group $\mathbb{G}_N\times\mathrm{Dih}_N$, where Dih_N is a subgroup of the Weyl group of the $U(N)$ gauge group and $\mathbb{G}_N$ is a (dihedral) symmetry group that is implied by a web of dualities of the little string theory (see [42] for more details). Since $\mathbb{G}_N\times\mathrm{Dih}_N$ leaves the free energy invariant, the results discussed above also hold when formulated in the new basis $(\hat{a}_1,\ldots,\hat{a}_N,S,R)$. For further convenience we also introduce

$$\begin{aligned}&Q_R=e^{2\pi iR}, \qquad Q_S=e^{2\pi iS},\\ &Q_{\hat{a}_j}=e^{2\pi i\hat{a}_j} \quad \forall\, j=1,\ldots,N.\end{aligned} \tag{2.16}$$

[7]Here $\mathcal{F}^{\mathrm{orb,NS}}_{N,1}$ is the NS limit of $\mathcal{F}^{\mathrm{orb}}_{N,1}$ in Eq. (2.12), i.e., $\mathcal{F}^{\mathrm{orb,NS}}_{N,1}(\rho,\tau,m,\epsilon_1)=\lim_{\epsilon_2\mapsto0}\frac{\epsilon_2}{\epsilon_1}\mathcal{F}^{\mathrm{orb}}_{N,1}(\rho,\tau,m,\epsilon_{1,2})$.

[8]We implicitly use $t_{N+1}=t_1$.

(b) Instead of the $\mathcal{F}^{\mathrm{plet}}_{N,1}$ in Eq. (2.4) (which only counts single-particle BPS states), we work with the full free energy

$$\mathcal{F}_{N,1}(a_{1,\dots,N}, S, R, \epsilon_{1,2}) = \ln \mathcal{Z}_{N,1}(\hat{a}_{1,\dots,N}, S, R, \epsilon_{1,2}). \tag{2.17}$$

$\mathcal{F}_{N,1}$ can be expanded in powers of Q_R,

$$\mathcal{F}_{N,1}(\hat{a}_{1,\dots,N}, S, R; \epsilon_{1,2}) = \sum_r Q_R^r P_N^{(r)}(\hat{a}_{1,\dots,N}, S; \epsilon_{1,2}), \tag{2.18}$$

where we can also expand the coefficients $P^{(r)}(\hat{a}_{1,\dots,N}, S; \epsilon_{1,2})$ in powers of $\epsilon_{1,2}$,

$$P_N^{(r)}(\hat{a}_{1,\dots,N}, S; \epsilon_{1,2}) = \sum_{s_1,s_2} \epsilon_1^{s_1-1}\epsilon_2^{s_2-1} P^{(r)}_{N,(s_1,s_2)}(\hat{a}_{1,\dots,N}, S). \tag{2.19}$$

We will be interested mostly in the NS limit, i.e., $s_2 = 0$ and $s_1 \in \mathbb{N}_{\mathrm{even}}$. Finally, we can expand the $P^{(r)}_{N,(s_1,s_2)}$ in powers of $Q_{\hat{a}_i} = e^{2\pi i \hat{a}_i}$

$$P^{(r)}_{N,(s_1,s_2)}(\hat{a}_{1,\dots,N}, S) = \sum_{n_1,\dots,n_N} Q_{\hat{a}_1}^{n_1}\dots Q_{\hat{a}_N}^{n_N} P^{(r),\{n_1,\dots,n_N\}}_{(s_1,s_2)}(S). \tag{2.20}$$

For later convenience, we will also use the notation $\underline{n} = \{n_1, \dots, n_N\}$. From the $P^{(r),\underline{n}}_{N,(s_1,s_2)}$ we construct the following (*a priori* formal) object

$$H^{(r),\{n_1,\dots,n_N\}}_{(s_1,s_2)}(\rho, S) = \sum_{k=0}^{\infty} Q_\rho^k P^{(r),\{n_1+k,n_2+k,\dots,n_N+k\}}_{N,(s_1,s_2)}(S), \tag{2.21}$$

where $Q_\rho = e^{2\pi i \sum_{j=1}^N \hat{a}_j}$. The $P^{(r)}_{N,(s_1,s_2)}(\hat{a}_{1,\dots,N}, S)$ in Eq. (2.20) are resummed as

$$P^{(r)}_{N,(s_1,s_2)}(\hat{a}_{1,\dots,N}, S) = H^{(r),\{0,\dots,0\}}_{(s_1,s_2)}(\rho, S) + \sum_{\underline{n}}{}' H^{(r),\underline{n}}_{(s_1,s_2)}(\rho, S) Q_{\hat{a}_1}^{n_1}\dots Q_{\hat{a}_N}^{n_N}. \tag{2.22}$$

Here the summation is over all $\underline{n} = \{n_1, \dots, n_N\} \in (\mathbb{N}\cup\{0\})^N$ such that at least one of the $n_i = 0$. Furthermore, we implicitly assume that $\hat{a}_N = \rho - \sum_{i=1}^{N-1} \hat{a}_i$.

In the remainder of this paper we identify a limit in which the NS limit of the free energy diverges but the residue of the second order pole counts BPS states of a symmetric orbifold theory: For example, the partition function for the case $N = 2$ is discussed in Appendix B. In this case the partition function has a pole at $\hat{a}_1 = \pm 2\epsilon_+$, while in the NS limit the free energy $\mathcal{F}_{N=2,1}$ has a pole at $\hat{a}_1 = \pm\epsilon_1$. Terms of different order in Q_R have different order poles at $\hat{a}_1 = \pm\epsilon_1$ with different residues. If we expand the NS free energy in powers of ϵ_1, then the coefficients have different order poles at $\hat{a}_1 = 0$ with residues now shifted because of the ϵ_1 expansion. The lowest order pole is of order 2.

On a technical level, just as in previous work, we rely on studying series expansions of examples for small values of N which reveal certain patterns. However, since the corresponding computations are rather technical, we will summarize our observations in the following section before presenting the computations for $N = 2$, $N = 3$, and $N = 4$, respectively.

III. SUMMARY OF RESULTS

Because of the technical nature of some of the results of this paper, we provide a short overview of our main observations. For a given N, we start by extracting the following N functions from the (expansion coefficients of the) free energy $P^{(r)}_{N,(2s,0)}(\hat{a}_{1,\dots,N}, S)$ in Eq. (2.19)

$$\mathcal{C}^{N,(r)}_{i,(2s,0)}(\rho, S) = \frac{1}{(2\pi i)^N r^{i-1}} \sum_{\ell=0}^{\infty} Q_\rho^\ell \oint_0 d\hat{a}_1 \hat{a}_1 \oint_{-\hat{a}_1} d\hat{a}_2(\hat{a}_1+\hat{a}_2)\dots \oint_{-\hat{a}_1-\dots-\hat{a}_{i-2}} d\hat{a}_{i-1}(\hat{a}_1+\dots+\hat{a}_{i-1}) \times \oint_0 \frac{dQ_{\hat{a}_i}}{Q_{\hat{a}_i}^{1+\ell}}\dots\oint_0 \frac{dQ_{\hat{a}_N}}{Q_{\hat{a}_N}^{1+\ell}} P^{(r)}_{N,(2s,0)}(\hat{a}_1, \dots, \hat{a}_N, S), \quad \forall\, i = 1, \dots, N. \tag{3.1}$$

The last can be resummed into a (formal) series expansion in ϵ_1[9]

[9]Although *a priori* it is a formal expansion, the $i = 1$ case given in Eq. (3.5) and the $i = 2$ example discussed in Appendix B for $N = 2$ [see Eq. (B24)] shows that it is a Jacobi form involving $\theta_1(\rho, z)$ and its derivatives.

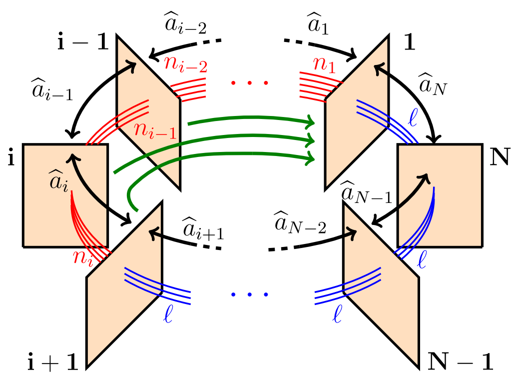

FIG. 4. Brane configuration for extracting $\mathcal{C}_i^{N,(r)}$.

$$\mathcal{C}_i^{N,(r)}(\rho,S,\epsilon_1)=\sum_{s=0}^{\infty}\epsilon_1^{2s-2i}\mathcal{C}_{i,(2s,0)}^{N,(r)}(\rho,S), \qquad (3.2)$$

which defines a Jacobi form of weight zero and index rN. The contour integrals in Eq. (3.1) are understood to be small circles centered around the specified points.[10] From a mathematical point of view, this extracts and combines specific coefficients in a mixed Fourier-Taylor expansion of $P_N^{(r)}$: with respect to the variables $\hat{a}_j$ for $1\leq j\leq i-1$ the prescription computes successively the coefficient of the second order pole at $\hat{a}_j=-\hat{a}_1-\cdots-a_{j-1}$. For the variables $\hat{a}_j$ for $i\leq j\leq N$, the prescription sums (weighted by Q_ρ^ℓ) the coefficients of the term $Q_{\hat{a}_i}^\ell Q_{\hat{a}_{i+1}}^\ell\ldots Q_{\hat{a}_N}^\ell$ in a Fourier series expansion in terms of $Q_{\hat{a}_j}=e^{2\pi i\hat{a}_j}$. From a physical perspective, the latter prescription combines contributions from M2-brane configurations (weighted by Q_ρ^ℓ), in which an equal number of ℓ M2-branes is stretched between the M5-branes j and $j+1$ for $i\leq j\leq N$[11] (see Fig. 4 for a schematic picture of a generic configuration, as well as Figs. 6, 8, and 10 for examples of the cases $N=2$, 3, 4). Concerning the remaining M5-branes, we consider a region in the moduli space in which they all form a stack on top of each other: from the resulting divergent expression, the contour integrals for $\hat{a}_j$ for $1\leq j\leq i-1$ in Eq. (3.1) extract successively the poles of the form $(\hat{a}_1+\hat{a}_2+\cdots+\hat{a}_{i-1})^{-2}$, $(\hat{a}_1+\hat{a}_2+\cdots+\hat{a}_{i-2})^{-2}$, …, $\hat{a}_1^{-2}$. The total order of the divergence selected in this fashion is $2i-2$: from all the examples that we have explicitly calculated, this seems to be the highest singularity that appears in the prepotential at leading instanton order. This matches the analysis of the singularities of the partition function in Appendix B.

[10]We implicitly assume here that we are in a generic point in the moduli space, such that the free energy has isolated poles with respect to the various variables. In this case, the precise form of the contour is not crucial, and the prescription is simply designed to extract their residues.

[11]In this notation it is understood that the M5 brane $N+1$ is in fact the M5-brane 1.

The brane configuration of parallel M5-branes labeled $1,\ldots,i-1$ stacked on top of each other (which is used to define the $\mathcal{C}_i^{N,(r)}$) can also be understood geometrically: we recall that the Calabi-Yau threefold $X_{N,1}$ dual to the generic M5-brane configuration has a resolved A_{N-1} singularity. The case in which $i-1$ M5-branes are on top of each other corresponds to a partial resolution of the A_{N-1} singularity so that it becomes a A_{i-1} singularity,

$$A_{N-1}\mapsto A_{i-2}\times\underbrace{A_{N-i+1}}_{\text{resolved}}. \qquad (3.3)$$

The functions $\mathcal{C}_{i=1}^{N,(r)}(\rho,S,\epsilon_1)$ were already studied in [46]. As already briefly discussed in the previous section, there (based on the study of numerous explicit examples) the following relation was observed (we are using the notation introduced above):

$$\mathcal{C}_{i=1}^{N,(r)}(\rho,S,\epsilon_1)=\mathcal{H}_r\Big[\mathcal{C}_{i=1}^{N,(1)}(\rho,S,\epsilon_1)\Big],\qquad\forall\, r\geq 1, \qquad (3.4)$$

where $\mathcal{H}_r$ is the rth Hecke operator [see Eq. (A10) in Appendix A for the definition] and

$$\mathcal{C}_{i=1}^{N,(1)}(\rho,S,\epsilon_1)=\frac{\theta_+\theta_-(\theta_+\theta_-'-\theta_-\theta_+')^{N-1}}{\eta(\rho)^{3N}\theta_1(\rho,\epsilon_1)^N}, \qquad (3.5)$$

with $\theta_\pm=\theta_1(\rho,-S+\rho\pm\frac{\epsilon_1}{2})$. In this paper, based on a detailed study of $\mathcal{C}_i^{N,(r)}(\rho,S,\epsilon_1)$ for $N=2$ and $N=3$ (and partially also for $N=4$), we provide evidence that Eq. (3.4) can be generalized to all the functions defined in Eq. (3.1):

$$\mathcal{C}_{i,(2s,0)}^{N,(r)}(\rho,S)=\mathcal{H}_r\Big[\mathcal{C}_{i,(2s,0)}^{N,(1)}(\rho,S)\Big],\qquad \begin{array}{l}\forall\, r\geq 1,\\ \forall\, i=1,\ldots,N,\, s\geq 0.\end{array} \qquad (3.6)$$

The $\mathcal{C}_{i,(2s,0)}^{N,(1)}$ are index N Jacobi forms of weight $2s-2i$ and are coefficients in the ϵ_1 expansion of $\mathcal{C}_i^{N,(1)}(\rho,S,\epsilon_1)$, which is a Jacobi form in two complex variables (S,ϵ_1) [67]. The action of the Hecke operator $\mathcal{H}_r$ extends to the case of multiple complex variables as given in Eq. (A11) in Appendix A. From Eq. (3.6) it then follows that

$$\mathcal{C}_i^{N,(r)}(\rho,S,\epsilon_1)=\mathcal{H}_r\Big[\mathcal{C}_i^{N,(1)}(\rho,S,\epsilon_1)\Big],\qquad \begin{array}{l}\forall\, r\geq 1,\\ \forall\, i=1,\ldots,N.\end{array} \qquad (3.7)$$

Assuming that this relation indeed holds for a generic r, a generating function can be formed capturing the degeneracies BPS states in this subsector

$$Z_i^{(N)}(R,\rho,S,\epsilon_1) = \exp\left(\sum_{r=1}^{\infty} Q_R^{Nr} \mathcal{C}_i^{N,(r)}(\rho,S,\epsilon_1)\right). \quad (3.8)$$

Equation (3.7) together with the fact that the "seed function" $\mathcal{C}_i^{N,(1)}(\rho,S,\epsilon_1)$ is a weight zero (index N) Jacobi form implies that $Z_i^{(N)}(R,\rho,S,\epsilon)$ can be interpreted as the partition functions of symmetric orbifold conformal field theories.[12] These symmetric orbifold theories arise from a special subsector of the full theory and are extracted using the contour integrals involving $a_{1,\ldots,i-1}$ in the NS limit. The fact that in this special subsector the moduli space of charge r instantons can be realized as a symmetric product of r charge 1 instantons suggests that this special subsector is getting contributions from well separated instantons only [68,69].

Furthermore, the study of the above-mentioned examples has brought to light numerous interesting patterns which suggest additional interesting properties of the functions $\mathcal{C}_i^{N,(r)}(\rho,S,\epsilon_1)$: it was already remarked in [47] that to order $\mathcal{O}(Q_R)$ (i.e., for $r=1$), the free energy can be decomposed into simpler building blocks, which are given by the expansion coefficients of the free energy for $N=1$ (see Appendix C 1 for the definition) as well as the expansion of the function W [see Eq. (C4) in Appendix C 2], which governs the counting of BPS states of a single M5-brane with single M2-branes ending on it on either side (see Secs. IV B and V B for details about the cases $N=2$ and $N=3$, respectively). This decomposition is also reflected at the level of the functions $\mathcal{C}_i^{N,(r=1)}(\rho,S,\epsilon_1)$ [and accordingly also for their expansion coefficients $\mathcal{C}_{i,(2s,0)}^{N,(r=1)}(\rho,S)$], for which we find, order by order in ϵ_1,

$$\mathcal{C}_i^{N,(r=1)}(\rho,S,\epsilon_1) = \varkappa_{i,N}^{(1)} (H_{N=1}^{(1)}(\rho,S,\epsilon_1))^i \times (W_{\mathrm{NS}}^{(1)}(\rho,S,\epsilon_1))^{N-i}, \quad \forall\, i=1,\ldots,N, \quad (3.9)$$

where $\varkappa_{i,N}^{(1)}$ is a numerical factor. From the study of the examples $N=2, 3, 4$ we conjecture that the factor $\varkappa_{i,N}^{(r)}$ depends only on i and N. Modulo the factor $\varkappa_{i,N}^{(1)}$ the functions $\mathcal{C}_i^{N,(r=1)}(\rho,S,\epsilon_1)$ satisfy the recursive relation

$$\begin{aligned}\mathcal{C}_i^{N+1,(r=1)}(\rho,S,\epsilon_1) &\sim \mathcal{C}_i^{N,(r=1)}(\rho,S,\epsilon_1) W_{NS}^{(1)}(\rho,S,\epsilon_1),\\ \mathcal{C}_{i+1}^{N+1,(r=1)}(\rho,S,\epsilon_1) &\sim \mathcal{C}_i^{N,(r=1)}(\rho,S,\epsilon_1) H_{N=1}^{(1)}(\rho,S,\epsilon_1).\end{aligned} \quad (3.10)$$

Starting from a configuration of $(N+1)$ M5-branes with i of them collapsed to form a stack, the first recursion relation suggests that the BPS states that contribute to the poles in $\mathcal{C}_i^{N+1,(r=1)}$ can be counted from a similar configuration, where we remove one of the M5-branes that is not part of the stack and it is related to $\mathcal{C}_i^{N,(r=1)}$ through multiplication with $W_{\mathrm{NS}}^{(1)}(\rho,S,\epsilon_1)$. Similarly, the second recursion relation suggests that the effect of removing an M5-brane from the stack of collapsed branes on the counting function $\mathcal{C}_i^{N+1,(r=1)}$ is described by multiplying $\mathcal{C}_i^{N,(r=1)}$ with the function $H_{N=1}^{(1)}(\rho,S,\epsilon_1)$.

To higher orders in Q_R (i.e., for $r>1$), the decomposition is more complicated. While we did not manage to identify all coefficients uniquely,[13] the examples we have studied suggest that

$$\mathcal{C}_i^{N,(r)}(\rho,S,\epsilon_1) = \varkappa_{i,N}^{(r)} (H_{N=1}^{(r)}(\rho,S,\epsilon_1))^i \times (W_{\mathrm{NS}}^{(r)}(\rho,S,\epsilon_1))^{N-i} + \mathfrak{R}_i^{N,(r)}(\rho,S,\epsilon_1), \quad (3.11)$$

where the remainder term $\mathfrak{R}_i^{N,(r)}$ itself can be decomposed into combinations of $H_{N=1}^{(r)}(\rho,S,\epsilon_1)$, where the coefficients are quasimodular forms that depend only on ρ and which can be written as harmonic polynomials in the Eisenstein series $(E_2(\rho),E_4(\rho),E_6(\rho))$ (see Appendix A for the definitions). Moreover, different such polynomials are related through derivatives with respect to the Eisenstein series E_2 in the style of holomorphic anomaly equations. We refer the reader to Secs. IV D and V D for details about the cases $N=2$ and $N=3$, respectively.

In the following sections we shall present detailed computations for $N=2$ and $N=3$ (and partially also $N=4$), which support the observations just outlined. After that we shall conclude in Sec. VII.

IV. LITTLE STRING THEORY WITH $N=2$

The simplest nontrivial example is to consider a model of little strings, which is engineered by two M5-branes on a circle that probe a flat $\mathbb{R}^4$ transverse space.[14]

A. Decomposition of the free energy

As explained above, the partition function and free energy of this LST is captured by the topological string on the toric Calabi-Yau threefold, $X_{2,1}$, whose web diagram is shown in Fig. 5. Here we use a basis of Kähler parameters

[12] The power Nr of Q_R in the summation in Eq. (3.8) is chosen such that $Z_i^{(N)}$ can be recognized more readily as a paramodular form with respect to the group Σ_N^* (see Appendix D).

[13] They are, however, implicitly given through the relation (3.6).

[14] The partition function of the case in which the transverse space is $\mathbb{C}^2/\mathbb{Z}_M$ is given in [25,26]. It would be interesting to see whether the Hecke structure we study in this paper is also present for $M>1$.

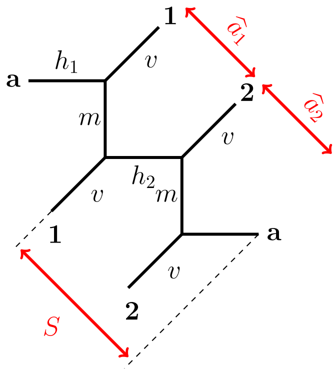

FIG. 5. Web diagram of $X_{2,1}$.

$(R, S, \rho, \hat{a}_1)$ where in addition to the parameters given in the figure, we have

$$\rho = \hat{a}_1 + \hat{a}_2, \qquad R - 2S = v - m. \tag{4.1}$$

Starting from the partition function $\mathcal{Z}_{2,1}$, we define the free energy

$$\mathcal{F}_{2,1}(\hat{a}_{1,2}, S, R, \epsilon_{1,2}) = \log \mathcal{Z}_{2,1}(\hat{a}_{1,2}, S, R, \epsilon_{1,2}). \tag{4.2}$$

We decompose the latter in terms of $H^{(r),\{n,0\}}_{(s_1,s_2)}(\rho, S)$ (for $n \in \mathbb{N} \cup \{0\}$) as described in Sec. II. Upon using the symmetries of the former, the summation in Eq. (2.22) becomes

$$P^{(r)}_{2,(s_1,s_2)}(\hat{a}_{1,2}, S) = H^{(r),\{0,0\}}_{(s_1,s_2)}(\rho, S) + \sum_{n=1}^{\infty} H^{(r),\{n,0\}}_{(s_1,s_2)}(\rho, S)\left(Q^n_{\hat{a}_1} + \frac{Q^n_\rho}{Q^n_{\hat{a}_1}}\right). \tag{4.3}$$

In the following we shall discuss only the so-called NS limit [44,45], i.e., we consider $s_2 = 0$. Only for $n = 0$ (which corresponds to the part of the free energy discussed in [46]) are the $H^{(r),\{n,0\}}_{(s,0)}$ (quasi-)Jacobi forms. For $n > 0$, the $H^{(r),\{n,0\}}_{(s,0)}$ are no longer modular objects. However, following [47,58], based on studying series expansions in Q_ρ (and exploiting certain patterns arising in the expansion coefficients) we can conjecture the following generic form[15]:

$$H^{(r),\{n,0\}}_{(2s,0)}(\rho, S) = \begin{cases} \mathfrak{h}^{(r)}_{0,(2s)}(\rho, S) & \text{for } n = 0 \\ \frac{1}{1-Q^n_\rho}\sum_{k=1}^{rs+1} n^{2k-1}\mathfrak{h}^{(r)}_{k,(2s)}(\rho, S) & \text{for } n > 0 \end{cases}, \tag{4.4}$$

where $\mathfrak{h}^{(r)}_{k,(2s)}$ is a (quasi-)Jacobi form of index $2r$ and weight $2s - 2 - 2k$. Using the standard Jacobi forms $\phi_{-2,1}$ and $\phi_{0,1}$ (see Appendix A for the definition), they can be cast into the form

$$\mathfrak{h}^{(r)}_{k,(2s)}(\rho, S) = \sum_{a=0}^{2r} h^{(r)}_{a,k,(2s)}(\phi_{-2,1}(\rho, S))^a(\phi_{0,1}(\rho, S))^{2r-a}, \tag{4.5}$$

where $h^{(r)}_{a,k,(2s)}$ are (quasi)modular forms of weight $2s - 2 - 2k + 2a$ and depth $sr + \delta_{k,0}$ which can be expressed as homogeneous polynomials of the Eisenstein series $\{E_{2,4,6}\}$. For later convenience, the coefficients $h^{(r)}_{a,k,(2s)}$ for $r = 1$, $r = 2$, and $r = 3$ are tabulated in Tables I–III, respectively.

B. Factorization at order $\mathcal{O}(Q_R)$

As conjectured in [47], the coefficients $H^{(r=1),\{n,0\}}_{(2s,0)}(\rho, S)$ can be factorized in terms of $H^{(1),\{0\}}_{(s,0)}$ and $W_{(s,0)}(\rho, S)$, i.e., the coefficients that appear in the expansion of the free energy for $N = 1$ and the function $W^{(1)}_{\rm NS}$ defined in Eq. (C8) (as reviewed in Appendix C 1), concretely,

$$H^{(r=1),\{0,0\}}_{(2s,0)}(\rho, S) = 2\sum_{a+b=s} H^{(1),\{0\}}_{(2a,0)}(\rho, S)W_{(2b,0)}(\rho, S),$$
$$H^{(r=1),\{n,0\}}_{(2s,0)}(\rho, S) = \frac{1}{1-Q^n_\rho}\sum_{a,b=0}^{s} H^{(1),\{0\}}_{(2a,0)}(\rho, S) \times H^{(1),\{0\}}_{(2b,0)}(\rho, S)\mathcal{M}^{(s)}_{ab}(n), \quad \forall\, n \geq 1. \tag{4.6}$$

Here $\mathcal{M}^{(s)}$ is a symmetric $[(s+1)\times(s+1)]$-dimensional matrix, that depends only on n, which in [47] was conjectured to take the form

$$\mathcal{M}^{(s)}_{ab} = -2\frac{(-1)^{s+a+b}n^{2s+1-2(a+b)}}{\Gamma(2s-2(a+b-1))}, \quad a, b \in \{0, \ldots, s\}, \tag{4.7}$$

where it is understood that $1/\Gamma(-m) = 0$ for $m \in \mathbb{N} \cup \{0\}$. The first few instances of $\mathcal{M}^{(s)}$ are

[15]We have verified that these expressions agree with an expansion of $P^{(r)}_{2,(s_1,s_2)}$ in Eq. (4.3) following from the general definition in Eqs. (2.17) and (2.18) in terms of the partition function defined in Eqs. (B1) and (B8), up to order $\mathcal{O}(Q^{30}_\rho)$ for $r = 1$, $\mathcal{O}(Q^{16}_\rho)$ for $r = 2$, and $\mathcal{O}(Q^{12}_\rho)$ for $r = 3$ and up to $2s = 8$.

TABLE I. Expansion coefficients $h^{(r=1)}_{a,k,(2s)}$.

s	k	$h^{(r=1)}_{0,k,(2s)}$	$h^{(r=1)}_{1,k,(2s)}$	$h^{(r=1)}_{2,k,(2s)}$
0	0	0	$-\frac{1}{12}$	$-\frac{E_2}{6}$
	1	0	0	-2
1	0	$\frac{1}{1152}$	0	$\frac{E_4-2E_2^2}{288}$
	1	0	$\frac{1}{24}$	$-\frac{E_2}{12}$
	2	0	0	$\frac{1}{3}$
2	0	$\frac{E_2}{55296}$	$\frac{5E_2^2-7E_4}{69120}$	$\frac{-10E_2^3-3E_2E_4+4E_6}{69120}$
	1	$-\frac{1}{4608}$	$\frac{E_2}{576}$	$-\frac{10E_2^2+13E_4}{5760}$
	2	0	$-\frac{1}{144}$	$\frac{E_2}{72}$
	3	0	0	$-\frac{1}{60}$
3	0	$\frac{E_4}{2211840}$	$\frac{35E_2^3-21E_2E_4-29E_6}{17418240}$	$\frac{-70E_2^4-168E_2^2E_4-8E_2E_6+123E_4^2}{34836480}$
	1	$-\frac{E_2}{110592}$	$\frac{E_2^2+E_4}{27648}$	$\frac{-70E_2^3-273E_2E_4-92E_6}{2903040}$
	2	$\frac{1}{27648}$	$-\frac{E_2}{3456}$	$\frac{10E_2^2+13E_4}{34560}$
	3	0	$\frac{1}{2880}$	$-\frac{E_2}{1440}$
	4	0	0	$\frac{1}{2520}$
4	0	$\frac{118E_6+105E_2E_4-70E_2^3}{13377208320}$	$\frac{175E_2^4+210E_2^2E_4-130E_2E_6-381E_4^2}{5573836800}$	$\frac{1682E_{10}-350E_2^5-2030E_2^3E_4-1000E_2^2E_6+177E_2E_4^2}{16721510400}$
	1	$\frac{-10E_2^2-7E_4}{53084160}$	$\frac{10E_2^3+30E_2E_4+11E_6}{19906560}$	$\frac{-350E_2^4-2730E_2^2E_4-1840E_2E_6-2283E_4^2}{1393459200}$
	2	$\frac{E_2}{663552}$	$\frac{-E_2^2-E_4}{165888}$	$\frac{70E_2^3+273E_2E_4+92E_6}{17418240}$
	3	$-\frac{1}{552960}$	$\frac{E_2}{69120}$	$\frac{-10E_2^2-13E_4}{691200}$
	4	0	$-\frac{1}{120960}$	$\frac{E_2}{60480}$
	5	0	0	$-\frac{1}{181440}$

TABLE II. Expansion coefficients $h^{(r=2)}_{a,k,(2s)}$.

s	k	$h^{(r=2)}_{0,k,(2s)}$	$h^{(r=2)}_{1,k,(2s)}$	$h^{(r=2)}_{2,k,(2s)}$	$h^{(r=2)}_{3,k,(2s)}$	$h^{(r=2)}_{4,k,(2s)}$
0	0	0	$-\frac{1}{4608}$	$-\frac{E_2}{1152}$	$-\frac{E_4}{1152}$	$\frac{E_6-E_2E_4}{144}$
	1	0	0	$-\frac{1}{96}$	0	$-\frac{E_4}{12}$
	2	0	0	0	$-\frac{1}{12}$	0
	3	0	0	0	0	$-\frac{1}{24}$
1	0	$\frac{1}{442368}$	0	$\frac{5E_4-4E_2^2}{55296}$	$\frac{4E_2E_4-3E_6}{6912}$	$\frac{-16E_2^2E_4-16E_2E_6+37E_4^2}{27648}$
	1	0	$\frac{1}{4608}$	$-\frac{E_2}{1152}$	$\frac{E_4}{128}$	$-\frac{E_2E_4+2E_6}{144}$
	2	0	0	$\frac{1}{128}$	$-\frac{E_2}{144}$	$\frac{53E_4}{1440}$
	3	0	0	0	$\frac{13}{576}$	$-\frac{E_2}{288}$
	4	0	0	0	0	$\frac{1}{120}$
2	0	$\frac{E_2}{32\cdot 24^4}$	$\frac{10E_2^2-21E_4}{13271040}$	$\frac{87E_6-20E_2^3-59E_2E_4}{6635520}$	$\frac{170E_2^2E_4+140E_2E_6-251E_4^2}{3317760}$	$\frac{E_4(746E_6-80E_2^3)-240E_2^2E_6-351E_2E_8}{3317760}$
	1	$\frac{-1}{884736}$	$\frac{E_2}{55296}$	$\frac{-20E_2^2-109E_4}{552960}$	$\frac{9E_2E_4+13E_6}{13824}$	$\frac{-80E_2^3E_4-320E_2E_6-721E_4^2}{276480}$
	2	0	$-\frac{5}{36864}$	$\frac{E_2}{1536}$	$\frac{-40E_2^2-593E_4}{138240}$	$\frac{371E_2E_4+535E_6}{120960}$
	3	0	0	$-\frac{301}{184320}$	$\frac{13E_2}{6912}$	$\frac{-20E_2^2-901E_4}{138240}$
	4	0	0	0	$-\frac{7}{2880}$	$\frac{E_2}{1440}$
	5	0	0	0	0	$-\frac{73}{120960}$

TABLE III. Expansion coefficients $h^{(r=3)}_{a,k,(2s)}$.

s	k	$h^{(r=3)}_{0,k,(2s)}$	$h^{(r=3)}_{1,k,(2s)}$	$h^{(r=3)}_{2,k,(2s)}$	$h^{(r=3)}_{3,k,(2s)}$	$h^{(r=3)}_{4,k,(2s)}$	$h^{(r=3)}_{5,k,(2s)}$	$h^{(r=3)}_{6,k,(2s)}$
0	0	0	$\frac{-1}{2985984}$	$-\frac{E_2}{497664}$	$-\frac{E_4}{124416}$	$\frac{22E_6-27E_2E_4}{186624}$	$\frac{8E_2E_6-9E_4^2}{20736}$	$\frac{E_4(20E_6-21E_2E_4)}{31104}$
	1	0	0	$-\frac{1}{41472}$	0	$-\frac{E_4}{576}$	$\frac{E_6}{216}$	$-\frac{7E_4^2}{864}$
	2	0	0	0	$-\frac{1}{1296}$	0	$-\frac{E_4}{90}$	$\frac{11E_6}{1134}$
	3	0	0	0	0	$-\frac{1}{432}$	0	$-\frac{7E_4}{1080}$
	4	0	0	0	0	0	$-\frac{1}{810}$	0
	5	0	0	0	0	0	0	$-\frac{1}{7560}$
1	0	$\frac{1}{36\cdot 24^5}$	0	$\frac{13E_4-6E_2^2}{23887872}$	$\frac{9E_2E_4-8E_6}{1119744}$	$\frac{-36E_2^2E_4-96E_2E_6+127E_4^2}{1990656}$	$\frac{9E_2^2E_6+42E_2E_4^2-53E_4E_6}{186624}$	$\frac{-378E_2^2E_4^2-1440E_2E_4E_6+1233E_4^3+544E_6^2}{4478976}$
	1	0	$\frac{1}{1990656}$	$-\frac{E_2}{331776}$	$\frac{E_4}{9216}$	$\frac{-27E_2E_4-94E_6}{124416}$	$\frac{24E_2E_6+139E_4^2}{41472}$	$-\frac{E_4(21E_2E_4+100E_6)}{20736}$
	2	0	0	$\frac{13}{248832}$	$-\frac{E_2}{10368}$	$\frac{59E_4}{25920}$	$-\frac{E_2E_4}{720}-\frac{97E_6}{18144}$	$\frac{11E_2E_6}{9072}+\frac{79E_4^2}{8640}$
	3	0	0	0	$\frac{5}{10368}$	$-\frac{E_2}{3456}$	$\frac{287E_4}{51840}$	$-\frac{7E_2E_4}{8640}-\frac{55E_6}{13608}$
	4	0	0	0	0	$\frac{7}{8640}$	$-\frac{E_2}{6480}$	$\frac{13E_4}{6480}$
	5	0	0	0	0	0	$\frac{353}{1088640}$	$-\frac{E_2}{60480}$
	6	0	0	0	0	0	0	$\frac{1}{34020}$

$$\mathcal{M}^{(0)} = (-2n), \qquad \mathcal{M}^{(1)} = \begin{pmatrix} \frac{n^3}{3} & -2n \\ -2n & 0 \end{pmatrix},$$
$$\mathcal{M}^{(2)} = \begin{pmatrix} -\frac{n^5}{60} & \frac{n^3}{3} & -2n \\ \frac{n^3}{3} & -2n & 0 \\ -2n & 0 & 0 \end{pmatrix}. \tag{4.8}$$

In [24] it was shown that the NS limit of the partition functions $\mathcal{Z}_{N,1}$ have a self-similar behavior[16] in a certain region of the Kähler moduli space, i.e., for $\hat{a}_1 = \hat{a}_2 = \cdots = \hat{a}_N = \frac{\rho}{N}$. Relations such as Eq. (2.10) allow one to infer nontrivial information about the free energy for generic N based only on the knowledge of the (much simpler) free energy for the configuration $N = 1$, albeit only at a specific point in the moduli space. From this perspective, Eq. (4.6) is similar in spirit to this self-similarity: they allow one to obtain nontrivial information about the $N = 2$ free energy at leading instanton order just from the configuration $N = 1$. We shall see that relations of this type also exist for $N > 2$ and (to some extent) also generalize to higher orders in Q_R.

C. Hecke structures

The coefficients $H^{(r),\{n,0\}}_{(2s,0)}(\rho, S)$ for $r > 1$ do not seem to exhibit simple factorizations of the type (4.6). We shall, however, in the following identify particular subsectors of the free energy [as introduced in Eq. (3.1) for generic N] that, in fact, do again factorize.

[16]The precise relation that was shown in [24] is Eq. (2.10).

To this end, we define the following contour integrals:

$$\mathcal{C}^{N=2,(r)}_{1,(2s,0)}(\rho, S) \coloneqq \frac{1}{(2\pi i)^2}\sum_{\ell=0}^{\infty} Q_\rho^\ell \oint_0 \frac{dQ_{\hat{a}_1}}{Q_{\hat{a}_1}^{1+\ell}} \times \oint_0 \frac{dQ_{\hat{a}_2}}{Q_{\hat{a}_2}^{1+\ell}} P^{(r)}_{2,(2s,0)}(\hat{a}_1, \hat{a}_2, S), \tag{4.9}$$

$$\mathcal{C}^{N=2,(r)}_{2,(2s,0)}(\rho, S) \coloneqq \frac{1}{(2\pi i)}\frac{1}{r}\oint_0 d\hat{a}_1 \hat{a}_1 P^{(r)}_{2,(2s,0)}(\rho, \hat{a}_1, S), \tag{4.10}$$

where all contours are small circles around the origin[17] and in Eq. (4.10) we have implicitly used $\hat{a}_2 = \rho - \hat{a}_1$. With these coefficient functions, we define the (*a priori* formal) series in ϵ_1,

$$\mathcal{C}^{N=2,(r)}_a(\rho, S, \epsilon_1) = \sum_{s=0}^{\infty} \epsilon_1^{2s-2a}\mathcal{C}^{N=2,(r)}_{a,(2s,0)}(\rho, S), \qquad \forall\, a = 1, 2. \tag{4.11}$$

From the perspective of the M-brane web, the functions (4.9) and (4.10) count certain BPS configurations of M2-branes stretched between two M5-branes on a circle. Because of the contour prescriptions, however, only certain configurations contribute, and they are depicted in Fig. 6:

(a) *Combination* $\mathcal{C}^{N=2,(r)}_{1,(2s,0)}$.—Upon writing $P^{(r)}_{2,(2s,0)}$ as a Fourier expansion in $Q_{\hat{a}_{1,2}}$ [similar to $H^{(r),\{n_1,\ldots,n_N\}}_{(2s,0)}$ in Eq. (2.20)]

[17]The integrals are in fact designed to precisely extract the residues in a Laurent series expansion.

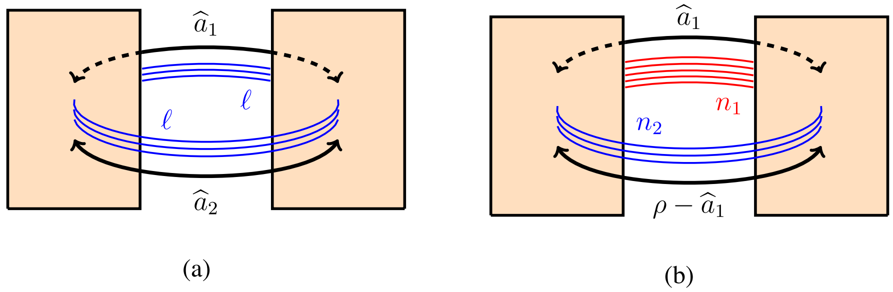

FIG. 6. Brane web configurations made up of $N=2$ M5-branes (drawn in orange) spaced out on a circle, with various M2-branes (drawn in red and blue) stretched between them. (a) An equal number ℓ of M2-branes is stretched between the M5-branes on either side of the circle. Configurations of this type are relevant for the computation of $\mathcal{C}_1^{N=2,(r)}$. (b) n_1 M2-branes are stretched on one side of the circle and n_2 ($\neq n_1$) on the other side of the circle. Configurations of this type are relevant for the contributions $\mathcal{C}_2^{N=2,(r)}$.

$$P^{(r)}_{2,(2,0)}(\hat{a}_{1,2},S,\epsilon_{1,2})=\sum_{n_1,n_2=0}^{\infty}Q^{n_1}_{\hat{a}_1}Q^{n_2}_{\hat{a}_2}P^{(r),\{n_1,n_2\}}_{(2s,0)}(S), \tag{4.12}$$

the contour prescriptions in Eq. (4.9) extract all terms with $n_1=n_2$. $\mathcal{C}^{N=2,(r)}_{1,(2s,0))}$ thus receives contributions only from those brane configurations, where an equal number of M2-branes is stretched between the two M5-branes on either side of the circle, as shown in Fig. 6(a). In fact, $\mathcal{C}^{N=2,(r)}_{1,(2s,0)}$ can equivalently be written as

$$\mathcal{C}^{N=2,(r)}_{1,(2s,0)}(\rho,S)=H^{(r),\{0,0\}}_{(2s,0)}(\rho,S), \tag{4.13}$$

and $\mathcal{C}^{N=2,(r)}_{1}(\rho,S,\epsilon_1)$ is in fact exactly the reduced free energy studied in [46]. Explicit expansions of $\mathcal{C}^{N=2,(r)}_{1,(2s,0)}$ for $r=1$, $r=2$, and $r=3$ can be recovered from Tables I–III, respectively, from the coefficients with $k=0$.

(b) *Combination* $\mathcal{C}^{N=2,(r)}_{2,(2s,0)}$.—The function $\mathcal{C}^{N=2,(r)}_{2,(2s,0)}$ in Eq. (4.10) receives contributions from configurations in which n_1 M2-branes are stretched between the M5-branes on one side of the circle and n_2 (with $n_2\neq n_1$) on the other side, as schematically shown in Fig. 6(b). Furthermore, from each of these contributions, the contour integral extracts the pole of the type $\hat{a}_1^{-2}$ (where it is important to write $\hat{a}_2=\rho-\hat{a}_1$).

In terms of the functions $H^{(r),\{n,0\}}_{(2s,0)}$ in Eq. (4.4), the contour prescription in fact extracts the contributions of $\mathfrak{h}^{(r)}_{k=1,(2s)}$,

$$\mathcal{C}^{N=2,(r)}_{2,(2s,0)}(\rho,S)=\frac{1}{r}\mathfrak{h}^{(r)}_{k=1,(2s)}. \tag{4.14}$$

To intuitively understand this result, we introduce [58]

$$\mathcal{I}_\alpha(\rho,\hat{a}_1)=D^{2\alpha}_{\hat{a}_1}\mathcal{I}_0=D^{2\alpha}_{\hat{a}_1}\sum_{n=1}^{\infty}\frac{n}{1-Q^n_\rho}\left(Q^n_{\hat{a}_1}+\frac{Q^n_\rho}{Q^n_{\hat{a}_1}}\right),$$
$$\text{with}\quad D_{\hat{a}_1}=Q_{\hat{a}_1}\frac{\partial}{\partial Q_{\hat{a}_1}}. \tag{4.15}$$

As argued in [58], $\mathcal{I}_0$ can be written in terms of Weierstrass's elliptic function $\wp$ and the second Eisenstein series (see Appendix A for the definitions)

$$\mathcal{I}_0(\rho,\hat{a}_1)=\frac{1}{(2\pi i)^2}[2\zeta(2)E_2(\rho)+\wp(\hat{a}_1;\rho)]. \tag{4.16}$$

Since Weierstrass's elliptic function affords the following series expansion:

$$\wp(z;\rho)=\frac{1}{z^2}+\sum_{k=1}^{\infty}2\zeta(2k+2)(2k+1)E_{2k+2}(\rho)z^{2k}, \tag{4.17}$$

we have for the contour integral

$$\oint d\hat{a}_1\hat{a}_1\mathcal{I}_\alpha(\rho,\hat{a}_1)=2\pi i\delta_{\alpha 0}, \tag{4.18}$$

such that with Eqs. (4.3) and (4.4) we have Eq. (4.14). The factor $1/r$ in the latter relation is simply a convenient normalization factor, as will become apparent later on.

A more direct way to arrive at Eq. (4.14) is to start from the decomposition (4.3) and exchange[18] the summations over k and n

$$P^{(r)}_{(2s,0)}(\hat{a}_{1,2},S)$$
$$=H^{(r),\{0,0\}}_{(2s,0)}(\rho,S)+\sum_{k=1}^{rs}\mathfrak{h}^{(r)}_{k,(2s)}(\rho,S)X_k(\hat{a}_{1,2}), \tag{4.19}$$

[18]This is possible since the sum over k is finite.

where we introduce the shorthand notation

$$X_k(\rho, \hat{a}_{1,2}) = \sum_{n=1}^{\infty} \frac{n^{2k-1}}{1-Q_\rho^n}(Q_{\hat{a}_1}^n + Q_{\hat{a}_2}^n)$$
$$= \sum_{n=1}^{\infty}\sum_{b=0}^{\infty} n^{2k-1} Q_\rho^{nb}(Q_{\hat{a}_1}^n + Q_{\hat{a}_2}^n). \quad (4.20)$$

We can express X_k in terms of the q-polygamma function $\psi_q^{(m)}(z)$,

$$\psi_q(z) = \frac{d\ln\Gamma_q(z)}{dz}$$
$$= -\ln(1-q) + \ln(q)\sum_{n=0}^{\infty}\frac{q^{n+z}}{1-q^{n+z}},$$
$$\psi_q^{(m)}(z) = \frac{d^m\psi_q(z)}{dz^m}, \quad (4.21)$$

where Γ_q is the q-gamma function

$$\Gamma_q(z) = (1-q)^{1-z}\prod_{n=0}^{\infty}\frac{1-q^{n+1}}{1-q^{n+z}}. \quad (4.22)$$

To this end, we interchange[19] the sums in the last expression in Eq. (4.20) and find with Eq. (4.21)

$$X_k(\rho, \hat{a}_{1,2}) = \frac{1}{\ln(Q_\rho)^{k+1}}\Big(\psi_{Q_\rho}^{(2k-1)}(\hat{a}_1/\rho)$$
$$+ \psi_{Q_\rho}^{(2k-1)}(\hat{a}_2/\rho)\Big) \quad \text{for } k \geq 1. \quad (4.23)$$

The q-gamma function $\Gamma_q(z)$ satisfies the identity $\Gamma_q(z+1) = \frac{1-q^z}{1-q}\Gamma_q(z)$ and, therefore, for $z \mapsto 0$ we obtain

$$\Gamma_q(z) = -\frac{1-q}{z\ln(q)} + \mathcal{O}(z^0). \quad (4.24)$$

[19]This is possible for $|Q_\rho| < 1$ and $|Q_{\hat{a}_{1,2}}| < 1$. To see this, we consider, for example,

$$\sum_{n=1}^{\infty}\sum_{b=0}^{\infty} n^{2k-1}|Q_\rho|^{nb}|Q_{\hat{a}_1}|^n \leq \sum_{n=1}^{\infty}\sum_{b=0}^{\infty} n^{2k-1}|Q_\rho|^{b}|Q_{\hat{a}_1}|^n$$
$$= \left(\sum_{n=1}^{\infty} n^{2k-1}|Q_{\hat{a}_1}|^n\right)\left(\sum_{b=0}^{\infty}|Q_\rho|^b\right)$$
$$= \frac{\mathrm{Li}_{1-2k}(|Q_{\hat{a}_1}|)}{1-|Q_\rho|},$$

where Li_{1-2k} denotes the polylogarithm. Thus, Eq. (4.20) is absolutely convergent, and therefore the summations can be interchanged.

Thus, the function $X_k(\rho, \hat{a}_{1,2})$ diverges for $\hat{a}_1 \mapsto 0$ and in fact has a pole of order $k+1$

$$X_k(\rho, \hat{a}_{1,2}) \sim -\frac{(2k-1)!}{\hat{a}_1^{2k}} + O(\hat{a}_1^0). \quad (4.25)$$

Therefore, the only contribution to the contour integral in Eq. (4.10) (which extracts the pole of order 2) stems from $X_1(\rho, \hat{a}_{1,2})$, thus yielding Eq. (4.14).

By comparing the explicit expressions for the contributions $\mathcal{C}_{1,(2s,0)}^{N=2,(r)}$ and $\mathcal{C}_{2,(2s,0)}^{N=2,(r)}$ to the free energy, we find that they satisfy the following recursion relation:

$$\mathcal{C}_{1,(2s,0)}^{N=2,(r)}(\rho, S) = \mathcal{H}_r\Big[\mathcal{C}_{1,(2s,0)}^{N=2,(1)}(\rho, S)\Big],$$
$$\mathcal{C}_{2,(2s,0)}^{N=2,(r)}(\rho, S) = \mathcal{H}_r\Big[\mathcal{C}_{2,(2s,0)}^{N=2,(1)}(\rho, S)\Big]. \quad (4.26)$$

The normalization factor $1/r$ appearing in the definition (4.10) was chosen to normalize the right-hand side of the second equation above.

D. Decomposition of $\mathcal{C}_{1,(2s,0)}^{N=2,(r)}$ and $\mathcal{C}_{2,(2s,0)}^{N=2,(r)}$

In Sec. IV B we have seen that the free energy in the NS limit factorizes to order $\mathcal{O}(Q_R)$ as in Eq. (4.6) with the basic building blocks given by the expansion coefficients of the free energy in the case $N=1$. While the complete free energy at higher orders $\mathcal{O}(Q_R^r)$ (for $r>1$) does not exhibit such a behavior, the particular contributions $\mathcal{C}_{1,(2s,0)}^{N=2,(r)}$ and $\mathcal{C}_{2,(2s,0)}^{N=2,(r)}$ defined in Eq. (4.10) lend themselves to a generalization of Eq. (4.6).

1. Factorization at order Q_R^1

The first step is to establish the factorization of $\mathcal{C}_{1,(2s,0)}^{N=2,(r=1)}$ and $\mathcal{C}_{2,(2s,0)}^{N=2,(r=1)}$, which are in fact induced by Eq. (4.6). Indeed, using Eq. (4.13) as well as Eq. (4.14), we have immediately

$$\mathcal{C}_{1,(2s,0)}^{N=2,(r=1)}(\rho, S) = 2\sum_{i,j=0}^{s}\delta_{s,i+j}H_{(2i,0)}^{(1),\{0\}}(\rho, S)W_{(2j,0)}(\rho, S), \quad (4.27)$$

$$\mathcal{C}_{2,(2s,0)}^{N=2,(r=1)}(\rho, S) = -2\sum_{i,j=0}^{s}\delta_{s,i+j}H_{(2i,0)}^{(1),\{0\}}(\rho, S)H_{(2j,0)}^{(1),\{0\}}(\rho, S). \quad (4.28)$$

Combining these expansion coefficients (in a series of ϵ_1), we can equivalently write the following relations for the (*a priori* formal) series expansions:

TABLE IV. Coefficients appearing in the expansion of the correction term $\mathfrak{R}_1^{(2)}$.

s	$\mathfrak{r}^{(2)}_{1,0,(2s,0)}$	$\mathfrak{r}^{(2)}_{1,1,(2s,0)}$	$\mathfrak{r}^{(2)}_{1,2,(2s,0)}$	$\mathfrak{r}^{(2)}_{1,3,(2s,0)}$	$\mathfrak{r}^{(2)}_{1,4,(2s,0)}$
0	0	0	0	0	$\frac{E_6-E_2E_4}{144}$
1	0	0	0	$\frac{E_2E_4-E_6}{3456}$	$\frac{E_2^2(-E_4)-E_2E_6+2E_4^2}{1728}$
2	0	0	$\frac{E_6-E_2E_4}{221184}$	$\frac{E_2^2E_4+E_2E_6-2E_4^2}{41472}$	$\frac{-20E_2^3E_4-60E_2^2E_6-69E_2E_4^2+149E_4E_6}{829440}$

TABLE V. Coefficients appearing in the expansion of the correction term $\mathfrak{R}_2^{(2)}$.

s	$\mathfrak{r}^{(2)}_{2,0,(2s,0)}$	$\mathfrak{r}^{(2)}_{2,1,(2s,0)}$	$\mathfrak{r}^{(2)}_{1,2,(2s,0)}$	$\mathfrak{r}^{(2)}_{2,3,(2s,0)}$	$\mathfrak{r}^{(2)}_{2,4,(2s,0)}$
0	0	0	0	0	$-\frac{E_4}{24}$
1	0	0	0	$\frac{E_4}{576}$	$-\frac{E_2E_4+2E_6}{288}$
2	0	0	$-\frac{E_4}{36864}$	$\frac{E_2E_4+2E_6}{6912}$	$-\frac{20E_2^2E_4+80E_2E_6+149E_4^2}{138240}$
3	0	$\frac{E_4}{5308416}$	$\frac{-E_2E_4-2E_6}{442368}$	$\frac{8E_2^2E_4+32E_2E_6+59E_4^2}{1327104}$	$\frac{-140E_2^3E_4-840E_2^2E_6-3129E_2E_4^2-5056E_4E_6}{34836480}$

$$C_1^{N=2,(r=1)}(\rho,S,\epsilon_1)=2H^{(1)}_{N=1}(\rho,S,\epsilon_1)W^{(1)}_{\rm NS}(\rho,S,\epsilon_1),$$
$$C_2^{N=2,(r=1)}(\rho,S,\epsilon_1)=-2[H^{(1)}_{N=1}(\rho,S,\epsilon_1)]^2, \tag{4.29}$$

where the coefficients $H^{(1)}_{N=1}$ and $W^{(1)}_{\rm NS}$ are defined in Eqs. (C3) and (C8), respectively.

2. Factorization at order Q_R^2

Based on Eqs. (4.27) and (4.28), the first attempt to factorize the function $C_{1,2}^{N=2,(r=2)}$ to order $\mathcal{O}(Q_R^2)$ would be to use a similar decomposition, except to replace $H^{(1)}_{N=1}$ and $W^{(1)}_{\rm NS}$ with their order $\mathcal{O}(Q_R^2)$ counterparts $H^{(2)}_{N=1}$ and $W^{(2)}_{\rm NS}$, respectively. However, this does not fully reproduce the correct answer; instead we have[20]

$$\begin{aligned}C_1^{N=2,(r=2)}(\rho,S,\epsilon_1)&=\frac{4}{3}H^{(2)}_{N=1}(\rho,S,\epsilon_1)W^{(2)}_{\rm NS}(\rho,S,\epsilon_1)\\&\quad+\mathfrak{R}_1^{(2)}(\rho,S,\epsilon_1),\\C_2^{N=2,(r=2)}(\rho,S,\epsilon_1)&=-\frac{4}{3}[H^{(2)}_{N=1}(\rho,S,\epsilon_1)]^2\\&\quad+\mathfrak{R}_2^{(2)}(\rho,S,\epsilon_1).\end{aligned} \tag{4.30}$$

The additional contributions $\mathfrak{R}_{1,2}^{(2)}$ are formal expansions in powers of ϵ_1,

$$\mathfrak{R}_a^{(2)}(\rho,S,\epsilon_1)=\sum_{s=0}^{\infty}\epsilon_1^{2s-2a}\mathfrak{R}^{(2)}_{a,(2s,0)}(\rho,S),\quad \forall\, a=1,2, \tag{4.31}$$

where the $\mathfrak{R}^{(2)}_{a,(2s,0)}(\rho,S)$ in turn can be decomposed as

$$\mathfrak{R}^{(2)}_{a,(2s,0)}(\rho,S)=\sum_{i=0}^{4}\mathfrak{r}^{(2)}_{a,i,(2,0)}(\rho)(\phi_{-2,1}(\rho,S))^i\phi_{0,1}(\rho,S))^{4-i}, \tag{4.32}$$

and the $\mathfrak{r}^{(2)}_{a,i,(2,0)}(\rho)$ are (quasi)modular forms of weight $2s-2+2i-2a$ and the first few expressions are tabulated for $a=1$ in Table IV, and for $a=2$ in Table V.

The functions $\mathfrak{R}_a^{(2)}$ can themselves again be factorized where the basic building blocks are $H^{(1)}_{N=1}$,

$$\mathfrak{R}_a^{(2)}(\rho,S,\epsilon_1)=\mathfrak{S}^{(2)}_{a,4}(\rho,\epsilon_1)[H^{(1)}_{N=1}(\rho,S,\epsilon_1)]^4. \tag{4.33}$$

The only novel feature is the appearance of the functions $\mathfrak{S}^{(2)}_{a,4}$, which are S-independent (quasi)Jacobi forms that are characterized through

$$\begin{aligned}\frac{d\mathfrak{S}^{(2)}_{1,4}}{dE_2}(\rho,\epsilon_1)&=\frac{\epsilon_1^2}{6}\mathfrak{S}^{(2)}_{2,4}(\rho,\epsilon_1),\\\mathfrak{S}_{2,4}(\rho,\epsilon_1)&=\sum_{s=1}^{\infty}\epsilon_1^{2s-6}\frac{(4^{s+1}-1)(2s+1)}{3\cdot 4^s\pi^{2(s+1)}}\\&\quad\times\zeta(2s+2)E_{2s+2}(\rho).\end{aligned} \tag{4.34}$$

While we cannot write a closed form expression for the holomorphic anomaly in $\mathfrak{R}_1^{(2)}$, we have

[20]The relation (4.30) as well as the remaining equations in this subsection are understood to hold order by order in an expansion in powers of ϵ_1, and we have checked it up to order ϵ_1^6. To save writing, however, in the following we state our results in terms of the formal series expansions.

$$\mathfrak{S}^{(2)}_{2,4}=\int dE_2\mathfrak{S}^{(2)}_{1,4}+\sum_{s=1}^{\infty}\epsilon_1^{2s-6}\frac{(4^{s+1}-1)(2s+1)}{18\cdot 4^s\pi^{2(s+1)}}\zeta(2s+2)\mathfrak{e}_{2s+4}(\rho), \tag{4.35}$$

where $\mathfrak{e}_{2s+4}$ is a polynomial in $E_{4,6}$ of weight $2s+4$, normalized such that $\mathfrak{e}_{2s+4}(\rho)=1+\mathcal{O}(Q_\rho)$.[21]

3. Factorization at order Q_R^r for $r>2$

Following the decomposition (4.30) at order Q_R^2, we can consider similar expressions to higher orders. From the explicit examples we find to order Q_R^3

$$\begin{aligned}
\mathcal{C}_1^{N=2,(r=3)}(\rho,S,\epsilon_1)&=\frac{3}{2}H^{(3)}_{N=1}W^{(3)}_{\rm NS}+(H^{(1)}_{N=1})^6\mathfrak{S}^{(r=3)}_{1,(6,0)}(\rho,\epsilon_1)\\
&\quad+(H^{(1)}_{N=1})^4H^{(2)}_{N=1}\mathfrak{S}^{(r=3)}_{1,(4,1)}(\rho,\epsilon_1)+(H^{(1)}_{N=1})^2(H^{(2)}_{N=1})^2\mathfrak{S}^{(r=3)}_{1,(2,2)}(\rho,\epsilon_1),\\
\mathcal{C}_2^{N=2,(r=3)}(\rho,S,\epsilon_1)&=-\frac{3}{2}H^{(3)}_{N=1}H^{(3)}_{N=1}+(H^{(1)}_{N=1})^6\mathfrak{S}^{(r=3)}_{2,(6,0)}(\rho,\epsilon_1)\\
&\quad+(H^{(1)}_{N=1})^4H^{(2)}_{N=1}\mathfrak{S}^{(r=3)}_{2,(4,1)}(\rho,\epsilon_1)+(H^{(1)}_{N=1})^2(H^{(2)}_{N=1})^2\mathfrak{S}^{(r=3)}_{2,(2,2)}(\rho,\epsilon_1),
\end{aligned}\tag{4.36}$$

and to order Q_R^4

$$\begin{aligned}
\mathcal{C}_1^{N=2,(r=4)}(\rho,S,\epsilon_1)&=\frac{8}{7}H^{(4)}_{N=1}W^{(4)}_{\rm NS}+(H^{(1)}_{N=1})^8\mathfrak{S}^{(r=4)}_{1,(8,0,0)}(\rho,\epsilon_1)+(H^{(1)}_{N=1})^6H^{(2)}_{N=1}\mathfrak{S}^{(r=4)}_{1,(6,1,0)}(\rho,\epsilon_1)\\
&\quad+(H^{(1)}_{N=1})^4(H^{(2)}_{N=1})^2\mathfrak{S}^{(r=3)}_{1,(4,2,0)}(\rho,\epsilon_1)+(H^{(1)}_{N=1})^2(H^{(2)}_{N=1})^3\mathfrak{S}^{(r=3)}_{1,(2,3,0)}(\rho,\epsilon_1)\\
&\quad+(H^{(1)}_{N=1})^2(H^{(3)}_{N=1})^2\mathfrak{S}^{(r=3)}_{1,(2,0,2)}(\rho,\epsilon_1),\\
\mathcal{C}_2^{N=2,(r=4)}(\rho,S,\epsilon_1)&=-\frac{8}{7}H^{(4)}_{N=1}H^{(4)}_{N=1}+(H^{(1)}_{N=1})^8\mathfrak{S}^{(r=4)}_{2,(8,0,0)}(\rho,\epsilon_1)+(H^{(1)}_{N=1})^6H^{(2)}_{N=1}\mathfrak{S}^{(r=4)}_{2,(6,1,0)}(\rho,\epsilon_1)\\
&\quad+(H^{(1)}_{N=1})^4(H^{(2)}_{N=1})^2\mathfrak{S}^{(r=3)}_{2,(4,2,0)}(\rho,\epsilon_1)+(H^{(1)}_{N=1})^2(H^{(2)}_{N=1})^3\mathfrak{S}^{(r=3)}_{2,(2,3,0)}(\rho,\epsilon_1)\\
&\quad+(H^{(1)}_{N=1})^2(H^{(3)}_{N=1})^2\mathfrak{S}^{(r=3)}_{2,(2,0,2)}(\rho,\epsilon_1).
\end{aligned}\tag{4.37}$$

Here $\mathfrak{S}^{(r),\ell}_{i,k,}(\rho,\epsilon_1)$ are independent of S and we find the following ϵ_1 expansions for $r=3$:

$$\begin{aligned}
\frac{1}{\epsilon_1^{10}}\mathfrak{S}^{(3)}_{1,(6,0)}&=\frac{E_4(E_6-E_2E_4)}{2592}+\frac{\epsilon_1^2(E_6^2-3E_2E_4E_6+2E_4^3)}{15552}+\frac{\epsilon_1^4(43E_4^2E_6-28E_2E_4^3-15E_2E_6^2)}{622080}+O(\epsilon_1^6),\\
\frac{1}{\epsilon_1^{8}}\mathfrak{S}^{(3)}_{1,(4,1)}&=\frac{E_4^2-E_2E_6}{162}-\frac{5\epsilon_1^2E_4(E_2E_4-E_6)}{1944}+\frac{\epsilon_1^4(-196E_2E_4E_6+123E_4^3+73E_6^2)}{233280}+O(\epsilon_1^6),\\
\frac{1}{\epsilon_1^{6}}\mathfrak{S}^{(3)}_{1,(2,2)}&=\frac{2(E_6-E_2E_4)}{81}+\frac{2\epsilon_1^2}{243}(E_4^2-E_2E_6)+\frac{\epsilon_1^4E_4}{405}(E_6-E_2E_4)+O(\epsilon_1^6),\\
\frac{1}{\epsilon_1^{8}}\mathfrak{S}^{(3)}_{2,(6,0)}&=-\frac{E_4^2}{648}-\frac{E_4E_6\epsilon_1^2}{1296}-\frac{\epsilon_1^4(28E_4^3+15E_6^2)}{155520}+O(\epsilon_1^6),\\
\frac{1}{\epsilon_1^{6}}\mathfrak{S}^{(3)}_{2,(4,1)}&=\frac{-2E_6}{81}-\frac{5E_4^2\epsilon_1^2}{486}-\frac{49E_4E_6\epsilon_1^4}{14580}+O(\epsilon_1^6),\qquad \frac{1}{\epsilon_1^{4}}\mathfrak{S}^{(3)}_{2,(2,2)}=\frac{-8E_4}{81}-\frac{8\epsilon_1^2E_6}{243}-\frac{4E_4^2\epsilon_1^4}{405}+O(\epsilon_1^6),
\end{aligned}$$

and for $r=4$

[21] Implicitly $\mathfrak{e}_{2s+4}(\rho)$ is, of course, fixed uniquely through the relation (4.26).

$$\begin{aligned}
\frac{1}{\epsilon_1^{14}}\mathfrak{S}^{(4)}_{1,(8,0,0)} &= \frac{(-21E_2E_4^3-31E_2E_6^2+52E_4^2E_6)}{1741824}-\frac{13\epsilon_1^2(E_4E_6(E_2E_4-E_6))}{497664}\\
&\quad+\frac{\epsilon_1^4(-2E_2(654E_4^4+4129E_4E_6^2)+6179E_4^3E_6+3387E_6^3)}{627056640}+O(\epsilon_1^6),\\
\frac{1}{\epsilon_1^{12}}\mathfrak{S}^{(4)}_{1,(6,1,0)} &= \frac{(-181E_2E_4E_6+129E_4^3+52E_6^2)}{108864}+\frac{\epsilon_1^2(-651E_2E_4^3-313E_2E_6^2+964E_4^2E_6)}{653184}\\
&\quad+\frac{\epsilon_1^4E_4(-12075E_2E_4E_6+5269E_4^3+6806E_6^2)}{13063680}+O(\epsilon_1^6),\\
\frac{1}{\epsilon_1^{10}}\mathfrak{S}^{(4)}_{1,(4,2,0)} &= -\frac{5}{756}(E_4(E_2E_4-E_6))+\frac{\epsilon_1^2(-919E_2E_4E_6+540E_4^3+379E_6^2)}{163296}\\
&\quad+\frac{\epsilon_1^4(-2118E_2E_4^3-1303E_2E_6^2+3421E_4^2E_6)}{979776}+O(\epsilon_1^6),\\
\frac{1}{\epsilon_1^{8}}\mathfrak{S}^{(4)}_{1,(2,3,0)} &= \frac{7(E_4^2-E_2E_6)}{243}-\frac{103\epsilon_1^2(E_4(E_2E_4-E_6))}{5103}\\
&\quad+\frac{\epsilon_1^4(5386E_6^2-13887E_2E_4E_6+8501E_4^3)}{1224720}+O(\epsilon_1^6),\\
\frac{1}{\epsilon_1^{6}}\mathfrak{S}^{(4)}_{1,(2,0,2)} &= -\frac{15(E_2E_4-E_6)}{128}+\frac{17\epsilon_1^2}{256}(E_4^2-E_2E_6)-\frac{1511\epsilon_1^4(E_4(E_2E_4-E_6))}{43008}+O(\epsilon_1^6),\\
\frac{1}{\epsilon_1^{12}}\mathfrak{S}^{(4)}_{2,(8,0,0)} &= -\frac{(21E_4^3+31E_6^2)}{580608}-\frac{13E_4^2E_6\epsilon_1^2}{165888}-\frac{\epsilon_1^4(654E_4^4+4129E_4E_6^2)}{104509440}+O(\epsilon_1^6),\\
\frac{1}{\epsilon_1^{10}}\mathfrak{S}^{(4)}_{2,(6,1,0)} &= -\frac{181E_4E_6}{36288}-\frac{\epsilon_1^2(651E_4^3+313E_6^2)}{217728}-\frac{115E_4^2E_6\epsilon_1^4}{41472}+O(\epsilon_1^6),\\
\frac{1}{\epsilon_1^{8}}\mathfrak{S}^{(4)}_{2,(4,2,0)} &= -\frac{5E_4^2}{252}-\frac{919E_4E_6\epsilon_1^2}{54432}-\frac{\epsilon_1^4(2118E_4^3+1303E_6^2)}{326592}+O(\epsilon_1^6),\\
\frac{1}{\epsilon_1^{6}}\mathfrak{S}^{(4)}_{2,(2,3,0)} &= -\frac{7E_6}{81}-\frac{103E_4^2\epsilon_1^2}{1701}-\frac{1543E_4E_6\epsilon_1^4}{45360}+O(\epsilon_1^6),\\
\frac{1}{\epsilon_1^{4}}\mathfrak{S}^{(4)}_{2,(2,0,2)} &= -\frac{45E_4}{128}-\frac{51\epsilon_1^2E_6}{256}-\frac{1511E_4^2\epsilon_1^4}{14336}+O(\epsilon_1^6).
\end{aligned}$$

Comparing these expressions suggests the following form:

$$\left.\begin{array}{l}\mathcal{C}_1^{N=2,(r)}\\ \mathcal{C}_2^{N=2,(r)}\end{array}\right\} = {\sum_{i_1,\ldots,i_r}}' \mathfrak{S}^{(r)}_{a,(i_1,\ldots,i_r)}(H^{(1)}_{N=1})^{i_1}\ldots(H^{(1)}_{N=1})^{i_r}+\frac{2r}{\sigma_1(r)}\begin{cases}H^{(r)}_{N=1}W^{(r)}_{\rm NS} & \text{for } a=1,\\ (-1)H^{(r)}_{N=1}H^{(r)}_{N=1} & \text{for } a=2.\end{cases} \tag{4.38}$$

Here the prime on the summation denotes the following conditions on $(i_1,\ldots,i_{r-1})$:

$$\sum_{j=1}^{r} j i_j = 2r, \quad i_1\in\mathbb{N}_{\rm even}, \quad i_1>0, \tag{4.39}$$

and $\mathfrak{S}^{(r)}_{a,(i_1,\ldots,i_{r-1})}$ are quasimodular forms depending on ρ and ϵ_1 which, in particular, satisfy

$$\frac{\partial\mathfrak{S}^{(r)}_{2,(i_1,\ldots,i_r)}}{\partial E_2(\rho)}(\rho,\epsilon_1)=0, \quad \mathfrak{S}^{(r)}_{2,(i_1,\ldots,i_r)}(\rho,\epsilon_1)=\frac{12}{r\epsilon_1^2}\frac{\partial\mathfrak{S}^{(r)}_{1,(i_1,\ldots,i_r)}}{\partial E_2(\rho)}(\rho,\epsilon_1), \quad \forall\, r>1. \tag{4.40}$$

This generalizes the first relation in Eq. (4.34) and also implies that $\mathfrak{S}^{(r)}_{2,(i_1,\ldots,i_r)}$ is a holomorphic Jacobi form. Notice also that, for all examples we have computed thus far, $\mathfrak{S}^{(r)}_{2,(i_1,\ldots,i_r)} = 0$ for $i_r > 0$.

V. HECKE STRUCTURE FOR $N = 3$

After discussing the free energy of the $N = 2$ LST, we continue with $N = 3$.

A. Decomposition of the free energy

The starting point is to compute the decomposition of the free energy. The web diagram representing $X_{3,1}$, which is relevant for the $N = 3$ free energy, is shown in Fig. 7. In addition to the Kähler parameters shown in the figure, we also have

$$\rho = \hat{a}_1 + \hat{a}_2 + \hat{a}_3, \quad R - 3S = m - 2v. \tag{5.1}$$

From the partition function $\mathcal{Z}_{3,1}$ we can compute the free energy

$$\mathcal{F}_{3,1}(\hat{a}_{1,2,3}, S, R, \epsilon_{1,2}) = \log \mathcal{Z}_{3,1}(\hat{a}_{1,2,3}, S, R, \epsilon_{1,2}).$$

As in the case of $N = 2$, we focus exclusively on the NS limit. In this case, following Eq. (2.22), we can decompose

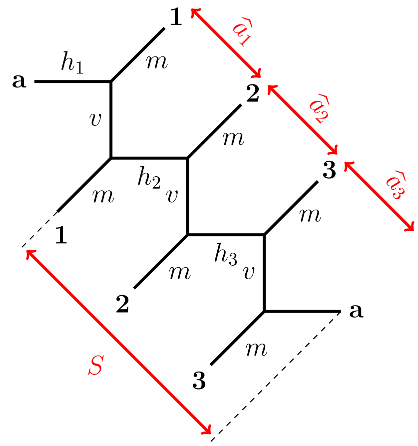

FIG. 7. Web diagram of $X_{3,1}$.

the free energy in terms of $H^{(r),\underline{n}}_{(2s,0)}$, where $\underline{n}$ can be either of the following triples:

$$\{0,0,0\}, \quad \{n,0,0\}, \quad \{n,n,0\}, \quad \{n_1+n_2, n_1, 0\}, \quad \text{with} \quad n, n_1, n_2 \in \mathbb{N}. \tag{5.2}$$

More concretely, we can write the following (*a priori* formal) decomposition:

$$\begin{aligned} P^{(r)}_{3,(2s,0)}(\hat{a}_{1,2,3}, S) &= H^{(r),\{0,0,0\}}_{(2s,0)}(\rho, S) + \sum_{n=1}^{\infty} H^{(r),\{n,0,0\}}_{(2s,0)}(\rho, S)\left(Q^n_{\hat{a}_1} + Q^n_{\hat{a}_2} + \frac{Q^n_\rho}{Q^n_{\hat{a}_1} Q^n_{\hat{a}_2}}\right) \\ &\quad + \sum_{n=1}^{\infty} H^{(r),\{n,n,0\}}_{(2s,0)}(\rho, S)\left(Q^n_{\hat{a}_1} Q^n_{\hat{a}_2} + \frac{Q^n_\rho}{Q^n_{\hat{a}_1}} + \frac{Q^n_\rho}{Q^n_{\hat{a}_2}}\right) \\ &\quad + \sum_{n_1,n_2=1}^{\infty} H^{(r),\{n_1+n_2,n_1,0\}}_{(2s,0)}(\rho, S)\left(Q^{n_1+n_2}_{\hat{a}_1} Q^{n_1}_{\hat{a}_2} + \frac{Q^{n_2}_{\hat{a}_1} Q^{n_1}_\rho}{Q^{n_1}_{\hat{a}_2}} + \frac{Q^{n_1+n_2}_\rho}{Q^{n_1+n_2}_{\hat{a}_1} Q^{n_2}_{\hat{a}_2}} + (\hat{a}_1 \leftrightarrow \hat{a}_2)\right). \end{aligned} \tag{5.3}$$

Comparing this with an explicit expansion of the free energy (2.18), we observe that the coefficients $H^{(r),\underline{n}}_{(2s,0)}$ can be written in the form

$$\begin{aligned} H^{(r),\{n,0,0\}}_{(2s,0)}(\rho, S) &= \frac{1}{1 - Q^n_\rho} \sum_{k=1}^{rs+1} n^{2k-1} \mathfrak{f}^{(r)}_{k,(2s)}(\rho, S) + \frac{Q^n_\rho}{(1 - Q^n_\rho)^2} \sum_{k=1}^{rs+1} n^{2k} \mathfrak{g}^{(r)}_{k,(2s)}(\rho, S), \\ H^{(r),\{n,n,0\}}_{(2s,0)}(\rho, S) &= \frac{1}{1 - Q^n_\rho} \sum_{k=1}^{rs+1} n^{2k-1} \mathfrak{f}^{(r)}_{k,(2s)}(\rho, S) + \frac{1}{(1 - Q^n_\rho)^2} \sum_{k=1}^{rs+1} n^{2k} \mathfrak{g}^{(r)}_{k,(2s)}(\rho, S), \\ H^{(r),\{n_1+n_2,n_1,0\}}_{(2s,0)}(\rho, S) &= \frac{n_2(n_2 + 2n_1)}{(1 - Q^{n_1}_\rho)(1 - Q^{n_2}_\rho)} \sum_{k=1}^{rs+1} \sum_{\ell} p^{(r)}_{\ell,k,(2s)}(n_1, -n_1 - n_2) \mathfrak{j}^{(r)}_{\ell,k,(2s)}(\rho, S) \\ &\quad + \frac{(n_1^2 - n_2^2)}{(1 - Q^{n_1}_\rho)(1 - Q^{n_1+n_2}_\rho)} \sum_{k=1}^{rs+1} \sum_{\ell} p^{(r)}_{\ell,k,(2s)}(n_1, n_2) \mathfrak{j}^{(r)}_{\ell,k,(2s)}(\rho, S), \end{aligned} \tag{5.4}$$

where $\mathfrak{f}^{(r)}_{k,(2s)}$ are (quasi-)Jacobi forms of index $3r$ and weight $2s-2-2k$, $\mathfrak{g}^{(r)}_{k,(2s)}$ are (quasi-)Jacobi forms of index $3r$ and weight $2s-4-2k$, and $\mathfrak{j}^{(r)}_{a,k,(2s)}$ are (quasi-)Jacobi forms of index $3r$ and weight $2s-4-2k$. They can be written in the following fashion:

$$\mathfrak{f}^{(r)}_{k,(2s)}(\rho,S) = -\sum_{a=0}^{3r} f^{(r)}_{a,k,(2s)}(\rho)(\phi_{-2,1}(\rho,S))^a(\phi_{0,1}(\rho,S))^{3r-a},$$

$$\mathfrak{g}^{(r)}_{k,(2s)}(\rho,S) = -\sum_{a=0}^{3r} g^{(r)}_{a,k,(2s)}(\rho)(\phi_{-2,1}(\rho,S))^a(\phi_{0,1}(\rho,S))^{3r-a},$$

$$\mathfrak{j}^{(r)}_{\ell,k,(2s)}(\rho,S) = -\sum_{a=0}^{3r} j^{(r)}_{a,\ell,k,(2s)}(\rho)(\phi_{-2,1}(\rho,S))^a(\phi_{0,1}(\rho,S))^{3r-a},$$

where $f^{(r)}_{a,k,(2s)}$, $g^{(r)}_{a,k,(2s)}$, and $j^{(r)}_{a,\ell,k,(2s)}$ are quasimodular forms of weight $2s-2-2k+2a$, $2s-4-2k+2a$ and $2s-4-2k+2a$, respectively. Similarly, we can expand

$$H^{(r),\{0,0,0\}}_{(2s,0)}(\rho,S) = -\sum_{a=0}^{3r} d^{(r)}_{a,(2s)}(\rho)(\phi_{0,1}(\rho,S))^a(\phi_{-2,1}(\rho,S))^{3r-a}, \tag{5.5}$$

where $d^{(r)}_{a,(2s)}$ are quasi-Jacobi forms of weight $2s+2k$. Furthermore, $p^{(r)}_{\ell,k,(2s)}(n_1,n_2)$ in Eq. (5.4) are homogeneous polynomials in $n_{1,2}$ of order $2(k-1)$, that are symmetric in the exchange of $n_1 \leftrightarrow n_2$. Explicit expressions for $d^{(r)}_{a,k,(2s)}$, $f^{(r)}_{a,k,(2s)}$, $g^{(r)}_{a,k,(2s)}$, and $j^{(r)}_{a,k,(2s)}$ as well as $p^{(r)}_{k,(2s)}(n_1,n_2)$ for low values of s are tabulated for $r=1$ in Tables VI–IX and $r=2$ in Tables X–XIII, respectively.

TABLE VI. Expansion coefficients $d^{(r=1)}_{a,(2s)}$.

s	$d^{(r=1)}_{0,(2s)}$	$d^{(r=1)}_{1,(2s)}$	$d^{(r=1)}_{2,(2s)}$	$d^{(r=1)}_{3,(2s)}$
0	0	$\frac{1}{192}$	$\frac{E_2}{48}$	$\frac{E_2^2}{48}$
1	$\frac{1}{18432}$	$\frac{E_2}{9216}$	$\frac{2E_4-3E_2^2}{4608}$	$\frac{2E_2E_4-3E_2^3}{2304}$
2	$\frac{E_2}{884736}$	$\frac{45E_2^2-43E_4}{4423680}$	$\frac{8E_6-21E_2E_4}{1105920}$	$\frac{-45E_2^4+21E_2^2E_4+16E_2E_6-10E_4^2}{1105920}$
3	$\frac{17E_4-5E_2^2}{424673280}$	$\frac{315E_2^3-63E_2E_4-248E_6}{1486356480}$	$\frac{315E_2^4-819E_2^2E_4-208E_2E_6+468E_4^2}{743178240}$	$\frac{152E_2^2E_6-315E_2^5-189E_2^3E_4+300E_2E_4^2-112E_4E_6}{371589120}$

TABLE VII. Expansion coefficients $f^{(r=1)}_{a,k,(2s)}$.

s	k	$f^{(r=1)}_{0,k,(2s)}$	$f^{(r=1)}_{1,k,(2s)}$	$f^{(r=1)}_{2,k,(2s)}$	$f^{(r=1)}_{3,k,(2s)}$
0	1	0	0	$\frac{1}{12}$	$\frac{E_2}{6}$
1	1	0	$\frac{1}{576}$	0	$\frac{E_4-3E_2^2}{288}$
	2	0	0	$\frac{1}{72}$	$\frac{E_2}{36}$
2	1	$\frac{-1}{110592}$	$\frac{E_2}{18432}$	$\frac{15E_2^2-17E_4}{92160}$	$\frac{-45E_2^3-9E_2E_4+8E_6}{138240}$
	2	0	$-\frac{1}{3456}$	0	$\frac{3E_2^2-E_4}{1728}$
	3	0	0	$-\frac{1}{1440}$	$-\frac{E_2}{720}$
3	1	$-\frac{E_2}{2654208}$	$\frac{E_4}{442368}$	$\frac{315E_2^3-189E_2E_4-136E_6}{46448640}$	$-\frac{16E_6E_2-255E_4^2+504E_2^2E_4+315E_2^4}{46448640}$
	2	$\frac{1}{663552}$	$-\frac{E_2}{110592}$	$\frac{17E_4-15E_2^2}{552960}$	$-\frac{8E_6-9E_4E_2-45E_2^3}{829440}$
	3	0	$\frac{1}{69120}$	0	$\frac{E_4-3E_2^2}{34560}$
	4	0	0	$\frac{1}{60480}$	$\frac{E_2}{30240}$

TABLE VIII. Expansion coefficients $g^{(r=1)}_{a,k,(2s)}$.

s	k	$g^{(r=1)}_{0,k,(2s)}$	$g^{(r=1)}_{1,k,(2s)}$	$g^{(r=1)}_{2,k,(2s)}$	$g^{(r=1)}_{3,k,(2s)}$
0	1	0	0	0	1
1	1	0	0	$\frac{1}{32}$	$-\frac{E_2}{16}$
	2	0	0	0	$\frac{1}{12}$
2	1	0	$-\frac{1}{3072}$	$\frac{E_2}{512}$	$\frac{-15E_2^2-13E_4}{7680}$
	2	0	0	$-\frac{1}{384}$	$\frac{E_2}{192}$
	3	0	0	0	$-\frac{1}{360}$
3	1	$\frac{1}{884736}$	$-\frac{E_2}{49152}$	$\frac{11E_4+15E_2^2}{245760}$	$-\frac{184E_6+819E_2E_4+315E_2^3}{7741440}$
	2	0	$\frac{1}{36864}$	$-\frac{E2}{6144}$	$\frac{13E_4+15E_2^2}{92160}$
	3	0	0	$\frac{1}{11520}$	$-\frac{E_2}{5760}$
	4	0	0	0	$\frac{1}{20160}$

B. Factorization at order $\mathcal{O}(Q_R)$

Following the results of [47] for $N=2$, which were reviewed in Sec. IV B, we expect that the free energy for the $N=3$ LSTs in the NS limit to order $\mathcal{O}(Q_R)$ can be decomposed in terms of $H^{(1),\{0\}}_{(2s,0)}$, as in Eq. (4.6). In the following we shall provide nontrivial evidence that such a decomposition is indeed possible.

From Tables VIII and IX we first notice that $g^{(r=1)}_{a,k,(2s)}(\rho) = j^{(r=1)}_{a,k,(2s)}(\rho)$ and, similarly, the polynomials $p^{(r=1)}_{\ell,k,(2s)}$ take the following simple form:

TABLE IX. Expansion coefficients $j^{(r=1)}_{a,k,(2s)}$.

s	k	ℓ	$j^{(r=1)}_{0,\ell,k,(2s)}$	$j^{(r=1)}_{1,\ell,k,(2s)}$	$j^{(r=1)}_{2,\ell,k,(2s)}$	$j^{(r=1)}_{3,\ell,k,(2s)}$	$p^{(r)}_{\ell,k,(2s)}$
0	1	1	0	0	0	1	1
1	1	1	0	0	$\frac{1}{32}$	$-\frac{E_2}{16}$	1
	2	1	0	0	0	$\frac{1}{12}$	$n_1^2+n_2^2$
2	1	1	0	$-\frac{1}{3072}$	$\frac{E_2}{512}$	$\frac{-15E_2^2-13E_4}{7680}$	1
	2	1	0	0	$-\frac{1}{384}$	$\frac{E_2}{192}$	$n_1^2+n_2^2$
	3	1	0	0	0	$-\frac{1}{360}$	$n_1^4+n_1^2n_2^2+n_2^4$
3	1	1	$\frac{1}{884736}$	$-\frac{E_2}{49152}$	$\frac{11E_4+15E_2^2}{245760}$	$-\frac{184E_6+819E_2E_4+315E_2^3}{7741440}$	1
	2	1	0	$\frac{1}{36864}$	$-\frac{E2}{6144}$	$\frac{13E_4+15E_2^2}{92160}$	$n_1^2+n_2^2$
	3	1	0	0	$\frac{1}{11520}$	$-\frac{E_2}{5760}$	$n_1^4+n_1^2n_2^2+n_2^4$
	4	1	0	0	0	$\frac{1}{20160}$	$n_1^6+n_1^4n_2^2+n_1^2n_2^4+n_2^6$

TABLE X. Expansion coefficients $d^{(r=2)}_{a,(2s)}$.

s	$d^{(r=2)}_{0,(2s)}$	$d^{(r=2)}_{1,(2s)}$	$d^{(r=2)}_{2,(2s)}$	$d^{(r=2)}_{3,(2s)}$	$d^{(r=2)}_{4,(2s)}$
0	0	$\frac{1}{1769472}$	$\frac{E_2}{221184}$	$\frac{2E_2^2+E_4}{221184}$	$\frac{3E_2E_4-2E_6}{55296}$
1	$\frac{1}{512\cdot 24^4}$	$\frac{E_2}{42467328}$	$\frac{5E_4-4E_2^2}{14155776}$	$\frac{7E_2E_4-2E_2^3-4E_6}{1769472}$	$\frac{108E_2^2E_4-208E_2E_6+123E_4^2}{10616832}$

s	$d^{(r=2)}_{5,(2s)}$	$d^{(r=2)}_{6,(2s)}$
0	$\frac{24E_2^2E_4-32E_2E_6+9E_4^2}{110592}$	$\frac{-E_6E_2^2+3E_2E_4^2-2E_4E_6}{6912}$
1	$\frac{67E_2E_4^2-24E_2^3E_4-8(E_6E_2^2+4E_6E_4)}{884736}$	$\frac{128E_6^2+48E_2^3E_6+72E_2^2E_4^2-384E_2E_4E_6+141E_4^3}{2654208}$

TABLE XI. Expansion coefficients $f^{(r=2)}_{a,k,(2s)}$.

s	k	$f^{(r=2)}_{0,k,(2s)}$	$f^{(r=2)}_{1,k,(2s)}$	$f^{(r=2)}_{2,k,(2s)}$	$f^{(r=2)}_{3,k,(2s)}$	$f^{(r=2)}_{4,k,(2s)}$	$f^{(r=2)}_{5,k,(2s)}$	$f^{(r=2)}_{6,k,(2s)}$
0	1	0	0	$\frac{1}{55296}$	$\frac{E_2}{13824}$	$\frac{E_4}{4608}$	$\frac{3E_2E_4-2E_6}{1728}$	$\frac{3E_4^2-2E_2E_6}{1728}$
	2	0	0	0	$\frac{1}{6912}$	$\frac{E_2}{1728}$	$\frac{E_4}{1728}$	$\frac{E_2E_4-E_6}{1440}$
	3	0	0	0	0	$\frac{1}{13824}$	$\frac{E_2}{3456}$	$\frac{E_4}{3456}$
1	1	0	$\frac{1}{2654208}$	0	$\frac{2E_4-E_2^2}{110592}$	$\frac{9E_2E_4-8E_6}{82944}$	$\frac{-12E_2^2E_4-12E_2E_6+19E_4^2}{55296}$	$\frac{E_2^2E_6+3E_2E_4^2-5E_4E_6}{6912}$
	2	0	0	$\frac{1}{73728}$	$\frac{7E_2}{165888}$	$\frac{59E_4-30E_2^2}{414720}$	$\frac{120E_2E_4-77E_6}{207360}$	$\frac{-126E_2^2E_4-274E_2E_6+771E_4^2}{1451520}$
	3	0	0	0	$\frac{13}{331776}$	$\frac{25E_2}{165888}$	$\frac{14E_4-3E_2^2}{82944}$	$\frac{33E_2E_4-38E_6}{207360}$
	4	0	0	0	0	$\frac{1}{69120}$	$\frac{E_2}{17280}$	$\frac{E_4}{17280}$

TABLE XII. Expansion coefficients $g^{(r=2)}_{a,k,(2s)}$.

s	k	$g^{(r=2)}_{0,k,(2s)}$	$g^{(r=2)}_{1,k,(2s)}$	$g^{(r=2)}_{2,k,(2s)}$	$g^{(r=2)}_{3,k,(2s)}$	$g^{(r=2)}_{4,k,(2s)}$	$g^{(r=2)}_{5,k,(2s)}$	$g^{(r=2)}_{6,k,(2s)}$
0	1	0	0	0	$\frac{1}{2304}$	0	$\frac{E_4}{96}$	$-\frac{E_6}{144}$
	2	0	0	0	0	$\frac{7}{1152}$	0	$\frac{11E_4}{480}$
	3	0	0	0	0	0	$\frac{5}{576}$	0
	4	0	0	0	0	0	0	$\frac{1}{720}$
1	1	0	0	$\frac{1}{73728}$	$-\frac{E_2}{18432}$	$\frac{5E_4}{6144}$	$\frac{-3E_2E_4-5E_6}{2304}$	$\frac{2E_2E_6+9E_4^2}{2304}$
	2	0	0	0	$\frac{5}{13824}$	$-\frac{7E_2}{9216}$	$\frac{83E_4}{11520}$	$-\frac{11E_2E_4}{3840}-\frac{29E_6}{6048}$
	3	0	0	0	0	$\frac{97}{55296}$	$-\frac{5E_2}{4608}$	$\frac{3E_4}{512}$
	4	0	0	0	0	0	$\frac{23}{17280}$	$-\frac{E_2}{5760}$
	5	0	0	0	0	0	0	$\frac{1}{6048}$

$$p^{(r=1)}_{\ell=1,k,(2s)}(n_1,n_2)=\sum_{\alpha=0}^{k-1} n_1^{2k-2-2\alpha}n_2^{2\alpha},\quad p^{(r=1)}_{\ell,k,(2s)}(n_1,n_2)=0,\quad \forall\, \ell>1. \tag{5.6}$$

Furthermore, we can summarize Tables VI–IX through the following decompositions:

$$\begin{aligned}\sum_{k=1}^{s+1} n^{2k-1}\mathfrak{f}^{(r=1)}_{k,(2s)}(\rho,S)&=-\sum_{a=0}^{s} W_{(2s-2a)}(\rho,S)\sum_{i,j=0}^{a} A^{(a)}_{ij}(n)H^{(1),\{0\}}_{(2i,0)}(\rho,S)H^{(1),\{0\}}_{(2j,0)}(\rho,S),\\ \sum_{k=1}^{s+1}\mathfrak{g}^{(r=1)}_{k,(2s)}(\rho,S)&=-\sum_{a=0}^{s} W_{(2s-2a)}(\rho,S)\sum_{i,j=0}^{a} B^{(a)}_{ij}(n)H^{(1),\{0\}}_{(2i,0)}(\rho,S)H^{(1),\{0\}}_{(2j,0)}(\rho,S),\\ \sum_{k=1}^{s+1}\sum_{\ell} p^{(r=1)}_{\ell,k,(2s)}(n_1,n_2)\mathfrak{j}^{(r)}_{\ell,k,(2s)}(\rho,S)&=-\sum_{a=0}^{s} W_{(2s-2a)}(\rho,S)\sum_{i,j=0}^{a} C^{(a)}_{ij}(n_1,n_2)H^{(1),\{0\}}_{(2i,0)}(\rho,S)H^{(1),\{0\}}_{(2j,0)}(\rho,S),\end{aligned} \tag{5.7}$$

which we conjecture to hold for generic values of s and where $A^{(a)}_{ij}$, $B^{(a)}_{ij}$, and $C^{(a)}_{ij}$ are $(a+1)\times(a+1)$ matrices whose components are given by

TABLE XIII. Expansion coefficients $j^{(r=2)}_{a,k,(2s)}$.

s	k	ℓ	$j^{(r=2)}_{0,\ell,k,(2s)}$	$j^{(r=2)}_{1,\ell,k,(2s)}$	$j^{(r=2)}_{2,\ell,k,(2s)}$	$j^{(r=2)}_{3,\ell,k,(2s)}$	$j^{(r=2)}_{4,\ell,k,(2s)}$	$j^{(r=2)}_{5,\ell,k,(2s)}$	$j^{(r=2)}_{6,\ell,k,(2s)}$	$p^{(r=2)}_{\ell,k,(2s)}$
0	1	1	0	0	0	$\frac{1}{4\cdot 24^2}$	0	0	0	1
0	1	2	0	0	0	0	0	$\frac{E_4}{96}$	0	1
0	1	3	0	0	0	0	0	0	$-\frac{E_6}{144}$	1
	2	1	0	0	0	0	$\frac{1}{1152}$	0	0	$7n_1^2+10n_1n_2+7n_2^2$
	2	1	0	0	0	0	0	0	$\frac{E_4}{480}$	$11n_1^2+20n_1n_2+11n_2^2$
	2	1	0	0	0	0	0	$\frac{1}{576}$	0	$(n_1^2+n_1n_2+n_2^2)(5n_1^2+9n_1n_2+5n_2^2)$
	3	1	0	0	0	0	0	0	$\frac{1}{1440}$	$(n_1+n_2)^2(2n_1^4+4n_1^3n_2+9n_1^2n_2^2+4n_1n_2^3+2n_2^4)$
1	1	1	0	0	$\frac{1}{73728}$	$-\frac{E_2}{18432}$	$\frac{5E_4}{6144}$	$\frac{-3E_2E_4-5E_6}{2304}$	$\frac{2E_2E_6+9E_4^2}{2304}$	1
	2	1	0	0	0	$\frac{5}{13824}$	0	0	0	$n_1^2+n_1n_2+n_2^2$
	2	2	0	0	0	0	$-\frac{E_2}{9216}$	0	0	$7n_1^2+10n_1n_2+7n_2^2$
	2	3	0	0	0	0	0	$\frac{E_4}{11520}$	0	$83n_1^2+121n_1n_2+83n_2^2$
	2	4	0	0	0	0	0	0	$-\frac{E_2E_4}{3840}$	$11n_1^2+20n_1n_2+11n_2^2$
	2	5	0	0	0	0	0	0	$-\frac{E_6}{6048}$	$29n_1^2+48n_1n_2+29n_2^2$
	3	1	0	0	0	0	$\frac{1}{55296}$	0	0	$(n_1^2+n_1n_2+n_2^2)(97n_1^2+141n_1n_2+97n_2^2)$
	3	2	0	0	0	0	0	$-\frac{E_2}{4608}$	$\frac{3E_4}{2560}$	$(n_1^2+n_1n_2+n_2^2)(5n_1^2+9n_1n_2+5n_2^2)$
	4	1	0	0	0	0	0	$\frac{1}{69120}$	0	$92n_1^6+344n_1^5n_2+751n_1^4n_2^2+974n_1^3n_2^3$ $+751n_1^2n_2^4+344n_1n_2^5+92n_2^6$
	4	2	0	0	0	0	0	0	$-\frac{E_2}{11520}$	$(n_1+n_2)^2(2n_1^4+4n_1^3n_2+9n_1^2n_2^2+4n_1n_2^3+2n_2^4)$
0	5	1	0	0	0	0	0	0	$\frac{1}{120960}$	$(n_1+n_2)^2(20n_1^6+60n_1^5n_2+165n_1^4n_2^2$ $+166n_1^3n_2^3+165n_1^2n_2^4+60n_1n_2^5+20n_2^6)$

$$
\begin{aligned}
A^{(a)}_{ij} &= -\frac{2(-1)^{a+i+j}n^{2a+1-2(i+j)}}{\Gamma(2a-2(i+j-1))},\\
B^{(a)}_{ij} &= \frac{2(-1)^{a+i+j}n^{2a+2-2(i+j)}}{\Gamma(2a+1-2(i+j-1))}\theta(a-i-j),\\
C^{(a)}_{ij} &= \frac{2(-1)^{a+i+j}}{\Gamma(2a+1-2(i+j-1))}\theta(a-i-j)\sum_{\alpha=0}^{a-(i+j)} n_1^{2\alpha}n_2^{2(a-1-\alpha)}, \qquad \forall\ \begin{matrix} a\in\{0,\ldots,s\}, \\ i,j\in\{0,\ldots,a\}. \end{matrix}
\end{aligned}
\tag{5.8}
$$

Explicitly, for low values of a we have

$$
\begin{aligned}
&A^{(0)} = (-2n), \quad A^{(1)} = \begin{pmatrix} \frac{n^3}{3} & -2n \\ -2n & 0 \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} -\frac{n^5}{60} & \frac{n^3}{3} & -2n \\ \frac{n^3}{3} & -2n & 0 \\ -2n & 0 & 0 \end{pmatrix},\\
&B^{(0)} = (-n^2), \quad B^{(1)} = \begin{pmatrix} \frac{n^4}{12} & -n^2 \\ -n^2 & 0 \end{pmatrix}, \quad B^{(2)} = \begin{pmatrix} -\frac{n^6}{360} & \frac{n^4}{12} & -n^2 \\ \frac{n^4}{12} & -n^2 & 0 \\ -n^2 & 0 & 0 \end{pmatrix},\\
&C^{(0)} = (1), \quad C^{(1)} = \begin{pmatrix} -\frac{n_1^2+n_2^2}{12} & 1 \\ 1 & 0 \end{pmatrix}, \quad C^{(2)} = \begin{pmatrix} \frac{n_1^4+n_1^2n_2^2+n_2^4}{360} & -\frac{n_1^2+n_2^2}{12} & 1 \\ -\frac{n_1^2+n_2^2}{12} & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\end{aligned}
\tag{5.9}
$$

Notice that these matrices are very closely related and satisfy, for example, $\partial_n B^{(a)}(n) = A^{(a)}(n)$. Moreover, Eq. (5.7) yields a complete decomposition of the free energy for the $N=3$ LSTs in the NS limit, where the building blocks are given only by the free energy of the $N = 1$ LST $H^{(1),\{0\}}_{(2s,0)}$ and the expansion coefficients of the NS limit of the function (C4) $W_{(2s)}(\rho, S)$.

C. Hecke structures

Following the discussion in Sec. IV C for the case of $N = 2$, we will search for subsectors of the $N = 3$ free energy which in the NS limit are related via Hecke transformations. We shall be able to identify three different contributions that are defined via certain contour integral prescriptions.

Generalizing Eqs. (4.27) and (4.28) to the case $N = 3$, we can define the following three subsectors of the $N = 3$ free energy:

$$\mathcal{C}^{N=3,(r)}_{1,(2s,0)}(\rho,S) \coloneqq \frac{1}{(2\pi i)^3}\sum_{\ell=0}^{\infty} Q_\rho^\ell \oint_0 \frac{dQ_{\hat{a}_1}}{Q_{\hat{a}_1}^{1+\ell}} \oint_0 \frac{dQ_{\hat{a}_2}}{Q_{\hat{a}_2}^{1+\ell}} \oint_0 \frac{dQ_{\hat{a}_3}}{Q_{\hat{a}_3}^{1+\ell}} P^{(r)}_{2,(2s,0)}(\hat{a}_1,\hat{a}_2,\hat{a}_3,S), \tag{5.10}$$

$$\mathcal{C}^{N=3,(r)}_{2,(2s,0)}(\rho,S) \coloneqq \frac{1}{(2\pi i)^3 r}\sum_{\ell=0}^{\infty} Q_\rho^\ell \oint_0 d\hat{a}_1 \hat{a}_1 Q_{\hat{a}_1}^{-\ell} \oint_0 \frac{dQ_{\hat{a}_2}}{Q_{\hat{a}_2}^{1+\ell}} \oint_0 \frac{dQ_{\hat{a}_3}}{Q_{\hat{a}_3}^{1+\ell}} P^{(r)}_{2,(2s,0)}(\hat{a}_1,\hat{a}_2,\hat{a}_3,S), \tag{5.11}$$

$$\mathcal{C}^{N=3,(r)}_{3,(2s,0)}(\rho,S,\epsilon_1) \coloneqq \frac{1}{(2\pi i)^2 r^2} \oint_0 d\hat{a}_1 \hat{a}_1 \oint_{-\hat{a}_1} d\hat{a}_2 (\hat{a}_1+\hat{a}_2) P^{(r)}_{2,(2s,0)}(\hat{a}_1,\hat{a}_2,\rho,S). \tag{5.12}$$

Here the contour integral $\oint_z$ is along a small circle around the point $z \in \mathbb{Z}$, in such a way as to extract the residue in a Laurent series. Furthermore, in the definition of $\mathcal{C}^{N=3,(r)}_{3,(2s,0)}(\rho,S)$ we have implicitly used $\hat{a}_3 = \rho - \hat{a}_1 - \hat{a}_2$. Finally, as in the case of $N = 2$, we also introduce the following series in powers of ϵ_1:

$$\mathcal{C}^{N=3,(r)}_{a}(\rho,S,\epsilon_1) = \sum_{s=0}^{\infty} \epsilon_1^{2s-2a} \mathcal{C}^{N=3,(r)}_{a,(2s)}(\rho,S), \quad \forall\, a = 1, 2.3. \tag{5.13}$$

In the case of $N = 3$, the free energy counts BPS states of three M5-branes separated on a circle with multiple M2-branes stretched between them. The functions $\mathcal{C}^{N=3,(r)}_a$ receive contributions only from certain such configurations, as is schematically shown in Fig. 8:

(i) *Combination* $\mathcal{C}^{N=3,(r)}_{1,(2s,0)}$.—$\mathcal{C}^{N=3,(r)}_{1,(2s,0)}$ can be described by extracting a particular class of terms in the Fourier expansion of $P^{(r)}_{3,(2,0s)}$ in powers of $Q_{\hat{a}_{1,2,3}}$. Indeed, upon writing

$$P^{(r)}_{3,(2s,0)}(\hat{a}_1,\hat{a}_2,\hat{a}_3,S) = \sum_{n_1,n_2,n_3=0}^{\infty} Q_{\hat{a}_1}^{n_1} Q_{\hat{a}_2}^{n_2} Q_{\hat{a}_3}^{n_3} P^{(r),\{n_1,n_2,n_3\}}_{(2s,0)}(S), \tag{5.14}$$

the contour prescriptions in Eq. (5.12) are designed to extract only those terms with $n_1 = n_2 = n_3$. Therefore, $\mathcal{C}^{N=3,(r)}_{1,(2s,0)}$ receives contributions only from those brane configurations, in which an equal number of M2-branes is stretched between any two adjacent M5-branes, as visualized in Fig. 8(a). Following the definition of $H^{(r),\underline{n}}_{(2s,0)}$ in Eq. (2.21), we find that $\mathcal{C}^{N=3,(r)}_{1,(2s,0)}$ can equivalently be written as

$$\mathcal{C}^{N=3,(r)}_{1,(2s,0)}(\rho,S) = H^{(r),\{0,0,0\}}_{(2s,0)}(\rho,S),$$
$$\mathcal{C}^{N=3,(r)}_{1}(\rho,S,\epsilon_1) = \sum_{s=0}^{\infty} \epsilon_1^{2s-2} H^{(r),\{0,0,0\}}_{(2s,0)}(\rho,S). \tag{5.15}$$

It is in fact the reduced free energy for $N = 3$ that was studied in [46]. Explicit expansions of $\mathcal{C}^{N=3,(r)}_{1,(2s,0)}$ for $r = 1$ and $r = 2$ can be recovered from Tables VI and X.

(ii) *Combination* $\mathcal{C}^{N=3,(r)}_{2,(2s,0)}$.—The function $\mathcal{C}^{N=3,(r)}_{2,(2s,0)}$ in Eq. (5.11) extracts specific coefficients in a mixed Fourier and Laurent series expansion of the free energy. Starting from the Fourier expansion (5.14), $\mathcal{C}^{N=3,(r)}_{2,(2s,0)}$ receives contributions only from coefficients with $n_1 \neq n_2 = n_3$. Physically, these correspond to configurations where an equal number ℓ of M2-branes is stretched between the second and third as well as the third and first M5-branes, while a different number $n \neq \ell$ of M2-branes is stretched between the first and second M5-branes. Such configurations are schematically shown in Fig. 8(b). Finally, the last contour integral in Eq. (5.11) over $\hat{a}_1$ extracts the second order pole in the Laurent expansion.

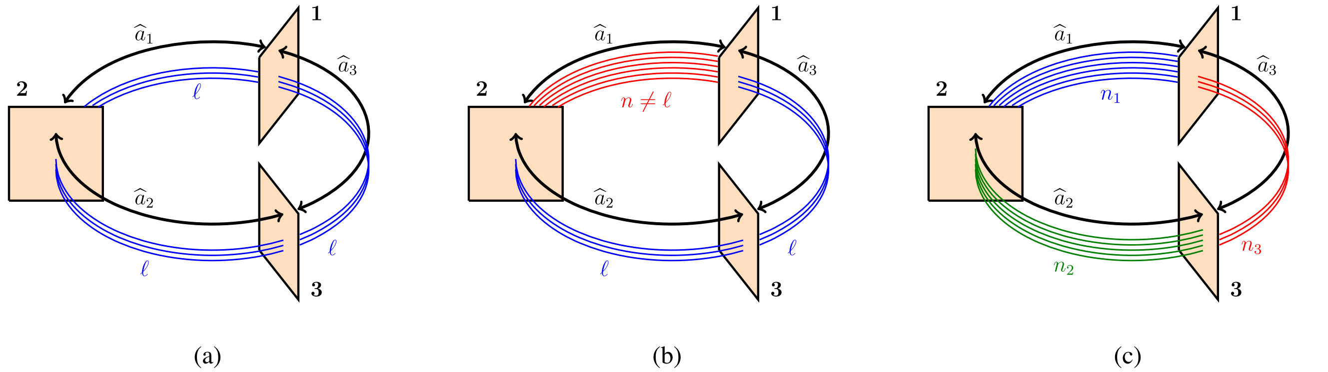

FIG. 8. Brane-web configurations made up of $N = 3$ M5-branes (drawn in orange) spaced out on a circle, with various M2-branes (drawn in red, blue, and green) stretched between them. (a) An equal number ℓ of M2-branes is stretched between any two neighboring M5-branes. Configurations of this type are relevant for $\mathcal{C}_1^{N=3,(r)}$. (b) ℓ M2-branes are stretched between M5-branes 2 and 3 as well as 3 and 1, while $n \neq \ell$ M2-branes between M5-branes 1 and 2. Configurations of this type are relevant for $\mathcal{C}_2^{N=3,(r)}$. (c) Different numbers $n_{1,2,3}$ (with $n_i \neq n_j$ for $i \neq j$) of M2-branes are stretched between any of the neighboring M5-branes. Configurations of this type are relevant for $\mathcal{C}_3^{N=3,(r)}$.

With respect to the decomposition (5.3), the coefficients $\mathcal{C}_{2,(2s,0)}^{N=3,(r)}$ can be written in the following form[22]:

$$\mathcal{C}_{2,(2s,0)}^{N=3,(r)}(\rho,S) = \frac{1}{(2\pi i)}\oint_0 d\hat{a}_1 \hat{a}_1 \sum_{n=1}^{\infty}\left[H_{(2s,0)}^{(r),\{n,0,0\}}(\rho,S) Q_{\hat{a}_1}^n + H_{(2s,0)}^{(r),\{n,n,0\}}(\rho,S)\frac{Q_\rho^n}{Q_{\hat{a}_1}^n}\right]. \tag{5.16}$$

In order to perform the final contour integration over $\hat{a}_1$, we can use the conjectured form (5.4) of $H_{(2s,0)}^{(r),\{n,0,0\}}$ and $H_{(2s,0)}^{(r),\{n,n,0\}}$ to write for the integrand

$$\begin{aligned}
\mathcal{I}_{\mathcal{C}_{2,(2s,0)}^{N=3,(r)}} &= \sum_{n=1}^{\infty}\left[H_{(2s,0)}^{(r),\{n,0,0\}}(\rho,S) Q_{\hat{a}_1}^n + H_{(2s,0)}^{(r),\{n,n,0\}}(\rho,S)\frac{Q_\rho^n}{Q_{\hat{a}_1}^n}\right] \\
&= \sum_{n=1}^{\infty}\sum_{k=1}^{rs+1}\left[\frac{n^{2k-1}\mathfrak{f}_{k,(2s)}^{(r)}}{1-Q_\rho^n}\left(Q_{\hat{a}_1}^n + \frac{Q_\rho^n}{Q_{\hat{a}_1}^n}\right) + \frac{n^{2k} Q_\rho^n \mathfrak{g}_{k,(2s)}^{(r)}}{(1-Q_\rho^n)^2}\left(Q_{\hat{a}_1}^n + \frac{1}{Q_{\hat{a}_1}^n}\right)\right] \\
&= \sum_{k=1}^{rs+1}\mathfrak{f}_{k,(2s)}^{(r)}\mathcal{I}_{k-1}(\rho,\hat{a}_1) + \sum_{k=1}^{rs+1}\mathfrak{g}_{k,(2s)}^{(r)}\sum_{n=1}^{\infty}\frac{n^{2k}Q_\rho^n}{(1-Q_\rho^n)^2}(Q_{\hat{a}_1}^n + Q_{\hat{a}_1}^{-n}),
\end{aligned} \tag{5.17}$$

where we have exchanged the order of summations and $\mathcal{I}_\alpha$ as defined in Eq. (4.15). With $\frac{x}{(1-x)^2} = \sum_{\ell=1}^{\infty}\ell x^\ell$, we can write for the sum over n in the last term in Eq. (5.17)

$$\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty} n^{2k}\ell Q_\rho^{n\ell}(Q_{\hat{a}_1}^n + Q_{\hat{a}_1}^{-n}) = D_{\hat{a}_1}^{2k}\sum_{n=1}^{\infty}Q_\rho^n\sum_{\ell|n}\frac{n}{\ell}(Q_{\hat{a}_1}^\ell + Q_{\hat{a}_1}^{-\ell}) \coloneqq \mathcal{J}_k(\rho,\hat{a}_1) = D_{\hat{a}_1}^{2k}\mathcal{J}_0(\rho,\hat{a}_1).$$

For $0 < |Q_\rho| < 1$ the function $\mathcal{J}_0$ is in fact regular at $\hat{a}_1$, such that $\oint_0 d\hat{a}_1\hat{a}_1\mathcal{J}_k(\rho,\hat{a}_1) = 0$ for $k \geq 0$. Furthermore, using Eq. (4.18), we get

$$\mathcal{C}_{2,(2s,0)}^{N=3,(r)}(\rho,S) = \frac{1}{r}\mathfrak{f}_{k=1,(2s)}^{(r)}(\rho,S), \quad \mathcal{C}_2^{N=3,(r)}(\rho,S,\epsilon_1) = \frac{1}{r}\sum_{s=0}^{\infty}\epsilon_1^{2s-1}\mathfrak{f}_{k=1,(2s)}^{(r)}(\rho,S). \tag{5.18}$$

[22]We remark in passing that contributions with $n = \ell$ in Fig. 8(b) would give rise to terms with $H_{(2s,0)}^{(r),\{0,0,0\}}$ in Eq. (5.16). The latter, however, depend only on ρ, not on $\hat{a}_1$, and thus do not contribute to the contour integral over $\hat{a}_1$ in $\mathcal{C}_{2,(2s)}^{N=3,(r)}$.

(iii) *Combination* $\mathcal{C}^{N=3,(r)}_{3,(2s,0)}$.—The function $\mathcal{C}^{N=3,(r)}_{3,(2s,0)}$ in Eq. (5.12) receives contributions from M-brane configurations with n_i M2-branes stretched between the ith and $(i+1)$st M5-brane (with $n_i \neq n_j$ for $i \neq j$). The contour integrals, however, extract the second order poles for the successive limits $\hat{a}_2 \to -\hat{a}_1$ and $\hat{a}_1 \to 0$. In the decomposition (5.3) only the terms with $\underline{n} = \{n_1 + n_2, n_1, 0\}$ for $n_{1,2} \geq 1$ contribute:

$$\mathcal{C}^{N=3,(r)}_{3,(2s,0)}(\rho, S) = \frac{1}{(2\pi i)^2} \oint_0 d\hat{a}_1 \hat{a}_1 \oint d\hat{a}_2 (\hat{a}_1 + \hat{a}_2) \sum_{n_1,n_2=1}^{\infty} H^{(r),\{n_1+n_2,n_1,0\}}_{(2s,0)}(\rho, S) X_{n_1,n_2}(\hat{a}_1, \hat{a}_2, \rho), \tag{5.19}$$

with

$$X_{n_1,n_2}(\hat{a}_1, \hat{a}_2, \rho) = Q_{\hat{a}_1}^{n_1+n_2} Q_{\hat{a}_2}^{n_1} + \frac{Q_{\hat{a}_1}^{n_2} Q_\rho^{n_1}}{Q_{\hat{a}_2}^{n_1}} + \frac{Q_\rho^{n_1+n_2}}{Q_{\hat{a}_1}^{n_1+n_2} Q_{\hat{a}_2}^{n_2}} + (\hat{a}_1 \leftrightarrow \hat{a}_2). \tag{5.20}$$

Using the conjectured form (5.4) of $H^{(r),\{n_1+n_2,n_1,0\}}_{(2s,0)}$, we can write

$$\begin{aligned}\mathcal{C}^{N=3,(r)}_{3,(2s,0)}(\rho, S) = \frac{1}{(2\pi i)^2} \oint_0 d\hat{a}_1 \hat{a}_1 \oint_{-\hat{a}_1} d\hat{a}_2 (\hat{a}_1 + \hat{a}_2) \sum_{n_1,n_2=1}^{\infty} \sum_{k=1}^{rs+1} \sum_{\ell} \sum_{k_1,k_2=0}^{\infty} X_{n_1,n_2}(\hat{a}_1, \hat{a}_2, \rho) \\ \times \Big[Q_\rho^{k_1 n_1 + k_2 n_2} n_2 (n_2 + 2n_1) p^{(r)}_{\ell,k,(2s)}(n_1, -n_1 - n_2) \mathfrak{j}^{(r)}_{\ell,k,(2s)}(\rho, S) \\ + Q_\rho^{k_1 n_1 + k_2 (n_1+n_2)} (n_1^2 - n_2^2) p^{(r)}_{\ell,k,(2s)}(n_1, n_2) \mathfrak{j}^{(r)}_{\ell,k,(2s)}(\rho, S) \Big].\end{aligned} \tag{5.21}$$

To further simplify this expression, let $\beta_{1,2} \in \mathbb{N}$ and consider

$$\Upsilon = \frac{1}{(2\pi i)^2} \oint_0 d\hat{a}_1 \hat{a}_1 \oint d\hat{a}_2 (\hat{a}_1 + \hat{a}_2) \sum_{n_1,n_2=1}^{\infty} n_1^{\beta_1} n_2^{\beta_2} \sum_{k_1,k_2=0}^{\infty} Q_\rho^{k_1 n_1 + k_2 n_2} X_{n_1,n_2}(\hat{a}_1, \hat{a}_2, \rho). \tag{5.22}$$

Assuming that $0 < |Q_\rho| < 1$, the factors Q_ρ act as regulators for the sum over $n_{1,2}$ in the limit $Q_{\hat{a}_{1,2}} \to 1$. The divergence that is relevant for the contour integrals therefore only stems from those terms where these factors are absent—namely, for $k_1 = k_2 = 0$,[23]

$$\begin{aligned}\Upsilon &= \frac{1}{(2\pi i)^2} \oint_0 d\hat{a}_1 \hat{a}_1 \oint d\hat{a}_2 (\hat{a}_1 + \hat{a}_2) \sum_{n_1,n_2=1}^{\infty} n_1^{\beta_1} n_2^{\beta_2} [(Q_{\hat{a}_1} Q_{\hat{a}_2})^{n_1} (Q_{\hat{a}_1}^{n_2} + Q_{\hat{a}_2}^{n_2})] \\ &= \frac{1}{(2\pi i)^2} \oint_0 d\hat{a}_1 \hat{a}_1 \oint d\hat{a}_2 (\hat{a}_1 + \hat{a}_2) \Big[D_{\hat{a}_2}^{\beta_1} (D_{\hat{a}_1} - D_{\hat{a}_2})^{\beta_2} \sum_{n_1,n_2=1}^{\infty} (Q_{\hat{a}_1} Q_{\hat{a}_2})^{n_1} Q_{\hat{a}_1}^{n_2} \\ &\quad + D_{\hat{a}_1}^{\beta_1} (D_{\hat{a}_2} - D_{\hat{a}_1})^{\beta_2} \sum_{n_1,n_2=1}^{\infty} (Q_{\hat{a}_1} Q_{\hat{a}_2})^{n_1} Q_{\hat{a}_2}^{n_2} \Big].\end{aligned} \tag{5.23}$$

Assuming that $0 < |Q_{\hat{a}_{1,2}}| < 1$ we can perform the sum over $n_{1,2}$ to find

$$\begin{aligned}\Upsilon = \frac{1}{(2\pi i)^2} \oint_0 d\hat{a}_1 \hat{a}_1 \oint d\hat{a}_2 (\hat{a}_1 + \hat{a}_2) \Big[D_{\hat{a}_2}^{\beta_1} (D_{\hat{a}_1} - D_{\hat{a}_2})^{\beta_2} \frac{Q_{\hat{a}_1}^2 Q_{\hat{a}_2}}{(1 - Q_{\hat{a}_1} Q_{\hat{a}_2})(1 - Q_{\hat{a}_1})} \\ + D_{\hat{a}_1}^{\beta_1} (D_{\hat{a}_2} - D_{\hat{a}_1})^{\beta_2} \frac{Q_{\hat{a}_1} Q_{\hat{a}_2}^2}{(1 - Q_{\hat{a}_1} Q_{\hat{a}_2})(1 - Q_{\hat{a}_2})} \Big].\end{aligned} \tag{5.24}$$

From the explicit series expansions

[23]Notice that, in order to have a pole for $\hat{a}_2 \to -\hat{a}_1$ and $\hat{a}_1 \to 0$, k_1 and k_2 have to vanish simultaneously.

$$\frac{Q_{\hat{a}_1}^2 Q_{\hat{a}_2}}{(1-Q_{\hat{a}_1}Q_{\hat{a}_2})(1-Q_{\hat{a}_1})} = \frac{1}{(2\pi i)^2 \hat{a}_1(\hat{a}_1+\hat{a}_2)}[1+\pi i(2\hat{a}_1+\hat{a}_2)+\mathcal{O}(\hat{a}_{1,2}^2)],$$
$$\frac{Q_{\hat{a}_1} Q_{\hat{a}_2}^2}{(1-Q_{\hat{a}_1}Q_{\hat{a}_2})(1-Q_{\hat{a}_2})} = \frac{1}{(2\pi i)^2 \hat{a}_2(\hat{a}_1+\hat{a}_2)}[1+\pi i(\hat{a}_1+2\hat{a}_2)+\mathcal{O}(\hat{a}_{1,2}^2)],$$
$$= -\frac{\left(1+\frac{\hat{a}_1+\hat{a}_2}{\hat{a}_1}+\frac{(\hat{a}_1+\hat{a}_2)^2}{\hat{a}_1^2}+\mathcal{O}((\hat{a}_1+\hat{a}_2)^3)\right)}{(2\pi i)^2\hat{a}_1(\hat{a}_1+\hat{a}_2)}[1+\pi i(\hat{a}_1+2\hat{a}_2)+\mathcal{O}(\hat{a}_{1,2}^2)], \tag{5.25}$$

it follows that Υ is nonvanishing only for $\beta_1+\beta_2=2$. To understand why no higher derivatives may contribute, we define $\hat{b}=\hat{a}_1+\hat{a}_2$ and consider, respectively, for the first and second terms in Eq. (5.24)

$$\frac{Q_{\hat{a}_{1,2}}}{1-Q_{\hat{a}_{1,2}}}\frac{Q_{\hat{b}}}{1-Q_{\hat{b}}} = \left[\frac{1}{2\pi i\hat{a}_{1,2}}+\frac{1}{2}+\frac{i\pi\hat{a}_{1,2}}{6}+\mathcal{O}(\hat{a}_1^2)\right]\left[\frac{1}{2\pi i\hat{b}}+\frac{1}{2}+\frac{i\pi\hat{b}}{6}+\mathcal{O}(\hat{b}^2)\right]$$
$$= \left[\frac{1}{2\pi i\hat{b}}+\frac{1}{2}+\frac{i\pi\hat{b}}{6}+\mathcal{O}(\hat{b}^2)\right]\times\begin{cases}\frac{1}{2\pi i\hat{a}_1}+\frac{1}{2}+\frac{i\pi\hat{a}_1}{6}+\mathcal{O}(\hat{a}_1^2)\\ \frac{(1+\frac{\hat{b}}{\hat{a}_1}+\frac{\hat{b}^2}{\hat{a}_1^2}+\mathcal{O}(\hat{b}^3))}{2\pi i\hat{a}_1}+\frac{1}{2}+\frac{i\pi(\hat{b}-\hat{a}_1)}{6}+\mathcal{O}(\hat{a}_1^2),\end{cases} \tag{5.26}$$

which only has poles of second order in $\hat{a}_{1,2}$ and $\hat{b}$ if hit with two derivatives. This implies that in Eq. (5.21) only terms with $k=1$ (in which case $p^{(r)}_{\ell,k=1,(2s)}=$ const are polynomials of order 0) contribute. Performing the explicit integrals, we obtain

$$\mathcal{C}^{N=3,(r)}_{3,(2s,0)}(\rho,S) = \frac{1}{r^2}\sum_{\ell} p^{(r)}_{\ell,k=1,(2s)}\mathbf{i}^{(r)}_{\ell,k=1,(2s)}(\rho,S),$$
$$\mathcal{C}^{N=3,(r)}_{3}(\rho,S,\epsilon_1) = \frac{1}{r^2}\sum_{s=0}^{\infty}\epsilon_1^{2s-4}\sum_{\ell} p^{(r)}_{\ell,k=1,(2s)}\mathbf{i}^{(r)}_{\ell,k=1,(2s)}(\rho,S). \tag{5.27}$$

By comparing the explicit expressions for the contributions $\mathcal{C}^{N=3,(r)}_{1,2,3,(2s)}$ for $r=1,2$ (and $r=3$) and s up to 4, we find the that they are related through Hecke operators in the following fashion:

$$\mathcal{C}^{N=3,(r)}_{a,(2s,0)}(\rho,S) = \mathcal{H}_r[\mathcal{C}^{N=3,(1)}_{1,(2s,0)}(\rho,S)], \quad \forall\, a=1,2,3. \tag{5.28}$$

The normalization factors $1/r$ and $1/r^2$ appearing in Eqs. (5.11) and (5.12) were chosen to normalize the right-hand side of Eq. (5.28).

D. Decomposition of $\mathcal{C}_i^{N=3,(r)}$

1. Factorization at order Q_R^1

As in the case of the entire free energy $P^{(r)}_{3,(2s,0)}(\hat{a}_{1,2,3},S)$ [see Eq. (5.7)], the functions $\mathcal{C}^{N=3,(r=1)}_{1,2,3,(2s)}$ can be decomposed into small building blocks. Based on the examples provided in Sec. VI A, we find the following decomposition:

$$\mathcal{C}^{N=3,(r=1)}_{1,(2s,0)} = 3\sum_{a=0}^{s} H^{(1),\{0\}}_{(2s-2a)} \sum_{i,j=0}^{a} \delta_{a,i+j} W_{(2i,0)} W_{(2j,0)},$$
$$\mathcal{C}^{N=3,(r=1)}_{2,(2s,0)} = -2\sum_{a=0}^{s} H^{(1),\{0\}}_{(2s-2a)}(\rho,S) \sum_{i,j=0}^{a} \delta_{a,i+j} W_{(2i,0)}(\rho,S) H^{(1),\{0\}}_{(2j,0)}(\rho,S),$$
$$\mathcal{C}^{N=3,(r=1)}_{3,(2s,0)} = \sum_{a=0}^{s} H^{(1),\{0\}}_{(2s-2a)} \sum_{i,j=0}^{a} \delta_{a,i+j} H^{(1),\{0\}}_{(2i,0)} H^{(1),\{0\}}_{(2j,0)}. \tag{5.29}$$

These expressions (as similar equations in the remainder of this subsection) are understood to hold order by order in an expansion of ϵ_1. Combining these expansion coefficients (in a series of ϵ_1), we can equivalently write

$$\mathcal{C}^{N=3,(r=1)}_{1}(\rho,S,\epsilon_1) = 3H^{(1)}_{N=1}(\rho,S,\epsilon_1) W^{(1)}_{\mathrm{NS}}(\rho,S,\epsilon_1) W^{(1)}_{\mathrm{NS}}(\rho,S,\epsilon_1),$$
$$\mathcal{C}^{N=3,(r=1)}_{2}(\rho,S,\epsilon_1) = -2\epsilon_1 H^{(1)}_{N=1}(\rho,S,\epsilon_1) H^{(1)}_{N=1}(\rho,S,\epsilon_1) W^{(1)}_{\mathrm{NS}}(\rho,S,\epsilon_1),$$
$$\mathcal{C}^{N=3,(r=1)}_{3}(\rho,S,\epsilon_1) = \epsilon_1^2 H^{(1)}_{N=1}(\rho,S,\epsilon_1) H^{(1)}_{N=1}(\rho,S,\epsilon_1) H^{(1)}_{N=1}(\rho,S,\epsilon_1), \tag{5.30}$$

where the coefficients $H^{(1)}_{N=1}$ and $W^{(1)}_{\mathrm{NS}}$ are as defined in Eqs. (C3) and (C8), respectively.

2. Factorization at order Q_R^2

Following the example of $N=2$ discussed in Sec. IV D 3, we expect a decomposition of $\mathcal{C}^{N=3,(r)}_{1,2,3}$ into more fundamental building blocks to also hold for $r>1$. Indeed, for $r=2$ we find

$$\mathcal{C}^{N=3,(2)}_{1} = \frac{4}{3} H^{(2)}_{N=1} W^{(2)}_{\mathrm{NS}} W^{(2)}_{\mathrm{NS}} + (H^{(1)}_{N=1})^6 \mathfrak{T}^{(2)}_{1,(6,0)} + (H^{(1)}_{N=1})^4 H^{(2)}_{N=1} \mathfrak{T}^{(2)}_{1,(4,1)} + (H^{(1)}_{N=1})^2 (H^{(2)}_{N=1})^2 \mathfrak{T}^{(2)}_{1,(2,2)},$$
$$\mathcal{C}^{N=3,(2)}_{2} = -\frac{8}{9} H^{(2)}_{N=1} H^{(2)}_{N=1} W^{(2)}_{\mathrm{NS}} + (H^{(1)}_{N=1})^6 \mathfrak{T}^{(2)}_{2,(6,0)} + (H^{(1)}_{N=1})^4 H^{(2)}_{N=1} \mathfrak{T}^{(2)}_{2,(4,1)} + (H^{(1)}_{N=1})^2 (H^{(2)}_{N=1})^2 \mathfrak{T}^{(2)}_{2,(2,2)},$$
$$\mathcal{C}^{N=3,(2)}_{3} = \frac{4}{9} H^{(2)}_{N=1} H^{(2)}_{N=1} H^{(2)}_{N=1} + (H^{(1)}_{N=1})^6 \mathfrak{T}^{(2)}_{3,(6,0)} + (H^{(1)}_{N=1})^4 H^{(2)}_{N=1} \mathfrak{T}^{(2)}_{3,(4,1)} + (H^{(1)}_{N=1})^2 (H^{(2)}_{N=1})^2 \mathfrak{T}^{(2)}_{3,(2,2)},$$

and for $r=3$

$$\begin{aligned}
\mathcal{C}^{N=3,(3)}_{1} &= \frac{27}{16} H^{(3)}_{N=1} W^{(3)}_{\mathrm{NS}} W^{(3)}_{\mathrm{NS}} + (H^{(1)}_{N=1})^9 \mathfrak{T}^{(3)}_{1,(9,0,0)} + (H^{(1)}_{N=1})^7 H^{(2)}_{N=1} \mathfrak{T}^{(3)}_{1,(7,1,0)} \\
&\quad + (H^{(1)}_{N=1})^5 (H^{(2)}_{N=1})^2 \mathfrak{T}^{(3)}_{1,(5,2,0)} + (H^{(1)}_{N=1})^3 (H^{(2)}_{N=1})^3 \mathfrak{T}^{(3)}_{1,(3,3,0)} \\
&\quad + H^{(1)}_{N=1} H^{(2)}_{N=1} (H^{(3)}_{N=1})^2 \mathfrak{T}^{(3)}_{1,(1,1,2)} + (H^{(1)}_{N=1})^2 (H^{(2)}_{N=1})^2 H^{(3)}_{N=1} \mathfrak{T}^{(3)}_{1,(2,2,1)}, \\
\mathcal{C}^{N=3,(3)}_{2} &= -\frac{9}{8} H^{(3)}_{N=1} H^{(3)}_{N=1} W^{(3)}_{\mathrm{NS}} + (H^{(1)}_{N=1})^9 \mathfrak{T}^{(3)}_{2,(9,0,0)} + (H^{(1)}_{N=1})^7 H^{(2)}_{N=1} \mathfrak{T}^{(3)}_{2,(7,1,0)} \\
&\quad + (H^{(1)}_{N=1})^5 (H^{(2)}_{N=1})^2 \mathfrak{T}^{(3)}_{2,(5,2,0)} + (H^{(1)}_{N=1})^3 (H^{(2)}_{N=1})^3 \mathfrak{T}^{(3)}_{2,(3,3,0)} \\
&\quad + H^{(1)}_{N=1} H^{(2)}_{N=1} (H^{(3)}_{N=1})^2 \mathfrak{T}^{(3)}_{2,(1,1,2)} + (H^{(1)}_{N=1})^2 (H^{(2)}_{N=1})^2 H^{(3)}_{N=1} \mathfrak{T}^{(3)}_{2,(2,2,1)}, \\
\mathcal{C}^{N=3,(3)}_{3} &= \frac{9}{16} H^{(3)}_{N=1} H^{(3)}_{N=1} H^{(3)}_{N=1} + (H^{(1)}_{N=1})^9 \mathfrak{T}^{(3)}_{3,(9,0,0)} + (H^{(1)}_{N=1})^7 H^{(2)}_{N=1} \mathfrak{T}^{(3)}_{3,(7,1,0)} \\
&\quad + (H^{(1)}_{N=1})^5 (H^{(2)}_{N=1})^2 \mathfrak{T}^{(3)}_{3,(5,2,0)} + (H^{(1)}_{N=1})^3 (H^{(2)}_{N=1})^3 \mathfrak{T}^{(3)}_{3,(3,3,0)} \\
&\quad + H^{(1)}_{N=1} H^{(2)}_{N=1} (H^{(3)}_{N=1})^2 \mathfrak{T}^{(3)}_{3,(1,1,2)} + (H^{(1)}_{N=1})^2 (H^{(2)}_{N=1})^2 H^{(3)}_{N=1} \mathfrak{T}^{(3)}_{3,(2,2,1)}.
\end{aligned}$$

Here $\mathfrak{T}^{(2)}_{a,(i_1,i_2)}$ are quasimodular forms that are independent of S which satisfy

$$\mathfrak{T}^{(2)}_{3,(i_1,i_2)} = \frac{3}{\epsilon_1^2}\frac{\partial \mathfrak{T}^{(2)}_{2,(i_1,i_2)}}{\partial E_2(\rho)} = \frac{6}{\epsilon_1^4}\frac{\partial^2 \mathfrak{T}^{(2)}_{1,(i_1,i_2)}}{\partial (E_2(\rho))^2}, \quad \mathfrak{T}^{(2)}_{2,(i_1,i_2)} = 2\epsilon_1^2 \frac{\partial \mathfrak{T}^{(2)}_{1,(i_1,i_2)}}{\partial E_2(\rho)},$$
$$\mathfrak{T}^{(3)}_{3,(i_1,i_2,i_3)} = \frac{2}{\epsilon_1^2}\frac{\partial \mathfrak{T}^{(3)}_{2,(i_1,i_2,i_3)}}{\partial E_2(\rho)} = \frac{8}{3\epsilon_1^4}\frac{\partial^2 \mathfrak{T}^{(3)}_{1,(i_1,i_2,i_3)}}{\partial (E_2(\rho))^2}, \quad \mathfrak{T}^{(3)}_{2,(i_1,i_2,i_3)} = \frac{4}{3\epsilon_1^2}\frac{\partial \mathfrak{T}^{(3)}_{1,(i_1,i_2,i_3)}}{\partial E_2(\rho)}, \tag{5.31}$$

and where $\mathfrak{T}^{(2)}_{3,\underline{i}}$ can be expanded in ϵ_1 as follows:

$$\frac{1}{\epsilon_1^6}\mathfrak{T}^{(2)}_{3,(6,0)} = \frac{E_6}{192} + \frac{E_4^2\epsilon_1^2}{768} + \frac{11E_4E_6\epsilon_1^4}{46080} + \frac{\epsilon_1^6(7E_4^3+4E_6^2)}{290304} + O(\epsilon_1^8),$$
$$\frac{1}{\epsilon_1^4}\mathfrak{T}^{(2)}_{3,(4,1)} = \frac{E_4}{8} + \frac{E_6\epsilon_1^2}{48} + \frac{17E_4^2\epsilon_1^4}{5760} + \frac{31E_4E_6\epsilon_1^6}{80640} + O(\epsilon_1^8), \quad \mathfrak{T}^{(2)}_{3,(2,2)} = 0. \tag{5.32}$$

Similarly, $\mathfrak{T}^{(2)}_{3,\underline{i}}$ can be expanded as[24]

$$\frac{1}{3\epsilon_1^{12}}\mathfrak{T}^{(3)}_{3,(9,0,0)} = \frac{9E_4^3+4E_6^2}{62208} + \frac{19E_4^2E_6\epsilon_1^2}{124416} + \frac{\epsilon_1^4(493E_4^4+583E_4E_6^2)}{14929920} + O(\epsilon_1^6),$$
$$\frac{1}{3\epsilon_1^{10}}\mathfrak{T}^{(3)}_{3,(7,1,0)} = \frac{5E_4E_6}{1296} + \frac{\epsilon_1^2(13E_4^3+7E_6^2)}{7776} + \frac{215E_4^2E_6\epsilon_1^4}{186624} + O(\epsilon_1^6),$$
$$\frac{1}{3\epsilon_1^8}\mathfrak{T}^{(3)}_{3,(5,2,0)} = \frac{5E_4^2}{324} + \frac{19\epsilon_1^2E_4E_6}{1944} + \epsilon_1^4\left(\frac{2E_4^3}{729} + \frac{73E_6^2}{46656}\right) + O(\epsilon_1^6), \quad \mathfrak{T}^{(3)}_{3,(1,1,2)} = 0.$$
$$\frac{1}{3\epsilon_1^6}\mathfrak{T}^{(3)}_{3,(3,3,0)} = \frac{20E_6}{243} + \frac{\epsilon_1^2E_4^2}{27} + \frac{19E_4E_6\epsilon_1^4}{1458} + O(\epsilon_1^6), \quad \frac{1}{3\epsilon_1^4}\mathfrak{T}^{(3)}_{3,(2,2,1)} = \frac{E_4}{3} + \frac{\epsilon_1^2E_6}{9} + \frac{\epsilon_1^4E_4^2}{30} + O(\epsilon_1^6). \tag{5.33}$$

These examples suggest the following general form:

$$\left.\begin{matrix} \mathcal{C}_1^{N=3,(r)} \\ \mathcal{C}_2^{N=3,(r)} \\ \mathcal{C}_3^{N=3,(r)} \end{matrix}\right\} = \sum_{i_1,\ldots,i_r}{}' \mathfrak{T}^{(r)}_{a,(i_1,\ldots,i_r)} (H^{(1)}_{N=1})^{i_1}\ldots(H^{(1)}_{N=1})^{i_r} + \left(\frac{r}{\sigma_1(r)}\right)^2 \begin{cases} 3H^{(r)}_{N=1}W^{(r)}_{\rm NS}W^{(r)}_{\rm NS} & \text{for} \quad a=1, \\ -2H^{(r)}_{N=1}H^{(r)}_{N=1}W^{(r)}_{\rm NS} & \text{for} \quad a=2, \\ H^{(r)}_{N=1}W^{(r)}_{\rm NS}W^{(r)}_{\rm NS} & \text{for} \quad a=3, \end{cases} \tag{5.34}$$

which generalizes Eq. (4.38). Here the summation in Eq. (5.34) is restricted to

$$\sum_{j=1}^r j i_j = 3r, \quad i_1 > 0, \tag{5.35}$$

and the coefficients $\mathfrak{T}^{(r)}_{a,(i_1,\ldots,i_r)}$ satisfy

$$\frac{\partial \mathfrak{T}^{(r)}_{3,(i_1,\ldots,i_r)}}{\partial E_2(\rho)} = 0, \quad \mathfrak{T}^{(r)}_{3,(i_1,\ldots,i_3)} = \frac{6}{r\epsilon_1^2}\frac{\partial \mathfrak{T}^{(r)}_{2,(i_1,\ldots,i_r)}}{\partial E_2(\rho)}, \quad \mathfrak{T}^{(r)}_{2,(i_1,\ldots,i_r)} = \frac{4}{r\epsilon_1^2}\frac{\partial \mathfrak{T}^{(r)}_{1,(i_1,\ldots,i_r)}}{\partial E_2(\rho)}, \quad \forall\, r>1. \tag{5.36}$$

The first equation in fact implies that $\mathfrak{T}^{(3)}_{3,(i_1,\ldots,i_r)}$ are (holomorphic) Jacobi forms.

[24]To keep the length of this paper manageable, we refrain from explicitly writing $\mathfrak{T}^{(2,3)}_{1,2,\underline{i}}$.

VI. HECKE STRUCTURE FOR $N = 4$

In this section we present some partial results for the LST with $N = 4$. Since in this case the free energy is much more complicated than for $N = 2$ or $N = 3$, we shall not be able to achieve a full characterization. However, the partial results we manage to extract fall in line with the patterns that we saw in the previous sections.

A. Decomposition of the free energy

As in the previous cases, the starting point is to compute the decomposition of the free energy. The web diagram representing $X_{4,1}$, which is relevant for the $N = 4$ free energy, is shown in Fig. 9. In addition to the Kähler parameters shown in the figure, we have

$$\rho = \hat{a}_1 + \hat{a}_2 + \hat{a}_3 + \hat{a}_4, \quad R - 4S = v - 3m. \tag{6.1}$$

From the partition function $\mathcal{Z}_{4,1}$ we can compute the free energy

$$\mathcal{F}_{4,1}(\hat{a}_{1,2,3,4}, S, R, \epsilon_{1,2}) = \log \mathcal{Z}_{4,1}(\hat{a}_{1,2,3,4}, S, R, \epsilon_{1,2}).$$

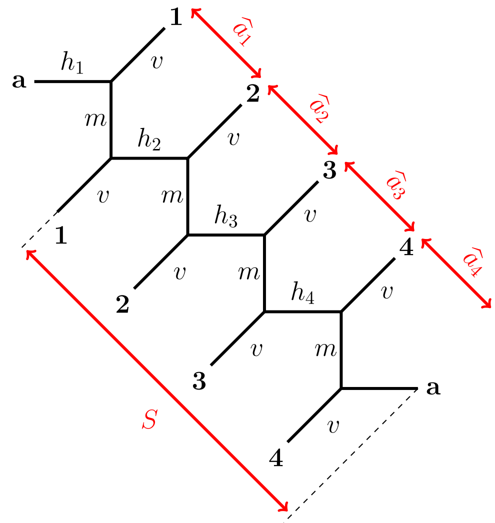

FIG. 9. Web diagram of $X_{4,1}$.

As in the cases of $N = 2$ and $N = 3$, we focus exclusively on the NS limit. In this case, following Eq. (2.22), we can decompose the free energy in terms of $H^{(r),\underline{n}}_{(2s,0)}$, where $\underline{n}$ can be any one of the following combinations

$$\begin{aligned}
&\{0,0,0,0\}, \quad \{n,0,0,0\}, \quad \{n,n,0,0\}, \\
&(n,0,n,0), \quad (n,n,n,0), \quad \{n_1+n_2, n_1, 0, 0\}, \\
&(n_1+n_2, 0, n_1, 0), \quad (n_1+n_2, n_1, n_1, 0), \quad (n_1+n_2, n_1, 0, n_1), \\
&(n_1+n_2, n_1+n_2, n_1, 0), \quad (n_1+n_2, n_1, n_1+n_2, 0), \quad (n_1+n_2+n_3, n_1+n_2, n_1, 0), \\
&(n_1+n_2+n_3, n_1, n_1+n_2, 0), \quad (n_1+n_2+n_3, n_1+n_2, 0, n_1),
\end{aligned} \tag{6.2}$$

with $n, n_1, n_2, n_3 \in \mathbb{N}$ as well as all combinations that can be obtained from Eq. (6.2) through the action of the dihedral group Dih_4, together with their cyclically permutations. The full free energy can then be written in the following somewhat symbolic fashion:

$$P^{(r)}_{4,(2s,0)}(\hat{a}_{1,2,3,4}, S) = \sum_{\underline{m}} H^{(r),\underline{m}}_{(2s,0)} \sum_{\mathrm{Dih}_4} Q^{m_1}_{\hat{a}_1} Q^{m_2}_{\hat{a}_2} Q^{m_3}_{\hat{a}_3} Q^{m_4}_{\hat{a}_4}. \tag{6.3}$$

Here the first sum $\underline{m} = (m_1, m_2, m_3, m_4)$ is over all combinations appearing in Eq. (6.2), while the second sum is over distinct orbits of Dih_4 acting on $(\hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_4)$.

Conjectures for the $H^{(r=1),\underline{m}}_{(2s,0)}$ for all $\underline{m}$ and $s = 0$, which were presented in [58], are of the form

$$H^{(r=1),\underline{m}}_{(2s,0)} = -\sum_{a=0}^{4} w^{(r=1),\underline{m}}_{a,(2s,0)}(\rho)(\phi_{-2,1}(\rho,S))^a(\phi_{0,1}(\rho,S))^{4-a}. \tag{6.4}$$

For the reader's convenience, we recall some of the $g^{(r=1),\underline{m}}_{(2s,0)}(\rho)$ in Table XIV, which turn out to be relevant to the continuing discussion.

TABLE XIV. Expansion coefficients $w^{(r=1),\underline{m}}_{a,(0,0)}$.

$\underline{m}$	$a=0$	$a=1$	$a=2$	$a=3$	$a=4$
(0, 0, 0, 0)	0	$\frac{1}{6\cdot 24^2}$	$\frac{E_2}{576}$	$\frac{E_2^2}{288}$	$\frac{E_2^3}{432}$
$(n,0,0,0)$	0	0	$\frac{n}{288(1-Q_\rho^n)}$	$\frac{nE_2}{72(1-Q_\rho^n)} + \frac{n^2 Q_\rho^n}{12(1-Q_\rho^n)^2}$	$\frac{nE_2^2}{72(1-Q_\rho^n)} + \frac{n^2 Q_\rho^n E_2}{6(1-Q_\rho^n)^2}$
$(n,n,n,0)$	0	0	$\frac{n}{288(1-Q_\rho^n)}$	$\frac{nE_2}{72(1-Q_\rho^n)} + \frac{n^2}{12(1-Q_\rho^n)^2}$	$\frac{nE_2^2}{72(1-Q_\rho^n)} + \frac{n^2 E_2}{6(1-Q_\rho^n)^2}$

While the structure of the $H^{(r=2),\underline{m}}_{(2s,0)}$ is in general more complicated, we have managed to identify particular patterns in some of them which allow us to conjecture the following expressions:

$$
\begin{aligned}
H^{(r),(0,0,0,0)}_{(0,0)} &= \sum_{a=0}^{8} v^{(r=2),(0,0,0,0)}_{a,(0,0)}(\rho)(\phi_{-2,1}(\rho,S))^a(\phi_{0,1}(\rho,S))^{8-a},\\
H^{(r),(n,0,0,0)}_{(0,0)} &= \frac{1}{1-Q_\rho^n}\sum_{k=1}^{3} n^{2k-1}\mathfrak{v}^{(2),(n,0,0,0)}_{1,k,(0)}(\rho,S) + \frac{Q_\rho^n}{(1-Q_\rho^n)^2}\sum_{k=1}^{4} n^{2k-2}\mathfrak{v}^{(2),(n,0,0,0)}_{2,k,(0)}(\rho,S)\\
&\quad + \frac{Q_\rho^n(1+Q_\rho^n)}{(1-Q_\rho^n)^3}\sum_{k=1}^{3} n^{2k+1}\mathfrak{v}^{(2),(n,0,0,0)}_{3,k,(0)}(\rho,S),\\
H^{(r),(n,n,n,0)}_{(0,0)} &= \frac{1}{1-Q_\rho^n}\sum_{k=1}^{3} n^{2k-1}\mathfrak{v}^{(2),(n,0,0,0)}_{1,k,(0)}(\rho,S) + \frac{1}{(1-Q_\rho^n)^2}\sum_{k=1}^{4} n^{2k-2}\mathfrak{v}^{(2),(n,0,0,0)}_{2,k,(0)}(\rho,S)\\
&\quad + \frac{1+Q_\rho^n}{(1-Q_\rho^n)^3}\sum_{k=1}^{3} n^{2k+1}\mathfrak{v}^{(2),(n,0,0,0)}_{3,k,(0)}(\rho,S),
\end{aligned}
\tag{6.5}
$$

where

$$
\begin{aligned}
\mathfrak{v}^{(2),(n,0,0,0)}_{a,k,(0)} &= \sum_{i=2}^{8} v^{(r=2),(n,0,0,0)}_{i,a,k,(0,0)}(\rho)(\phi_{-2,1}(\rho,S))^i\\
&\quad\times(\phi_{0,1}(\rho,S))^{8-i},
\end{aligned}
\tag{6.6}
$$

where the coefficients $v^{(r=2),(0,0,0,0)}_{a,(0,0)}$ are tabulated in Table XV and $v^{(r=2),(0,0,0,0)}_{a,k,(0,0)}$ are tabulated in Table XVI.

We have found evidence that other configurations in Eq. (6.2) afford similar expansions. However, owing to the increased complexity, it is difficult to make conjectures based on the limited expansion of the free energy.[25]

B. Hecke structures

Since the factorization of the free energy for $N = 4$ at order Q_R and $s = 0$ in the fundamental building blocks $H^{(1)}_{N=1}$ and $W^{(1)}_{\rm NS}$ was already commented upon in [47], we directly turn to the extraction of contributions that are related through Hecke transformations.

1. Contour prescription

As in Eqs. (4.27) and (4.28) for $N = 2$ and Eqs. (5.10)–(5.12) for $N = 3$, we define the following three subsectors of the $N = 4$ free energy:

TABLE XV. Expansion coefficients $\mathfrak{v}^{(r=2),(0,0,0,0)}_{a,(0,0)}$.

a	$v^{(r=2),(0,0,0,0)}_{a,(0,0)}$
0	0
1	$-\frac{1}{4\cdot 24^6}$
2	$-\frac{E_2}{8\cdot 24^5}$
a	$\mathfrak{v}^{(r=2),(0,0,0,0)}_{a,(0,0)}$
3	$\frac{-4E_2^2-E_4}{8\cdot 24^5}$
4	$\frac{3E_6-2E_2^3-6E_2E_4}{3\cdot 24^5}$
5	$\frac{32E_2E_6-28E_2^2E_4-9E_4^2}{2\cdot 24^5}$
a	$\mathfrak{v}^{(r=2),(0,0,0,0)}_{a,(0,0)}$
6	$\frac{8E_4E_6-16E_2^3E_4+28E_2^2E_6-21E_2E_4^2}{12\cdot 24^4}$
7	$\frac{64E_2^3E_6-216E_2^2E_4^2+192E_2E_4E_6-9E_4^3-32E_6^2}{36\cdot 24^4}$
8	$\frac{-4E_2^3E_4^2+14E_2^2E_4E_6-9E_2E_4^3-8E_2E_6^2+7E_4^2E_6}{36\cdot 24^3}$

[25] At order Q_R^2 and for $s = 0$, we managed to compute coefficients up to $\mathcal{O}(Q_{\hat{a}_i}^{24})$.

TABLE XVI. Expansion coefficients $v^{(2),(n,0,0,0)}_{i,a,k,(0,0)}$.

a	k	$v^{(r=2),(n,0,0,0)}_{2,a,k,(0,0)}$	$v^{(r=2),(n,0,0,0)}_{3,a,k,(0,0)}$	$v^{(r=2),(n,0,0,0)}_{4,a,k,(0,0)}$	$v^{(r=2),(n,0,0,0)}_{5,a,k,(0,0)}$	$v^{(r=2),(n,0,0,0)}_{6,a,k,(0,0)}$
1	1	$-\frac{1}{4\cdot 24^5}$	$-\frac{E_2}{12\cdot 24^4}$	$\frac{-E_2^2-E_4}{6\cdot 24^4}$	$\frac{4E_6-7E_2E_4}{3\cdot 24^4}$	$\frac{56E_2E_6-3E_4(16E_2^2+7E_4)}{6\cdot 24^4}$
	2	0	$-\frac{1}{12\cdot 24^4}$	$-\frac{E_2}{36\cdot 24^3}$	$\frac{-2E_2^2-E_4}{36\cdot 24^3}$	$\frac{3E_6-13E_2E_4}{90\cdot 24^3}$
	3	0	0	$-\frac{1}{24^5}$	$-\frac{E_2}{3\cdot 24^4}$	$\frac{-2E_2^2-E_4}{3\cdot 24^4}$
2	1	0	$-\frac{1}{2\cdot 24^4}$	$-\frac{E_2}{12\cdot 24^3}$	$-\frac{7E_4}{12\cdot 24^3}$	$\frac{27E_6-52E_2E_4}{15\cdot 24^3}$
	2	0	0	$-\frac{7}{24^4}$	$-\frac{7E_2}{6\cdot 24^3}$	$-\frac{17E_4}{180\cdot 24^2}$
	3	0	0	0	$-\frac{5}{12\cdot 24^3}$	$-\frac{5E_2}{3\cdot 24^3}$
	3	0	0	0	0	$-\frac{1}{15\cdot 24^3}$
3	1	0	0	0	0	$-\frac{E4}{4320}$
	2	0	0	0	0	$-\frac{1}{13824}$
	3	0	0	0	0	$-\frac{7}{34560}$
	4	0	0	0	0	0
	5	0	0	0	0	0

a	k	$v^{(r=2),(n,0,0,0)}_{7,a,k,(0,0)}$	$v^{(r=2),(n,0,0,0)}_{8,a,k,(0,0)}$
1	1	$\frac{4E_6(E_2^2+E_4)-9E_2E_4^2}{9\cdot 24^3}$	$\frac{-3E_4^2(4E_2^2+3E_4)+28E_2E_4E_6-8E_6^2}{18\cdot 24^3}$
	2	$\frac{-12E_2^2E_4+12E_2E_6-5E_4^2}{90\cdot 24^3}$	$\frac{E_4(E_6-E_2E_4)}{180\cot 24^2}$
	3	$-\frac{E_2E_4}{18\cdot 24^3}$	$-\frac{E_4^2}{36\cdot 24^3}$
2	1	$\frac{40E_2E_6-47E_4^2}{252\cdot 24^2}$	$\frac{E_4(27E_6-22E_2E_4)}{90\cdot 24^2}$
	2	$\frac{10E_6-21E_2E_4}{60\cdot 24^2}$	$\frac{20E_2E_6-97E_4^2}{420\cdot 24^2}$
	3	$-\frac{5E_4}{3\cdot 24^3}$	$\frac{E_6-E_2E_4}{15\cdot 24^2}$
	4	$-\frac{E_2}{90\cdot 24^2}$	$-\frac{E_4}{90\cdot 24^2}$
3	1	$\frac{E6}{1512}$	$-\frac{E4^2}{1080}$
	2	$-\frac{5E4}{3456}$	$\frac{19E6}{18144}$
	3	0	$-\frac{E4}{1440}$
	4	$-\frac{1}{21\cdot 24^2}$	0
	5	0	$-\frac{1}{315\cdot 24^2}$

$$\mathcal{C}^{N=4,(r)}_{i,(2s,0)}(\rho,S) = \frac{1}{(2\pi i)^4 r^{i-1}} \sum_{\ell=0}^{\infty} Q_\rho^\ell \oint_0 d\hat{a}_1 \hat{a}_1 \oint_{-\hat{a}_1} d\hat{a}_2(\hat{a}_1+\hat{a}_2)\ldots \oint_{-\hat{a}_1-\cdots-\hat{a}_{i-2}} d_{\hat{a}_{i-1}}(\hat{a}_1+\cdots+\hat{a}_{i-1})$$
$$\times \oint_0 \frac{dQ_{\hat{a}_i}}{Q_{\hat{a}_i}^{1+\ell}} \cdots \oint_0 \frac{dQ_{\hat{a}_4}}{Q_{\hat{a}_4}^{1+\ell}} P^{(r)}_{4,(2s,0)}(\hat{a}_1,\hat{a}_2,\hat{a}_3,\hat{a}_4,S), \quad \forall\, i=1,2,3,4. \tag{6.7}$$

In the following, we shall exclusively focus on $\mathcal{C}^{N=4,(r)}_{1,(2s,0)}$ and $\mathcal{C}^{N=4,(r)}_{2,(2s,0)}$, for which the functions presented in Eq. (6.5) are relevant:

(d) *Combination* $\mathcal{C}^{N=4,(r)}_{1,(2s,0)}$.—As before, $\mathcal{C}^{N=4,(r)}_{1,(2s,0)}$ can be described by extracting a particular class of terms in the Fourier expansion of $P^{(r)}_4$ in powers of $Q_{\hat{a}_{1,2,3,4}}$. Indeed, upon writing

$$P^{(r)}_{4,(2s,0)}(\hat{a}_1,\hat{a}_2,\hat{a}_3,\hat{a}_4,S) = \sum_{n_1,n_2,n_3,n_4=0}^{\infty} Q_{\hat{a}_1}^{n_1} Q_{\hat{a}_2}^{n_2} Q_{\hat{a}_3}^{n_3} Q_{\hat{a}_4}^{n_4} P^{(r),\{n_1,n_2,n_3,n_4\}}_{(2s,0)}(S), \tag{6.8}$$

the contour prescriptions for $\mathcal{C}^{N=4,(r)}_1$ in Eq. (6.7) are designed to extract only those terms with $n_1=n_2=n_3=n_4$. Therefore, $\mathcal{C}^{N=3,(r)}_{1,(2s,0)}$ receives contributions only from those brane configurations in which an equal number of M2-branes is stretched between any two adjacent M5-branes, as shown in Fig. 10(a). Following the definition

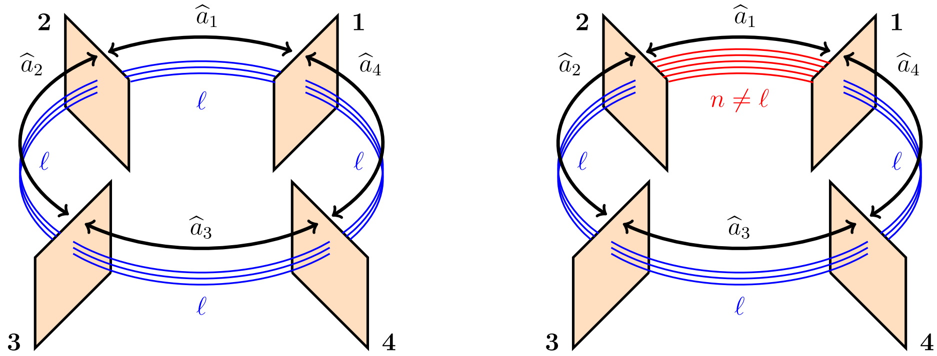

FIG. 10. Brane-web configurations made up of $N = 4$ M5-branes (shown in orange) spaced out on a circle, with various M2-branes (shown in red and blue) stretched between them. (a) An equal number ℓ of M2-branes is stretched between any two neighboring M5-branes. Configurations of this type are relevant for the computation of $\mathcal{C}_1^{N=4,(r)}$. (b) ℓ M2-branes are stretched between M5-branes 2 and 3, 3 and 4, and 4 and 1, while $n \neq \ell$ M2-branes are stretched between M5-branes 1 and 2. Configurations of this type are relevant for the computation of $\mathcal{C}_2^{N=4,(r)}$.

of $H^{(r),\underline{n}}_{(2s,0)}$ in Eq. (2.21), we find that $\mathcal{C}^{N=3,(r)}_{1,(2s,0)}$ can equivalently be written as

$$\mathcal{C}^{N=4,(r)}_{1,(2s,0)}(\rho,S) = H^{(r),\{0,0,0,0\}}_{(2s,0)}(\rho,S),$$
$$\mathcal{C}^{N=4,(r)}_{1}(\rho,S,\epsilon_1) = \sum_{s=0}^{\infty} \epsilon_1^{2s-1} H^{(r),\{0,0,0,0\}}_{(2s,0)}(\rho,S). \quad (6.9)$$

This is in fact the reduced free energy for $N = 4$ that was studied in [46]. Explicit expansions of $\mathcal{C}_1^{N=4,(r)}$ for $r = 1$ and $r = 2$ can be recovered from Tables XIV and XV.

(e) *Combination* $\mathcal{C}_2^{N=4,(r)}$.—The function $\mathcal{C}_2^{N=4,(r)}$ in Eq. (6.7) extracts specific coefficients in a mixed Fourier and Laurent series expansion of the free energy. Starting from the Fourier expansion (6.8) $\mathcal{C}_2^{N=4,(r)}$ receives contributions only from coefficients with $n_1 \neq n_2 = n_3 = n_4$. From the brane-web picture, these correspond to configurations where an equal number ℓ of M2-branes is stretched between the M5 branes 2 and 3, 3 and 4, and 4 and 1, while a different number $n \neq \ell$ of M2-branes is stretched between the first and second M5-branes. Such configurations are schematically shown in Fig. 10(b). Finally, the last contour integral in Eq. (5.11) over $\hat{a}_1$ extracts the second order pole in the Laurent expansion.

With respect to the decomposition (6.3), the coefficients $\mathcal{C}^{N=3,(r)}_{2,(2s,0)}$ can be written in the following form:

$$\mathcal{C}^{N=4,(r)}_{2,(2s,0)}(\rho,S) = \frac{1}{2\pi i}\oint_0 d\hat{a}_1 \hat{a}_1 \sum_{n=1}^{\infty}\left[H^{(r),\{n,0,0,0\}}_{(2s,0)}(\rho,S) Q^n_{\hat{a}_1} + H^{(r),\{n,n,n,0\}}_{(2s,0)}(\rho,S)\frac{Q^n_\rho}{Q^n_{\hat{a}_1}}\right]. \quad (6.10)$$

In order to perform the final contour integration over $\hat{a}_1$, we can use the conjectured form (6.5) of $H^{(r),\{n,0,0,0\}}_{(0,0)}$ and $H^{(r),\{n,n,n,0\}}_{(0,0)}$ for $s = 0$ and $r = 1$ and $r = 2$ to write for the integrand

$$\begin{aligned}
\mathcal{I}_{\mathcal{C}^{N=4,(r)}_{2,(0)}} &= \sum_{n=1}^{\infty}\left[H^{(r),\{n,0,0,0\}}_{(0,0)}(\rho,S) Q^n_{\hat{a}_1} + H^{(r),\{n,n,n,0\}}_{(0,0)}(\rho,S)\frac{Q^n_\rho}{Q^n_{\hat{a}_1}}\right] \\
&= \sum_{n=1}^{\infty}\sum_{k=1}^{3}\frac{n^{2k-1}\mathfrak{b}^{(r),(n,0,0,0)}_{1,k,(0)}}{1-Q^n_\rho}\left(Q^n_{\hat{a}_1} + \frac{Q^n_\rho}{Q^n_{\hat{a}_1}}\right) + \sum_{n=1}^{\infty}\sum_{k=1}^{4}\frac{n^{2k-2}Q^n_\rho \mathfrak{b}^{(r),(n,0,0,0)}_{2,k,(0)}}{(1-Q^n_\rho)^2}\left(Q^n_{\hat{a}_1} + \frac{1}{Q^n_{\hat{a}_1}}\right) \\
&\quad + \sum_{n=1}^{\infty}\sum_{k=1}^{3}\frac{n^{2k+1}Q^n_\rho(1+Q^n_\rho)}{(1-Q^n_\rho)^3}\mathfrak{b}^{(r),(n,0,0,0)}_{3,k,(0)}\left(Q^n_{\hat{a}_1} + \frac{1}{Q^n_{\hat{a}_1}}\right) \\
&= \sum_{k=1}^{3}\mathfrak{b}^{(r),(n,0,0,0)}_{2,k,(0)}\mathcal{I}_{k-1}(\rho,\hat{a}_1) + \sum_{k=1}^{4}\mathfrak{b}^{(r),(n,0,0,0)}_{2,k,(0)}\sum_{n=1}^{\infty}\frac{n^{2k-2}Q^n_\rho}{(1-Q^n_\rho)^2}(Q^n_{\hat{a}_1} + Q^{-n}_{\hat{a}_1}) \\
&\quad + \sum_{k=1}^{3}\mathfrak{b}^{(r),(n,0,0,0)}_{3,k,(0)}\sum_{n=1}^{\infty}\frac{n^{2k+1}Q^n_\rho(1+Q^n_\rho)}{(1-Q^n_\rho)^3}(Q^n_{\hat{a}_1} + Q^{-n}_{\hat{a}_1}),
\end{aligned} \quad (6.11)$$

where we have exchanged the summation over k and n and $\mathcal{I}_\alpha$ is as defined in Eq. (4.15). Using the geometric series

$$\frac{x}{(1-x)^2} = \sum_{\ell=1}^{\infty} \ell x^\ell, \quad \frac{x(1+x)}{(1-x)^3} = \sum_{\ell=1}^{\infty} \ell^2 x^\ell, \quad \text{for } |x| < 1, \tag{6.12}$$

we can write for the sum over n in the last two terms of Eq. (6.11)

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{n^{2k-2} Q_\rho^n}{(1-Q_\rho^n)^2} (Q_{\hat{a}_1}^n + Q_{\hat{a}_1}^{-n}) &= \sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty} n^{2k-2} \ell Q_\rho^{n\ell} (Q_{\hat{a}_1}^n + Q_{\hat{a}_1}^{-n}) \\ &= D_{\hat{a}_1}^{2k-2} \sum_{n=1}^{\infty} Q_\rho^n \sum_{\ell|n} \frac{n}{\ell} (Q_{\hat{a}_1}^\ell + Q_{\hat{a}_1}^{-\ell}) = D_{\hat{a}_1}^{2k-2} \mathcal{J}_0^{(1)}(\rho, \hat{a}_1),\end{aligned} \tag{6.13}$$

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{n^{2k+1} Q_\rho^n (1+Q_\rho^n)}{(1-Q_\rho^n)^3} (Q_{\hat{a}_1}^n + Q_{\hat{a}_1}^{-n}) &= \sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty} n^{2k+1} \ell^2 Q_\rho^{n\ell} (Q_{\hat{a}_1}^n + Q_{\hat{a}_1}^{-n}) \\ &= D_{\hat{a}_1}^{2k+1} \sum_{n=1}^{\infty} Q_\rho^n \sum_{\ell|n} \left(\frac{n}{\ell}\right)^2 (Q_{\hat{a}_1}^\ell + Q_{\hat{a}_1}^{-\ell}) = D_{\hat{a}_1}^{2k+1} \mathcal{J}_0^{(2)}(\rho, \hat{a}_1).\end{aligned} \tag{6.14}$$

For $0 < |Q_\rho| < 1$ the functions $\mathcal{J}_0^{(1)}$ and $\mathcal{J}_0^{(2)}$ are in fact regular at $\hat{a}_1 = 0$, such that $\oint_0 d\hat{a}_1 \hat{a}_1 \mathcal{D}_{\hat{a}_1}^k \mathcal{J}_0^{(1,2)}(\rho, \hat{a}_1) = 0$ for $k \geq 0$. Furthermore, using Eq. (4.18), we get

$$\mathcal{C}_{2,(2s=0)}^{N=4,(r)}(\rho, S) = \frac{1}{r} \mathfrak{v}_{1,1,(0)}^{(r),(n,0,0,0)}, \quad \mathcal{C}_2^{N=4,(r)}(\rho, S, \epsilon_1) = \frac{1}{r} \sum_{s=0}^{\infty} \epsilon_1^{2s-1} \mathfrak{v}_{1,1,(0)}^{(r),(n,0,0,0)}(\rho, S). \tag{6.15}$$

While we leave the study of the other functions for future work, we remark in passing that the $\hat{a}_i$ of which we extract the pole $Q_{\hat{a}_i}^{-2}$ in Eq. (6.7) are consecutive. An interesting question is whether it makes sense to define more general functions such as, for example,

$$\tilde{\mathcal{C}}_{3,(2s,0)}^{N=4,(r)}(\rho, S) = \frac{1}{r^{i-1}} \sum_{\ell=0}^{\infty} Q_\rho^\ell \oint_0 d\hat{a}_1 \hat{a}_1 \oint_{-\hat{a}_1} d\hat{a}_3 (\hat{a}_1 + \hat{a}_3) \oint_0 \frac{dQ_{\hat{a}_1}}{Q_{\hat{a}_1}^{1+\ell}} \oint_0 \frac{dQ_{\hat{a}_4}}{Q_{\hat{a}_4}^{1+\ell}} P_{4,(2s,0)}^{(r)}(\hat{a}_{1,\dots,4}, S), \tag{6.16}$$

Both $\mathcal{C}_3^{N=4,(r)}$ and $\tilde{\mathcal{C}}_{3,(2s,0)}^{N=4,(r)}$ receive contributions from slightly different M2-brane configurations (they are schematically shown in Fig. 11). With respect to the list in Eq. (6.2), the precise configurations are, respectively,

$$\begin{aligned}&\mathcal{C}_{3,(2s,0)}^{N=4,(r)}: \{n_1+n_2, n_1, 0, 0\}, \quad \{n_1+n_2, n_1, n_1, 0\} \quad \{n_1+n_2, n_1+n_2, n_1, 0\}, \quad \{n, n, 0, 0\}, \\ &\tilde{\mathcal{C}}_{3,(2s,0)}^{N=4,(r)}: \{n_1+n_2, 0, n_1, 0\}, \quad \{n_1+n_2, n_1, 0, n_1\}, \quad \{n_1+n_2, n_1, n_1+n_2, 0\}, \quad \{n, 0, n, 0\}.\end{aligned}$$

From [58], one can see that the corresponding $H_{(0,0)}^{(r=1),\underline{n}}$ in the case of $\tilde{\mathcal{C}}_{3,(2s,0)}^{N=4,(r)}$ all involve polynomials of the $\underline{n}$ of order 3 or higher (while there are contributions with polynomials of order 2 in the case of $\mathcal{C}_3^{N=4,(r)}$). Following the knowledge gained in the previous sections, this suggests that (at least to leading order in Q_R) $\tilde{\mathcal{C}}_{3,(2s,0)}^{N=4,(r=1)}$ in fact may be vanishing.

2. *Factorization and Hecke relations*

In the cases $N = 2$ and $N = 3$ we have observed that the functions $\mathcal{C}_a^{N=2,(r=1)}$ and $\mathcal{C}_a^{N=3,(r=1)}$ can be factorized as in Eqs. (4.28) and (5.29), respectively. Analyzing $\mathcal{C}_{i,(2s,0)}^{N=4,(r=1)}$ for $s > 0$ is very complicated; however, based on the expansions presented above, we find that

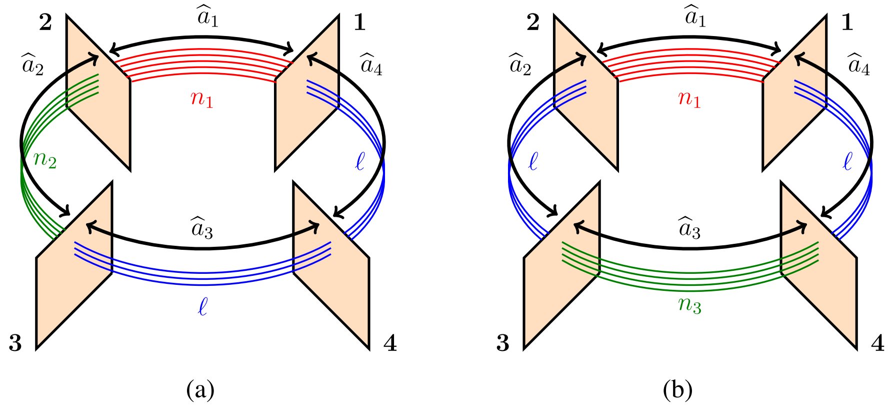

FIG. 11. Brane-web configurations made up of $N = 4$ M5-branes (shown in orange) spaced out on a circle, with various M2-branes (shown in red and blue) stretched between them. (a) An equal number ℓ of M2-branes is stretched between the neighboring M5-branes 3 and 4 as well as 4 and 1, while different numbers of M2-branes $n_1 \neq n_2 \neq \ell$ are stretched between the M5-branes 1 and 2 as well as 2 and 3. Configurations of this type are relevant for the computation of $\mathcal{C}_3^{N=4,(r)}$. (b) An equal number ℓ of M2-branes is stretched between the neighboring M5-branes 2 and 3 as well as 4 and 1, while different numbers of M2-branes $n_1 \neq n_3 \neq \ell$ are stretched between the M5-branes 1 and 2 as well as 3 and 4. Configurations of this type are relevant for the computation of $\tilde{\mathcal{C}}_{3,(2s,0)}^{N=4,(r)}$.

$$\mathcal{C}_{1,(0,0)}^{N=4,(r=1)}(\rho,S) = 4H_{(0,0)}^{(1),\{0\}}(\rho,S)(W_{(0,0)}(\rho,S))^3,$$
$$\mathcal{C}_{2,(0,0)}^{N=4,(r=1)}(\rho,S) = \left(H_{(0,0)}^{(1),\{0\}}(\rho,S)\right)^2\left(W_{(0,0)}(\rho,S)\right)^2, \tag{6.17}$$

which are indeed in agreement with the general conjectured form (3.9). Moreover, by comparing the explicit expressions for the contributions $\mathcal{C}_{1,(0,0)}^{N=4,(r)}$ and $\mathcal{C}_{2,(0,0)}^{N=4,(r)}$ for $r = 1$ and $r = 2$ to the free energy, we find that they satisfy the following recursion relation:

$$\mathcal{C}_{1,(0,0)}^{N=4,(r=2)}(\rho,S) = \mathcal{H}_2\left[\mathcal{C}_{1,(0,0)}^{N=4,(1)}(\rho,S)\right],$$
$$\mathcal{C}_{2,(0,0)}^{N=4,(r=2)}(\rho,S) = \mathcal{H}_2\left[\mathcal{C}_{2,(0,0)}^{N=4,(1)}(\rho,S)\right], \tag{6.18}$$

which generalizes the relations (4.26) and (5.28) to $N = 4$. In view of the results of the previous sections, we conjecture that this result in fact generalizes not only for $r > 2$ and $s > 0$ but also to all functions $\mathcal{C}_i^{N=4,(r)}$ for $i = 1, \ldots, 4$.

VII. CONCLUSIONS AND INTERPRETATION

Although the observations of the previous sections were only for the specific cases $N = 2$ and $N = 3$ (as well as partially for $N = 4$) and for limited values of the order of Q_R (indicated by the subscript r) as well as ϵ_1 (indicated by the subscript $2s$), the fact that they exhibit a rather clear-cut pattern leads us to believe that they hold in general (i.e., for generic N and generic values of r and s). To be concrete, we therefore conjecture that for given N, to any instanton order[26] r we can extract at every order ϵ_1^{2s-2} (for $s \in \mathbb{N}$) N different functions $\mathcal{C}_{i,(2s,0)}^{N,(r)}(\rho,S)$ [see Eq. (3.1) for the definitions] for $i = 1, \ldots, N$ from the NS limit of the free energy $P_N^{(r)}(\hat{a}_{1,\ldots,N})$ that count very specific BPS states from the perspective of the M-brane webs. Indeed, focusing on configurations where the same number of M2-branes is stretched between $N - i$ neighboring M5-branes, they extract a particular polar part of the free energy when the remaining M5-branes are collapsed on top of each other. Viewed order by order in Q_R, the formal series $\mathcal{C}_i^{N,(r)}(\rho,S,\epsilon_1)$ for different values of r are related through Hecke transformations [see Eq. (3.6)]. This generalizes the observation made in [46], which in our language is the specific case $\mathcal{C}_{i=1}^{N,(r)}(\rho,S,\epsilon_1)$. Furthermore, following the logic put forward in [43,46], the Hecke relation (3.6) suggests that the BPS states counted by $\mathcal{C}_{i=1}^{N,(r)}(\rho,S,\epsilon_1)$ can be arranged in the form of a symmetric torus orbifold CFT and we can define the corresponding CFT partition functions,

$$Z_i^{(N)}(R,\rho,S,\epsilon_1) = \exp\left(\sum_{r\geq 1} Q_R^{Nr}\mathcal{C}_i^{N,(r)}(\rho,S,\epsilon_1)\right). \tag{7.1}$$

The relation (3.6) $\mathcal{C}_i^{N,(r)}(\rho,S,\epsilon_1) = \mathcal{H}_r(\mathcal{C}_i^{N,(1)}(\rho,S,\epsilon_1))$ then implies [70]

$$Z_i^{(N)}(R,\rho,S,\epsilon_1) = \exp\left(\sum_{r\geq 1} Q_R^{Nr}\mathcal{H}_r(\mathcal{C}_i^{N,(1)}(\rho,S,\epsilon_1))\right) \tag{7.2}$$

$$= \sum_{r\geq} Q_R^r \chi_{\text{ell}}(\text{Sym}^r(\mathcal{M}_i)). \tag{7.3}$$

[26]Here we are taking the point of view of the $U(N)$ gauge theory that is engineered from the Calabi-Yau threefold $X_{N,1}$,

Here Q_R keeps track of the symmetric products and we conjecture the existence of spaces $\mathcal{M}_i$ with equivariant elliptic genus $\mathcal{C}_i^{N,(1)}(\rho, S, \epsilon_1)$. We have defined only those terms with instanton order, in the language of the dual supersymmetric gauge theory, $\mathcal{O}(Q_R^r)$ with $r > 0$, i.e., we have not included the terms coming from the little string partition function with $Q_R = 0$ which correspond to perturbative corrections in the dual gauge theory. Furthermore, to make a paramodular symmetry of the CFT partition function more manifest [43], we have used Q_R^N as the generating function parameter rather than Q_R as in Eq. (7.3): indeed, the Hecke structure of Eq. (4.26) implies that $Z_i^{(N)}(R, \rho, S, \epsilon_1)$ is the partition function of a symmetric orbifold conformal field theory on the torus that is invariant under the paramodular group $\Sigma_N^* \subset Sp(4, \mathbb{R})$ (see Appendix D for the definition). To make invariance under Σ_N^* more manifest, we remark that the Hecke structure of Eq. (4.26) in $Z_i^{(N)}$ can be expressed in a product form [70]

$$Z_i^{(N)} = \prod_{r,k,\ell} (1 - Q_R^{Nr} Q_\rho^k Q_S^\ell q^p)^{-c_i(kr,\ell,p)}, \tag{7.4}$$

where $c_i(k, \ell, p)$ are the Fourier coefficients of the seed function $\mathcal{C}_i^{N,(1)}(\rho, S, \epsilon_1)$,

$$\mathcal{C}_i^{N,(1)}(\rho, S, \epsilon_1) = \sum_{k,\ell,p} c_i(k, \ell, p) Q_\rho^k Q_S^\ell q^p. \tag{7.5}$$

Thus, the partition function $Z_i^{(N)}(R, \rho, S, \epsilon_1)$ is an exponential lift of the Jacobi form $\mathcal{C}_i^{N,(1)}(\rho, S, \epsilon_1)$ that is related[27] to a paramodular form of the group Σ_N^* satisfying the property [71],

$$Z_i^{(N)}(R, \rho, S, \epsilon_1) = Z_i^{(N)}\left(\frac{\rho}{N}, NR, S, \epsilon_1\right). \tag{7.6}$$

Finally, we remark that Σ_N^* acts on $\mathbb{H}_2$ the space of 2×2 matrices with a positive imaginary part as in Eq. (D3). The quotient $\Sigma_N^* \backslash \mathbb{H}_2$ is the moduli space of Abelian surfaces with polarization $(1, N)$ [72,73]. These Abelian surfaces are precisely the ones appearing in the F theory forming the fibers of the double elliptically fibered Calabi-Yau threefolds [27]. It would be very interesting to have a clearer geometric interpretation of this result, for example, understanding the target space of this CFT. We leave this question for future work.

The functions $\mathcal{C}_i^{N,(r=1)}(\rho, S, \epsilon_1)$ at leading instanton order $\mathcal{O}(Q_R)$ exhibit a factorization into simpler building blocks which go beyond the known self-similarity and recursive structure (see Sec. II for a review of both) of the free energy and extend the preliminary results in [47]: indeed, $\mathcal{C}_i^{N,(r=1)}(\rho, S, \epsilon_1)$ can be written as the product (3.9), where the building blocks $H_{N=1}^{(1)}(\rho, S, \epsilon_1)$ and $W_{\rm NS}^{(1)}(\rho, S, \epsilon_1)$ either stem from the expansion of the free energy for $N = 1$ or govern the BPS counting of a single M5-brane with single M2-branes attached to it on either side (for a review see Appendix C). To higher order in Q_R, remnants of such a factorization persist, but new elements appear as well [see Eq. (3.11)]. It is difficult to conjecture a closed form expression of the latter; however, we have succeeded in showing for $N = 2$ and $N = 3$ that they are governed by differential equations that are very similar to holomorphic anomaly equations.

The $\mathcal{C}_i^{N,(r=1)}(\rho, S, \epsilon_1)$ discussed in this work are specific contributions to the BPS free energy of LSTs of type A. It would be very interesting to understand the geometric reason that makes these states special relative to others, such that they can be interpreted as part of the spectrum of a symmetric torus orbifold. This could give us the key to understanding whether there are further sectors in the spectrum of the LSTs of A type which exhibit similar properties. Furthermore, this may also give us a hint as to whether these various orbifold CFTs can in any way be connected via a duality transformation.

Another interesting observation is the fact that the $\mathcal{C}_i^{N,(r=1)}(\rho, S, \epsilon_1)$ (except for $i = 1$) are obtained through contour integrals from the free energy $P_N^{(r)}(\hat{a}_{1,\ldots,N})$ that select the coefficient of a pole or poles in $\hat{a}_{1,\ldots,N-1}$. In [74] the BPS counting of supersymmetric black holes has been discussed. It has been pointed out that the phenomenon of *wall crossing* can be attributed to the polar part of a meromorphic Jacobi form that counts multi-centered black holes whose number can jump when crossing a wall. It would be interesting to analyze if a similar phenomenon takes place for the BPS counting functions discussed in this paper when we cross the loci $\hat{a}_i = 0$. In the dual $U(1)^N$ gauge theory $\hat{a}_i$ are inverse coupling constants for each of the $U(1)$ factors and crossing the $\hat{a}_i = 0$ locus corresponds to passing through the infinite coupling region [75,76]. It would be interesting to understand what happens in this case to the BPS states that are counted by $\mathcal{C}_i^{N,(r=1)}(\rho, S, \epsilon_1)$. We leave this question for future work.

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[27]If the terms with $Q_R = 0$ are included in the definition (7.3) of the reduced partition function then $Z_i^{(N=2)}$ is precisely the paramodular form for Σ_N^*.

APPENDIX A: MODULAR FORMS

Throughout this work we use various modular objects. This Appendix compiles the definitions of all objects that are used in the body of this article, as well as additional useful information and identities. For a more comprehensive review, we relegate the reader to the literature, e.g., [77–79].

A weak Jacobi form of the modular group $\Gamma \cong SL(2,\mathbb{Z})$ of index $m \in \mathbb{Z}$ and weight $w \in \mathbb{Z}$ is a holomorphic function of the type

$$\begin{aligned} \phi\colon\ & \mathbb{H}\times\mathbb{C}\to\mathbb{C} \\ & (\rho,z)\mapsto\phi(\rho;z) \end{aligned} \tag{A1}$$

(where $\mathbb{H}$ is the upper complex plane), which behaves in the following manner under transformations of Γ:

$$\begin{aligned} &\phi\left(\frac{a\rho+b}{c\rho+d};\frac{z}{c\rho+d}\right)=(c\rho+d)^w e^{\frac{2\pi i m c z^2}{c\rho+d}}\phi(\rho;z), \\ &\qquad \forall \begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\Gamma, \\ &\phi(\rho;z+\ell_1\rho+\ell_2)=e^{-2\pi i m(\ell_1^2\rho+2\ell_1 z)}\phi(\rho;z), \\ &\qquad \forall\, \ell_{1,2}\in\mathbb{N}. \end{aligned} \tag{A2}$$

Such functions allow a Fourier expansion of the form

$$\phi(z,\rho)=\sum_{n=0}^{\infty}\sum_{\ell\in\mathbb{Z}}c(n,\ell)Q_\rho^n e^{2\pi i z\ell},\quad \text{with}\quad Q_\rho=e^{2\pi i\rho}. \tag{A3}$$

The Jacobi forms encountered throughout this work can be decomposed in terms of two basis functions, i.e., for index m and weight w, we can write

$$\phi(\rho;z)=\sum_{a=0}^{m}f_a(\rho)(\phi_{0,1}(\rho,z))^a(\phi_{-2,1}(\rho,z))^{m-a}. \tag{A4}$$

Here $\phi_{-2,1}$ and $\phi_{0,1}$ are Jacobi forms of index 1 and weight -2 and 0, respectively, which are defined as[28]

$$\phi_{0,1}(\rho,z)=8\sum_{a=2}^{4}\frac{\theta_a^2(z;\rho)}{\theta_a^2(0,\rho)},\quad \phi_{-2,1}(\rho,z)=\frac{\theta_1^2(z;\rho)}{\eta^6(\rho)}, \tag{A5}$$

with $\theta_{a=1,2,3,4}(z;\rho)$ the Jacobi theta functions and $\eta(\rho)$ the Dedekind eta function. Furthermore, the $f_a(\rho)$ in Eq. (A4) are modular forms of weight $w+2a$. In practice, the $f_a(\rho)$ can be written as homogeneous polynomials in the Eisenstein series E_{2n}, which are modular forms of weight $2n$ and which are defined as

$$E_{2k}(\rho)=1-\frac{4k}{B_{2k}}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)Q_\rho^n,\quad \forall\, k\in\mathbb{N}, \tag{A6}$$

where B_{2k} are the Bernoulli numbers, while $\sigma_k(n)$ is the divisor function. We shall sometimes also use the differently normalized functions

$$\begin{aligned} G_{2k}(\rho)&=2\zeta(2k)+2\frac{(2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)Q_\rho^n \\ &=2\zeta(2k)E_{2k}(\rho). \end{aligned} \tag{A7}$$

The holomorphic Eisenstein series (i.e., the E_{2n} for $n>1$) form a ring, which is generated by $\{E_4,E_6\}$. Furthermore, most of the examples we encounter in this paper are in fact quasi-Jacobi forms, in the sense that the $f_a(\rho)$ in their decomposition (A4) also depend on the Eisenstein series E_2: the latter is strictly speaking not a modular form. However, one can define the following nonholomorphic object:

$$\hat{E}_2(\rho,\bar\rho)=E_2(\rho)-\frac{6i}{\pi(\rho-\bar\rho)}, \tag{A8}$$

which transforms with weight 2 under modular transformations.

Another object that we encounter in the body of this paper is the Weierstrass elliptic function

$$\wp(z;\rho)=\frac{1}{z^2}+\sum_{k=1}^{\infty}(2k+1)G_{2k+2}(\rho)z^{2k}, \tag{A9}$$

which has a pole of order 2 in z.

Finally, many of the results found in this paper use Hecke operators: these are maps from the space $J_{w,m}(\Gamma)$ of Jacobi forms of index m and weight w to the space $J_{w,km}(\Gamma)$ of Jacobi forms of index km and weight w for $k\in\mathbb{N}$:

$$\begin{aligned} &\mathcal{H}_k\colon\ J_{w,m}(\Gamma)\to J_{w,km}(\Gamma) \\ &\phi(\rho;z)\mapsto\mathcal{H}_k(\phi(\rho;z))=k^{w-1}\sum_{\substack{d|k \\ b \bmod d}}d^{-w}\phi\left(\frac{k\rho+bd}{d^2};\frac{kz}{d}\right). \end{aligned} \tag{A10}$$

Hecke transformations of this type can also be extended to Jacobi forms that depend on more than one variable: let $f_{w,\vec{m}}(\rho,\vec{z})\colon \mathbb{H}\times\mathbb{C}^n\to\mathbb{C}$ be a Jacobi form with index vector $\vec{m}$. We then define

[28] $\phi_{0,1}(\rho,z)$ defined below differs by a factor of 2 from its usual definition in the literature [77]. As defined it is equal to the elliptic genus of $K3$.

$$\mathcal{H}_k\colon\ f_{w,\vec{m}}(\rho,\vec{z}) \mapsto \mathcal{H}_k(f_{w,\vec{m}}(\rho,\vec{z})) = k^{w-1}\sum_{\substack{d|k \\ b \bmod d}} d^{-w} f_{w,\vec{m}}\left(\frac{k\rho+bd}{d^2};\frac{k\vec{z}}{d}\right). \quad \text{(A11)}$$

For further use, we consider the case that f allows for a Laurent series expansion in one of the variables: let $(z_1,\ldots,z_n) = (\vec{z},z_n)$ (where $\vec{z}\in\mathbb{C}^{n-1}$), and let $(m_1,\ldots,m_n) = (\vec{m},m_n)$ be the index vector of a Jacobi form that affords the following (convergent) Laurent series:

$$f_{w,(m_1,\ldots,m_n)}(\rho;\vec{z},z_n) = \sum_a z_n^a f_{w+a,\vec{m}}(\rho,\vec{z}), \quad \text{(A12)}$$

where $f_{w+a,\vec{m}}(\rho,\vec{z})$ are Jacobi forms of weight $w+a$ and index vector $\vec{m}$. We then find for the action of the Hecke operator

$$\begin{aligned}\mathcal{H}_k(f_{w,(m_1,\ldots,m_n)}(\rho;\vec{z},z_n)) &= k^{w-1}\sum_{\substack{d|k \\ b \bmod d}} d^{-w} f_{w,(m_1,\ldots,m_n)}\left(\frac{k\rho+bd}{d^2};\frac{k\vec{z}}{d},\frac{kz_n}{d}\right)\\ &= \sum_a z_n^a k^{w+a-1}\sum_{\substack{d|k \\ b \bmod d}} d^{-w-a} f_{w+a,\vec{m}}\left(\frac{k\rho+bd}{d^2};\frac{k\vec{z}}{d}\right)\\ &= \sum_a z_n^a \mathcal{H}_k(f_{w+a;\vec{m}}(\rho,\vec{z})). \end{aligned} \quad \text{(A13)}$$

APPENDIX B: $(N,1)$ PARTITION FUNCTIONS

The topological string partition function of the Calabi-Yau threefold $X_{N,1}$ is given by [21,22,26]

$$\mathcal{Z}_{N,1}(\tau,\hat{\mathbf{a}},m,\epsilon_{1,2}) = \sum_{\lambda_1\cdots\lambda_N} Q_\tau^{|\lambda_1|+\cdots+|\lambda_N|} Z_{\lambda_1\cdots\lambda_N}(\hat{\mathbf{a}},m,\epsilon_{1,2}), \quad \text{with} \quad \hat{\mathbf{a}} = \{\hat{a}_1,\ldots,\hat{a}_N\}, \quad \text{(B1)}$$

and where the sum is over N-tuples of partitions of non-negative integers. The parts of the partition λ_α are denoted by $\lambda_{\alpha,i}$ with $\lambda_{\alpha,1}\geq\lambda_{\alpha,2}\geq\lambda_{\alpha,3}\geq\cdots$. Each partition λ_α corresponds to a Young diagram which is obtained by putting $\lambda_{\alpha,i}$ boxes in the ith column such that a box in the Young diagram can be assigned a coordinate (i,j) as long as $1\leq i\leq\ell(\lambda_\alpha)$, $1\leq j\leq\lambda_{\alpha,i}$. The transpose of a partition λ_α is denoted by λ_α^t and is defined as the partition corresponding to the Young diagram obtained by interchanging rows and columns of the Young diagram corresponding to λ_α. If we denote by $\ell(\lambda_\alpha)$ the total number of nonzero parts of the partition λ_α, we then define

$$|\lambda_\alpha| = \sum_{i=1}^{\ell(\lambda_\alpha)}\lambda_{\alpha,i}, \quad ||\lambda_\alpha||^2 = \sum_{i=1}^{\ell(\lambda_\alpha)}\lambda_{\alpha,i}^2. \quad \text{(B2)}$$

As discussed in the body of the paper, the topological string partition function (B1) also captures the partition function of a supersymmetric gauge theories. Furthermore, from a geometric point of view, the instanton part of $\mathcal{Z}_{N,1}$ is the generating function of equivariant elliptic genera of the instanton moduli space $M(N,k)$,

$$\mathcal{Z}_{N,1} = Z_0\sum_k Q_\tau^k \chi_{\text{ell}}(M(N,k)), \quad \text{(B3)}$$

where $\chi_{\text{ell}}(X)$ denotes the equivariant elliptic genus of any manifold X,

$$\chi_{\text{ell}}(X) = \text{Tr}_{\mathcal{H}(X)}(-1)^{F_L+F_R} y^{F_L} q^H e^{2\pi i\hat{\mathbf{a}}\cdot\mathbf{h}}. \quad \text{(B4)}$$

Here the trace is over the $R{-}R$ sector, $F_{L,R}$ are the left and the right moving fermion numbers and h_i are the Cartan generators of the symmetry group G which acts on X (and $\hat{\mathbf{a}}\cdot\mathbf{h} = \sum_{i=1}^N \hat{a}_i h_i$). The path integral representation of the above reduces to an index calculation,

$$\chi_{\text{ell}}(X) = \int_X \text{ch}(E_{Q_\tau,y})\text{Td}(X) = \int_X \prod_{i=1}^{\dim(X)} 2\pi i\xi_i \frac{\vartheta(\tau,m+\xi_i)}{\vartheta(\tau,\xi_i)}, \quad \text{(B5)}$$

where ($y = e^{2\pi i m}$)

$$E_{Q_\tau,y} = y^{-\frac{d}{2}} \otimes_{\ell\geq1} [\wedge_{-yQ_\tau^{\ell-1}} T_X \otimes \wedge_{-y^{-1}Q_\tau^\ell} \overline{T_X} \otimes S_{Q_\tau^\ell} T_X \otimes S_{Q_\tau^\ell}\overline{T_X}] \quad \text{(B6)}$$

and x_i are the formal roots of the Chern polynomial. The relation between $Z_{\lambda_1\cdots\lambda_N}$ and $\chi_{\text{ell}}(M(N,k))$ is given by

$$\chi_{\text{ell}}(M(N,k)) = \sum_{|\lambda_1|+\cdots+|\lambda_N|=k} Z_{\lambda_1\cdots\lambda_N}/Z_0. \quad \text{(B7)}$$

The function $Z_{\lambda_1\cdots\lambda_N}(\hat{\mathbf{a}},m,\epsilon_{1,2})$ in Eq. (B1) is defined as

$$Z_{\lambda_1\cdots\lambda_N}(\hat{\mathbf{a}},m,\epsilon_{1,2}) = Z_0\prod_{\alpha=1}^N \frac{\vartheta_{\lambda_\alpha\lambda_\alpha}(Q_m)}{\vartheta_{\lambda_\alpha\lambda_\alpha}\left(\sqrt{\frac{t}{q}}\right)} \prod_{1\leq\alpha<\beta\leq N} \frac{\vartheta_{\lambda_\alpha\lambda_\beta}(Q_{\alpha\beta}Q_m)\vartheta_{\lambda_\alpha\lambda_\beta}(Q_{\alpha\beta}Q_m^{-1})}{\vartheta_{\lambda_\alpha\lambda_\beta}\left(Q_{\alpha\beta}\sqrt{\frac{t}{q}}\right)\vartheta_{\lambda_\alpha\lambda_\beta}\left(Q_{\alpha\beta}\sqrt{\frac{q}{t}}\right)}, \quad \text{(B8)}$$

where $Q_{\alpha\beta} = e^{2\pi i(\hat{a}_\alpha-\hat{a}_\beta)}$ and

$$\vartheta_{\lambda\mu}(\rho,z) = \prod_{(i,j)\in\lambda}\theta_1(\rho;z^{-1}t^{-\mu_j^t+i-\frac{1}{2}}q^{-\lambda_i+j-\frac{1}{2}}) \times \prod_{(i,j)\in\mu}\theta_1(\rho;z^{-1}t^{\lambda_j^t-i+\frac{1}{2}}q^{\mu_i-j+\frac{1}{2}}), \quad \text{(B9)}$$

with $\theta_1(\rho, z)$ the Jacobi theta function and $\rho = \sum_{\alpha=1}^{N} \hat{a}_\alpha$. The factor Z_0 in Eqs. (B3) and (B8) is independent of Q_τ and is given by

$$Z_0 = \prod_{n=1}^{\infty}(1-Q_\rho^n)^{-1}\left[\prod_{1\le\alpha<\beta\le N} F_{\alpha\beta}\right]\left[\prod_{\alpha,\beta=1}^{N} H_{\alpha\beta}\right], \quad \text{(B10)}$$

where (with $\tilde{Q}_{\alpha\beta} = Q_1 Q_2 \cdots Q_a Q_1^{-1} \cdots Q_b^{-1} Q_m^{a-b}$)

$$F_{\alpha\beta} = \prod_{i,j=1}^{\infty} \frac{(1-Q_{\alpha\beta}Q_m^{-1}t^{i-\frac{1}{2}}q^{j-\frac{1}{2}})(1-Q_{\alpha\beta}Q_m t^{i-\frac{1}{2}}q^{j-\frac{1}{2}})}{(1-Q_{\alpha\beta}t^i q^{j-1})(1-Q_{\alpha\beta}t^{i-1}q^j)},$$

$$H_{\alpha\beta} = \prod_{n,i,j=1}^{\infty} \frac{(1-Q_\rho^n \tilde{Q}_{\alpha\beta}Q_m^{-1}t^{i-\frac{1}{2}}q^{j-\frac{1}{2}})(1-Q_\rho^n\tilde{Q}_{\alpha\beta}Q_m t^{i-\frac{1}{2}}q^{j-\frac{1}{2}})}{(1-Q_\rho^n\tilde{Q}_{\alpha\beta}t^i q^{j-1})(1-Q_\rho^n\tilde{Q}_{\alpha\beta}t^{i-1}q^j)}. \quad \text{(B11)}$$

1. Modular transformation

To understand how the partition function $\mathcal{Z}_{N,1}$ transforms under the modular transformation,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z})\colon\ (\rho, m, \epsilon_{1,2}, \hat{a}_{\alpha\beta}) \mapsto \left(\frac{a\rho+b}{c\rho+d}, \frac{m}{c\rho+d}, \frac{\epsilon_{1,2}}{c\rho+d}, \frac{\hat{a}_{\alpha\beta}}{c\rho+d}\right), \quad \text{(B12)}$$

which is generated by

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{(B13)}$$

we need to determine the transformation properties of $\vartheta_{\lambda\mu}(\rho, z)$. Although $\vartheta_{\lambda\mu}(\rho, z)$ is not invariant under the T transformation, it is easy to see that the ratio $\frac{\vartheta_{\lambda\mu}(\rho,z_1)}{\vartheta_{\lambda\mu}(\rho,z_2)}$ is invariant for any $z_{1,2}$. In view of the structure of Eq. (B8), this implies that $\mathcal{Z}_{N,1}$ is invariant under the T transformation. The ratio $\frac{\vartheta_{\lambda\mu}(\rho,z_1)}{\vartheta_{\lambda\mu}(\rho,z_2)}$, however, is not invariant under the S transformation,

$$\frac{\vartheta_{\lambda\mu}(-\frac{1}{\rho}, \frac{z_1}{\rho})}{\vartheta_{\lambda\mu}(-\frac{1}{\rho}, \frac{z_2}{\rho})} = e^{\frac{2\pi i K_{\lambda\mu}}{\tau}} \frac{\vartheta_{\lambda\mu}(\rho, z_1)}{\vartheta_{\lambda\mu}(\rho, z_2)}$$

$$K_{\lambda\mu}(h_1,h_2) = \frac{1}{2}(h_1^2-h_2^2)(|\lambda|+|\mu|) + (h_2-h_1)\left(\sum_{(i,j)\in\lambda}\left(\epsilon_2\left(\mu_j^t - i + \frac{1}{2}\right) - \epsilon_1\left(\lambda_i - j + \frac{1}{2}\right)\right) + \sum_{(i,j)\in\mu}\left(-\epsilon_2\left(\lambda_j^t - i + \frac{1}{2}\right) + \epsilon_1\left(\mu_i - j + \frac{1}{2}\right)\right)\right), \quad \text{(B14)}$$

where $z_{1,2} = e^{2\pi i h_{1,2}}$. Using the following identities,

$$\sum_{(i,j)\lambda}\mu_j^t = \sum_{(i,j)\in\mu}\lambda_j^t, \quad \sum_{(i,j)\lambda}\left(\lambda_i - j + \frac{1}{2}\right) = \frac{||\lambda||^2}{2}, \quad \text{(B15)}$$

$K_{\lambda\mu}(h_1, h_2)$ appearing in Eq. (B14) can be simplified,

$$K_{\lambda\mu}(h_1,h_2) = \frac{1}{2}(h_1^2-h_2^2)(|\lambda|+|\mu|) + (h_1-h_2) \times \left[\epsilon_1\frac{||\lambda||^2-||\mu||^2}{2} + \epsilon_2\frac{||\lambda^t||^2-||\mu^t||^2}{2}\right]. \quad \text{(B16)}$$

Notice that in the unrefined case $(\epsilon_2 = -\epsilon_1 = \epsilon) K_{\lambda\mu}(h_1, h_2)$ simplifies,

$$K_{\lambda\mu}(h_1,h_2) = \frac{1}{2}(h_1^2-h_2^2)(|\lambda|+|\mu|) + (h_1-h_2)\epsilon[\kappa(\lambda)-\kappa(\mu)], \quad \text{(B17)}$$

where $\kappa(\lambda) = \frac{||\lambda||^2-||\lambda^t||^2}{2}$.

Thus, the function $Z_{\lambda_1\cdots\lambda_N}$ given in Eq. (B8) transforms as

$$Z_{\lambda_1\cdots\lambda_N} \mapsto e^{\frac{2\pi i K}{\rho}} Z_{\lambda_1\cdots\lambda_N}, \quad \text{(B18)}$$

with

$$K_{\lambda_1\cdots\lambda_N}(\hat{a}_{\alpha\beta}, m, \epsilon_+) = \sum_{\alpha=1}^{N} K_{\lambda_\alpha\lambda_\alpha}(m, -\epsilon_+) + \sum_{1\le\alpha<\beta\le N}[K_{\lambda_\alpha\lambda_\beta}(\hat{a}_{\alpha\beta}+m, \hat{a}_{\alpha\beta}-\epsilon_+) + K_{\lambda_\alpha\lambda_\beta}(\hat{a}_{\alpha\beta}-m, \hat{a}_{\alpha\beta}+\epsilon_+)].$$

Here we have defined $\hat{a}_{\alpha\beta} = \hat{a}_\alpha - \hat{a}_\beta$. Thus, the partition function is not invariant under modular transformations

(B12) but can be made invariant at the expense of introducing a holomorphic anomaly [59].

2. Singularities

The function $\vartheta_{\lambda\mu}(\rho, z)$ has some interesting properties. In the unrefined case it becomes proportional to a Kronecker delta function for $z = 1$ [80] and $t = q$,

$$\vartheta_{\lambda\mu}(\rho,1) = \delta_{\lambda\mu}\prod_{(i,j)\in\lambda}\theta_1(\rho, q^{h(i,j)})\theta_1(\rho, q^{-h(i,j)}) = (-1)^{|\lambda|}\delta_{\lambda\mu}\prod_{(i,j)\in\lambda}\theta_1(\rho, q^{h(i,j)})^2. \quad \text{(B19)}$$

Since the partition function $\mathcal{Z}_{N,1}$ is a sum over all partitions, from Eqs. (B8) and (B9) it follows that the partition function will have a pole whenever $a_{\alpha\beta} \in \mathcal{S}^1_{\lambda_\alpha\lambda_\beta} \cup \mathcal{S}^2_{\lambda_\alpha\lambda_\beta}$,

$$\mathcal{S}^1_{\lambda_\alpha\lambda_\beta} = \left\{\epsilon_1\left(-\lambda_{\alpha,i} + j - \frac{1}{2}\right) + \epsilon_2\left(\lambda^t_{\beta,j} - i + \frac{1}{2}\right) \pm \epsilon_+ | (i,j) \in \lambda_\alpha\right\},$$
$$\mathcal{S}^2_{\lambda_\alpha\lambda_\beta} = \left\{\epsilon_1\left(\lambda_{\beta,i} - j + \frac{1}{2}\right) + \epsilon_2\left(-\lambda^t_{\alpha,j} + i - \frac{1}{2}\right) \pm \epsilon_+ | (i,j) \in \lambda_\beta\right\}. \quad \text{(B20)}$$

Thus, the total order of the poles in $\hat{a}_{\alpha\beta}$ (counting with possible multiplicity) is $2(|\lambda_\alpha| + |\lambda_\beta|)$. The poles in the variable $\hat{a}_{\alpha\beta}$ depend only on the shape of the pair of partitions $(\lambda_\alpha, \lambda_\beta)$, and therefore the pole structure for $N > 2$ in the variables $\hat{a}_{\alpha\beta}$ follows from the pole structure for the $N = 2$ case in the variable $\hat{a}_{12} = \hat{a}$.

The poles in the variable $\hat{a}$ for the $N = 2$ case form a nested sequence, i.e., the set of poles at order Q_R^k are contained in the set of poles at order Q_R^{k+1}. To see this, consider a pair of partitions (λ_1, λ_2), with $|\lambda_1| + |\lambda_2| = k$, giving the set of poles $\mathcal{S}_{\lambda_1\lambda_2}$. For the case of $N = 2$ consider the pair of partitions $(\lambda_1, \lambda_2) = ((k_1), 1^{k-k_1})$ which contribute to the coefficient of Q_ρ^k for all $k_1 = 0, 1, \ldots, k$. With this choice of the partitions the set of possible poles in Eq. (B20) becomes ($\sigma = 0, 1$)

$$\hat{a}_{12} = \hat{a} \in \{-(k_1 - 1)\epsilon_1 + k_2\epsilon_2 - 2\sigma\epsilon_+\} \cup \{-j\epsilon_1 - 2\sigma\epsilon_+ | j = 0, \ldots, k_1 - 2\} \times \{(i-2)\epsilon_2 + 2\epsilon_+\sigma | i = 1, \ldots, k - k_1\}. \quad \text{(B21)}$$

The free energy $\ln(\mathcal{Z}_{N,1})$ is a power series in $\epsilon_{1,2}$ with coefficients which are refined genus g amplitudes. Once the expansion in $\epsilon_{1,2}$ has been carried out the coefficients, refined genus g amplitudes, now have poles at $\hat{a}_{\alpha\beta} = 0$. In this paper we study the poles of the refined genus g amplitudes at $\hat{a}_{\alpha\beta} = 0$ rather than the poles of the partition function which occur at various locations in the (ϵ_1, ϵ_2) plane.

Example.—Let us consider the case $N = 2$ to first order in Q_τ. The free energy (2.17) is given by

$$\mathcal{F}_{2,1} = \ln(Z_0) + Q_\tau\frac{\vartheta_{(1)(1)}(Q_m)}{\vartheta_{(1)(1)}(\sqrt{\frac{t}{q}})}\left[\frac{\vartheta_{(1)(0)}(Q_{12}Q_m)\vartheta_{(1)(0)}(Q_{12}Q_m^{-1})}{\vartheta_{(1)(0)}(Q_{12}\sqrt{\frac{t}{q}})\vartheta_{(1)(0)}(Q_{12}\sqrt{\frac{q}{t}})} + \frac{\vartheta_{(0)(1)}(Q_{12}Q_m)\vartheta_{(0)(1)}(Q_{12}Q_m^{-1})}{\vartheta_{(0)(1)}(Q_{12}\sqrt{\frac{t}{q}})\vartheta_{(0)(1)}(Q_{12}\sqrt{\frac{q}{t}})}\right] + \cdots$$
$$= \ln(Z_0) + Q_\tau\frac{\theta_1(\rho, m+\epsilon_-)\theta_1(m-\epsilon_-)}{\theta_1(\rho,\epsilon_1)\theta_1(\rho,\epsilon_2)}\left[\frac{\theta_1(\rho,\hat{a}_1+m+\epsilon_+)\theta_1(\rho,\hat{a}_1-m+\epsilon_+)}{\theta_1(\rho,\hat{a}_1)\theta_1(\rho,\hat{a}_1+2\epsilon_+)} + \frac{\theta_1(\rho,\hat{a}_1+m-\epsilon_+)\theta_1(\hat{a}_1-m-\epsilon_+)}{\theta_1(\rho,\hat{a}_1-2\epsilon_+)\theta_1(\hat{a}_1)}\right] + \cdots. \quad \text{(B22)}$$

From Eqs. (B10) and (B11) we see that as a function of $\hat{a}_1$

$$\ln(Z_0) = A\ln(\hat{a}_1) + \cdots, \quad \text{(B23)}$$

where A is independent of $\hat{a}_1$. Thus, the free energy diverges like

$$\mathcal{F}_{2,1} = A\ln(\hat{a}_1) - Q_\tau\frac{\theta_1(\rho,m+\epsilon_-)\theta_1(m-\epsilon_-)\theta_1(\rho,m+\epsilon_+)\theta_1(\rho,m-\epsilon_+)}{\theta_1(\rho,\epsilon_1)\theta_1(\rho,\epsilon_2)}\left[\frac{2}{(\hat{a}_1+2\epsilon_+)(\hat{a}_1-2\epsilon_+)} + \cdots\right]. \quad \text{(B24)}$$

Thus, we see that there is a pole at $\hat{a}_1 = \pm 2\epsilon_+$. However, if we first expand in $\epsilon_{1,2}$, then we get a single pole $\hat{a}_1 = 0$ of order 2. This persists at higher order in Q_τ and we see poles at $\hat{a}_1 = 0$ of various even orders.

APPENDIX C: EXPANSION COEFFICIENTS OF THE BASIC BUILDING BLOCKS

In this Appendix we collect explicit expressions for the expansions of the free energy for $N=1$, $N=2$, and $N=3$.

1. Coefficients of the $N=1$ free energy

Because of their frequent use throughout the body of this article, we tabulate the coefficients $H^{(r),\{0\}}_{(2s,0)}(\rho,S)$ that appear in the expansion of the free energy to leading orders r and s. To this end, we decompose the former in the following fashion:

$$H^{(r),\{0\}}_{(2s,0)}(\rho,S)=\sum_{i=0}^{r}\mathfrak{b}^{(r)}_{i,(2s,0)}(\rho)(\phi_{-2,1}(\rho,S))^i(\phi_{0,1}(\rho,S))^{r-i},\tag{C1}$$

where $\mathfrak{b}^{(r)}_{i,(2s,0)}$ is a quasimodular form of weight $2s+2i-2$, which can be written as a polynomial in the Eisenstein series $\{E_2,E_4,E_6\}$. For $r=1$, $r=2$, and $r=3$ the expansion coefficients are tabulated in Tables XVII–XIX, respectively.

Following [46], the coefficients $H^{(r),\{0\}}_{(2s,0)}$ with $r>1$ can be recovered from those with $r=1$ through Hecke transformations, i.e.,

TABLE XVII. Coefficients in the expansion of $\mathfrak{b}^{(r=1)}_{i,(2s,0)}(\rho,S)$.

s	$\mathfrak{b}^{(r=1)}_{0,(2s,0)}$	$\mathfrak{b}^{(r=1)}_{1,(2s,0)}$
0	0	-1
1	$\frac{1}{96}$	$-\frac{E_2}{48}$
2	$\frac{E_2}{4608}$	$-\frac{5E_2^2+13E_4}{23040}$
3	$\frac{5E_2^2+7E_4}{2211840}$	$-\frac{35E_2^3+273E_2E_4+184E_6}{23224320}$
4	$\frac{35E_2^3+147E_2E_4+124E_6}{2229534720}$	$-\frac{175E_2^4+2730E_2^2E_4+3680E_2E_6+5583E_4^2}{22295347200}$

$$H^{(r),\{0\}}_{(s,0)}(\rho,S)=\mathcal{H}_r(H^{(1),\{0\}}_{(s,0)}(\rho,S)).\tag{C2}$$

The relations (4.26) for $N=2$, (5.28) for $N=3$, and (6.18) for $N=4$ can be understood as generalizations of Eq. (C2). Finally, for use in the body of this paper, we also introduce

$$H^{(r)}_{N=1}(\rho,S,\epsilon_1)=\sum_{s=0}^{\infty}\epsilon_1^{2s-2}H^{(r),\{0\}}_{(s,0)}(\rho,S).\tag{C3}$$

2. Expansion of $W(\rho,S,\epsilon_1,\epsilon_2)$

In [22,46] the (quasi-)Jacobi form

$$W(\rho,S,\epsilon_{1,2})=\frac{\theta_1(\rho,S+\epsilon_+)\theta_1(\rho,S-\epsilon_+)-\theta_1(\rho,S+\epsilon_-)\theta_1(\rho,S-\epsilon_-)}{\theta_1(\rho,\epsilon_1)\theta_2(\rho,\epsilon_2)},\quad \text{with}\quad \epsilon_\pm=\frac{\epsilon_1\pm\epsilon_2}{2},\tag{C4}$$

was introduced, which governs the BPS counting of a single M5-brane with an M2-brane ending on it on either side. In the NS limit, expanding the latter in powers of ϵ_1, we define

TABLE XVIII. Coefficients in the expansion of $\mathfrak{b}^{(r=2)}_{i,(2s,0)}(\rho,S)$.

s	$\mathfrak{b}^{(r=2)}_{0,(2s,0)}$	$\mathfrak{b}^{(r=2)}_{1,(2s,0)}$	$\mathfrak{b}^{(r=2)}_{2,(2s,0)}$
0	0	$-\frac{1}{16}$	0
1	$\frac{1}{1536}$	$-\frac{E_2}{384}$	$\frac{5E_4}{384}$
2	$\frac{E_2}{36864}$	$-\frac{5E_2^2+27E_4}{92160}$	$\frac{5(E_2E_4+2E_6)}{9216}$
3	$\frac{5E_2^2+13E_4}{8847360}$	$-\frac{35E_2^3+567E_2E_4+1066E_6}{46448640}$	$\frac{5E_2^2E_4+20E_2E_6+53E_4^2}{442368}$
4	$\frac{70E_2^3+546E_2E_4+1067E_6}{8918138880}$	$-\frac{175E_2^4+5670E_2^2E_4+21320E_2E_6+54303E_4^2}{22295347200}$	$\frac{70E_2^3E_4+420E_2^2E_6+2226E_2E_4^2+5393E_4E_6}{445906944}$

TABLE XIX. Coefficients in the expansion of $\mathfrak{b}^{(r=3)}_{i,(2s,0)}(\rho,S)$.

s	$\mathfrak{b}^{(r=3)}_{0,(2s,0)}$	$\mathfrak{b}^{(r=3)}_{1,(2s,0)}$	$\mathfrak{b}^{(r=3)}_{2,(2s,0)}$	$\mathfrak{b}^{(r=3)}_{3,(2s,0)}$
0	0	$-\frac{1}{432}$	0	$-\frac{E_4}{36}$
1	$\frac{1}{41472}$	$-\frac{E_2}{6912}$	$\frac{E_4}{384}$	$-\frac{9E_2E_4+32E_6}{5184}$
2	$\frac{E_2}{663552}$	$-\frac{15E_2^2+151E_4}{3317760}$	$\frac{27E_2E_4+88E_6}{165888}$	$-\frac{45E_2^2E_4+320E_2E_6+1333E_4^2}{829440}$
3	$\frac{5E_2^2+23E_4}{106168320}$	$\frac{-105E_2^3-3171E_2E_4-10088E_6}{1114767360}$	$\frac{405E_2^2E_4+2640E_2E_6+10103E_4^2}{79626240}$	$\frac{-315E_2^3E_4-3360E_2^2E_6-27993E_2E_4^2-103400E_4E_6}{278691840}$

$$W_{\mathrm{NS}}^{(1)}(\rho,S,\epsilon_1)=\lim_{\epsilon_2\to 0}W(\rho,S,\epsilon_{1,2})=\sum_{s=0}^{\infty}\epsilon_1^{2s}W_{(2s)}(\rho,S), \tag{C5}$$

where to low orders in s, we find

$$\begin{aligned} W_{(0)}&=\frac{1}{24}(\phi_{0,1}+2E_2\phi_{-2,1}),\\ W_{(2)}&=-\frac{1}{576}(E_4-E_2^2)\phi_{-2,1},\\ W_{(4)}&=\frac{5(E_4-E_2^2)\phi_{0,1}+2(5E_2^3+3E_2E_4-8E_6)\phi_{-2,1}}{552960},\\ W_{(6)}&=\frac{\phi_{-2,1}(35E_2^4+168E_2^2E_4+16E_2E_6-219E_4^2)-7\phi_{0,1}(5E_2^3+3E_2E_4-8E_6)}{278691840}. \end{aligned} \tag{C6}$$

While not a function of R, following the free energy for $N=1$ discussed in the previous Appendix C 1, we can define an extension of $W_{(2s)}$ to higher orders through

$$W_{(2s)}^{(r)}(\rho,S)=\mathcal{H}_r(W_{(2s)}(\rho,S)), \tag{C7}$$

along with the building block

$$W_{\mathrm{NS}}^{(r)}(\rho,S,\epsilon_1)=\sum_{s=0}^{\infty}\epsilon_1^{2s}W_{(2s)}^{(r)}(\rho,S). \tag{C8}$$

For convenience we can give explicit expressions for the first few instances of $W_{(2s)}^{(r)}$. To this end, we introduce the decomposition

$$W_{(2s)}^{(r)}(\rho,S)=\sum_{i=0}^{r}\mathfrak{l}_{i,(2s)}^{(r)}(\rho)(\phi_{-2,1}(\rho,S))^i(\phi_{0,1}(\rho,S))^{r-i}, \tag{C9}$$

where $\mathfrak{l}_{i,(2s)}^{(r)}$ is a quasimodular form of weight $2s+2i$, which can be written as a polynomial in the Eisenstein series. For $r=1$, the expression (C6) can be tabulated as

s	$\mathfrak{l}_{0,(2s)}^{(r=1)}$	$\mathfrak{l}_{1,(2s)}^{(r=1)}$
0	$\frac{1}{24}$	$\frac{E_2}{12}$
1	0	$-\frac{E_4-E_2^2}{576}$
2	$\frac{E_4-E_2^2}{110592}$	$\frac{5E_2^3+3E_2E_4-8E_6}{276480}$
3	$\frac{-5E_2^3-3E_2E_4+8E_6}{39813120}$	$\frac{35E_2^4+168E_2^2E_4+16E_2E_6-219E_4^2}{278691840}$
4	$\frac{-35E_2^4-126E_2^2E_4-16E_2E_6+177E_4^2}{35672555520}$	$\frac{175E_2^5+2030E_2^3E_4+2000E_2^2E_6+1203E_2E_4^2-5408E_4E_6}{267544166400}$

For $r=2$ we obtain

s	$\mathfrak{l}_{0,(2s)}^{(r=2)}$	$\mathfrak{l}_{1,(2s)}^{(r=2)}$	$\mathfrak{l}_{2,(2s)}^{(r=2)}$
0	$\frac{1}{384}$	$\frac{E_2}{96}$	$\frac{E_4}{96}$
1	0	$\frac{E_2^2-E_4}{2304}$	$\frac{E_6-E_2E_4}{576}$
2	$\frac{E_4-E_2^2}{442368}$	$\frac{5E_2^3+17E_2E_4-22E_6}{552960}$	$\frac{-3E_2^2E_4-4E_2E_6+7E_4^2}{36864}$
3	$\frac{-5E_2^3-12E_2E_4+17E_6}{79626240}$	$\frac{35E_2^4+462E_2^2E_4+604E_2E_6-1101E_4^2}{278691840}$	$\frac{-7E_2^3E_4-24E_2^2E_6-48E_2E_4^2+79E_4E_6}{3981312}$

and for $r=3$ we find

s	$\mathfrak{l}^{(r=3)}_{0,(2s)}$	$\mathfrak{l}^{(r=3)}_{1,(2s)}$	$\mathfrak{l}^{(r=3)}_{2,(2s)}$	$\mathfrak{l}^{(r=3)}_{3,(2s)}$
0	$\frac{1}{10368}$	$\frac{E_2}{1728}$	$\frac{E_4}{864}$	$\frac{9E_2E_4-8E_6}{1296}$
1	0	$\frac{E_2^2-E_4}{27648}$	$\frac{E_6-E_2E_4}{1728}$	$\frac{3E_2^2E_4+8E_2E_6-11E_4^2}{6912}$
2	$\frac{E_4-E_2^2}{5308416}$	$\frac{15E_2^3+121E_2E_4-136E_6}{13271040}$	$\frac{-51E_2^2E_4-128E_2E_6+179E_4^2}{1327104}$	$\frac{45E_2^3E_4+280E_2^2E_6+1003E_2E_4^2-1328E_4E_6}{3317760}$

APPENDIX D: THE PARAMODULAR GROUP Σ_N^*

Let $N \in \mathbb{N}$ with $N > 1$. The degree 2 paramodular groups are subgroups of the symplectic group $Sp(4,\mathbb{Q})$ labeled by an integer N and defined as [81,82]

$$\Sigma_N = \left\{ \begin{pmatrix} \star & N\star & \star & \star \\ \star & \star & \star & \star/N \\ \star & N\star & \star & \star \\ N\star & N\star & N\star & \star \end{pmatrix} \in Sp(4,\mathbb{Q}), |\star \in \mathbb{Z} \right\}. \tag{D1}$$

Σ_N has the interesting property that $\Sigma_N \Gamma_N \subset \Gamma_N$, where Γ_N is the lattice $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus N\mathbb{Z}$ in $Sp(4,\mathbb{Q})$ and Σ_N acts through simple matrix multiplication. A very useful review of the degree n paramodular groups was given in [83].

In order to define the action of Σ_N on the free energies discussed in the body of this paper, we introduce the period matrix

$$\Omega = \begin{pmatrix} \rho & S \\ S & R \end{pmatrix} \in \mathbb{H}(2), \tag{D2}$$

where $\mathbb{H}(2)$ is the space of 2×2 matrices with a positive imaginary part. We then define the action of Σ_N by

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Sigma_N \colon \ \Omega \mapsto \Omega' = g \circ \Omega = (A\cdot\Omega + B)\cdot(C\cdot\Omega + D)^{-1}, \tag{D3}$$

where A, B, C, and D are 2×2 matrices.

Following [82,84,85] one can define an extension of Σ_N to a subgroup of $Sp(4,\mathbb{R})$. To this end we introduce

$$h_N = \begin{pmatrix} U_N & 0 \\ 0 & U_N^T \end{pmatrix} \subset Sp(4,\mathbb{R}), \qquad U_N = \frac{1}{\sqrt{N}}\begin{pmatrix} 0 & N \\ 1 & 0 \end{pmatrix} \tag{D4}$$

and define

$$\Sigma_N^* = \Sigma_N \cup \Sigma_N h_N \subset Sp(4,\mathbb{R}). \tag{D5}$$

Notice that h_N in Eq. (D4) acts as

$$h_N \colon \ \Omega \mapsto \Omega' = h_N \circ \Omega = \begin{pmatrix} NR & S \\ S & \frac{\rho}{N} \end{pmatrix}, \tag{D6}$$

which implies the symmetry $f(R,\rho,S) = f(\frac{\rho}{N}, NR, S)$ for paramodular forms.

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