Holographic entanglement entropy for $Lif_4^{(2)} \times S^1 \times S^5$ spacetime with string excitations

Sabyasachi Maulik* and Harvendra Singh†

Theory Division, Saha Institute of Nuclear Physics, 1/AF, Bidhannagar, West Bengal 700064, India and Homi Bhabha National Institute, Anushakti Nagar, Mumbai 400094, India

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The $(F1, D2, D8)$ brane configuration with $Lif_4^{(2)} \times S^1 \times S^5$ geometry is a known Lifshitz vacua supported by massive $B_{\mu\nu}$ field in type IIA theory. This system allows exact IR excitations which couple to massless modes of the fundamental string. Due to these massless modes the solutions have a flow to a dilatonic $Lif_4^{(3)} \times S^1 \times S^5$ vacua in IR. We study the entanglement entropy on the boundary of this spacetime for the strip and the disk subsystems. To our surprise net entropy density of the excitations at first order is found to be independent of the typical size of subsystems. We interpret our results in light of the first law of entanglement thermodynamics.

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I. INTRODUCTION

The gauge-gravity correspondence [1–3] has got a nonrelativistic version where strongly coupled quantum theories at critical points can be studied [4–23]. Some of these quantum systems involve strongly coupled fermions at finite density or it may simply be a gas of ultracold atoms [4,5]. In the studies involving "nonrelativistic" Schrödinger spacetimes the four-dimensional spacetime geometry generally requires a supporting Higgs-like field such as a massive vector field [4,6,10] or a tensor field. The spacetimes possessing a Lifshitz symmetry provide a similar holographic dual description of nonrelativistic quantum theories living on their boundaries [11]; see [23] for a review.

In this work we shall mainly study entanglement entropy of the excitations in asymptotically $Lif_4^{(a=2)} \times S^1 \times S^5$ background. The latter is a Lifshitz vacua in massive type IIA (mIIA) theory [20,21] with the dynamical exponent of time being $a = 2$. The massive type IIA theory [24] is a ten-dimensional maximal supergravity where the antisymmetric tensor field is explicitly massive. The theory also includes a positive cosmological constant related to the mass parameter. Due to this structure the mIIA theory provides a unique setup to study Lifshitz solutions. Particularly the $Lif_4^{(2)} \times S^1 \times S^5$ solution is a background generated by the bound state of $(F1, D2, D8)$ branes [20]

$$ds^2 = L^2\left(-\frac{dt^2}{z^4} + \frac{dx_1^2 + dx_2^2}{z^2} + \frac{dz^2}{z^2} + \frac{dy^2}{q^2} + d\Omega_5^2\right),$$
$$e^{\phi} = g_0, \qquad C_{(3)} = -\frac{1}{g_0}\frac{L^3}{z^4}dt \wedge dx_1 \wedge dx_2,$$
$$B_{(2)} = \frac{L^2}{qz^2}dt \wedge dy. \tag{1}$$

The metric and the form fields have explicit invariance under constant scalings (dilatation); $z \to \lambda z$, $t \to \lambda^2 t$, $x_i \to \lambda x_i$, and $y \to y$. The dynamical exponent of time is 2 here. The background describes a strongly coupled nonrelativistic quantum theory at the UV critical point.[1]

It is worthwhile to study excitations of the $Lif_4^{(2)} \times S^1 \times S^5$ vacua as it immediately provides us a prototype $Lif_4^{(2)}$ background in four dimensions which is holographic dual to three-dimensional Lifshitz theory on its boundary. The excitations would tell us how this Lifshitz theory behaves near its critical point. Particularly we shall study a class of stringlike excitations which themselves form solutions of massive IIA supergravity and explicitly involve the B field [21]. These also induce running of dilaton as well. It is observed that the resulting renormalization group flow in the deep IR can be described simply by ordinary type IIA theory. The reason for this is due to the fact that the contributions of massive stringy modes decouple from the

*sabyasachi.maulik@saha.ac.in
†h.singh@saha.ac.in

[1] Analogous T-dual solutions do also exist in type IIB theory with constant axion flux switched on [14].

low-energy dynamics of the theory in the IR, far away from the UV critical point [21].

In this report we aim to study holographic entanglement entropy (HEE) [25] of the excited Lifshitz subsystems which are either a disk or a strip in a perturbative framework. A critical observation is that for small-sized systems the entanglement entropy density remains constant at first order. That is, the first-order contributions to the entropy density remain independent of the size (ℓ) of the subsystem. This is a peculiarity and quite unlike relativistic CFTs where usually the entropy density (of excitations) is linearly proportional to the typical size of the subsystem [26]. We discover that the resolution lies in the nature of the chemical potential (μ_E) for the Lifshitz system. We gather evidence that suggests that energy density (of excitations) falls off with the size of system as $\propto 1/\ell^2$. Furthermore the $1/\ell^2$ dependence is exactly same as the entanglement temperature behavior in the Lifshitz theory. Notwithstanding these peculiarities, the entropy of excitations consistently follows the first law of entanglement thermodynamics [26,27] up to first order.

In addition, we also carry out a calculation of entanglement entropy at second order for both disk and strip subsystems. Contributions arising at this order bestow an explicit ℓ dependence upon the entropy. We argue how the first law can still be obeyed by modifying our chemical potential (μ_E) and entanglement temperature (T_E). A similar argument was put forward in [28] for asymptotically AdS spacetime.

The unusual symmetry of Lifshitz spacetime makes it a good background to study novel features of entanglement in a nonrelativistic quantum theory at zero temperature [4,5,11]. It is well known that for such systems, e.g., a particle in a one-dimensional box, the momentum of the particle scales with the length as $p \propto \frac{1}{\ell}$ and the energy $\mathcal{E} \propto \frac{1}{\ell^2}$; our calculations of entanglement entropy also support this explicit size dependence of energy, as shown in Eq. (16). We hope our work will help shed some light on holographic treatment of nonrelativistic quantum systems at strong coupling that are often interesting in e.g., condensed matter theory.

The rest of the paper is organized as follows: in Sec. II we review salient features of $Lif_4^{(2)} \times S^1 \times S^5$ vacua with IR excitations in mIIA theory. The holographic entanglement entropy for a disk subsystem is calculated in Sec. III. In Sec. IV we carry out a similar analysis for strip subsystem at first and second orders, and Sec. V contains the conclusion.

II. $Lif_4^{(2)} \times S^1 \times S^5$ VACUA AND EXCITATIONS

The massive type IIA supergravity theory is the only known maximal supergravity in ten dimensions which allows a massive string $B_{\mu\nu}$ field and a mass-dependent cosmological constant [24]. The cosmological constant generates a nontrivial potential term for the dilaton field. The mIIA theory does not admit flat Minkowski solutions. Nonetheless the theory gives rise to well-known Freund-Rubin-type vacua $AdS_4 \times S^6$ [24], the supersymmetric domain- walls or D8-branes [29–33], $(D6, D8)$, $(D4, D6, D8)$ bound states [34,35] and Galilean-AdS geometries [12,13]. In all of these massive tensor field plays a key role. Under the "massive" T duality [30] the D8-branes can be mapped over to the axionic D7-branes of type IIB string theory and vice versa. The B field also plays an important role in obtaining nonrelativistic Lifshitz solutions [20,21]. The latter solutions are of no surprise in mIIA theory, as an observed feature in four-dimensional AdS gravity theories has been that in order to obtain nonrelativistic solutions one needs to include massive (Proca) gauge fields in the gravity theory [4]. Other different situations where massless vector fields can give rise to nonrelativistic vacua involve boosted black Dp-branes compactified along the light cone direction [15,16]. These latter class of solutions are also called hyperscaling (or conformally) Lifshitz vacua [17].

Particularly the $a = 2$ Lifshitz vacua with IR excitations in mIIA theory can be written as [21]

$$ds^2 = L^2\left(-\frac{dt^2}{z^4 h} + \frac{dx_1^2 + dx_2^2}{z^2} + \frac{dz^2}{z^2} + \frac{dy^2}{q^2 h} + d\Omega_5^2\right),$$
$$e^{\phi} = g_0 h^{-1/2}, \qquad C_{(3)} = -\frac{1}{g_0}\frac{L^3}{z^4} dt \wedge dx_1 \wedge dx_2,$$
$$B_{(2)} = \frac{L^2}{q z^2} h^{-1} dt \wedge dy, \tag{2}$$

where the harmonic function $h(z) = 1 + \frac{z^2}{z_I^2}$. The parameter z_I is related to the charge of the NS-NS strings. The excitations involve g_{tt} and g_{yy} metric components, leaving the x_1, x_2 plane (world volume directions of D2-branes) unaffected.[2] The excitations do also induce a running of dilaton field. The B_{ty} component of the string field is also coupled to the excitations. Since $h \sim 1$ as $z \to 0$, these excitations form normalizable modes (z_I would correspond to adding relevant operators in the boundary Lifshitz theory). The solution (2) asymptotically flows to a weakly coupled regime in the UV (note that the string coupling $g_0 < 1$). While in the deep IR region, with

[2] Here $L = \frac{2}{g_0 m l_s}$, m being the mass parameter in the mIIA action. (We would set $l_s = 1$ and $g_0 = 1$.) The constant q is a free (length) parameter and g_0 is weak string coupling. Note L is a dimensionless parameter; it determines overall radius of curvature of the spacetime. Therefore Romans' theory with $m \ll \frac{2}{g_0 l_s}$ would be preferred here so that $L \gg 1$ in the solutions (2); else these classical vacua cannot be trusted. Also, from the D8-brane and domain-wall correspondence in [30], one typically expects $m \approx \frac{g_0 N_{D8}}{l_s}$, a value which is definitely well within $\frac{2}{g_0 l_s}$ for a finite number of D8-branes, N_{D8}, in these backgrounds.

$z \gg z_I$ where $h \approx \frac{z^2}{z_I^2}$, the vacua is driven to another weakly coupled Lifshitz regime. For $z \gg z_I$, the IR geometry transforms to a dilatonic $Lif_4^{(3)} \times S^1 \times S^5$ solution. This solution enables us to study the effect of the excitations in $a = 2$ Lifshitz theory. Note the z_I-dependent excitations at zero temperature are mainly in the form of charge excitations, along with nontrivial entanglement chemical potential, as we would see next.

III. ENTANGLEMENT OF A DISK SUBSYSTEM

For asymptotically AdS spacetime dual to a CFT, the entanglement entropy can be calculated by the Ryu-Takayanagi formula [25]. We assume the same is true for an asymptotically Lifshitz spacetime, dual to a nonrelativistic field theory with Lifshitz scaling symmetry. We consider a round disk of radius ℓ at the center of the x_1, x_2 plane with its boundary identified with the corresponding boundary of $2d$ Ryu-Takayanagi surface lying inside the Lifshitz bulk geometry (2). We shall assume y is a compactified direction:

$$y \sim y + 2\pi r_y. \tag{3}$$

In radial coordinates $(r = \sqrt{x_1^2 + x_2^2})$ the Ryu-Takayanagi area functional [25] for static bulk surface is given by

$$\mathcal{A}_\gamma = 2\pi L^2 \int_\epsilon^{z_*} dz \frac{r\sqrt{1+r'^2}}{z^2} h^{1/2}, \tag{4}$$

where $r' = \frac{dr}{dz}$, $h(z) = (1 + \frac{z^2}{z_I^2})$ and $\epsilon \ll \ell$ is UV cutoff of the Lifshitz theory. We need to extremize the area integral by solving the Euler-Lagrange equation for $r(z)$:

$$\begin{aligned} &2zrr''h(z) - 4rr'^3h(z) - 4rr'h(z) - 2zr'^2h(z) - 2zh(z) \\ &\quad - zrr'^3h'(z) - zrr'h'(z) = 0. \end{aligned} \tag{5}$$

It is impossible to analytically calculate the full area integral (4). To facilitate our job, therefore, we restrict ourselves to small subsystems, with $\ell \ll z_I$. In this domain, we can make a perturbative expansion and obtain solutions order by order in the dimensionless ratio $\frac{\ell}{z_I}$, such that $r(z) = r_{(0)} + r_{(1)} + \cdots$, and correspondingly we would write

$$\mathcal{A}_\gamma = \mathcal{A}_0 + \mathcal{A}_1 + \cdots,$$

for small ℓ. Our immediate interest is in calculating terms up to leading order and first order only in the $\frac{\ell}{z_I}$ expansion.

The equation at zeroth order is

$$zr_{(0)}r''_{(0)} - 2r_{(0)}r'^3_{(0)} - 2r_{(0)}r'_{(0)} - zr'^2_{(0)} - z = 0, \tag{6}$$

for which $r_{(0)} = \sqrt{\ell^2 - z^2}$ defines the extremal surface (half circle) [25,36] with the boundary conditions $r_{(0)}(0) = \ell$ and $r_{(0)}(z_*) = 0$, where $z = z_*$ is the point of return that lies at $z_* = \ell$. One then finds that the area

$$\mathcal{A}_0 = 2\pi L^2 \int_\epsilon^{z_*} dz \frac{r_{(0)}\sqrt{1+r'^2_{(0)}}}{z^2} = 2\pi L^2\left(\frac{\ell}{\epsilon} - 1\right). \tag{7}$$

$\mathcal{A}_0$ being a ground state contribution it obviously remains independent of the parameter z_I of the bulk geometry. This only means that there is no effect of excitations on the leading term. As explained in [36], the first-order contribution can be evaluated using only the tree-level embedding function and is given by

$$\mathcal{A}_1 = 2\pi L^2 \int_\epsilon^{z_*} dz r_{(0)} \frac{\sqrt{1+r'^2_{(0)}}}{2z_I^2} = \pi L^2\left(\frac{\ell^2}{z_I^2}\right). \tag{8}$$

From here the complete expression of entanglement entropy of a disk-shaped subsystem up to first order becomes

$$S_E^{\rm Disk}[\ell, z_I] \equiv \frac{\mathcal{A}_\gamma}{4G_4} = S_E^{(0)} + \frac{\pi L^2}{4G_4}\left(\frac{\ell^2}{z_I^2}\right), \tag{9}$$

where the Newton's constant in 4D and 5D are related to the ten-dimensional Newton's constant by $\frac{1}{G_4} = \frac{L 2\pi r_y}{G_5}$ and $\frac{1}{G_5} \equiv \frac{L^5 {\rm Vol}(S^5)}{G_{10}}$. We shall be using G_4 and G_5 back and forth in our calculation.

The ground state entropy contribution is

$$S_E^{(0)} = \frac{\pi L^2}{2G_4}\left(\frac{\ell}{\epsilon} - 1\right). \tag{10}$$

Equation (9) is a meaningful expression for entanglement entropy only if we maintain $\ell \ll z_I$. The first-order term explicitly depends on z_I, so small fluctuations of the bulk quantities, like δz_I, would result in a corresponding change in entropy. For a fixed size ℓ, one could express these variations of the entropy density as

$$\delta s_E^{\rm Disk} = \frac{\delta S_E^{\rm Disk}}{\pi \ell^2} = \frac{L^2}{4G_4}\delta\left(\frac{1}{z_I^2}\right), \tag{11}$$

where $\pi\ell^2$ is the disk area. Equation (11) provides a complete expression up to first order. At second order the entropy will receive new z_I-dependent contributions.

Next, we note that the right-hand side of Eq. (11) is actually independent of the disk size ℓ. On first-hand observation this appears very surprising because, according to the first law of entanglement thermodynamics [26], we expected that the entropy density of excitations would have had ℓ^2 dependence, namely in the form of inverse temperature (usually entanglement temperature goes as $T_E^{-1} \propto \ell^a$; and the dynamical exponent of time in our Lifshitz background is $a = 2$). Especially this aspect of the first law has been found to remain true in a variety of relativistic CFTs, where entanglement temperature is given by $T_E \propto \frac{1}{\pi\ell}$; see for example [22,26–28,37–39]. What, then, is so different for the Lifshitz system described by Eq. (11)? To understand this phenomenon we first need to get an estimate of the energy associated with the excitations in our system.

A. Energy, winding charge and chemical potential

We now turn to find the energy of excitations of the massive strings due to which we have a configuration in Eq. (2), where we can express $B_{ty} \simeq B_{ty}^{\text{massive}} + B_{ty}^{\text{excitation}}$. Note that we are treating y as a compact direction. The Scherk-Schwarz compactification [40,41] of the Lifshitz background (2) on a circle along y gives rise to the following 1-form potential:

$$A_{(1)} = \frac{L^2}{qz^2}\left(1 + \frac{z^2}{z_I^2}\right)^{-1} dt. \tag{12}$$

It represents a gauge field in the lower-dimensional supergravity whose only nonzero component is A_t. It can be determined from here that due to string excitations the net change in the U(1) charge (due to winding strings) is

$$\triangle\rho = \frac{N}{V_2} = \frac{\triangle Q}{2\pi r_y V_2} = \frac{2L}{G_5 z_I^2}, \tag{13}$$

where V_2 is the area element of the x_1, x_2 plane; see a calculation in the Appendix. The entanglement chemical potential, with the prescription in [28], can be obtained by measuring the gauge field at the turning point, namely

$$\mu_E \equiv A_t|_{z=z_*} = \frac{L^2 r_y}{qz_*^2} + \cdots, \tag{14}$$

where ellipses denote subleading terms which are not required at first order. This is a logical guess inspired by black hole thermodynamics, where the value of the 1-form at the black hole horizon is known to give the chemical potential conjugate to the U(1) charge. Even for backgrounds with nonrelativistic conformal symmetry as considered in [9], the Kaluza-Klein gauge field measured at the horizon produces the correct thermal chemical potential. There is no horizon in our bulk spacetime; instead, we use the critical point z_* associated with the entanglement wedge.

At leading order we have $z_* \simeq \ell$; hence, essentially this thermodynamic variable gets uniquely fixed by the Lifshitz ground state (1). So for small $\ell(>0)$ the chemical potential remains quite important, and we obtain

$$\mu_E \cdot \triangle\rho \simeq \frac{L^2}{\pi G_4}\frac{1}{z_I^2\ell^2}. \tag{15}$$

There are no other excitations except the winding strings; the energy density due to the excitations can be estimated to be

$$\triangle\mathcal{E} = \mathcal{E} - \mathcal{E}_0 \simeq \frac{1}{2}\mu_E\triangle\rho = \frac{L^3 r_y}{qG_5}\frac{1}{z_I^2\ell^2} = \frac{L^2}{2\pi G_4}\frac{1}{z_I^2\ell^2}, \tag{16}$$

where $\mathcal{E}_0$ is the (normalized) energy of the ground state of our Lifshitz theory.[3] This is the only meaningful deduction we can make from here, particularly in the absence of a direct method to evaluate full stress-energy tensor of the Lifshitz theory.[4] Assuming that the entanglement temperature of the three-dimensional $a = 2$ Lifshitz system faithfully behaves as [26]

$$T_E = \frac{4}{\pi\ell^2}, \tag{17}$$

we determine that the ratio

$$\frac{\mu_E}{T_E} = \frac{\pi L^2 r_y}{4q}$$

is indeed independent of ℓ. Essentially this ratio seems to get uniquely fixed by the Lifshitz ground state (1) at the leading order. Note that the excitations seem to have no effect on it. The analysis also implies that the energy density and the entanglement temperature both fall off with the system size ℓ at the same rate, and the ratio

$$\frac{\triangle\mathcal{E}}{T_E} = \frac{\pi L^3 r_y}{4qG_5 z_I^2} \equiv \frac{1}{2}\frac{k_E N}{V_2} \tag{18}$$

stays fixed for small disks. However this ratio does depend on the excitations namely through z_I. In the second equality we have preferred to view dimensionless quantity $k_E = \frac{\pi L^2 r_y}{8q}$ as being analogous to the Boltzmann constant in usual thermodynamics. (For example, we could have expressed total energy of the disk as $\triangle E = \frac{1}{2}Nk_E T_E$ without affecting anything.) *Hence it can be concluded that the entanglement entropy per unit disk area is fixed for small disks of radii*

[3]We do notice an explicit dependence of energy density on the system size, which is unlike relativistic CFT but is a familiar feature in nonrelativistic theories, the particle in a box being an immediate example.

[4]There is an early work [42] but it does not include dilatonic scalar field excitations like in our background. In contrast in asymptotically AdS spacetimes one knows how to obtain the stress-energy tensor by doing Fefferman-Graham expansion near the AdS boundary [43]. Perhaps something similar could also be done in the Lifshitz case involving a dilaton field.

$\ell \ll z_I$. It is also confirmed that the entropy of excitations (11) follows the first law relation [26–28,37–39,44–46]

$$\delta s_E = \frac{1}{T_E}\left(\delta\Delta\mathcal{E} + \frac{1}{2}\mu_E\delta\Delta\rho\right), \tag{19}$$

under infinitesimal changes in the bulk quantity, δz_I.

We summarize our main observations at first order:

$$T_E \propto \frac{1}{\ell^2}, \qquad \triangle s_E = \text{fixed}, \qquad \mu_E \propto r_y T_E,$$
$$\triangle\mathcal{E} \propto N T_E, \qquad \Delta\rho = \text{fixed}, \tag{20}$$

at a given entanglement temperature.

B. Entanglement entropy of a disk at second order

Let us now consider corrections to holographic entanglement entropy at next higher order. It is somewhat easier to calculate when one chooses $z(r)$ parameterization, so let us rewrite the integral as

$$\mathcal{A}_\gamma = 2\pi L^2 \int_0^1 dr \frac{r\sqrt{1+z'^2}}{z^2} h^{1/2}, \tag{21}$$

where we rescaled r and z to the dimensionless variables $\frac{r}{\ell}$ and $\frac{z}{\ell}$. It suffices to obtain the embedding up to first order to get the entanglement at second order [36,39]. So, we expand $z(r)$ as $z(r) = z_{(0)} + z_{(1)} + \cdots$, where $z_{(0)} = \sqrt{1-r^2}$ and $z_{(1)}$ satisfies the equation

$$z''_{(1)} + \frac{1-2r^2}{r(1-r^2)} z'_{(1)} - \frac{2}{(1-r^2)^2} z_{(1)} = \frac{1}{\sqrt{1-r^2}}, \tag{22}$$

with the boundary conditions $z'_{(1)}(0) = 0$ and $z_{(1)}(\ell) = 0$. One can check that a consistent solution to Eq. (22) is

$$z_{(1)} = -\frac{1 - r^2 - 2\sqrt{1-r^2} + 2\ln(1+\sqrt{1-r^2})}{2\sqrt{1-r^2}}. \tag{23}$$

Therefore, the area integral now acquires a new contribution $\mathcal{A}_\gamma = \mathcal{A}_0 + \mathcal{A}_1 + \mathcal{A}_2$, where

$$\mathcal{A}_2 = 2\pi L^2 \frac{\ell^4}{z_I^4}\left(\frac{5}{8} - \ln 2\right), \tag{24}$$

which is negative as expected. The area difference from pure AdS at both orders is plotted in Fig. 1. Total entropy of the disk at this order will be

$$S_E^{(2)} = S_E^{(0)} + \frac{\pi L^2}{4G_4}\frac{\ell^2}{z_I^2}\left(1 + \frac{\ell^2}{z_I^2}\left(\frac{5}{4} - 2\ln 2\right)\right). \tag{25}$$

So the variation of entropy density, at second order, becomes

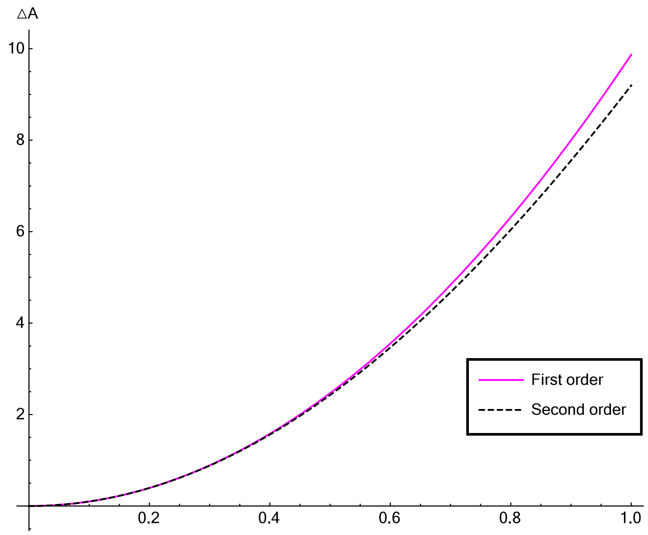


FIG. 1. Area difference from AdS ground state for spherical subsystem; the second-order correction is negative. Plot drawn by choosing $z_I^2 = 2$ and $L = r_y = q = 1$.

$$\delta s_E^{(2)} = \frac{L^2}{4G_4}\left(1 + \frac{\ell^2}{z_I^2}\left(\frac{5}{2} - 4\ln 2\right)\right)\delta(z_I^{-2}). \tag{26}$$

As previous, we wish to express (26) as a "first law"–like relationship. We find that one way to achieve this is to absorb all second-order corrections to a modified temperature and chemical potential; this method was first used in [28] although they worked with differences rather than variation as we do. To this end, we first note that the turning point z_* should be corrected at $\mathcal{O}(\frac{\ell^2}{z_I^2})$ as

$$z_* \equiv z(0) = \ell + \frac{\ell^3}{z_I^2}\left(\frac{1}{2} - \ln 2\right).$$

The chemical potential, defined in Eq. (14), can be expressed including $\mathcal{O}(\frac{\ell^2}{z_I^2})$ corrections as

$$\mu_E^{(1)} \simeq \frac{L^2 r_y}{q\ell^2}\left(1 + \frac{\ell^2}{z_I^2}\left(\frac{1}{2} - \ln 2\right)\right)^{-2}\left(1 + \frac{\ell^2}{z_I^2}\right)^{-1}$$
$$= \frac{L^2 r_y}{q\ell^2}\left(1 - \frac{\ell^2}{z_I^2}(2 - 2\ln 2)\right). \tag{27}$$

So we get

$$\mu_E^{(1)}\delta\Delta\rho = \frac{2L^3 r_y}{qG_5\ell^2}\left(1 - \frac{\ell^2}{z_I^2}(2 - 2\ln 2)\right)\delta(z_I^{-2})$$
$$= \frac{L^2}{\pi G_4 \ell^2}\left(1 - \frac{\ell^2}{z_I^2}(2 - 2\ln 2)\right)\delta(z_I^{-2}),$$

while the energy remains the same as defined in (16). From Eq. (26), a bit of paperwork then leads to the following result:

$$\delta s_E^{(2)} = \frac{1}{T_E^{(2)}} \left(\delta\Delta\mathcal{E} + \frac{1}{2}\mu_E^{(1)} \delta\Delta\rho \right), \quad (28)$$

where $T_E^{(2)}$ denotes the "entanglement temperature" at second order, which is given by

$$\begin{aligned} T_E^{(2)} &= \frac{\delta\Delta\mathcal{E} + \frac{1}{2}\mu_E^{(1)}\delta\Delta\rho}{\delta\Delta s_E^{(2)}} \\ &= \frac{\frac{L^2}{\pi G_4 \ell^2}[1 - \frac{\ell^2}{z_I^2}(1 - \ln 2)]}{\frac{L^2}{4G_4}[1 - \frac{\ell^2}{z_I^2}(4\ln 2 - \frac{5}{2})]} \\ &\simeq T_E^{(1)} \left[1 + \frac{\ell^2}{z_I^2} \left(5\ln 2 - \frac{7}{2} \right) \right], \end{aligned} \quad (29)$$

where $T_E^{(1)}$ stands for the first-order temperature, defined in Eq. (17). The term in parentheses is a negative number, so second-order correction to entanglement temperature results in its sharper fall. See Fig. 2 for an illustration of this behavior.

Some comments are in order to justify Eq. (28). We have seen that, for small enough subsystem size ($\ell \ll z_I$), the change in entanglement entropy at first order in our perturbative calculation follows a relationship akin to the first law of thermodynamics. If one considers this relationship an actual "law" for entanglement entropy, one must find a consistent way to describe new contributions at higher orders. Equation (29) proposes that at second order the chemical potential as well as the entanglement temperature should be corrected to keep the law intact. In fact, we expect this procedure to work at all higher orders. It could be thought that a more accurate measure of these quantities is obtained as one climbs the perturbation ladder.

IV. ENTANGLEMENT ENTROPY OF NARROW STRIP

We now consider a striplike subsystem with coordinate width $-\ell/2 \le x_1 \le \ell/2$ and the range of $x_2 \in [0, l_2]$, such that $l_2 \gg \ell$. The straight line boundary of the two-dimensional strip is identified with the boundary of the RT surface in the bulk at constant time. The area functional of this static surface is

$$\mathcal{A}_\gamma = 2L^2 l_2 \int_\epsilon^{z_*} dz \frac{\sqrt{1 + x_1'^2}}{z^2} h^{1/2}. \quad (30)$$

For small width $\ell \ll z_I$, we make a perturbative expansion of the integrand. The extremal surface satisfies the following equation:

$$x_1' = \frac{z^2}{z_*^2} \frac{1}{\sqrt{\frac{h}{h_*} - \frac{z^4}{z_*^4}}}, \quad (31)$$

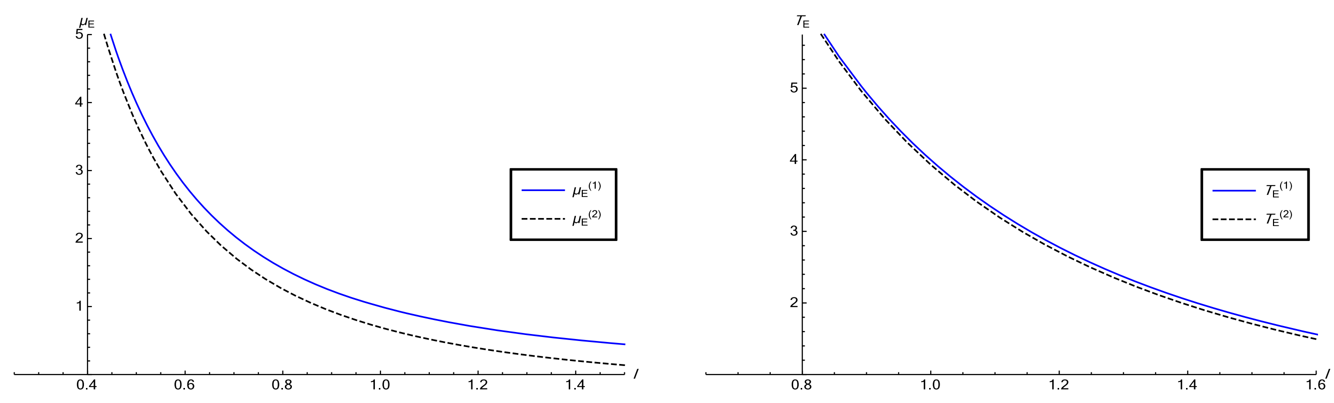
where $h_* \equiv h(z_*)$. We have specific boundary conditions such that near the spacetime boundary $x_1|_{z=0} = \ell/2$ and the turning point is given by $x_1|_{z\sim z_*} = 0$. This leads to the first integral of the following type:

$$\ell = 2\int_0^{z_*} dz \frac{z^2}{z_*^2} \frac{1}{\sqrt{\frac{h}{h_*} - \frac{z^4}{z_*^4}}}, \quad (32)$$

which gives rise to a perturbative expansion in $\frac{z_*}{z_I}$:

$$\ell = z_* \left(b_0 + \frac{z_*^2}{2z_I^2} I_1 + \cdots \right), \quad (33)$$

where coefficients are expressible as Beta functions $b_0 = \frac{1}{4}B(\frac{3}{4}, \frac{1}{2})$ and $I_1 = \frac{1}{4}(B(\frac{3}{4}, -\frac{1}{2}) - B(\frac{5}{4}, -\frac{1}{2}))$. Equation (33)



(a) μ_E vs l (b) T_E vs l

FIG. 2. The unbroken and dashed curves display the behavior of the uncorrected and corrected quantities, respectively; both the entanglement temperature and chemical potential decrease due to higher-order corrections. The plots were drawn by setting $z_I^2 = 2$ and $L = r_y = q = 1$.

can be inverted and expressed as a perturbative expansion of the turning point

$$z_* = z_*^{(0)}\left(1 - \frac{z_*^{(0)2}}{z_I^2}\frac{I_1}{2b_0} + \cdots\right), \tag{34}$$

where $z_*^{(0)} \equiv \frac{\ell}{2b_0}$ is the turning point in the absence of excitations.

The leading area of strip can be evaluated using the tree-level values

$$\begin{aligned}\mathcal{A}_0 &= 2L^2 l_2 \int_\epsilon^{z_*^{(0)}} dz \frac{\sqrt{1+x'^2_{1(0)}}}{z^2} \\ &= \frac{2L^2 l_2}{z_*^{(0)}}\int_{\epsilon/z_*^{(0)}}^1 d\zeta \frac{1}{\zeta^2\sqrt{1-\zeta^4}} \\ &= 2L^2 l_2\left(\frac{1}{\epsilon} - \frac{2(b_0)^2}{\ell}\right),\end{aligned} \tag{35}$$

while the first-order contribution is evaluated as

$$\mathcal{A}_1 = 2L^2 l_2 \int_0^{z_*} dz \frac{\sqrt{1+x'^2_{1(0)}}}{2z_I^2} = L^2 l_2\left(\frac{a_1 z_*^{(0)}}{z_I^2}\right), \tag{36}$$

where the coefficient $a_1 = \frac{1}{4}B(\frac{1}{4}, \frac{1}{2})$. The entanglement entropy of a small strip up to first order is then given by

$$S_E^{\text{strip}} = \frac{\mathcal{A}_0 + \mathcal{A}_1}{4G_5} = \frac{L^2 l_2}{2G_4}\left(\frac{1}{\epsilon} - \frac{2b_0^2}{\ell} + \frac{a_1}{4b_0}\frac{\ell}{z_I^2}\right). \tag{37}$$

Now any small change in the bulk parameter (δz_I) will necessarily affect the entanglement entropy at first order. For a fixed width ℓ, we find the change in entropy per unit area of the strip as

$$\delta s_E^{\text{strip}} \equiv \frac{\delta S_E^{\text{strip}}}{l_2 \ell} = \frac{L^2}{8G_4}\frac{a_1}{b_0}\delta(z_I^{-2}), \tag{38}$$

which is a complete expression up to first order. Once again we find that the right-hand side is independent of ℓ, as it was also in the case of a disk. Following from the disk case in the previous section, the effective chemical potential for the strip subregion is

$$\mu_E = \frac{L^2 r_y}{q z_*^2} \simeq \frac{4b_0^2 L^2 r_y}{q\ell^2}. \tag{39}$$

From here and (13), let us define for the strip

$$\triangle\mathcal{E} \equiv \frac{1}{2}\mu_E . \triangle\rho = \frac{4L^3 r_y}{G_5 q}\frac{b_0^2}{z_I^2\ell^2} = \frac{2L^2}{\pi G_4}\frac{b_0^2}{z_I^2\ell^2}. \tag{40}$$

This is like the disk result in (16), i.e., $\triangle\mathcal{E} \propto T_E$. Using (40) we conclude that the entanglement entropy density (38) of the strip subsystems also conforms to the first law relation

$$\delta s_E = \frac{1}{T_E}\left(\delta\Delta\mathcal{E} + \frac{1}{2}\mu_E \delta\Delta\rho\right), \tag{41}$$

where, for the strip, entanglement temperature is defined as $T_E = \frac{8b_0^3}{a_1}\frac{4}{\pi\ell^2}$ in three-dimensional Lifshitz theory.

A. Strip entropy at second order

It is instructive to find out the change in entanglement entropy at higher orders in $\frac{\ell^2}{z_I^2}$ and interpret its thermodynamic property; here we include the results at $\mathcal{O}(\frac{\ell^4}{z_I^4})$.

The turning point z_*, as discussed before in (32) and (33), could be related to the strip width ℓ as

$$z_* = \frac{z_*^{(0)}}{1 + \frac{z_*^{(0)2}}{2z_I^2}\frac{I_1}{b_0} - \frac{z_*^{(0)4}}{8z_I^4}\left(\frac{I_2}{b_0} + \frac{4I_1^2}{b_0^2}\right)}, \tag{42}$$

where the new coefficient I_2 can be expressed as $I_2 = \frac{1}{8}(2B(\frac{3}{4}, -\frac{3}{2}) - 3B(\frac{5}{4}, -\frac{3}{2}))$. With the help of (42), the area integral (30) now reads $\mathcal{A}_\gamma = \mathcal{A}_0 + \mathcal{A}_1 + \mathcal{A}_2$, where $\mathcal{A}_0$ and $\mathcal{A}_1$ are as obtained before. The second-order contribution is

$$\mathcal{A}_2 = -\frac{2L^2 l_2}{z_*^{(0)}}\frac{z_*^{(0)4}}{8z_I^4}\left(\frac{4a_0 I_1^2}{b_0^2} + \frac{2I_1 J_1}{b_0}\right). \tag{43}$$

The new coefficients introduced in above expression are listed below:

$$\begin{aligned} a_0 &= -\frac{1}{4}B\left(\frac{3}{4}, \frac{1}{2}\right) = -b_0, \\ J_1 &= \frac{1}{4}\left(B\left(\frac{3}{4}, -\frac{1}{2}\right) + 3B\left(\frac{1}{4}, -\frac{1}{2}\right)\right). \end{aligned}$$

After some simplification the contribution to the area of the RT surface at second order turns out to be

$$\mathcal{A}_2 = -\frac{L^2 l_2 \ell}{64}\frac{\ell^2}{z_I^4}\frac{1}{b_0^2}\left(\frac{a_1^2}{b_0^2} - 1\right). \tag{44}$$

The coefficient a_1 has already been defined in Eq. (36). Hence, the total entanglement entropy density, at second order in perturbation theory, becomes

$$s_E^{(2)} = s_E^{(0)} + \frac{L^2}{8G_4}\frac{1}{z_I^2}\frac{a_1}{b_0}\left(1 - \frac{\ell^2}{z_I^2}\frac{1}{32b_0^2}\left(\frac{a_1^2}{b_0^2} - 1\right)\right). \tag{45}$$

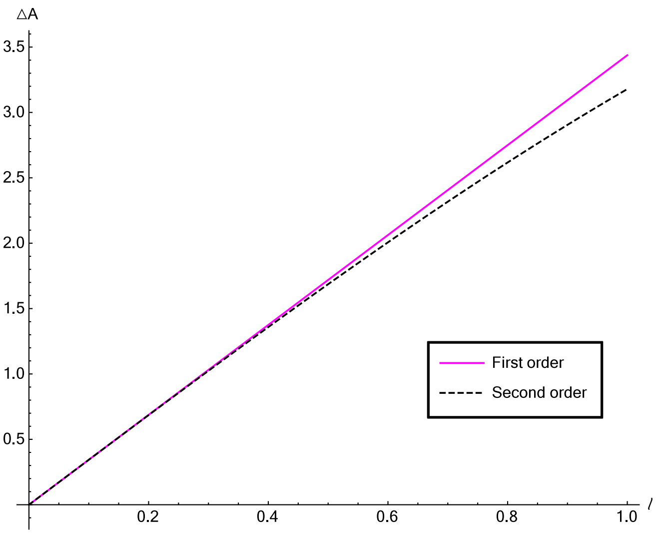


FIG. 3. The area difference at first and second order of perturbation analysis for the strip subsystem, plots drawn by choosing $z_I^2 = 2$ and $L = r_y = q = l_2 = 1$.

The area difference including second-order correction has been shown in Fig. 3. To write down the "first law" we need to rewrite the expression for $s_E^{(2)}$ in terms of variation in $\mathcal{E}$ and $\mu_E \Delta\rho$; recall that the chemical potential was defined as the value of the gauge potential at the turning point. Here, it is sufficient to compute μ_E up to first order:

$$\mu_E^{(1)} \simeq \frac{L^2}{z_*^2}\left(1 - \frac{z_*^2}{z_I^2}\right) = \frac{L^2 r_y}{q z_*^{(0)2}}\left(1 + \frac{z_*^{(0)2}}{z_I^2}\left(\frac{I_1}{b_0^2} - 1\right)\right).$$

So that

$$\mu_E^{(1)}\delta\Delta\rho = \frac{L^3 r_y}{qG_5}\frac{8b_0^2}{\ell^2}\left[1 + \frac{\ell^2}{z_I^2}\frac{1}{8b_0^2}\left(\frac{a_1}{b_0} - 3\right)\right]\delta(z_I^{-2})$$
$$= \frac{L^2}{2\pi G_4}\frac{8b_0^2}{\ell^2}\left[1 + \frac{\ell^2}{z_I^2}\frac{1}{8b_0^2}\left(\frac{a_1}{b_0} - 3\right)\right]\delta(z_I^{-2}).$$

A little effort, then, allows us to write

$$\delta s_E^{(2)} = \frac{1}{T_E^{(2)}}\left(\delta\Delta\mathcal{E} + \frac{1}{2}\mu_E^{(1)}\delta\Delta\rho\right). \tag{46}$$

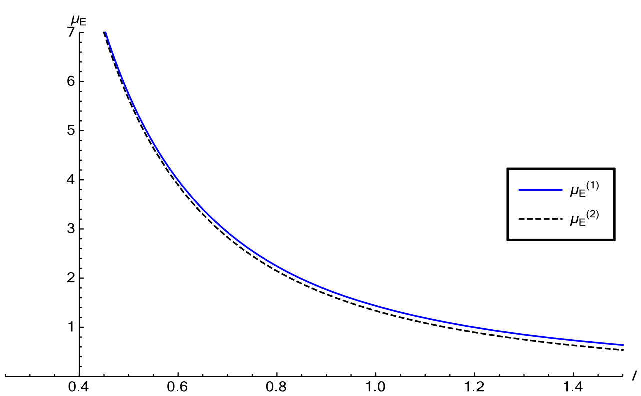
Here, $T_E^{(2)}$ stands for the modified entanglement temperature at second order:

$$T_E^{(2)} = \frac{\delta\Delta\mathcal{E} + \frac{1}{2}\mu_E^{(1)}\delta\Delta\rho}{\delta\Delta s_E^{(2)}}$$
$$= \frac{4}{\pi\ell^2}\frac{8b_0^3}{a_1}\left[1 + \frac{\ell^2}{z_I^2}\frac{1}{16b_0^2}\left(\left(\frac{a_1}{b_0} - 3\right) + \left(\frac{a_1^2}{b_0^2} - 1\right)\right)\right]$$
$$= T_E^{(1)}\left[1 + \frac{\ell^2}{z_I^2}\frac{1}{16b_0^2}\left(\left(\frac{a_1}{b_0} - 1\right)\left(\frac{a_1}{b_0} + 2\right) - 2\right)\right], \tag{47}$$

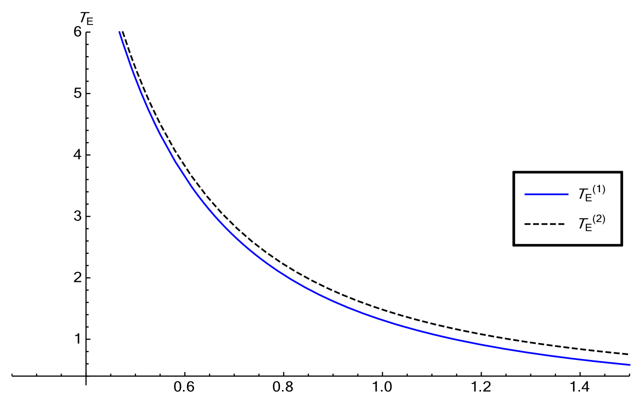
where, by $T_E^{(1)}$, we refer to the temperature at first order defined in Eq. (41), the numerical value of $\frac{a_1}{b_0} \approx 2.188$, so the correction at this order results in an increase of T_E, albeit by a tiny amount. The uncorrected and corrected temperatures are plotted in Fig. 4.

B. Numerical results for strip subsystem

We end this section with a comparison of our perturbative results with some numerical analysis. For the numerical computation we chose $z_I = 4$ and used (32) to obtain corresponding lengths ℓ of the subregion for different choices of the turning point z_*. We also obtain the area



(a) μ_E vs ℓ



(b) T_E vs ℓ

FIG. 4. The unbroken and dashed curves display the behavior of the uncorrected and corrected quantities, respectively; the entanglement temperature is found to increase due to higher-order corrections while the chemical potential decreases. The plots were drawn by setting $z_I = 2$ and $L = r_y = q = G_5 = 1$.

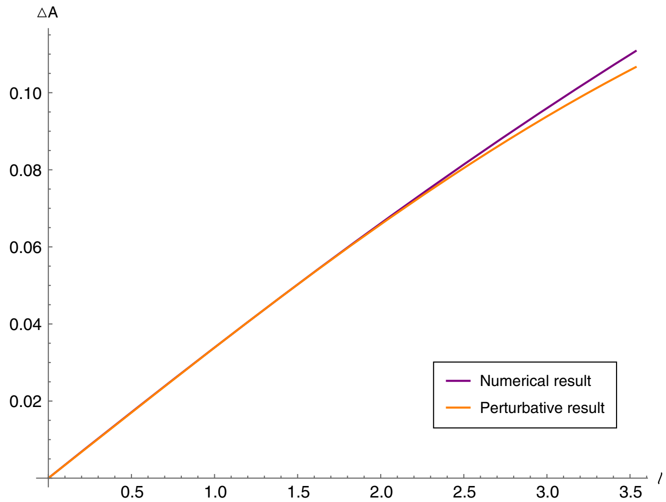


FIG. 5. Numerical plot of area difference from AdS ground state for the strip subsystem and comparison with second-order perturbation series analysis. The prefactor in (30) was ignored in the plot.

difference $\Delta\mathcal{A}$ from (30) for the same z_* values and plot the two sets against each other. The output is summarized in Fig. 5. From the graph we conclude that a second-order perturbation series analysis is trustworthy for small strip width.

V. CONCLUSION

The Lifshitz background $Lif_4^{(2)} \times S^1 \times S^5$ of the massive type IIA theory allows exact excitations which couple to massless modes of string in the IR. We calculated the entanglement entropy of the theory at the boundary of these spacetimes, both for strip as well as disk-shaped systems. At leading order, we found that the entropy density of the excitations remains fixed and does not grow with ℓ, the subsystem size, so long as $\ell \ll z_I$. We find that this behavior is consistent with the fact that energy density of the excitations itself behaves as $\triangle\mathcal{E} \propto 1/\ell^2$, which is in agreement with $\triangle\mathcal{E} \simeq \frac{1}{2}\mu_E \triangle\rho$. Note that the entanglement temperature itself goes as $T_E \propto \frac{1}{\ell^2}$.

But this entanglement behavior is quite different in comparison to the relativistic CFTs, where the entropy density of excitations grows linearly with the subsystem size, while the energy density of excitations remains fixed. Nevertheless we have found that the first law of entanglement thermodynamics

$$\delta s_E = \frac{1}{T_E}\left(\delta\Delta\mathcal{E} + \frac{1}{2}\mu_E \delta\Delta\rho\right) \tag{48}$$

holds good if we accept the hypothesis that the energy of a subsystem in the Lifshitz background (2) is given by

$$\triangle E \simeq \mu_E N \simeq \frac{1}{2} N k_E T_E.$$

Our results appear to indicate an equipartition nature of the entanglement thermodynamics for a nonrelativistic Lifshitz system. But this is perhaps true only for the high entanglement temperature regime (i.e., small $\ell \ll z_I$).

Further, we studied what happens to the first law of entanglement if we assume it to remain valid beyond the leading order. There is lack of consensus on this aspect, despite there being enough evidence for it to be a natural feature at first order. We discussed how the first law could be extended up to second order by making use of an appropriately modified chemical potential and entanglement temperature. We think this is necessary because, otherwise, we need to look for a new quantity at each higher order to account for the corrections; while the entanglement entropy, like its thermal counterpart, should depend only on the energy and conserved charges of the theory. Such redefinition should work at all orders, thereby allowing the first law of entanglement thermodynamics to be obeyed quite generally, irrespective of the degree of perturbation theory.

It would be interesting to obtain the HEE numerically for ball subsystems and compare with our perturbative results. This, however, involves solving the boundary value problem and proves to be nontrivial. Another interesting problem is to consider shape dependence of holographic entanglement entropy in a similar spirit to [47,48]. We hope to return to these problems in the future.

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APPENDIX: THE WINDING STRING CHARGE IN MASSIVE LIFSHITZ VACUA

Here we would like to know the winding number of the string excitations. The circle compactification of the background (2) along the y direction gives rise to the following nine-dimensional fields (we set $g_0 = 1, \alpha' = 1$):

$$ds^2_{D=9} = L^2\left(-\frac{dt^2}{z^4 h} + \frac{dx_1^2 + dx_2^2}{z^2} + \frac{dz^2}{z^2} + d\Omega_5^2\right),$$
$$e^{2\bar{\phi}} = \frac{1}{h\sqrt{G_{yy}}}, \qquad A_t = \frac{L^2}{qz^2}h^{-1}, \tag{A1}$$

where $G_{yy} = \frac{L^2}{q^2 h}$ and $h(z) = 1 + \frac{z^2}{z_I^2}$. The $\bar{\phi}$ is a nine-dimensional dilaton field. The corresponding gauge field strength $F_{(2)} = dA$ gives rise to the winding charge

$$
\begin{aligned}
Q &= \frac{\pi r_y}{G_{10}} \int e^{-4\bar{\phi}/7} G^{yy} ({}_{*9}F_{(2)}) \\
&= \frac{\pi L^6 \omega_5 r_y}{G_{10}} \int dx_1 dx_2 \left(\frac{2}{z^2} + \frac{4}{z_I^2}\right) \\
&= \frac{\pi L r_y V_2}{G_5} \left(\frac{2}{z^2} + \frac{4}{z_I^2}\right) \\
&\equiv Q_{\text{groundstate}} + \triangle Q, \qquad \text{(A2)}
\end{aligned}
$$

where ω_5 is the size of unit 5-sphere. The total charge Q, of course, depends on scale z, because we are in asymptotically (nonflat) Lifshitz spacetime. However, the contribution purely due to string excitations is given by $\triangle Q$. The second term in (A2) is not affected by z and remains constant. Therefore the net contribution of string excitations is

$$
\triangle Q = Q - Q_{\text{groundstate}} = \frac{2\pi L r_y V_2}{G_5} \left(\frac{2}{z_I^2}\right) \simeq Q|_{z=\infty}. \qquad \text{(A3)}
$$

Alternatively the charge due to string excitations can also be measured near $z \sim \infty$, where the massive mode gets completely decoupled and only massless strings survive which contribute to the charge. The net winding number of these strings is quantized in the units $N = \frac{\triangle Q}{r_y}$, where N is an integer.

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