Holography of *pp* waves in conformal gravity

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We consider holography of two pp-wave metrics in conformal gravity, their one-point functions, and asymptotic symmetries. One of the metrics is a generalization of the standard pp waves in Einstein gravity to conformal gravity. The holography of this metric shows that within conformal gravity one can have a realized solution which has a nonvanishing partially massless response tensor even for a vanishing subleading term in the Fefferman-Graham expansion (i.e., Neumann boundary conditions) and vice versa.

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I. INTRODUCTION

Conformal gravity (CG) is a higher derivative theory of gravity which has a recurrent appearance in literature. It is power-counting renormalizable and highly symmetric which makes it interesting for studying [1,2]. The main argument against the theory is its nonunitarity, which manifests for example in two-point correlation functions [3]. That issue is addressed via known methods [1,4], or the theory is considered as a toy model for its symmetry properties. From phenomenological aspects CG explains galactic rotation curves without the addition of dark matteer [5], and it was also stated to be an exact solution to perturbative cosmology in the recombination era [6]. The analysis of the asymptotic symmetries of CG in $3+1$ dimensions allows for classification of the asymptotic solutions [7]. There is no classification of the global cosmological solutions in CG; however, a number of Einstein gravity (EG) solutions have been generalized to CG [8,9]. Four-dimensional cosmological Solutions of EG, have of course been most studied [10], and most classified. The most popular classifications to date are Bianchi classification and Petrov [11] classification, which often uses Newman-Penrose formalism. Here, we calculate two general solutions of the pp wave with and without a cylinder in CG and analyze their asymptotics.

CG holography has in the earlier studies showed that in the framework of AdS/CFT there are two holographic stress-energy tensors at the boundary. One of them is analogous to the Brown-York stress-energy tensor and another is called the partially massless response (PMR), which does not have an analog in EG. Holographic analyses of a Schwarzschild solution in EG, Manheimm-Kazanas-Riegert solution in CG [5], and rotating black hole solution in AdS with Rindler hair [8] showed that their PMR vanishes when generalized Fefferman-Graham boundary conditions reduce to standard Fefferman-Graham boundary conditions used in EG.[1] The pp-wave solutions which we analyze here show that it is possible to have vanishing PMR for the generalized Fefferman-Graham (FG) boundary conditions and that it is possible to have nonvanishing PMR for standard FG boundary conditions. Vanishing of the PMR also implies vanishing of the corresponding two-point correlation functions. Besides the holography of the solution, we consider its Killing vectors, charges, and asymmptotic symmetry algebra (ASA) as well as speculate the possibility of using the metric as a cosmological background for string quantization.

II. CONFORMAL GRAVITY

Given a manifold $\mathcal{M}$ and the coordinates x_i which we take to be (u, v, x, r), action of conformal gravity is defined by

$$S_{CG} = \alpha_{CG} \int_{\mathcal{M}} d^4x \sqrt{-g} C^{\alpha}{}_{\beta\gamma\rho} C_{\alpha}{}^{\beta\gamma\rho} \tag{2.1}$$

for the $C^{\alpha}{}_{\beta\gamma\rho}$ Weyl term, α_{CG} the dimensionless constant, and $g_{\mu\nu}$ the conformally invariant metric. The equation of motion of the action (2.1) is called the Bach equation,

$$\left(\nabla^{\delta}\nabla_{\gamma} + \frac{1}{2} R^{\delta}{}_{\gamma}\right) C^{\gamma}{}_{\alpha\delta\beta} = 0. \tag{2.2}$$

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[1] Generalized Fefferman-Graham boundary conditions allow for the subleading term in the expansion in a holographic coordinate, around the boundary of the manifold. In standard Fefferman-Graham expansion this term is set to zero [9].

The equation is fourth order in derivatives, and as a subset, it contains solutions of Einstein gravity. We want to consider the pp-wave metric which solves (2.2). A similar global solution of (2.2) from [7] showed that there can be interesting holography directly related to unitarity of the theory.

A. *Ansätze* 1 and 2

We consider a metric of the form

$$ds^2 = \frac{f(r)}{h(r)} du^2 + \frac{2}{h(r)} du dv + k(r) dr^2 + k(r) dx^2 \quad (2.3)$$

which solves the Bach equation for

$$k = \frac{c_2 e^{-c_1 r}}{h(r)} \quad \text{and}$$
$$f = \frac{c_1 c_2 - 2c_3 + r c_3 c_1}{c_1^3} + e^{-c_1 r}(c_4 + r c_5), \quad (2.4)$$

where one can recognize the $\frac{1}{h(r)}$ as a conformal factor. Due to conformal invariance of the Bach equation, each metric with an arbitrary conformal factor is also a solution. To investigate symmetries of this solution we examine its Killing vectors (KV). For arbitrary $h(r)$ the conformal Killing equation $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = \frac{1}{2} g_{\mu\nu} \nabla_\alpha \xi^\alpha$ is satisfied by KVs of translations

$$\xi_x = (0,0,1,0), \qquad \xi_u = (1,0,0,0) \qquad \xi_v = (0,1,0,0), \quad (2.5)$$

while in the special case of a conformally flat metric, when $c_1 = 1$, $c_3 = 0$, $c_5 = 0$ (*Ansatz* 2), there are two additional KVs,

$$\xi_1 = a_1 \left(0, e^{\frac{1}{2}(u+r)} \cos\left(\frac{1}{2}x\right), -e^{\frac{1}{2}(u-r)} \sin\left(\frac{1}{2}x\right), - e^{\frac{1}{2}(u-r)} \cos\left(\frac{1}{2}x\right)\right),$$
$$\xi_2 = a_1 \left(0, -e^{\frac{1}{2}(u+r)} \sin\left(\frac{1}{2}x\right), - e^{\frac{1}{2}(u-r)} \cos\left(\frac{1}{2}x\right), e^{\frac{1}{2}(u-r)} \sin\left(\frac{1}{2}x\right)\right). \quad (2.6)$$

This indicates that the solution in that special case becomes a plane wave. The KVs define commutation relations

$$[\xi_x, \xi_2] = \frac{1}{2}\xi_1, \ [\xi_x, \xi_1] = \frac{1}{2}\xi_2, \ [\xi_u, \xi_2] = \frac{1}{2}\xi_2, \ [\xi_u, \xi_1] = \frac{1}{2}\xi_1 \quad (2.7)$$

which can be recognized as two separate algebras. Redefining $\tilde{\xi}_x = 2\xi_x$ the first two commutation relations in (2.7) close Bianchi V algebra [12], while using $\tilde{\xi}_u = 2\xi_u$ the latter two close the Bianchi VII algebra. Some examples of Bianchi universes of the type V can be found in [13], and types IV, VI_h, VII_h in [14]. For comparison to other Bianchi types, one can look at type I in [15], type III in [16], and type II, VIII, IX in [17].

The solution (2.3) is a type N solution in the Petrov classification which we calculate using the *Mathematica* program RGTC.[2] Conformal gravity solutions, particularly of Petrov N type, have been studied in [18]. The studies of the gravitational waves in quadratic curvature gravity using Newman-Penrose formulation have been studied for Petrov D solutions in [19].

III. ASYMPTOTIC ANALYSIS

To analyze holography of (2.3) we transform coordinates $u \to aq - cy$ and $v \to bq + dy$, and take $h(r) = r^2$, obtaining the metric

$$ds^2 = \frac{1}{r^2}(c_2 dr^2 + dq dy(2H(r) + 2a^2 e^{-c_1 r}(c_5 r - 1)) + dq^2(H(r) + e^{-c_1 r}(a^2(c_5 r - 1) - 1)) + dy^2(H(r) + e^{-c_1 r}(a^2(c_5 r - 1) + 1)) + c_2 dx^2) \quad (3.1)$$

where $H(r) = a^2(\frac{c_3 r}{c_1^2} + \frac{c_2}{c_1^2} - \frac{2c_3}{c_1^3})$. The metric has a Ricci scalar equal to $R = -\frac{3(c_1^2 r^2 + 4c_1 r + 8)}{2c_2}$ which cannot take a Ricci flat form by suitable choice of parameters. The Ricci scalar is inversely proportional to c_2 which if sent to infinity would cause the metric to diverge. We expand (3.1) to a Fefferman-Graham form $ds^2 = \frac{1}{r^2}(dr^2 + \gamma_{ij} dx^i dx^j)$, for the γ_{ij} metric at the boundary, and r the holographic coordinate. The metric at the boundary is expanded in terms of the small perturbations around $r = 0$, such that $\gamma_{ij} = \gamma_{ij}^{(0)} + \gamma_{ij}^{(1)} r + \gamma_{ij}^{(2)} r^2 + \gamma_{ij}^{(3)} r^3$, for $\gamma_{ij}^{(I)}$, $I = 0, 1, 2, 3$ matrices given in the expansion. When $b \to -\frac{a^2(c_4+1)+1}{2a}$, $d \to -\frac{1}{2}a(c_4+1) + \frac{1}{2a}$, and $c \to -a$ we can choose the matrix $\gamma_{ij}^{(0)} = \text{diag}(-1,1,1)$ to be a Minkowski metric, where the time coordinate is q. That defines the $\gamma_{ij}^{(1)}$ matrix to be

$$\gamma_{ij}^{(1)} = \begin{pmatrix} (\frac{c_1 - c_3}{c_1^2} + c_5)a^2 + c_1 & \frac{a^2(c_5 c_1^2 + c_1 - c_3)}{c_1^2} & 0 \\ \frac{a^2(c_5 c_1^2 + c_1 - c_3)}{c_1^2} & a^2(\frac{c_1 - c_3}{c_1^2} + c_5) - c_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3.2)$$

This matrix can be compared with the $\gamma_{ij}^{(1)}$ term from the FG expansion for the Manheimm-Kazanas-Riegert (MKR)

[2] RGTC denotes Riemannian Geometry & Tensor Calculus program for Mathematica.

solution [20–22]. The MKR solution is different from (3.1); however, we can use its properties to better understand the meaning of parameters which appear in our case. In the FG expansion of the MKR solution, the $\gamma_{ij}^{(1)}$ matrix depends entirely on the term that describes Rindler acceleration. If we are drawing an analogous conclusion in (3.2), this role is played by the combination of c_1, c_3, c_5 parameters. Parameter a from (3.1) can be absorbed in the coordinate, so it does not carry physical meaning. The $\gamma_{ij}^{(2)}$ matrix of MKR solution does not show explicit dependence on the mass parameter when the Rindler acceleration parameter vanishes, so we consider matrix $\gamma_{ij}^{(3)}$

$$\gamma_{ij}^{(3)} = \begin{pmatrix} c_1^3 + a^2(3c_5c_1^2 + c_1 - 2c_3) & a^2(3c_5c_1^2 + c_1 - 2c_3) & 0 \\ a^2(3c_5c_1^2 + c_1 - 2c_3) & a^2(3c_5c_1^2 + c_1 - 2c_3) - c_1^3 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.3}$$

If the Rindler parameter in the MKR solution is zero, its $\gamma_{ij}^{(3)}$ matrix is given solely in terms of the mass parameter [22]. This implies that the combination of parameters in (3.3) carries physical meanings of mass and Rindler acceleration. Now, one can compute the holographic stress-energy tensors of (3.1) τ_{ij} and P_{ij} by inserting $\gamma_{ij}^{(1)}, \gamma_{ij}^{(2)}$, and $\gamma_{ij}^{(3)}$ in the τ_{ij} and P_{ij} [22]. The stress-energy tensor τ_{ij} and PMR are given by

$$\tau_{ij} = a^2c_1^2c_5 \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad P_{ij} = \frac{a^2(c_1^2c_5 - c_3)}{c_1} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.4}$$

respectively. The Ward identity of CG is satisfied with $2\tau_{ij}\gamma^{(0)ij} + P_{ij}\gamma^{(1)ij} = 0$. For $\chi^{(0)k}$ asymptotic KV, the current $J^i = Q^{ij}\chi_j^{(0)}$ is conserved $\mathcal{D}_iJ^i = 0$. The corresponding charge $Q_{ij} = 2\tau_{ij} + P_{ik}\gamma^{(1)k}{}_j + P_{ki}\gamma_j^{(1)k}$ for (3.1)

$$Q_{ij} = 2a^2c_3 \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.5}$$

is given in terms of c_3. For the specific case of the metric

$$ds^2 = \frac{1}{r^2}(c_2dr^2 + 2a^2(c_2 - e^{-r})dqdy + dq^2(a^2c_2 - (a^2+1)e^{-r}) + dy^2(c_2(a^2+1) + (1-a^2)e^{-r}) + c_2dx^2), \tag{3.6}$$

with five KVs (2.5)–(2.6), when $c_1 = 1, c_3 = c_5 = 0$, the stress tensors $\tau_{ij} = P_{ij} = Q_{ij} = 0$ and charges exactly vanish, as we can see from the general expressions for the stress-energy tensors (3.4) and charge (3.5). This is expected from the highly symmetric solution, which also describes the conformally flat metric. That choice of parameters still does not give a metric which satisfies Einstein equations. Interestingly, there is no such choice of parameters which would make solution (3.1) satisfy Einstein vacuum equation.

It is well known that conformal gravity is nonunitary. In [3] the nonunitarity of conformal gravity was shown via two-point correlation functions. It manifests through the negative sign of the correlation function with PMR. Since PMR is zero for (3.6), that issue is avoided. However, we have also vanishing τ_{ij} and a vanishing Q_{ij}. The conformal flatness means that the entropy of the solution (visible also from Weyl squared) is going to be zero. From (3.4) we see that P_{ij} will vanish for $c_3 = c_5$ when $c_1 = 1$. This choice of metric has only three global KVs and it is not conformally flat. The stress-energy tensor τ_{ij} and charge Q_{ij} are visible from (3.4) and (3.5) and they do not vanish. This appears as well in [22] for the example of a rotating black hole [8], where $\gamma_{ij}^{(1)}$ does not vanish while P_{ij} vanishes. If in our example (3.1) we demand $\gamma_{ij}^{(1)}$ to be zero, that implies P_{ij} (3.4) is as well zero. The vanishing of $\gamma_{ij}^{(1)}$, however, does not always automatically imply vanishing of the P_{ij}. On the example of the pp-wave solution [7],

$$ds^2 = \frac{1}{r^2}(dr^2 + (-1 + f(r))dx^2 + 2f(r)dxdy + (1 + f(r))dy^2 + dz^2), \tag{3.7}$$

where $f(r) = c_1 + c_2 r + c_3 r^2 + c_4 r^3$ one can fix the $\gamma_{ij}^{(1)}$ to be zero (setting $c_2 = 0$), without affecting the P_{ij} which becomes defined solely by the $\gamma_{ij}^{(2)}$ and c_3.

The charges that we calculated express stress-energy tensors and charge in a sense of [23] which differ from the other such charges (for example those defined by the Hamiltonian method as in [24]) only by a "constant offset" determined by boundary fields alone. The algebra generated by the charges in conformal gravity is equivalent to the Lie algebra of the transformations preserving boundary conditions, i.e., asymptotic symmetry algebra [25].

Asymptotic symmetry algebra for conformal gravity has been studied in [7].To obtain ASA for (3.1) one has to study the expansion of the conformal Killing equation (CKE) in the coordinate r. The metric at the boundary was chosen to be $\gamma_{ij}^{(0)} = \mathrm{diag}(-1, 1, 1)$ Minkowski metric, which leads to full conformal algebra in the leading order of expansion of the CKE. The subleading order of the CKE equation [7]

$$\pounds_{\xi^{(0)}} \gamma_{ij}^{(1)} = \frac{1}{3} \mathcal{D}_k \xi^k_{(0)} \gamma_{ij}^{(1)} \tag{3.8}$$

defines the ASA

$$[\xi_t, \xi_{L_1}] = \xi_x, \qquad [\xi_t, \xi_{L_2}] = \xi_y + \frac{1}{2}\xi_t,$$
$$[\xi_y, \xi_{L_1}] = -\xi_x, \qquad [\xi_y, \xi_{L_2}] = \frac{1}{2}\xi_t + \xi_y, \tag{3.9}$$

$$[\xi_x, \xi_{L_1}] = \xi_t + \xi_y, \qquad [\xi_x, \xi_{L_2}] = \xi_x, \qquad [\xi_{L_1}, \xi_{L_2}] = \frac{1}{2}\xi_{L_1} \tag{3.10}$$

with KVs

$$\xi_t = (1, 0, 0), \qquad \xi_x = (0, 0, 1) \qquad \xi_y = (0, 1, 0) \tag{3.11}$$

$$\xi_{L_1} = (x, x, t - y) \qquad \xi_{L_2} = \left(t + \frac{1}{2}y, \frac{1}{2}t + y, x\right). \tag{3.12}$$

The ASA is unaffected by the choice of the parameters c_i, $i = 1, \ldots, 5$, and it is equal for each of the special cases of the solution (3.1). It belongs to the ASA $a_{5,4}^a$ for $a = \frac{1}{2}$ from [26]. The classes of five-dimensional ASAs have been encountered in CG [7].

A. Applications of the metric

If we look at the metric as an Einstein solution with additional matter, we can have the following considerations. After the transformation of coordinates $e^{c_1 r/2} \to z$ and choice for conformal factor $h(r) = 1/(4z^2 \ln(z))$ and $c_1 = 1$, one can relate the metric (2.3) with the metric

$$ds^2 = 2dudv + f(z)du^2 + z^2 dx^2 + dz^2 \tag{3.13}$$

which solves the Bach equation for $f(z) = (\frac{1}{4} - 2c_3)z^2 + c_4 + 2c_5 \log z + 2z^2 c_3 \log z$, and $c_2 = \frac{1}{4}$, $x = 2\tilde{x}$, where we omit "$\sim$" for simplicity. This solution is similar to the metric considered in [27]. There, the metric

$$ds^2 = 2dudv - \lambda(u)x^2 du^2 + dx^i dx^i \tag{3.14}$$

was studied for the propagation of string modes and a first-quantized point particle in this time-dependent background, where $dx^i dx^i = dx^2 + dz^2$ is the Euclidean metric.

B. *Ansatz* 3

Generalization of the metric (3.13) by multiplying $f(z)$ with $\lambda(u)$ does not influence the solvability of the Bach equation. One may wonder if further simple generalizations are possible. We consider the metric

$$ds^2 = 2dudv + f(u, x, z)du^2 + dx^2 + dz^2 \tag{3.15}$$

where we immediately crossed from cylindrical to Euclidean coordinates. The generalization by introducing the dependency on z so that $f(z) \to f(x, z)$ in (3.13) leads to the fourth-order equation which can be decomposed into $(-\partial_z + i\partial_x)^2 (\partial_z + i\partial_x)^2 f(x, z) = 0$. The solution to this equation is

$$f(x, z) \to (d_1 + d_2 x + d_3 z) f_1(-ix + z) + (d_4 + d_5 x + d_5 z) f_2(ix + z) \tag{3.16}$$

and it can obviously become of the interesting form for trigonometric and exponential functions (we will mention specific cases later).[3]

The generalization of the function $f(x, z)$ so that it also has dependency on u, $f(x, z) \to f(u, x, z)$ will generalize the solution (3.16) into

$$f(u, x, z) = (d_1 + d_2 x + d_3 z) f_1(u, -ix + z) + (d_4 + d_5 x + d_5 z) f_2(u, ix + z), \tag{3.18}$$

where we took into account that each of the functions depending on (x, z) can be multiplied by an arbitrary

[3] We can bring the solution (3.18) to the form of the metric studied in [14] by considering the transformation $x \to ix_1 + \frac{i}{2}x_2$ and $z \to x_1 + \frac{1}{2}x_2$. The obtained metric reads

$$ds^2 = H_1(x_1, x_2)du^2 + 2dudv - 2dx_1 dx_2. \tag{3.17}$$

The general form of the solution obtained from the Bach equation would lead to $H_1 = f_1(x_2) + x_1 f_2(x_1) + f_3(x_1) + x_2 f_4(x_1)$. Only keeping f_1 and f_3 satisfies the Einstein solution and can be cast into the form studied in [14], while the CG solution involves all four functions.

function of u. The metric (3.15) is completely equal to the *Ansatz* metric in [28] after appropriate transformation of the coordinates. The statement that Einstein equations in vacuum are satisfied for every harmonic function f which is a function of x and z, whatever was the dependence on u, is now generalized. The Bach equations in vacuum are satisfied for every harmonic function f which is a function of x and z multiplied by an arbitrary function of u and by the $(d_1 + d_2x + d_3z)$ or $(d_4 + d_5x + d_6z)$ for d_1, d_2, d_3, d_4, d_5, d_6 which are arbitrary, or defined, depending on the function we want to express. For example, note the following:

(i) $f(u,x,z) = 2(d_1 + d_2x + d_3z)\arctan(\frac{z}{x})$, for $f_1 = i\log(x - iz)$, $f_2 = i\log(x - iz)$, and $d_1 = -d_4$, $d_2 = -d_5$ and $d_3 = -d_6$. This is a term from rotation of a metric analogous to (3.13) from a cylindrical to a Euclidean coordinate system.

(ii) $f(u,x,z) = b_2(x^2 + z^2)\log(x^2 + z^2)$, for $f_1 = (x - iz)\log(x - iz)$, $f_2 = (x + iz)\log(x + iz)$, and $d_3 = id_2, d_5 = d_2$, and $d_6 = -id_2$. This is an additional term in generalization of a plane wave metric (3.14) [27].

The above metric (3.15) conserves only one KV, that is ∂_v. If one would like to consider the solution (3.15) [with $f(u,x,z)$ from (3.18)] as the background for the string propagation, they need to transform it to Rosen coordinates, following the procedure of [27]. For functions f_1 and f_2 such that the metric (3.15) is $l(u)(x^2 + z^2)$ reduces to a case studied earlier in [27]. Here we focus on the study of the situation when f_1, f_2 are more general. In the asymptotic analysis we are going to keep writing $f(u,x,z)$ in terms of functions dependent on $x + iz$ and $x - iz$; however, this is only to keep the functional dependence. One needs to keep in mind that for each specific case the metric needs to be of course real.

C. Asymptotic analysis

Transformations $u \to -\frac{1}{2a_3}(q + y)$, $v \to a_3(q - y)$ where we can choose for simplicity $a_3 = -\frac{1}{2}$, lead from (3.15) and (3.18) to

$$ds^2 = \frac{1}{z^2}((-1 + g_1(q + y,x,z))dq^2 + 2g_1(q + y,x,z)dydq + (1 + g_1(q + y,x,z))dy^2 + dx^2 + dz^2) \tag{3.19}$$

for $g_1(q + y,x,z) = (d_1 + d_2x + d_3z)f_1(q + y,x - iz) + (d_3 + d_5x + d_6z)f_2(q + y,x + iz)$. The Ricci scalar of the metric is -12, while it is zero for the metric (3.15). The FG expansion of the metric (3.19) in a z coordinate, done analogously to the expansion of (3.1), requires $d_1 = d_3, d_5 = d_2$ and $f_1(q + y,x) = -f_2(q + y,x)$, and it results with the $\gamma_{ij}^{(1)}$ and $\gamma_{ij}^{(2)}$ matrices

$$\gamma_{ij}^{(1)} = \begin{pmatrix} h_1 & h_1 & 0 \\ h_1 & h_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \gamma_{ij}^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.20}$$

for $h_1 \equiv h_1(q+y,x) = (d_6 - d_3)f_2(q+y,x) - 2i(d_5x + d_4)\times f_2{}^{(0,1)}(q+y,x)$ and $h_2 \equiv h_2(q + y,x) = \frac{1}{3}i(d_3 + d_6)\times f_2{}^{(0,3)}(q + y,x)$. From h_1 and h_2 and comparison to (3.2) and (A1), respectively, we can see that the Rindler parameter and mass are given by a combination of the parameters d_6, d_3, d_5, and d_4. Expressing the stress tensors in terms of the function $f_2(q + y,x)$ and its derivatives allows one to see functional dependence from the metric directly in the response functions and charge. The stress tensor τ_{ij} is given by

$$\tau_{ij} = \begin{pmatrix} h_1h_2 + h_3 & -\frac{6}{7}h_1h_2 & 0 \\ -\frac{6}{7}h_1h_2 & -\frac{3}{4}h_1h_2 + \frac{3}{2}h_3 & 0 \\ 0 & 0 & \frac{6}{5}h_1h_2 + \frac{6}{5}h_3 \end{pmatrix} \tag{3.21}$$

for $h_3 \equiv h_3(q + y,x) = \frac{1}{60}(5(d_6 - d_3)f_2{}^{(0,4)}(q + y,x) - 2i(d_4 + d_5x)f_2{}^{(0,5)}(q + y,x))$ and

$$P_{ij} = -\frac{1}{3}h_2, \tag{3.22}$$

while definition of the charge is given in terms of h_1, h_2, and h_3; see Eq. (A2). It is important to notice that for this metric, one can choose $d_6 = d_3$ and $d_5 = d_4 = 0$ which will lead to vanishing of the $\gamma_{ij}^{(1)}$, while the PMR tensor will not vanish. This is due to proportionality of P_{ij} to $\gamma_{ij}^{(2)}$ and nonvanishing $\gamma_{ij}^{(2)}$. This is a specific property of the solution, observed only for the pp-wave solution in [7].

By choosing $g_1(q + y,x,z) = -4zf(q + y,x,z)$ we can set the metric (3.19) to become

$$ds^2 = \frac{1}{z^2}(-(1 + 4zf(q + y,x,z))dq^2 - 8zf(q + y,x,z)dqdy + (1 - 4zf(q + y,x,z))dy^2 + dx^2 + dz^2). \tag{3.23}$$

Further specification $f(q+y,x,z) = b(5x^4 - 10z^2x^2 + z^4)\times \cos(q+y)$ leads to a metric that has stress-energy tensor

$$\tau_{ij} = \mathrm{diag}(-8b\cos(q + y), - 16b\cos(q + y), 8b\cos(q + y))$$

defined only from nonvanishing matrices in the FG expansion, $\gamma_{ij}^{(1)}$ and $\gamma_{ij}^{(3)}$ [given in Eq. (A3)]. Since $\gamma_{ij}^{(2)}$ is zero, the partially massless response tensor P_{ij} vanishes, which results with $Q_{ij} = 2\tau_{ij}$. For the only parameter b which we have here, we can conclude to have a role similar to the Rindler parameter.

The asymptotic symmetry algebra for this metric is three dimensional, consisting of KVs $\xi_1 = (0, 0, 2a_1)$, $\xi_2 = (-a_2, a_2, 0)$, $\xi_3 = (-\frac{x}{2}, \frac{x}{2}, -\frac{a_3}{2}(q + y))$ closing the algebra $[\xi_1, \xi_2] = \frac{a_1}{a_2}\xi_3$. For $a_1 = a_2 = 1$ that defines Bianchi II algebra, which is also called Heisenberg-Weyl algebra.

Metric (3.19) can be reduced to a Ricci flat metric by multiplying it with z^2. That metric can be transformed to a flat metric which one can naturally write in the Rosen form.

IV. CONCLUSION

We have studied the pp-wave solution of conformal gravity and its symmetries. The most general form of the solution admits three translational Killing vectors, while by choosing specific parameters the symmetries can be increased to five KVs. Via asymptotic analysis we calculate holographic stress-energy tensors of conformal gravity. The most symmetric solution has both stress-energy tensors vanishing, as well as vanishing Weyl tensor and charge. For the specific choice of parameters we find vanishing PMR for a metric which is not conformally flat, and does not have vanishing charge or a Brown-York stress tensor. Zero PMR does not imply that the global solution becomes an Einstein solution. The interesting thing is that nonunitarity of the conformal gravity manifests through PMR; when PMR is zero, this is not the case, which renders this solution important. The second pp-wave solution we study is also the most general solution of its respective form, and it is a generalization of the pp waves in Einstein gravity studied in [28]. The holography of this solution shows that one can have a vanishing subleading term in the FG expansion and nonvanishing PMR, which makes it one of the first examples of its kind.

We also considered possible application of these pp-wave metrics. In future studies it would be interesting to use these metrics as a background for the calculations as string propagation. This is based on the fact that the supersymmetric analog of conformal gravity appears in twistor string theory [29]. It would be also interesting to use pp-wave metrics considered here, on the calculation as in [4]; i.e., it would be interesting to see if one could impose restrictions on a partially massless response function in order to avoid presence of the ghost.

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APPENDIX: QUANTITIES AT THE BOUNDARY

The matrix $\gamma_{ij}^{(2)}$ in the FG expansion of (3.1) is

$$\gamma_{ij}^{(2)} = \begin{pmatrix} a^2(\frac{2c_3}{c_1} - 2c_1c_5 - 1) - c_1^2 & -\frac{a^2(2c_5c_1^2+c_1-2c_3)}{c_1} & 0 \\ -\frac{a^2(2c_5c_1^2+c_1-2c_3)}{c_1} & (\frac{2c_3}{c_1} - 2c_1c_5 - 1)a^2 + c_1^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{A1}$$

The charge defined by the (3.21) and (3.22) is

$$Q_{ij} = \begin{pmatrix} -\frac{3}{4}h_2h_1 + \frac{3}{2}h_3 & -\frac{3}{11}h_2h_1 & 0 \\ -\frac{3}{11}h_2h_1 & -\frac{1}{4}h_2h_1 + h_3 & 0 \\ 0 & 0 & \frac{3}{5}h_2h_1 + h_3 \end{pmatrix}. \tag{A2}$$

The $\gamma_{ij}^{(k)}$ for $k = 1, 3$ matrices for the metric (3.23) are given by

$$\gamma_{ij}^{(1)} = -20bx^4\cos(q+y)\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \gamma_{ij}^{(3)} = -24b\cos(q+y)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{A3}$$

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