Thermodynamics of Kerr-AdS black holes in general Poincaré gauge theory

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A Hamiltonian variational approach is used to study asymptotic charges and entropy of Kerr–anti–de Sitter black holes in the general Poincaré gauge theory, with both even and odd parity modes. The results turn out to be the same as those found earlier in the sector of parity invariant Lagrangians.

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I. INTRODUCTION

The Poincaré gauge (PG) theory is a well-founded approach to gravity based on gauging the Poincaré group of spacetime symmetries [1]. The dynamical content of PG is expressed in terms of a Riemann-Cartan (RC) geometry of spacetime, characterized by the presence of two field strengths, the torsion T^i and the curvature R^{ij}; for a comprehensive analysis of the subject, see the reader [2], the lectures [3], and/or the monographs [4].

In the last half a century, various dynamical and geometric aspects of PG, as well as its relation to physics, have been intensively studied. Thus, successes in constructing a number of black hole solutions [2] inspired a detailed analysis of their *asymptotic charges*, energy and angular momentum; for an advanced exposition of the subject, see Ref. [5]. However, it is somewhat surprising that systematic studies of black hole entropy were rather neglected in the literature. Quite recently, the subject came to life in the work [6], where the idea that "black hole entropy is the Noether charge" [7] was given a natural *Hamiltonian extension*. In the papers [8–10], this approach was used to study both the asymptotic charges and entropy of stationary black holes in PG with *parity invariant* Lagrangians. The results obtained for spherically and axially symmetric solutions on the Minkowski or anti–de Sitter (AdS) background confirmed the validity of the first law of black hole thermodynamics. On the other hand, in the last decade, one can notice an increased interest in exploring various dynamical aspects of the general *parity violating* PG, such as cosmological applications, exact solutions, and particle spectrum [11–17]. In the present paper, our attention is focused on exploring energy, angular momentum, and entropy of Kerr-AdS black holes in the general PG, with both even and odd parity modes.

The paper is organized as follows. In Sec. II, we give a short account of the general PG [2,17], as well as an outline of the variational Hamiltonian approach to the thermodynamic charges of black holes with torsion [6]. In Sec. III, we analyze geometric and dynamical aspects of Kerr-AdS black holes, as a preparation for studying their thermodynamic charges. Then, in Secs. IV and V, we apply the Hamiltonian approach to calculate the Kerr-AdS asymptotic charges and entropy. The results are summarized in Sec. VI, and the Appendix is devoted to some technical details.

Our general conventions remain the same as in Refs. [9,10]. The Latin indices $(i, j, \ldots)$ refer to the local Lorentz frame, the Greek indices $(\mu, \nu, \ldots)$ refer to the coordinate frame, b^i is the orthonormal coframe (tetrad) dual to the frame h_i, with $h_i \rfloor b^k = \delta_i^k$, the local Lorentz metric is $\eta_{ij} = (1, -1, -1, -1)$, and ω^{ij} is a metric compatible (antisymmetric) connection. The exterior product of forms is implicit, the volume 4-form is $\hat{\epsilon} = b^0 b^1 b^2 b^3$, the Hodge dual of a form α is ${}^\star\alpha$, ${}^\star 1 = \hat{\epsilon}$, and the totally antisymmetric symbol is normalized to $\varepsilon_{0123} = +1$.

II. PG DYNAMICS AND BOUNDARY TERMS

Basic dynamical variables of PG are the tetrad field b^i and the antisymmetric spin connection ω^{ij} (1-forms), the gauge potentials associated to the translation, and Lorentz subgroups of the Poincaré group, respectively. The corresponding field strengths are the torsion $T^i = db^i + \omega^i{}_k b^k$ and the curvature $R^{ij} = d\omega^{ij} + \omega^i{}_k \omega^{kj}$, and the underlying structure of spacetime is characterized by a Riemann-Cartan geometry. In the absence of matter, dynamical properties of PG are determined by the gravitational Lagrangian $L_G(b^i, T^i, R^{ij})$ (4-form), which is assumed to be *at most quadratic* in the field strengths.

The gravitational field equations are obtained by varying L_G with respect to b^i and ω^{ij}. They can be written in a compact form as

$$\delta b^i: \quad \nabla H_i + E_i = 0, \tag{2.1a}$$

$$\delta \omega^{ij}: \quad \nabla H_{ij} + E_{ij} = 0, \tag{2.1b}$$

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were $H_i := \partial L_G/\partial T^i$ and $H_{ij} := \partial L_G/\partial R^{ij}$ (2-forms) are the covariant momenta, and $E_i := \partial L_G/\partial b^i$ and $E_{ij} := \partial L_G/\partial \omega^{ij}$ (3-forms) are the corresponding energy-momentum and spin currents, respectively.

The content of the field equations (2.1) depends on the structure of the Lagrangian. In the present work, the PG Lagrangian is assumed to have the most general form, with all possible parity even and parity odd terms,

$$L_G = L_G^+ + L_G^-, \tag{2.2a}$$

$$L_G^+ := -{}^\star(a_0 R + 2\Lambda) + T^i \sum_{n=1}^{3} {}^\star(a_n{}^{(n)}T_i) + \frac{1}{2} R^{ij} \sum_{n=1}^{6} {}^\star(b_n{}^{(n)}R_{ij}), \tag{2.2b}$$

$$L_G^- := -\bar{a}_0{}^\star X + T^i \sum_{n=1}^{3} \bar{a}_n{}^{(n)}T_i + \frac{1}{2} R^{ij} \sum_{n=1}^{6} \bar{b}_n{}^{(n)}R_{ij}. \tag{2.2c}$$

Here, (a_n, b_n, Λ_0) and $(\bar{a}_n, \bar{b}_n)$ are the coupling constant, R and X are the scalar and pseudoscalar curvatures, ${}^\star R = {}^\star(b_i b_j)R^{ij}$ and ${}^\star X = b_i b_j R^{ij}$, and ${}^{(n)}T^i$ and ${}^{(n)}R^{ij}$ are the irreducible components of the torsion and the curvature, respectively; see Ref. [6]. Because some terms in L_G^- are the same, the corresponding coupling constants are not independent; in particular, one can choose $\bar{a}_2 = \bar{a}_3$, $\bar{b}_2 = \bar{b}_4$, and $\bar{b}_3 = \bar{b}_6$; see [16,17]. Further freedom in the choice of parameters follows from the existence of three topological invariants [13].

With the above form of L_G, the explicit expressions for the covariant momenta read

$$H_i = 2 \sum_{n=1}^{3} [{}^\star(a_n{}^{(n)}T_i) + \bar{a}_n T^i],$$
$$H_{ij} = -2a_0{}^\star(b_i b_j) - 2\bar{a}_0(b_i b_j) + H'_{ij}, \tag{2.3a}$$

where

$$H'_{ij} := 2 \sum_{n=1}^{6} [{}^\star(b_n{}^{(n)}R_{ij}) + \bar{b}_n{}^{(n)}R_{ij}]. \tag{2.3b}$$

They play a crucial role not only in the structure of the field equations, but also, as we shall see, in the analysis of the conserved charges and entropy.

Following the ideas of Regge and Teitelboim [18], asymptotic charges can be introduced as certain boundary terms Γ associated to the naive canonical gauge generator G, which is weakly vanishing. Namely, if G is not regular (differentiable), it can be improved by adding a suitable surface term Γ, $\tilde{G} := G + \Gamma$, such that

$$\delta\tilde{G} = \delta G + \delta\Gamma = \text{regular}. \tag{2.4}$$

In Ref. [6], this construction, combined with Wald's identification of entropy as the Noether charge on horizon [7], is used to propose a *unified approach* to both the asymptotic charges and black hole entropy in PG.

Next, consider a stationary black hole, such that its spatial section Σ has a boundary with two components, one at infinity and the other at horizon, $\partial\Sigma = S_\infty \cup S_H$. The corresponding boundary integral has two parts, $\Gamma = \Gamma_\infty - \Gamma_H$ (the minus sign reflects a different orientation of S_H), which are determined by the variational equations

$$\delta\Gamma_\infty = \oint_{S_\infty} \delta B(\xi), \qquad \delta\Gamma_H = \oint_{S_H} \delta B(\xi), \tag{2.5a}$$

$$\delta B(\xi) := (\xi \rfloor b^i)\delta H_i + \delta b^i (\xi \rfloor H_i) + \frac{1}{2}(\xi \rfloor \omega^{ij})\delta H_{ij} + \frac{1}{2}\delta\omega^{ij}(\xi \rfloor \delta H_{ij}). \tag{2.5b}$$

Here, ξ is the Killing vector with values ∂_t and ∂_φ on S_∞, and a linear combination thereof on S_H, such that $\xi^2 = 0$. Moreover, a consistent interpretation of these equations is based on the following simple rules:

(r1) On the boundary S_∞, the variation δ acts on the parameters of a black hole solution, but not on the parameters of the background configuration.

(r2) On S_H, the variation δ must keep surface gravity constant.

The boundary conditions must be chosen so as to ensure the solutions for Γ_∞ and Γ_H to exist and be finite (δ integrability). When these requirements are satisfied, Γ_∞ and Γ_H are interpreted as the asymptotic charges and entropy, respectively, of a stationary black hole.

Note that each covariant momentum is given as a sum of parity even and parity odd terms. This allows us to make the corresponding decomposition for the thermodynamic charges (2.5) and simplify their calculation.

According to the variational equations (2.5), the boundary terms $\delta\Gamma_\infty$ and $\delta\Gamma_H$ are *a priori* independent quantities. However, since $\delta\Gamma = \delta\Gamma_\infty - \delta\Gamma_H$ is introduced to ensure the regularity of the canonical gauge generator G, see Eq. (2.4), one can conclude that if G is regular then $\delta\Gamma = 0$ by construction. Since the inverse statement is also true ($\delta\Gamma = 0$ implies G is regular), it follows that

$$G \text{ is regular} \Leftrightarrow \delta\Gamma \equiv \delta\Gamma_\infty - \delta\Gamma_H = 0. \tag{2.6}$$

The statement $\delta\Gamma_\infty = \delta\Gamma_H$ is nothing but the *first law* of black hole thermodynamics.

In the previous paper [10], we studied the asymptotic charges and entropy of the Kerr-AdS black holes with torsion, found by Baekler *et al.* [19] in the parity even sector of PG. In the present work, we extend these considerations to the general PG Lagrangian (2.2). The corresponding Kerr-AdS black holes were constructed recently by Obukhov [16].

III. KERR-AdS SOLUTIONS IN PG

We find it convenient to introduce the symbols PG^+ and PG^- referring to the parity sectors of PG, defined by the Lagrangians L_G^+ and L_G^-, respectively.

A. Geometry

The general Kerr-AdS solution in PG [16] and its PG^+ counterpart Kerr-AdS^+ [19,10] are defined by the same tetrad field; in Boyer-Lindquist coordinates, it has the form

$$
\begin{aligned}
&b^0 = N\left(dt + \frac{a}{\alpha}\sin^2\theta d\varphi\right), \qquad b^1 = \frac{dr}{N},\\
&b^2 = P d\theta, \qquad b^3 = \frac{\sin\theta}{P}\left[a dt + \frac{(r^2+a^2)}{\alpha}d\varphi\right],
\end{aligned} \tag{3.1a}
$$

where

$$
\begin{aligned}
&N = \sqrt{\Delta/\rho^2}, \qquad \rho^2 = r^2 + a^2\cos^2\theta,\\
&\Delta = (r^2+a^2)(1+\lambda r^2) - 2mr, \qquad \alpha = 1-\lambda a^2,\\
&P = \sqrt{\rho^2/f}, \qquad f = 1-\lambda a^2\cos^2\theta.
\end{aligned} \tag{3.1b}
$$

Here, m and a are the parameters of the solution, $0 \le \theta < \pi$, $0 \le \varphi < 2\pi$, and $a_0\lambda = -\Lambda/3$.

The metric $ds^2 = \eta_{ij} b^i \otimes b^j$, which is stationary and axially symmetric, admits the Killing vectors ∂_t and ∂_φ. The metric characteristics of Kerr-AdS black holes remain the same as for Kerr-AdS^+. In particular, this holds, respectively, for the location of the outer horizon $r = r_+$, the horizon area A_H, the angular velocity ω_+, and the surface gravity κ,

$$
\Delta(r_+) \equiv (r_+^2 + a^2)(1+\lambda r_+^2) - 2mr_+ = 0, \tag{3.2a}
$$

$$
A_H = \int_{r_+} b^2 b^3 = 4\pi\frac{r_+^2 + a^2}{\alpha}, \tag{3.2b}
$$

$$
\omega_+ = \frac{g_{t\varphi}}{g_{\varphi\varphi}}\bigg|_{r_+} = \frac{a\alpha}{r_+^2 + a^2}, \tag{3.2c}
$$

$$
\kappa = \frac{[\partial\Delta]_{r_+}}{2(r_+^2 + a^2)}. \tag{3.2d}
$$

By construction, the Kerr-AdS and Kerr-AdS^+ black holes also have the same torsion,

$$
\begin{aligned}
T^0 &:= \frac{1}{N}(-V_1 b^0 b^1 - 2V_4 b^2 b^3) + \frac{1}{N^2} b^-(V_2 b^2 + V_3 b^3)\\
&=: T^1,\\
T^2 &:= \frac{1}{N} b^-(V_5 b^2 + V_4 b^3),\\
T^3 &:= \frac{1}{N} b^-(-V_4 b^2 + V_5 b^3),
\end{aligned} \tag{3.3a}
$$

where $b^- := b^0 - b^1$ and the torsion functions V_n are

$$
\begin{aligned}
&V_1 = \frac{m}{\rho^4}(r^2 - a^2\cos^2\theta), \qquad V_2 = -\frac{m}{\rho^4 P} r a^2 \sin\theta\cos\theta,\\
&V_3 = \frac{m}{\rho^4 P} r^2 a\sin\theta, \qquad V_4 = \frac{m}{\rho^4} r a\cos\theta, \qquad V_5 = \frac{m}{\rho^4} r^2.
\end{aligned} \tag{3.3b}
$$

The third irreducible part of T^i vanishes.

For a given torsion, one can introduce the RC connection 1-form by

$$
\omega^{ij} := \tilde\omega^{ij} + K^{ij}, \tag{3.4}
$$

where $\tilde\omega^{ij}$ is the Riemannian connection and K_{ij} the contortion 1-form,

$$
K^{ij} = \frac{1}{2}[h^i \rfloor T^j - h_j \rfloor T^i - (h^i \rfloor (h^j \rfloor T^k)) b_k]. \tag{3.5}
$$

Clearly, the connection is the same for both Kerr-AdS and Kerr-AdS^+.

The corresponding RC curvature $R^{ij} = d\omega^{ij} + \omega^i{}_k \omega^{kj}$ has only two nonvanishing irreducible parts; with $A = (0,1)$ and $c = (2,3)$, they are

$$
{}^{(6)}R^{ij} = \lambda b^i b^j, \qquad {}^{(4)}R^{Ac} = \frac{\lambda m r}{\Delta} b^- b^c. \tag{3.6}
$$

The quadratic invariants are regular,

$$
R^{ij\star} R_{ij} = 12\lambda^2 \hat\epsilon, \qquad T^{i\star} T_i = 0. \tag{3.7}
$$

B. Dynamics

The Lagrangian parameters of Kerr-AdS solutions are restricted by the conditions [16]

$$
2a_1 + a_2 = 0, \quad a_0 - a_1 - \lambda(b_4 + b_6) = 0, \quad a_0\lambda = -\Lambda/3, \tag{3.8a}
$$

$$
\bar a_2 - \bar a_1 = 0, \qquad \bar a_0 - \bar a_1 + \lambda(\bar b_4 - \bar b_6) = 0, \tag{3.8b}
$$

imposed by the field equations.

Although the dynamical variables (b^i, ω^{ij}) have the same form for both Kerr-AdS and Kerr-AdS^+, the corresponding Lagrangians L_G and L_G^+ are different. To clarify dynamical aspects of this difference, recall that each of the covariant momenta H_i and H_{ij}, defined in Eqs. (2.3), contains two terms, one coming from L_G^+ and the other from L_G^-. As a consequence, the formulas (2.5) for the boundary terms imply the following:

(i) The thermodynamic charges of Kerr-AdS black holes can be obtained by summing up the contributions stemming from L_G^+ and L_G^-.

Geometrically, the first term is associated to Kerr-AdS$^+$ black holes.[1] Since the thermodynamic charges for Kerr-AdS$^+$ are already known, see Ref. [10], it remains only to calculate the contributions stemming from L_G^-. The related covariant momenta are determined by the effective form of the parity odd Lagrangian,

$$L_G^- = -\bar{a}_0{}^\star X + T^i(\bar{a}_1{}^{(1)}T_i + \bar{a}_2{}^{(2)}T_i) + \frac{1}{2}R^{ij}(\bar{b}_4{}^{(4)}R_{ij} + \bar{b}_6{}^{(6)}R_{ij}), \tag{3.9}$$

containing only the nonvanishing irreducible parts of the field strengths. Thus,

$$\begin{aligned} \bar{H}_i &= 2(\bar{a}_1{}^{(1)}T_i + \bar{a}_2{}^{(2)}T_i) = 2\bar{a}_1 T_i, \\ \bar{H}_{ij} &= -2(\bar{a}_0 - \lambda\bar{b}_6)b_i b_j + 2\bar{b}_4{}^{(4)}R_{ij}. \end{aligned} \tag{3.10}$$

The dynamical difference between Kerr-AdS and Kerr-AdS$^+$, including the values of their thermodynamic charges, is hidden just in the above expressions.

IV. ASYMPTOTIC CHARGES

Asymptotic conditions determine the behavior of dynamical variables on the boundary S_∞ where the asymptotic charges are calculated. Hence, a precise definition of the asymptotic charges requires to have a definite choice of the background configuration. For a Kerr-AdS black hole, the background is defined by $m = 0$ and interpreted as the standard AdS spacetime, with vanishing torsion and constant curvature. Note, however, that the AdS metric in the Boyer-Lindquist coordinates depends on the parameter a, which complicates the variational procedure introduced in Sec. II. Namely, according to the rule (r1), the variation over parameter a appearing in the AdS configuration should be avoided. How to recognize those unneeded δa terms? Technically, this can be taken care of by an improved version of the rule (r1), specifically designed for Kerr-AdS black holes, which is as follows:

(r1′) In the variational equation (2.5) for Γ_∞, the variation δ is first applied to all the parameters (m, a) appearing in $B(\xi)$. Then, those δa terms that survive the limit $m = 0$ have to be disregarded, as they stem from the variation of the AdS background.

[1] A comment on the geometric aspects of the second one is given in the last section.

By a careful analysis of the asymptotic states, Henneaux and Teitelboim [20] concluded that the Kerr-AdS metric in Boyer-Lindquist coordinates does not obey the asymptotic conditions compatible with the standard AdS background; see also Carter [21]. The problem was resolved using a suitable coordinate transformation which brings the metric to a manifestly asymptotically AdS form. In our approach, based on the variational equations (2.5), the inadequacy of the Boyer-Lindquist coordinates becomes visible through the lack of δ integrability. As we argued in [9,10], the problem can be solved by going over to the "untwisted" coordinates

$$T = t, \qquad \phi = \varphi - \lambda a t. \tag{4.1a}$$

Namely, if $\delta E_t \coloneqq \delta\Gamma_\infty(\partial_t)$ and $\delta E_\varphi \coloneqq \delta\Gamma_\infty(\partial_\varphi)$ are taken as the naive expressions for the asymptotic charges, then their (T, ϕ) transforms,

$$\delta E_T = \delta E_t + \lambda a \delta E_\varphi, \qquad \delta E_\phi = \delta E_\varphi, \tag{4.1b}$$

become δ integrable.

In further analysis, we shall focus on the unknown thermodynamic charges associated to the parity odd sector PG$^-$. Using the rule (r1′), we will first calculate the naive expressions for δE_φ and δE_t, whereupon (4.1b) will produce the final, δ-integrable results.

A. Angular momentum

Consider the expression $\bar{E}_\varphi$, defined by the variational equation $\delta\bar{E}_\varphi \coloneqq \delta\Gamma_\infty(\partial_\varphi)$, where the covariant momenta are restricted to the PG$^-$ sector. To calculate $\delta\bar{E}_\varphi$, we rewrite it in the form $\delta\bar{E}_\varphi = \delta\bar{E}_{\varphi 1} + \delta\bar{E}_{\varphi 2}$, where

$$\begin{aligned} \delta\bar{E}_{\varphi 1} &\coloneqq \frac{1}{2}\omega^{ij}{}_\varphi \delta\bar{H}_{ij} + \frac{1}{2}\delta\omega^{ij}\bar{H}_{ij\varphi}, \\ \delta\bar{E}_{\varphi 2} &\coloneqq b^i{}_\varphi \delta\bar{H}_i + \delta b^i \bar{H}_{i\varphi}, \end{aligned} \tag{4.2}$$

and the integration over S_∞ is implicitly understood. Using the relations (A1), one finds that the nontrivial content of $\delta\bar{E}_{\varphi 1}$ is given by

$$\delta\bar{E}_{\varphi 1} = \delta(\omega^{02}{}_\varphi \bar{H}_{02\theta\varphi} + \omega^{12}{}_\varphi \delta\bar{H}_{12\theta\varphi} + \omega^{23}{}_\varphi \delta\bar{H}_{23\theta\varphi}) d\theta d\varphi.$$

For large r, $\delta\bar{E}_{\varphi 1}$ is quadratically divergent,

$$\delta\bar{E}_{\varphi 1} = \alpha_2 \lambda r^2 + \alpha_0 + O_1. \tag{4.3}$$

Since the coefficients α_2 and α_0 contain only *m-independent* δa terms, the rule (r1′) implies that their contribution should be disregarded. Hence, the integration over S_∞ yields effectively $\delta\bar{E}_{\varphi 1} = 0$. Similar analysis of

$$\delta\bar{E}_{\varphi 2} = \delta(b^0{}_\varphi \bar{H}_{0\theta\varphi} + b^3{}_\varphi \bar{H}_{3\theta\varphi}) d\theta d\varphi$$

yields $\delta\bar{E}_{\varphi 2}=0$. Transition to the new coordinates (T,ϕ) implies $\delta\bar{E}_\phi=\delta\bar{E}_\varphi$. Hence:

(ii) The variation of angular momentum associated to the PG$^-$ sector vanishes,

$$\delta\bar{E}_\phi=\delta\bar{E}_{\varphi 1}+\delta\bar{E}_{\varphi 2}=0. \tag{4.4}$$

B. Energy

Following the procedure used for Kerr-AdS$^+$ black holes [9,10], energy stemming from the PG$^-$ sector is calculated in two steps. First, consider the naive expression $\bar{E}_t\coloneqq\delta\Gamma_\infty(\partial_t)$, where the covariant momenta are restricted to the PG$^-$ sector and represented it as the sum of two terms,

$$\begin{aligned}\delta\bar{E}_{t1}&=\frac{1}{2}\omega^{ij}{}_t\delta\bar{H}_{ij}+\frac{1}{2}\delta\omega^{ij}\bar{H}_{ijt},\\ \delta\bar{E}_{t2}&=b^i{}_t\delta\bar{H}_i+\delta b^i\bar{H}_{it}.\end{aligned} \tag{4.5}$$

Then, the relations (A2) allow one to reduce the form of $\delta\bar{E}_{t1}$ as

$$\begin{aligned}\delta\bar{E}_{t1}=(&\omega^{23}{}_t\delta\bar{H}_{23\theta\varphi}+\delta\omega^{03}{}_\theta\bar{H}_{03t\varphi}+\delta\omega^{13}{}_\theta\bar{H}_{13t\varphi}\\ &-\delta\omega^{02}{}_\varphi\bar{H}_{02t\theta}-\delta\omega^{23}{}_\varphi\bar{H}_{23t\theta})d\theta d\varphi.\end{aligned}$$

For large r, $\delta\bar{E}_{t1}$ is found to be

$$\delta\bar{E}_{t1}=\beta_2\lambda r^2+\beta_0+O_1, \tag{4.6}$$

where β_2 and β_0 are m independent and proportional to δa; thus, the integration over S_∞ yields $\delta\bar{E}_{t1}=0$. In a similar manner, the relation

$$\begin{aligned}\delta\bar{E}_{t2}=(&b^0{}_t\delta\bar{H}_0+b^3{}_t\delta\bar{H}_3+\delta b^0\bar{H}_{0t}+\delta b^2\bar{H}_{2t}+\delta b^3\bar{H}_{3t})\\ &\times d\theta d\varphi\end{aligned}$$

implies that $\delta\bar{E}_{t2}$ also vanishes. Hence, $\delta\bar{E}_t\equiv\delta\bar{E}_{t1}+\delta\bar{E}_{t2}=0$.

In the second step, after going over to the (T,ϕ) coordinates, one finds the following:

(iii) The variation of energy associated to the PG$^-$ sector also vanishes,

$$\delta\bar{E}_T=\delta\bar{E}_t+\lambda a\delta\bar{E}_\varphi=0. \tag{4.7}$$

V. ENTROPY

Consider the variational equation for $\delta\Gamma_H$ restricted to the PG$^-$ sector, where

$$\xi\coloneqq\partial_T-\Omega_+\partial_\varphi,\qquad \Omega_+\coloneqq\omega_++\lambda a=\frac{a(1+\lambda r_+^2)}{r_+^2+a^2}, \tag{5.1a}$$

and Ω_+ is the angular velocity in the new coordinates (T,ϕ). Moreover, we use the notation

$$\begin{aligned}&\bar{A}_0\coloneqq\bar{a}_0-\lambda\bar{b}_6,\\ &Y^A{}_\xi\coloneqq\xi\rfloor Y^A,\quad\text{where } Y^A=(b^i,\omega^{ij},H_i,H_{ij}).\end{aligned} \tag{5.1b}$$

For convenience, the expression $\delta\Gamma_H$ is divided into two parts, $\delta\Gamma_1$ and $\delta\Gamma_2$, and in further calculations, we rely on Eq. (A3).

A. $\delta\Gamma_1=\frac{1}{2}\omega^{ij}{}_\xi\delta H_{ij}+\frac{1}{2}\delta\omega^{ij}H_{ij\xi}$

The only nontrivial contributions to the first term in $\delta\Gamma_1$ are

$$\omega^{23}{}_\xi\delta H_{23\theta\varphi}=K^{23}{}_\xi\delta H_{23\theta\varphi}=2\bar{A}_0\cdot\frac{2amr_+}{(r_+^2+a^2)\rho_+^2}\delta\left(\frac{r_+^2+a^2}{\alpha}\right)\sin\theta\cos\theta, \tag{5.2a}$$

$$\begin{aligned}\omega^{02}{}_\xi\delta H_{02\theta\varphi}+\omega^{12}{}_\xi\delta H_{12\theta\varphi}&=\tilde{\omega}^{02}{}_\xi\delta H_{02\theta\varphi}+K^{02}{}_\xi\delta(H_{02\theta\varphi}+H_{12\theta\varphi})\\ &=\left[2\lambda\bar{b}_4\frac{Na^2}{P(r_+^2+a^2)}\delta\left(\frac{mr_+}{N\rho_+^2}P\frac{a}{\alpha}\right)-2\bar{A}_0\frac{a^2mr_+}{NP(r_+^2+a^2)\rho_+^2}\delta\left(PN\frac{a}{\alpha}\right)\right]\sin^3\theta\cos\theta.\end{aligned} \tag{5.2b}$$

The second term in $\delta\Gamma_1$ is determined by

$$\begin{aligned}\delta\omega^{03}H_{03\xi}+\omega^{13}\delta H_{13\xi}&=\delta\tilde{\omega}^{03}{}_\theta H_{03\xi\varphi}+\delta K^{03}{}_\theta(H_{03\xi\varphi}+H_{13\xi\varphi})\\ &=\left[-2\lambda\bar{b}_4\delta\left(\frac{aNP}{\rho_+^2}\right)\frac{mr_+}{NP\alpha}+2\bar{A}_0\delta\left(\frac{amPr_+}{N\rho_+^4}\right)\frac{N\rho_+^2}{P\alpha}\right]\sin\theta\cos\theta,\end{aligned} \tag{5.3a}$$

$$\begin{aligned}\delta\omega^{02}H_{02\xi}+\delta\omega^{12}H_{12\xi}&=-\delta\tilde{\omega}^{02}{}_\varphi H_{02\xi\theta}-\delta K^{02}{}_\varphi(H_{02\xi\theta}+H_{12\xi\theta})\\ &=\left[-2\lambda\bar{b}_4\delta\left(\frac{aN}{P\alpha}\right)\frac{mr_+P}{N(r_+^2+a^2)}+2\bar{A}_0\delta\left(\frac{amr_+}{NP\alpha\rho_+^2}\right)\frac{NP\rho_+^2}{r_+^2+a^2}\right]\sin\theta\cos\theta.\end{aligned} \tag{5.3b}$$

B. $\delta\Gamma_2 = b^i{}_\xi \delta H_i + \delta b^i H_{i\xi}$

The analysis of the nontrivial content of $\delta\Gamma_2$ yields

$$b^0{}_\xi \delta H_0 = b^0{}_\xi \delta H_{0\theta\varphi} = -2\bar{a}_1 \frac{N\rho_+^2}{r_+^2 + a^2} \cdot \delta\left[\frac{amr_+}{N\alpha\rho_+^4}(r_+^2 + a^2 + \rho_+^2)\right] \sin\theta\cos\theta, \tag{5.4a}$$

$$\delta b^0 H_{0\xi} = -\delta b^0{}_\varphi H_{0\xi\theta} = 2\bar{a}_1 \delta\left(\frac{Na}{\alpha}\right) \cdot \frac{a^2 m r_+}{N(r_+^2 + a^2)\rho_+^2} \sin^3\theta\cos\theta, \tag{5.4b}$$

$$\delta b^2 H_{2\xi} = \delta P H_{2\xi\varphi} = -2\bar{a}_1 \delta P \cdot \frac{amr_+}{P\alpha\rho_+^2} \sin\theta\cos\theta, \tag{5.4c}$$

$$\delta b^3 H_{3\xi} = -\delta b^3{}_\varphi H_{3\xi\theta} = -2\bar{a}_1 \delta\left(\frac{r_+^2 + a^2}{P\alpha}\right) \cdot \frac{amr_+ P}{(r_+^2 + a^2)\rho_+^2} \sin\theta\cos\theta. \tag{5.4d}$$

C. Calculations and the result

In order to obtain the entropy associated to the PG$^-$ sector, one could now apply the systematic procedure formulated in Ref. [10] to calculate $\delta\Gamma_H \equiv \delta\Gamma_1 + \delta\Gamma_2$. However, the procedure can be enormously shortened by noting that each term in Eqs. (5.2)–(5.4) is given as an integral of the form

$$I = \int_0^\pi d\theta f(\cos^2\theta)\cos\theta\sin\theta. \tag{5.5}$$

Then, the change of variables $x = \cos\theta$ implies $I = 0$, and consequently the following:

(iv) The variation of entropy associated to the PG$^-$ sector vanishes,

$$\delta\Gamma_H \equiv T\delta\bar{S} = 0. \tag{5.6}$$

VI. CONCLUDING REMARKS

In the present paper, we performed a Hamiltonian analysis of the thermodynamic charges for Kerr-AdS black holes in the general PG, with both even and odd parity modes.

Our methodology is compactly formulated by the variational equations (2.5), accompanied by the basic set of two rules, (r1) and (r2), for the variation δ. When the background configuration is an AdS spacetime, the validity of the rule (r1) is ensured by an additional instruction on how the variation of the background should be avoided; see (r1$'$) in Sec. IV. These rules are a variational counterpart of the asymptotic conditions used in Ref. [20], as well as in $[5]_2$, in their analyses of Kerr-AdS spacetimes.

Kerr-AdS solutions in PG can be understood as a superposition of two contributions, associated to the PG$^+$ and PG$^-$ sectors of PG. The thermodynamic charges originating from PG$^-$ are found to be vanishing. Thus:

(v) Asymptotic charges and entropy of Kerr-AdS black holes in PG [16] coincide with the corresponding expressions for Kerr-AdS$^+$ black holes, found earlier in PG$^+$ [10]. With $T \coloneqq \kappa/2\pi$, we have

$$E_T = 16\pi a_1 \frac{m}{\alpha^2}, \qquad E_\phi = 16\pi a_1 \frac{ma}{\alpha^2}, \qquad S = 16\pi a_1 \frac{A_H}{4}. \tag{6.1}$$

Using the Pontryagin and Nieh-Yan topological invariants, the effective form of L_G^- in (3.9) can be reduced to just two terms, ${}^\star X$ and $R^{ij(6)}R_{ij} \sim {}^\star X$. Thus, in spite of the fact that the thermodynamic charges in the parity odd sector vanish, the sector itself is not dynamically trivial—it is essentially equivalent to the term ${}^\star X$.

Kerr-AdS solutions cannot be consistently reduced to the pure parity odd sector PG$^-$. Namely, for $a_0, \Lambda = 0$, the parameter λ would remain undetermined (since $3a_0\lambda = \Lambda$), which would be a degenerate situation. However, they can be consistently restricted to the PG$^-$ sector extended by the nonvanishing (a_0, Λ). Although the resulting solution (Kerr-AdS$^-$)$'$ is interesting in its own right, the Kerr-AdS spacetime is not a proper superposition of Kerr-AdS$^+$ and (Kerr-AdS$^-$)$'$, as their parameters overlap.

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APPENDIX: USEFUL FORMULAS

In this appendix, we present technical formulas which are used in deriving the expressions for the asymptotic charges and entropy, in Secs. IV and V,

$$H^{01}{}_{\theta\varphi} = H^{03}{}_{\theta\varphi} = H^{13}{}_{\theta\varphi} = 0, \qquad b^1{}_\varphi = b^2{}_\varphi = 0, \qquad \tilde{\omega}^{03}{}_\varphi = \tilde{\omega}^{12}{}_\varphi = 0, \tag{A1}$$

$$\begin{gathered} H^{01}{}_{t\varphi} = H^{02}{}_{t\varphi} = H^{12}{}_{t\varphi} = H^{23}{}_{t\varphi} = 0, \\ H^{01}{}_{t\theta} = H^{03}{}_{t\theta} = H^{13}{}_{t\theta} = 0, \\ b^1{}_t = b^2{}_t = 0, \qquad \omega^{03}{}_t = \omega^{12}{}_t = 0, \\ \tilde{\omega}^{01}{}_\theta = \tilde{\omega}^{02}{}_\theta = \tilde{\omega}^{13}{}_\theta = \tilde{\omega}^{23}{}_\theta = 0. \end{gathered} \tag{A2}$$

$$
\begin{aligned}
&{b^0}_\xi = N\frac{\rho_+^2}{r_+^2+a^2} \qquad {b^a}_\xi|_{r_+} = 0 \quad a = 1,2,3;\\
&{H^{01}}_\xi = {H^{23}}_\xi = 0, \qquad {H^{03}}_{\xi\theta} = {H^{13}}_{\xi\theta} = 0, \qquad {H^{02}}_{\xi\varphi} = {H^{12}}_{\xi\varphi} = 0,\\
&{\tilde\omega^{03}}_\xi = {\tilde\omega^{12}}_\xi = {\tilde\omega^{23}}_\xi = 0, \qquad {\tilde\omega^{23}}_\xi = O(N^2),\\
&{\tilde\omega^{12}}_\varphi = 0, \qquad {\tilde\omega^{13}}_\theta = 0, \qquad {K^{01}}_\theta = {K^{23}}_\theta = 0.
\end{aligned} \tag{A3}
$$

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