

Cooling process of brown dwarfs in Palatini $f(\mathcal{R})$ gravity

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(Received 11 January 2021; accepted 15 February 2021; published 19 March 2021)

We present an analytical model for the evolution of brown dwarfs in quadratic Palatini $f(\mathcal{R})$ gravity. We improve previous studies by adopting a more realistic description of the partially degenerate state that characterizes brown dwarfs. Furthermore, we take into account the hydrogen metallic-molecular phase transition between the interior of the brown dwarf and its photosphere. For such an improved model, we revise the cooling process of substellar objects.

DOI: [10.1103/PhysRevD.103.064032](https://doi.org/10.1103/PhysRevD.103.064032)

I. INTRODUCTION

Dark Matter (DM) provides a consistent explanation of gravitational phenomena at spatial scales that roughly span up to 10 orders of magnitude. It explains the cosmic microwave background power spectrum [1] and the formation of structures in the Universe [2]. Furthermore, DM is invoked to explain the mismatch between the observed dynamical mass and that inferred from observations of the visible component in Galaxy clusters [3], elliptical and spiral galaxies [4], and in dwarf and ultrafaint dwarf galaxies [5,6]. Nonetheless, the nature of the DM remains unknown and none of the proposed candidates has been detected so far.

An alternative proposal to explain the observed discrepancy in the data is to modify the theory of gravity. Several modifications of gravity have been proposed in the literature (e.g., [7–13]). These proposals are able to describe gravitational phenomena at different scales, such as the rotation curve in spiral galaxies [14,15], the accelerating expansion of the Universe [16], or issues related to stellar structure (e.g., [17–20]). However, it is uncertain whether modifications of gravity are able to provide a coherent explanation at all scales. In this work, we focus on Palatini $f(\mathcal{R})$ gravity, and in particular on the Starobinsky (quadratic) model. For this theory of gravity, we present an analytical study of the evolution of brown dwarfs (BDs).

In vacuum, Palatini gravity—independently of the $f(\mathcal{R})$ model—turns out to be Einstein’s theory with a cosmological constant [11,12,21]. Moreover, the field equations are second order partially differential equations (PDE) with respect to the metric $f(R)$ gravity (the metric formalism

gives fourth order PDE with respect to the metric¹). There is no extra degree of freedom and, the most important feature for our purposes, is that the stellar equations are changed. Furthermore, Palatini gravity alters the early universe physics, explains the acceleration expansion of the late universe, provides different black hole solutions, or produces wormholes with no exotic fluid [23–44]. It passes current solar system tests [45] since the modifications of energy and momentum appearing in the Euler equation turn out not to be sensitive enough to the experiments performed for the solar system orbits [46]. The situation may, however, change when experiments on an atomic level will be available [47–49].

Apart from compact stars [50–56] and black holes, whose properties are not yet fully understood [57,58], there exists a class of stellar and substellar objects which turn out to be well suited to test gravitational theories (e.g., [59]). In particular, BDs are substellar objects that are not massive enough to sustain stable hydrogen burning, and thus they cool down as they age. The gravitational force in BDs is balanced by electron degeneracy pressure in their cores and thermal pressure in their atmospheres. These partially degenerate objects have masses smaller than the hydrogen-minimum mass which, according to evolutionary models in General Relativity (GR), is $\sim 0.075 M_{\odot}$ for a solar composition [60]. BDs emit mainly in the infrared. Since they have low luminosities, they are difficult to observe and they are mostly detected in the solar neighborhood. With the advent of wide-field surveys, large and homogeneous samples of BDs have been recently constructed [61–65], thus enabling statistical analyses of BDs that could constrain structural properties of our Galaxy [64], the substellar mass function, or sub-GeV DM particle models [66].

¹However, the field equations can be rewritten as second order PDE with respect to the metric, and a dynamical equation for a scalar field, which gives an extra degree of freedom (e.g., [10,22]).

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In this work, we analytically study the time evolution of BDs in quadratic Palatini $f(\mathcal{R})$ gravity. We improve previous studies by including a better description of the partially degenerate state that characterizes BDs. The structure of the paper is as follows: in Sec. II we discuss the basic elements of Palatini $f(\mathcal{R})$ gravity, and the analytical model for the time evolution of BDs is presented in Sec. III. Finally, we conclude in Sec. IV.

II. PALATINI $f(\mathcal{R})$ CHEAT SHEET

Let us briefly recall the basic elements of Palatini gravity, which is the simplest example of metric-affine theories of gravity. Instead of taking the linear in \mathcal{R} Lagrangian, we will consider an arbitrary, but analytical [46], functional $f(\mathcal{R})$. The action is then written as

$$S = S_g + S_m = \frac{1}{2\kappa} \int \sqrt{-g} f(\mathcal{R}) d^4x + S_m[g_{\mu\nu}, \psi_m], \quad (1)$$

where $\mathcal{R} = \mathcal{R}^{\mu\nu} g_{\mu\nu}$ is the Ricci scalar constructed with the metric $g_{\mu\nu}$ and the Ricci tensor $\mathcal{R}^{\mu\nu}$. The latter is a function of the independent connection $\hat{\Gamma}$. Adopting this, we abandon the common assumption on g -metricity, and thus, the connection $\hat{\Gamma}$ might be independent of the metric $g_{\mu\nu}$. Before going further, let us comment that we use the $(-+++)$ metric signature convention and we follow Weinberg's $\kappa = -\frac{8\pi G}{c^4}$ [67].

The variation of the action (1) with respect to the metric $g_{\mu\nu}$ provides the following field equations:

$$f'(\mathcal{R})\mathcal{R}_{\mu\nu} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} = \kappa T_{\mu\nu}, \quad (2)$$

where $T_{\mu\nu}$ is the energy momentum tensor of the matter field, i.e.,

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g_{\mu\nu}}. \quad (3)$$

The perfect fluid energy tensor will be assumed to describe low-mass stars. The prime in Eq. (2) denotes derivation with respect to the function's argument, that is, $f'(\mathcal{R}) = \frac{df(\mathcal{R})}{d\mathcal{R}}$.

On the other hand, varying the action with respect to the connection $\hat{\Gamma}$ provides

$$\hat{\nabla}_\beta(\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}) = 0, \quad (4)$$

which indicates that $\hat{\nabla}_\beta$ is the covariant derivative obtained with respect to $\hat{\Gamma}$. Following this, we notice that there exists a conformal metric

$$h_{\mu\nu} = f'(\mathcal{R})g_{\mu\nu}, \quad (5)$$

for which $\hat{\Gamma}$ is the Levi-Civita connection.

The trace of Eq. (2) taken with respect to $g_{\mu\nu}$ provides the structural equation

$$f'(\mathcal{R})\mathcal{R} - 2f(\mathcal{R}) = \kappa T, \quad (6)$$

where T is the trace of the energy-momentum tensor $T_{\mu\nu}$. When a suitable functional form of $f(\mathcal{R})$ is chosen, it is possible to solve the structural equation (6) in order to obtain the relation between the Palatini-Ricci curvature scalar \mathcal{R} and the energy momentum trace T , i.e., $\mathcal{R} = \mathcal{R}(T)$. An important feature of Palatini gravity, independently of the $f(\mathcal{R})$ form, is that in vacuum—as derived from Eq. (6)—the Einstein's vacuum solution with the cosmological constant is recovered.

One can rewrite the field equations (2) as dynamical equations for the conformal metric $h_{\mu\nu}$ [22,25] and the undynamic scalar field denoted as $\Phi = f'(\mathcal{R})$:

$$\bar{R}_{\mu\nu} - \frac{1}{2}h_{\mu\nu}\bar{R} = \kappa\bar{T}_{\mu\nu} - \frac{1}{2}h_{\mu\nu}\bar{U}(\Phi), \quad (7a)$$

$$\Phi\bar{R} - (\Phi^2\bar{U}(\Phi))' = 0, \quad (7b)$$

where $\bar{U}(\Phi) = \frac{\mathcal{R}\Phi - f(\mathcal{R})}{\Phi^2}$ and the energy momentum tensor in Einstein's frame is $\bar{T}_{\mu\nu} = \Phi^{-1}T_{\mu\nu}$. It was demonstrated in several works [40,68–71] that this representation of the Palatini $f(\mathcal{R})$ gravity simplifies examinations of physical problems.

In our work we will focus on the quadratic (Starobinsky) functional form of $f(\mathcal{R})$, i.e.,

$$f(\mathcal{R}) = \mathcal{R} + \beta\mathcal{R}^2, \quad (8)$$

where β is the Starobinsky parameter with dimension $[m^{-2}]$. Later on, we will introduce the parameter α which is related to the Starobinsky parameter β .

III. BROWN DWARF'S MODEL

In this section we improve the brown dwarf's analytical model for Palatini $f(\mathcal{R})$ gravity considered in [72]. The main difference with respect to the previous work is related to the equation of state (EoS). The model discussed in [72] used the polytropic EoS which works well in the degenerate and ideal gas extremes, being, however, a poor description for the intermediate zone, when one deals with a mixture of these two gases [73]. In this work we consider the EoS first presented in [74] which better describes a mixture of degenerate and ideal gas states at finite temperature. We then provide a simple cooling model for these substellar objects.

A. Equation of state for a partially degenerate Fermi gas

The barotropic EoS $p = p(\rho)$ —where p and ρ are the pressure and energy density, respectively—which accounts

for a mixture of a degenerate Fermi gas of electrons at a finite temperature and a gas of ionized hydrogen and helium is given as follows [74]:

$$p = C \left(\frac{\rho}{\mu_e} \right)^{\frac{5}{3}} \left[1 - \frac{5}{16} \Psi \ln(1 + e^{-1/\Psi}) + \frac{15}{8} \Psi^2 \left(\frac{\pi^2}{3} + \text{Li}_2[-e^{-1/\Psi}] \right) + a \Psi \right], \quad (9)$$

with $C = 10^{13} \text{ cm}^4 \text{ g}^{-2/3} \text{ s}^{-2}$, Li_2 denotes the second order polylogarithm function and the number of baryons per electron is given by $1/\mu_e = X + Y/2$, where X and Y are the mass fractions of hydrogen and helium, respectively. The degeneracy parameter Ψ is defined as

$$\Psi = \frac{k_B T}{\mu_F} = \frac{2m_e k_B T}{(3\pi^2 \hbar^3)^{2/3}} \left(\frac{\mu_e}{\rho_c N_A} \right)^{2/3}, \quad (10)$$

where μ_F is the electron's Fermi energy in the degenerate limit, T is the gas temperature, ρ_c is the density of the BDs' core, and the rest constants have the usual meaning. Finally, the quantity $a = \frac{5}{2} \mu_e \mu_1^{-1}$ with μ_1 defined as

$$\frac{1}{\mu_1} = (1 + x_{H^+})X + \frac{Y}{4}, \quad (11)$$

where x_{H^+} is the ionization fraction of hydrogen. This fraction changes from the completely ionized core to the surface of the BD, which is composed of molecular hydrogen and helium [74]. Its values depend on the phase transition points [75] to which we will come back later.

The EoS (9) has a familiar polytropic form for $n = 3/2$,

$$p = K \rho^{1+\frac{1}{n}} \quad (12)$$

with $K = C \mu_e^{-\frac{5}{3}} (1 + b + a \Psi)$ and

$$b = -\frac{5}{16} \Psi \ln(1 + e^{-1/\Psi}) + \frac{15}{8} \Psi^2 \left(\frac{\pi^2}{3} + \text{Li}_2[-e^{-1/\Psi}] \right), \quad (13)$$

which takes into account the corrections due to the finite temperature of the gas. Since we are interested in BDs, the polytropic models with $n = 3/2$ together with such improvements are well suited to describe these substellar objects.

The hydrostatic equilibrium equations in the Einstein frame were shown to be in our case [68]

$$-\bar{r}^2 \Phi(\bar{r}) \frac{d}{d\bar{r}} p = G \mathcal{M}(\bar{r}) \rho(\bar{r}), \quad \mathcal{M}(\bar{r}) \approx \int_0^{\bar{r}} 4\pi \rho \bar{r}^2 d\bar{r}, \quad (14)$$

which can be further rewritten, after introducing the standard dimensionless variables

$$\bar{r} = r_c \bar{\xi}, \quad \rho = \rho_c \theta^n, \quad p = p_c \theta^{n+1}, \quad (15)$$

$$r_c^2 = \frac{(n+1)p_c}{4\pi G \rho_c^2}, \quad (16)$$

as the Lane-Emden equation, which for quadratic Palatini $f(\mathcal{R})$ gravity after coming back to the Jordan frame ($\bar{\xi}^2 = \Phi \xi^2$) is given by

$$\frac{1}{\xi} \frac{d^2}{d\xi^2} \left[\sqrt{\Phi} \xi \left(\theta - \frac{4\kappa^2 c^2 \rho_c \alpha}{5} \theta^{\frac{5}{2}} \right) \right] = -\frac{(\Phi + \frac{1}{2} \xi \frac{d\Phi}{d\xi})^2}{\sqrt{\Phi}} \theta^{\frac{3}{2}}, \quad (17)$$

where $\Phi = 1 + 2\alpha \theta^{\frac{5}{2}}$, $\alpha = \kappa c^2 \beta \rho_c$, and $\kappa = -\frac{8\pi G}{c^4}$ [67]. For simplification, from now on we will use the parameter α instead of the Starobinsky parameter β . Moreover, the range of α is $(-0.5; +\infty)$. The parameters ρ_c and p_c stands for central density and central pressure, respectively. In such a framework, the temperature can be expressed as $T = T_c \theta(\xi)$, with T_c being the central temperature, while the density is $\rho = \rho_c \theta^{3/2}(\xi)$. The function $\theta(\xi)$ is the solution of the (modified) Lane-Emden equation with respect to the radial coordinate $\xi = r \rho_c \sqrt{8\pi G} / (2p_c)$. The solution $\theta(\xi)$ crosses zero at ξ_R which corresponds to the dimensionless BD's radius. For a more detailed discussion see e.g., [68].

The BD's radius R , central density ρ_c , and pressure p can be obtained by numerically solving Eq. (17). For an arbitrary polytropic parameter n , these parameters are expressed as

$$R = \gamma_n \left(\frac{K}{G} \right)^{\frac{n}{3-n}} M^{\frac{n-1}{n-3}}, \quad (18)$$

$$\rho_c = \delta_n \left(\frac{3M}{4\pi R^3} \right), \quad (19)$$

$$p = K \rho_c^{\frac{n+1}{n}} \theta^{n+1}. \quad (20)$$

The values of the parameters γ_n and δ_n depend on the adopted theory of gravity. In the case of Palatini $f(\mathcal{R})$ gravity, they take the following forms [76]:

$$\gamma_n = (4\pi)^{\frac{1}{n-3}} (n+1)^{\frac{n}{n-3}} \omega_n^{\frac{n-1}{n-3}} \xi_R, \quad (21)$$

$$\delta_n = -\frac{\xi_R}{3 \frac{\Phi^{\frac{1}{2}}}{1 + \frac{1}{2} \xi \frac{d\Phi}{d\xi}} \Big|_{\xi=\xi_R}}, \quad (22)$$

$$\omega_n = -\frac{\xi^2 \Phi^{\frac{3}{2}}}{1 + \frac{1}{2} \xi \frac{d\Phi}{d\xi}} \Big|_{\xi=\xi_R}. \quad (23)$$

Using the above definitions, the stars' external and internal characteristics can be obtained for the EoS given by Eq. (9)

as functions of the mass M and the degeneracy parameter Ψ . That is,²

$$R = 1.19141 \times 10^9 \gamma \left(\frac{M_\odot}{M} \right)^{\frac{1}{3}} \mu_e^{-\frac{5}{3}} (1 + b + a\Psi) \text{ [cm]}, \quad (24)$$

$$\rho_c = 2.80791 \times 10^5 \frac{\delta}{\gamma^3} \left(\frac{M}{M_\odot} \right)^2 \frac{\mu_e^5}{(1 + b + a\Psi)^3} \text{ [g/cm}^3\text{]}, \quad (25)$$

$$p_c = 1.20403 \times 10^9 \frac{\delta^{5/3}}{\gamma^5} \left(\frac{M}{M_\odot} \right)^{10/3} \frac{\mu_e^{20/3}}{(1 + b + a\Psi)^4} \text{ [Mbar]}. \quad (26)$$

Furthermore, the central temperature takes the following form:

$$T_c = 1.29396 \times 10^9 \frac{\delta^{2/3}}{\gamma^2} \left(\frac{M}{M_\odot} \right)^{4/3} \frac{\mu_e^{8/3}}{\Psi(1 + b + a\Psi)^2} \text{ [K]} \quad (27)$$

when one uses Eq. (25) together with the definition of the degeneracy parameter (10). Let us just emphasize that in the above formulas, the values of γ and δ depend on the solution of the modified Lane-Emden equation (17) with respect to the value of α .

B. Brown dwarfs' surface properties

Although BDs are simpler than compact objects, there are still missing elements in the theoretical and numerical modeling of the BD's interior. This introduces sizeable uncertainties in the predicted surface temperature. Despite this, there exist models which allow one to express the surface temperature in an analytical form—that is based on the isentropic BD's interior and the phase transition between the interior and the photosphere—which is convenient for our purposes. In [75] it was shown that a first order phase transition for the metallization of hydrogen happens for pressure and temperatures suitable for giant planets and BDs. The effective temperature T_{eff} can be written then in terms of the degeneracy parameter Ψ and the photospheric density ρ_{ph} as [74]

$$T_{\text{eff}} = b_1 \times 10^6 \rho_{\text{ph}}^{0.4} \Psi^\nu \text{ K}, \quad (28)$$

where the values of the parameters b_1 and ν depend on the specific model adopted for describing the phase transition between a metallic hydrogen and helium state that characterizes the BD's interior and the photosphere, which is composed of molecular hydrogen and helium. We adopt different models presented in [75] which are summarized in Table I.

The surface temperature, given by Eq. (28), is obtained from matching the entropy in the BD's interior,

TABLE I. Different metallic-molecular phase transition points taken from [75].

Model	x_{H^+}	b_1	ν
A	0.240	2.87	1.58
B	0.250	2.70	1.59
C	0.250	2.26	1.59
D	0.255	2.00	1.60
E	0.260	1.68	1.61
F	0.250	1.29	1.59
G	0.165	0.60	1.44
H	0.090	0.40	1.30

$$S_{\text{interior}} = \frac{3 k_B N_A}{2 \mu_{1\text{mod}}} (\ln \Psi + 12.7065) + C_1, \quad (29)$$

where C_1 is an integration constant of the first law of thermodynamics while

$$\frac{1}{\mu_{1\text{mod}}} = \frac{1}{\mu_1} + \frac{3 x_{H^+} (1 - x_{H^+})}{2 - x_{H^+}}, \quad (30)$$

with the photospheric entropy of nonionized molecular hydrogen and helium mixture [74]. The detailed derivation of this temperature can be found in [74] and in [77,78].

In order to estimate the surface luminosity, we will follow the approach presented in [77]. The surface of a star can be assumed to lie at the photosphere which is defined at the radius for which the optical depth equals $2/3$, i.e.,

$$\tau(r) = \kappa_R \int_r^\infty \rho dr = \frac{2}{3}, \quad (31)$$

where $\kappa_R (= 0.01 \text{ cm}^2/\text{g})$ is Rosseland's mean opacity. Since the radius of the photosphere is very close to the stellar radius, we approximate the surface gravity as a constant. That is,

$$g \equiv \frac{Gm(r)}{r^2} \sim \frac{GM}{R^2} = \text{const}, \quad (32)$$

where $M = m(R)$. Then, the hydrostatic equilibrium for the Palatini quadratic model is written as [72]

$$p' = -g\rho(1 + \kappa c^2 \beta [r\rho' - 3\rho]), \quad (33)$$

where now $' \equiv d/dr$ in the Jordan frame. The mass function of a nonrelativistic star in our model can be approximated to the familiar form $m'(r) = 4\pi r^2 \rho(r)$ allowing us to write

$$m'' = 8\pi r \rho + 4\pi r^2 \rho', \quad (34)$$

where the second derivation of the mass m is given by differentiating equation (32). Using this in Eq. (33) we may write

$$p' = -g\rho \left(1 + 8\beta \frac{g}{c^2 r} \right), \quad (35)$$

²We drop the subindex $3/2$ from $\gamma_{3/2}$ and $\delta_{3/2}$ for convenience.

which with the help of Eq. (31) we integrate to the following form:

$$p_{\text{ph}} = \frac{2}{3\kappa_R} \frac{GM(1 + 8\beta \frac{GM}{c^2 R^3})}{R^2}. \quad (36)$$

The EoS (9) near to the photosphere, where the degeneracy is negligible, provides the photospheric pressure in the ideal gas form

$$p_{\text{ph}} = \frac{\rho_{\text{ph}} N_A k_B T_{\text{eff}}}{\mu_2} \quad (37)$$

with $1/\mu_2 = X/2 + Y/4$. Using the radius given by Eq. (24) and the central density in Eq. (25), the photospheric pressure can be written as a function of mass M and the degeneracy parameter Ψ as

$$p_{\text{ph}} = \frac{62.3488}{\kappa_R \gamma^2} \left(\frac{M}{M_\odot} \right)^{5/3} \frac{\mu_e^{10/3} (1 - 1.33 \frac{\alpha}{\delta})}{(1 + b + a\Psi)^2} \text{ bar}. \quad (38)$$

Combining this with Eq. (37) and the surface temperature given by (28), we may express, in a similar way, the photospheric density as a function of the mass and the degeneracy parameter, i.e.,

$$\rho_{\text{ph}} = \frac{62.3488}{\kappa_R \gamma^2 N_A k_B} \left(\frac{M}{M_\odot} \right)^{5/3} \frac{\mu_e^{10/3} \mu_2 (1 - 1.33 \frac{\alpha}{\delta})}{(1 + b + a\Psi)^2 b_1 \Psi^\nu} \frac{\text{g}}{\text{cm}^3}. \quad (39)$$

Finally, the photospheric temperature has the following form:

$$T_{\text{eff}} = \frac{2.558 \times 10^4 \text{ K}}{\kappa_R^{0.286}} \left(\frac{M}{M_\odot} \right)^{0.4764} \frac{\Psi^{0.714\nu}}{(1 + b + a\Psi)^{0.571}} \times b_1^{0.714} \left(1 - 1.33 \frac{\alpha}{\delta} \right)^{0.286} \gamma^{-0.572}, \quad (40)$$

where $\mu_e = 1.143$ and $\mu_2 = 2.286$ were used.

Assuming black body radiation and using the Stefan-Boltzman law $L = 4\pi R^2 \sigma T_{\text{eff}}^4$, where σ is the Stefan-Boltzmann constant, we easily obtain the BDs' luminosity as a function of mass and the degeneracy parameter in the following form:

$$L = \frac{0.0721 L_\odot}{\kappa_R^{1.1424}} \left(\frac{M}{M_\odot} \right)^{1.239} \frac{\Psi^{2.856\nu}}{(1 + b + a\Psi)^{0.2848}} \times b_1^{2.856} \left(1 - 1.33 \frac{\alpha}{\delta} \right)^{1.143} \gamma^{-0.286}. \quad (41)$$

C. Cooling model for brown dwarfs

In order to express the luminosity (41) as a function of time t , we need to find out an evolutionary equation for the degeneracy parameter Ψ . Following the steps of [73,78], together with the improved EoS first obtained in [74] and

used in this work, the evolution of the luminosity of BDs as a function of time for Palatini gravity can be found.

Applying the energy equation from the first and the second laws of thermodynamics we may describe the pace of cooling and contraction related to such objects as

$$\frac{dE}{dt} + p \frac{dV}{dt} = T \frac{dS}{dt} = \dot{\epsilon} - \frac{\partial L}{\partial M}, \quad (42)$$

where S is the entropy per unit mass while other symbols have standard meaning. BDs are not massive enough to sustain stable hydrogen burning; therefore, they cool as they age and the energy generation term $\dot{\epsilon}$ can be ignored. Integrating the previous equation over mass one has

$$\frac{d\sigma}{dt} \left[\int N_A k_B T dM \right] = -L, \quad (43)$$

where L is a surface luminosity and $\sigma = S/k_B N_A$. Using Eq. (10) and the polytropic relation given by (12) in order to get rid of T and ρ , we may write

$$\frac{d\sigma}{dt} \frac{N_A A \mu_e \Psi}{C(1 + b + a\Psi)} \int p dV = -L, \quad (44)$$

where $A = (3\pi \hbar^3 N_A)^{2/3} / (2m_e) \approx 4.166 \times 10^{-11}$ while the integral in the Jordan frame is given by

$$\int p dV = \frac{2}{7} \Omega G \frac{M^2}{R} \quad (45)$$

with $\Omega = (\Phi^{3/2} / (1 + \frac{1}{2} \xi \Phi' / \Phi))^{-4/3}$. It can be shown that $\Omega = 1$ for $n = 3/2$ [43,76].

From the entropy formula (29), the entropy rate is given simply by

$$\frac{d\sigma}{dt} = \frac{1.5}{\mu_{1\text{mod}}} \frac{1}{\Psi} \frac{d\Psi}{dt} \quad (46)$$

and together with the luminosity (41) we may finally write down the evolutionary equation for the degeneracy parameter Ψ

TABLE II. Numerical values of ξ_R obtained from $\theta(\xi_R) = 0$, and the associated values of the functions $\gamma_{3/2}$, $\omega_{3/2}$, and $\delta_{3/2}$ for different values of $\alpha = \kappa c^2 \beta \rho_c$.

α	ξ_R	$\omega_{3/2}(\xi_R)$	$\gamma_{3/2}(\xi_R)$	$\delta_{3/2}(\xi_R)$
-0.400	3.16	1.40	1.63	7.53
-0.100	3.64	2.39	2.25	6.67
-0.010	3.65	2.68	2.35	6.09
0 (GR)	3.65	2.71	2.36	5.97
0.006	3.66	2.73	2.36	5.95
0.010	3.66	2.75	2.37	5.93
0.015	3.66	2.77	2.46	5.89

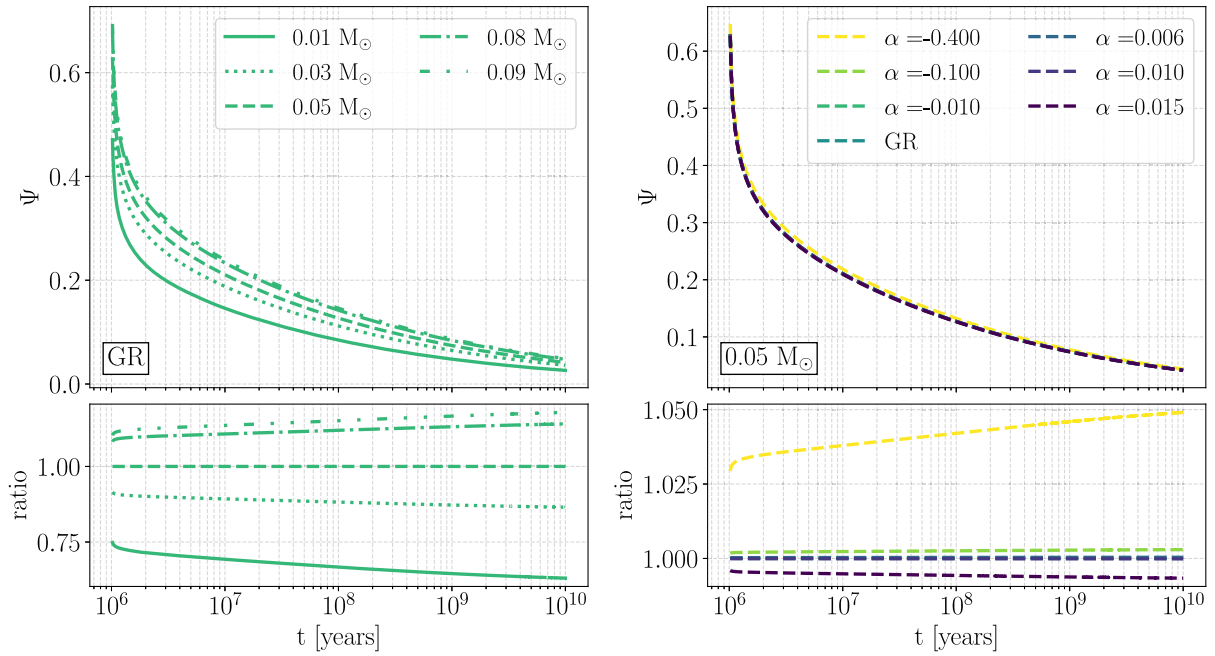


FIG. 1. The time evolution of the degeneracy parameter Ψ in the case of GR for different BD masses (left panels) and for $M = 0.05 M_{\odot}$ and different values of the α parameter (right panels). In the bottom panels, we show the ratio with respect to the curve with $M = 0.05 M_{\odot}$ in the left. In the right bottom panel, the ratio of the time evolution in different Palatini models versus GR is depicted.

$$\frac{d\Psi}{dt} = -\frac{1.1634 \times 10^{-18} b_1^{2.856} \mu_{1\text{mod}} \left(\frac{M_{\odot}}{M}\right)^{1.094}}{\kappa_R^{1.1424} \mu_e^{8/3}} \Psi^{2.856\nu} (1 + b + a\Psi)^{1.715} \frac{\gamma^{0.7143}}{\Omega} \left(1 - 1.33 \frac{\alpha}{\delta}\right)^{1.143}. \quad (47)$$

We have numerically solved the above ordinary differential equation by assuming that $\Psi = 1$ at $t = 0$ for the parameters' values given in Table II. The code for solving this differential uses the GNU Scientific Library [79] and can be found at this github repository [80].

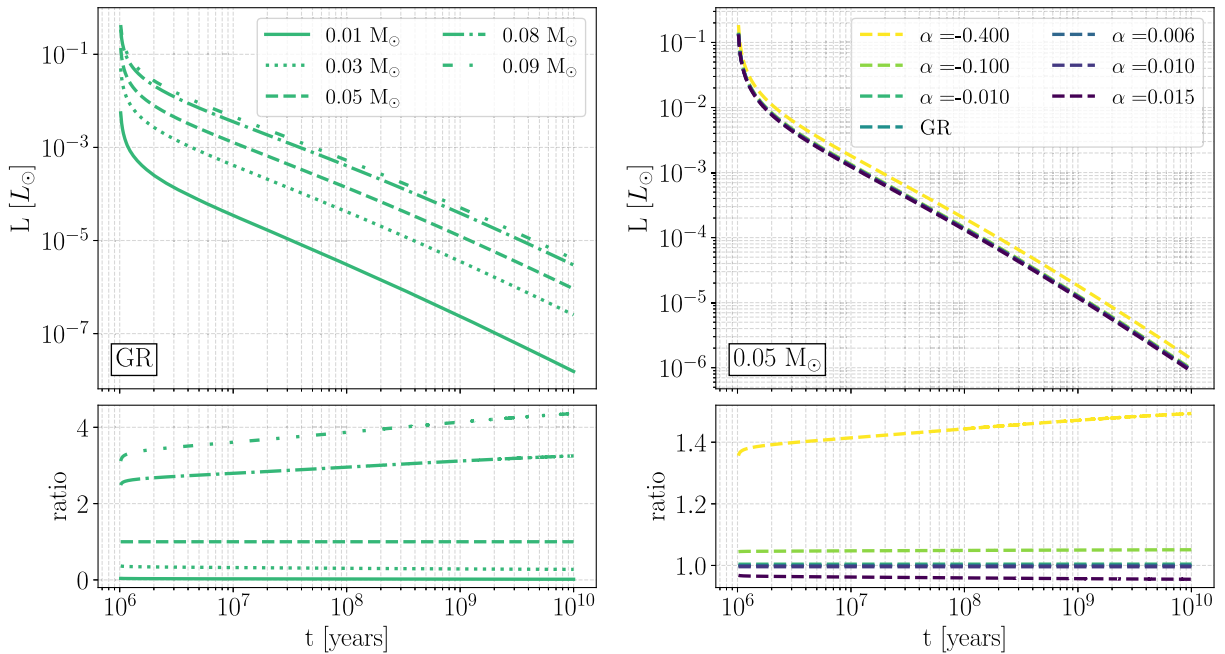


FIG. 2. Time evolution of the BDs' luminosity. Left panels: assuming GR and different masses. In the bottom panel, we show the ratio with respect to our fiducial mass $M = 0.05 M_{\odot}$. Right panels: for fixed $M = 0.05 M_{\odot}$ and different values of the parameter α . In the bottom panel we show the ratio with respect to the GR case.

As it turns out, for the examined range of the parameter α —which is summarized in Table II—the evolution of the degeneracy parameter does not differ significantly from GR ($\alpha = 0$), as it can be seen from Fig. 1. By plugging the numerically obtained $\Psi(t)$ into Eq. (41) we obtain the evolution of the luminosity of BDs as a function of time. This can be seen in Fig. 2.

IV. CONCLUSIONS

In this work we have updated the analytical model of BDs considered in [72] by adopting the EoS first presented in [74]. In addition, we have provided a cooling model for substellar objects in quadratic Palatini gravity. This more realistic EoS describes the BDs' interior as a mixture of a degenerate Fermi gas and ions of hydrogen and helium. Our model further includes a proper treatment of the hydrogen's phase transition from the photosphere, in which the hydrogen is in its molecular form, and the interior of BDs, where the hydrogen is ionized. For this improved description of these partially degenerate substellar objects, we conclude that

- (i) The time evolution of the degeneracy parameter in quadratic Palatini gravity—for the values of α adopted in this work—differs with respect to that in GR by $\lesssim 2.5\%$ when the BD is about 1 Myr and by $\lesssim 5\%$ at 10 Gyr.
- (ii) The difference in the estimated BDs' luminosity between Palatini gravity and GR slightly increases with the age of the BD. Furthermore, BDs have a

lower luminosity in Palatini gravity with positive α values ($\beta < 0$) than in GR. On the other hand, for negative α -values ($\beta > 0$), gravity inside BDs is weaker and the luminosity of BDs in Palatini is larger with respect to that estimated in GR.

- (iii) For α -values smaller than 0.1 in absolute value, the difference between the luminosity predicted in GR and the one in Palatini gravity is smaller than 6%. For $\alpha = -0.4$, this difference increases up to 50%. Although this difference is significant, it is smaller than the estimated difference obtained by varying the BD's mass. For instance, a variation in a mass of 40% produces a change in luminosity larger than a factor of 2.

To sum up, BDs could constrain modifications of gravity and, in particular, the observed luminosity of BDs might be used to constrain the β Starobinsky parameter, as shown in this work. Current and future wide-field surveys will provide large and homogeneous samples of BDs that could be used in this respect. Our study is a first analytical step in this direction, and we leave to future work the comparison of our results with more complete numerical models.

ACKNOWLEDGMENTS

The authors thank Rain Kipper for his comments. This work was supported by the EU through the European Regional Development Fund CoE program TK133 “The Dark Side of the Universe.” M.B. is supported by the Estonian Research Council PRG803 grant.

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