Extensions of the Penrose inequality with angular momentum

Wojciech Kulczycki and Edward Malec

Instytut Fizyki im. Mariana Smoluchowskiego, Uniwersytet Jagielloński, Łojasiewicza 11, 30-348 Kraków, Poland

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We numerically investigate the validity of recent modifications of the Penrose inequality that include angular momentum. Formulations expressed in terms of asymptotic mass and asymptotic angular momentum are contradicted. We analyze numerical solutions describing polytropic stationary toroids around spinning black holes.

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I. INTRODUCTION

The cosmic censorship hypothesis, originally formulated by Roger Penrose more than half a century ago [1], can be understood as a statement that classical general relativity is self-contained when describing regions exterior to black holes. Penrose has argued that the cosmic censorship hypothesis cannot be true if a body collapsing to a black hole fails to satisfy the inequality

$$M_{\mathrm{ADM}} \geq \sqrt{\frac{S}{16\pi}}, \tag{1}$$

where S is the area of the outermost apparent horizon that surrounds the body and M_{ADM} is the asymptotic mass of the spacetime [2]. (Herein and in what follows, we always assume asymptotic flatness of a spacetime.) Failure to satisfy Eq. (1) would imply the existence of a "naked singularity" and a loss of predictability in a collapsing system. There is probably no exaggeration in saying that this idea has shaped the development of classical and mathematical general relativity in the last five decades.

The Penrose inequality has been proven or checked numerically in a number of special cases—conformally flat systems with matter [3], Brill gravitational waves in Weyl geometries [4], and various foliations of spherically symmetric systems [5,6]. Most remarkably, it was proven in the important so-called Riemannian case, when the apparent horizon coincides with a minimal surface [7–9]. There exist scenarios for a general proof [10,11], but there are no easy prospects for their implementation. For more information, see specialized reviews, for instance Refs. [12,13].

Christodoulou introduced the concept of an irreducible mass $M_{\mathrm{irr}} = \sqrt{S/(16\pi)}$ [14], where S is the area of the event horizon. It appears that for the Kerr spacetime endowed with the asymptotic mass M_{ADM} and the angular momentum J_{ADM}, one has the relation $M_{\mathrm{ADM}}^2 = M_{\mathrm{irr}}^2 + \frac{J_{\mathrm{ADM}}^2}{4M_{\mathrm{irr}}^2} = \frac{S}{16\pi} + \frac{4\pi J_{\mathrm{ADM}}^2}{S}$. An analog of this formula might be used in order to define the quasilocal mass of a black hole (assuming that the linear momentum of the black hole vanishes) in terms of its area and quasilocal angular momentum J_{BH}: $M_{\mathrm{BH}} = \sqrt{M_{\mathrm{irr}}^2 + \frac{J_{\mathrm{BH}}^2}{4M_{\mathrm{irr}}^2}} = \sqrt{\frac{S}{16\pi} + \frac{4\pi J_{\mathrm{BH}}^2}{S}}$. This concept of the quasilocal mass of a black hole is commonly used in the literature.

There exist formulations of the Penrose inequality that involve the asymptotic mass and quasilocal angular momentum [13,15],

$$M_{\mathrm{ADM}} \geq \left(\frac{S}{16\pi} + \frac{4\pi J_{\mathrm{BH}}^2}{S}\right)^{1/2}; \tag{2}$$

here S and J_{BH} are the area and quasilocal angular momentum of the outermost apparent horizon. Anglada [16] and Khuri [17] have proved other versions of Eq. (2) under the assumption of axial symmetry,

$$M_{\mathrm{ADM}} \geq \left(\frac{S}{16\pi} + \frac{J_{\mathrm{BH}}^2}{\tilde{R}^2(S)}\right)^{1/2}, \tag{3}$$

where $\tilde{R}(S)$ is some linear measure of the boundary of the black hole.

In what follows, we shall study numerically the validity of Eq. (2) and related inequalities in a class of stationary configurations consisting of a black hole and a torus. The order is as follows: Section II contains a short description of equations and relevant quantities. Section III gives a concise summary of the numerical procedure. The main results are reported in Sec. IV. We conclude the paper with a summary.

II. EQUATIONS

We assume a *stationary* metric of the form

$$ds^2 = -\alpha^2 dt^2 + r^2 \sin^2\theta \psi^4 (d\varphi + \beta dt)^2 + \psi^4 e^{2q}(dr^2 + r^2 d\theta^2). \tag{4}$$

Here t is the time coordinate, and r, θ, φ are spherical coordinates. In this paper, the gravitational constant $G = 1$ and the speed of light $c = 1$. We assume axial symmetry and employ the stress-momentum tensor

$$T^{\alpha\beta} = \rho h u^\alpha u^\beta + p g^{\alpha\beta},$$

where ρ is the baryonic rest-mass density, h is the specific enthalpy, and p is the pressure. Metric functions α, ψ, q, and β in Eq. (4) depend on r and θ only.

The forthcoming Einstein equations have been found in Ref. [18] and checked by the authors of Refs. [19,20]; the present formulation follows closely the description of Refs. [19,20].

Below, K_{ij} denotes the extrinsic curvature of the $t =$ const. hypersurface. The conformal extrinsic curvature $\hat{K}_{ij}$ is defined as $\hat{K}_{ij} = \psi^2 K_{ij}$. The only nonzero component β of the shift vector is separated into two parts: $\beta = \beta_{\rm K} + \beta_{\rm T}$. Here $\beta_{\rm T}$ is a part of the shift vector related to the rotating torus [18]. Functions $\beta_{\rm K}$ and $\beta_{\rm T}$ are determined as follows. The nonzero components of $\hat{K}_{ij}$ can be written in the form

$$\hat{K}_{r\varphi} = \frac{H_{\rm E}\sin^2\theta}{r^2} + \frac{\psi^6}{2\alpha} r^2 \sin^2\theta \partial_r \beta_{\rm T},$$
$$\hat{K}_{\theta\varphi} = \frac{H_{\rm F}\sin\theta}{r} + \frac{\psi^6}{2\alpha} r^2 \sin^2\theta \partial_\theta \beta_{\rm T}.$$

As in Ref. [18], we choose the functions $H_{\rm E}$ and $H_{\rm F}$ to be expressed by the formulas obtained for the Kerr metric of mass m and the spin parameter a, written in the form of Eq. (4). In explicit terms, they read [21]

$$H_{\rm E} = \frac{ma[(r_{\rm K}^2 - a^2)\Sigma_{\rm K} + 2r_{\rm K}^2(r_{\rm K}^2 + a^2)]}{\Sigma_{\rm K}^2},$$

$$H_{\rm F} = -\frac{2ma^3 r_{\rm K}\sqrt{r_{\rm K}^2 - 2mr_{\rm K} + a^2}\cos\theta\sin^2\theta}{\Sigma_{\rm K}^2},$$

where

$$r_{\rm K} = r\left(1 + \frac{m}{r} + \frac{m^2 - a^2}{4r^2}\right)$$

and

$$\Sigma_{\rm K} = r_{\rm K}^2 + a^2\cos^2\theta.$$

It appears that for the Kerr metric, one has

$$\hat{K}_{r\varphi} = \frac{H_{\rm E}\sin^2\theta}{r^2}$$

and

$$\hat{K}_{\theta\varphi} = \frac{H_{\rm F}\sin\theta}{r}.$$

The function $\beta_{\rm K}$ has to be computed from the relation [18]

$$\frac{\partial\beta_{\rm K}}{\partial r} = \frac{2H_{\rm E}\alpha}{r^4\psi^6}. \tag{5}$$

The function $\beta_{\rm T}$, with suitable boundary conditions (see Sec. III), is found from the differential equation (6d) below.

In what follows, we apply the puncture method as implemented in Ref. [18]. Let $\Phi = \alpha\psi$, and assume the puncture at $r = 0$. Define $r_{\rm s} = \frac{1}{2}\sqrt{m^2 - a^2}$, and

$$\psi = \left(1 + \frac{r_{\rm s}}{r}\right)e^\phi, \qquad \Phi = \left(1 - \frac{r_{\rm s}}{r}\right)e^{-\phi}B.$$

Then the surface $r = r_{\rm s}$ is an apparent horizon.

The Einstein equations read

$$\left[\partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta}\right]q = S_q, \tag{6a}$$

$$\left[\partial_{rr} + \frac{2r}{r^2 - r_{\rm s}^2}\partial_r + \frac{1}{r^2}\partial_{\theta\theta} + \frac{\cot\theta}{r^2}\partial_\theta\right]\phi = S_\phi, \tag{6b}$$

$$\left[\partial_{rr} + \frac{3r^2 + r_{\rm s}^2}{r(r^2 - r_{\rm s}^2)}\partial_r + \frac{1}{r^2}\partial_{\theta\theta} + \frac{2\cot\theta}{r^2}\partial_\theta\right]B = S_B, \tag{6c}$$

$$\left[\partial_{rr} + \frac{4r^2 - 8r_{\rm s}r + 2r_{\rm s}^2}{r(r^2 - r_{\rm s}^2)}\partial_r + \frac{1}{r^2}\partial_{\theta\theta} + \frac{3\cot\theta}{r^2}\partial_\theta\right]\beta_{\rm T} = S_{\beta_{\rm T}}, \tag{6d}$$

where the source terms S_ϕ, S_B, $S_{\beta_{\rm T}}$, S_q are

$$S_q = -8\pi e^{2q}\left(\psi^4 p - \frac{\rho h u_\varphi^2}{r^2\sin^2\theta}\right) + \frac{3A^2}{\psi^8} + 2\left[\frac{r - r_{\rm s}}{r(r + r_{\rm s})}\partial_r + \frac{\cot\theta}{r^2}\partial_\theta\right]b + \left[\frac{8r_{\rm s}}{r^2 - r_{\rm s}^2} + 4\partial_r(b - \phi)\right]\partial_r\phi + \frac{4}{r^2}\partial_\theta\phi\partial_\theta(b - \phi), \tag{7a}$$

$$S_\phi = -2\pi e^{2q}\psi^4\left[\rho_{\mathrm{H}} - p + \frac{\rho h u_\varphi^2}{\psi^4 r^2 \sin^2\theta}\right] - \frac{A^2}{\psi^8} - \partial_r\phi\partial_r b - \frac{1}{r^2}\partial_\theta\phi\partial_\theta b - \frac{1}{2}\left[\frac{r-r_{\mathrm{s}}}{r(r+r_{\mathrm{s}})}\partial_r b + \frac{\cot\theta}{r^2}\partial_\theta b\right], \tag{7b}$$

$$S_B = 16\pi B e^{2q}\psi^4 p, \tag{7c}$$

$$S_{\beta_{\mathrm{T}}} = \frac{16\pi\alpha e^{2q}\tilde{J}}{r^2\sin^2\theta} - 8\partial_r\phi\partial_r\beta_{\mathrm{T}} + \partial_r b\partial_r\beta_{\mathrm{T}} - 8\frac{\partial_\theta\phi\partial_\theta\beta_{\mathrm{T}}}{r^2} + \frac{\partial_\theta b\partial_\theta\beta_{\mathrm{T}}}{r^2}, \tag{7d}$$

and

$$A^2 = \frac{\hat{K}_{r\varphi}^2}{r^2\sin^2\theta} + \frac{\hat{K}_{\theta\varphi}^2}{r^4\sin^2\theta},$$
$$\rho_{\mathrm{H}} = \rho h(\alpha u^t)^2 - p,$$
$$\tilde{J} = \rho h\alpha u^t u_\varphi,$$
$$B = e^b.$$

In the rest of this paper, we will deal with polytropes $p(\rho) = K\rho^\gamma$. Then, one has the specific enthalpy

$$h(\rho) = 1 + \frac{\gamma p}{(\gamma - 1)\rho}.$$

The four-velocity $(u^\alpha) = (u^t, 0, 0, u^\varphi)$ is normalized, $g_{\alpha\beta}u^\alpha u^\beta = -1$. The coordinate angular velocity reads

$$\Omega = \frac{u^\varphi}{u^t}. \tag{8}$$

We define the angular momentum per unit inertial mass ρh [22]:

$$j \equiv u_\varphi u^t. \tag{9}$$

It has been known since the early 1970s that general-relativistic Euler equations are solvable under the condition that $j \equiv j(\Omega)$ [23,24]. Within the fluid region, the Euler equations $\nabla_\mu T^{\mu\nu} = 0$ can be integrated,

$$\int j(\Omega)d\Omega + \ln\left(\frac{h}{u^t}\right) = C, \tag{10}$$

where C is a constant.

We shall use in this paper the rotation laws obtained in Ref. [25]:

$$j(\Omega) \equiv -\frac{1}{1-3\delta}\frac{d}{d\Omega}\ln\left(1 - (a\Omega)^2 - \kappa w^{1-\delta}\Omega^{1+\delta}(1 - a\Omega)^{1-\delta}\right). \tag{11}$$

Here, J_{BH} and $a = J_{\mathrm{BH}}/m$ are the angular momentum and the spin parameter of the central black hole, respectively. δ is a free parameter, and $\kappa = \frac{1-3\delta}{1+\delta}$. Let us remark that Eq. (11) supplements former rotation recipes that have been formulated in Refs. [26,27] and (in the case of the Keplerian rotation around spinning black holes) Refs. [19,20].

The Keplerian rotation corresponds to the parameters $\delta = -1/3$ and $\kappa = 3$.

The rotation curves—angular velocities as functions of spatial coordinates $\Omega(r, \theta)$—can be recovered from Eq. (9):

$$j(\Omega) = \frac{V^2}{(\Omega + \beta)(1 - V^2)}. \tag{12}$$

Here, the squared linear velocity is given by

$$V^2 = r^2\sin^2\theta(\Omega + \beta)^2\frac{\psi^4}{\alpha^2}.$$

The central black hole is defined by the puncture method [21]. The black hole is surrounded by a minimal two-surface S (the horizon), embedded in a fixed hypersurface of constant time, and located at $r = r_{\mathrm{s}} = \sqrt{m^2 - a^2}/2$, where m is a mass parameter. Its area is denoted as S, and its angular momentum J_{BH} follows from the Komar expression:

$$J_{\mathrm{BH}} = \frac{1}{4}\int_0^{\pi/2}\frac{r^4\psi^6}{\alpha}\partial_r\beta\sin^3\theta d\theta. \tag{13}$$

We would like to point out that the angular momentum is given rigidly on the event horizon S, in terms of data taken from the Kerr solution and independently of the content of mass in a torus, $J_{\mathrm{BH}} = ma$ [18]. The mass of the black hole is then defined in terms of its area and the angular momentum:

$$M_{\mathrm{BH}} = \sqrt{\frac{S}{16\pi} + \frac{4\pi J_{\mathrm{BH}}^2}{S}}. \tag{14}$$

Asymptotic (total) mass M_{ADM} and angular momentum J_{ADM} can be defined as appropriate Arnowitt-Deser-Misner charges, and they can be computed by means of corresponding volume integrals [18]. Thus, we have

$$M_{\mathrm{ADM}} = \sqrt{m^2 - a^2} - 2\int_{r_s}^{\infty}(r^2 - r_s^2)dr\int_0^{\pi/2}\sin\theta d\theta S_\phi,$$
$$J_{\mathrm{T}} = 4\pi\int_{r_s}^{\infty} r^2 dr\int_0^{\pi/2}\sin\theta d\theta\rho\alpha u^t\psi^6 e^{2q}hu_\varphi,$$
$$J_{\mathrm{ADM}} = J_{\mathrm{BH}} + J_{\mathrm{T}}. \tag{15}$$

Here, J_{T} is the angular momentum deposited within the torus. A circumferential radius corresponding to the coordinate $r = \mathrm{const.}$ on the symmetry plane $\theta = \pi/2$ is given by

$$r_{\mathrm{C}}(r) = r\psi^2(r, \theta = \pi/2). \tag{16}$$

One can define an alternative mass of the apparent horizon in terms of $r_{\mathrm{C}}(r)$:

$$M_{\mathrm{C}} \equiv \frac{r_{\mathrm{C}}(r_s)}{2}. \tag{17}$$

In the Kerr spacetime, one has exactly $M_{\mathrm{C}} = M_{\mathrm{BH}} = m$. It is known that in numerically obtained spacetimes with gaseous toroids, the first equality holds with good accuracy, albeit depending on spin [18–20]. The second equality is true only approximately for relatively light disks, and it is not true for heavy tori.

III. DESCRIPTION OF NUMERICS

The numerical method is based on Ref. [18], and it has been presented in more detail in Ref. [20]. Here we use a more general rotation law [Eq. (11)] and different linear algebra routines—the PARDISO library [28] instead of LAPACK [29]. In what follows, we briefly summarize the main points.

The solutions are found iteratively. In each iteration, one solves the Einstein equations (6) and (7a)–(7d), Eq. (5), and two hydrodynamic equations, (10) and (12). Equations (6a)–(6d), with their source terms in Eqs. (7a)–(7d), are solved using a fixed-point method (we use the PARDISO library [28]) with respect to the functions ϕ, B, β_{T}, and q. The function β_{K} is computed by the integration of Eq. (5). Equations (10) and (12) are used to calculate the specific enthalpy h and the angular velocity Ω, respectively. Constants (C and w) that appear in these two equations are computed by solving them at boundary points $(r_1, \pi/2)$ and $(r_2, \pi/2)$, using the Newton-Raphson method. Here r_1 and r_2 are the values of the inner and outer radii of the torus, respectively; they are given *a priori*.

The free hydrodynamic data consist of the maximal baryonic density $\rho_{\max}$ and the polytropic index γ. We assume from now on that $\gamma = 4/3$. The baryonic density ρ is calculated from the specific enthalpy h using the polytropic formula

$$\rho = \left[\frac{h-1}{4K}\right]^{\frac{1}{1/3}}.$$

This yields (in each iteration) the constant K as a function of the maximal value of the specific enthalpy $h_{\max}$ and $\rho_{\max}$:

$$K = \frac{h_{\max} - 1}{4\rho_{\max}^{1/3}}.$$

We have assumed axial and equatorial symmetry and the puncture method with the puncture at $r = r_s \equiv \sqrt{m^2 - a^2}/2$. We should add that the mass m and the spin a [which appears also in Eq. (11)] are given *a priori*. Thus, it suffices that the numerical grid covers the region $r_s \le r < \infty$ and $0 \le \theta \le \pi/2$ with suitable boundary conditions [18]. We have at the equator ($\theta = \pi/2$) the conditions $\partial_\theta\phi = \partial_\theta B = \partial_\theta\beta_{\mathrm{T}} = \partial_\theta q = 0$. The regularity conditions along the axis ($\theta = 0$) read $\partial_\theta\phi = \partial_\theta B = \partial_\theta\beta_{\mathrm{T}} = 0$. It is required that $q(\theta = 0) = 0$; this is due to the local flatness of the metric. The puncture formalism implies the boundary conditions at the horizon $r = r_s$: $\partial_r\phi = \partial_r B = \partial_r\beta_{\mathrm{T}} = \partial_r q = 0$ and $\partial_{rr}\beta_{\mathrm{T}} = \partial_{rrr}\beta_{\mathrm{T}} = 0$. These last two conditions on β_{T} follow from a careful analysis of Eq. (6d) that yields stringent conditions at $r = r_s$ [18].

At the outer boundary of the numerical domain, the boundary conditions are obtained from the multipole expansion and the conditions of asymptotic flatness. Thus, we have for $r \to \infty$

$$\phi \sim \frac{M_1}{2r}, \qquad B \sim 1 - \frac{B_1}{r^2},$$
$$\beta_{\mathrm{T}} \sim -\frac{2J_1}{r^3}, \qquad q \sim \frac{q_1 \sin^2\theta}{r^2}. \tag{18}$$

Herein, the constants M_1, B_1, J_1, and q_1 are given by

$$M_1 = -2\int_{r_s}^{\infty}(r^2 - r_s^2)dr\int_0^{\pi/2}\sin\theta d\theta S_\phi, \tag{19}$$

$$B_1 = \frac{2}{\pi}\int_{r_s}^{\infty}dr\frac{(r^2 - r_s^2)^2}{r}\int_0^{\pi/2}d\theta\sin^2\theta S_B, \tag{20}$$

$$J_1 = 4\pi\int_{r_s}^{\infty}r^2 dr\int_0^{\pi/2}\sin\theta d\theta\rho\alpha u^t\psi^6 e^{2q}hu_\varphi, \tag{21}$$

$$q_1 = \frac{2}{\pi}\int_{r_s}^{\infty}drr^3\int_0^{\pi/2}d\theta\cos(2\theta)S_q$$
$$-\frac{4}{\pi}r_s^2\int_0^{\pi/2}d\theta\cos(2\theta)q(r_s, \theta). \tag{22}$$

Finally, we add that the Kerr solution emerges in our method as a vacuum limit $\rho_{\max} \to 0$.

IV. THE PENROSE INEQUALITY IN STATIONARY BLACK HOLE–TORUS SYSTEMS

In the rest of the paper, we always assume that $\Omega > 0$ and the mass parameter $m = 1$. Corotating disks have $a > 0$, while counterrotating disks have negative spins: $a < 0$. The disk's boundaries are numerically defined by the condition that the specific enthalpy is $h = 1$. The results of numerical calculations are provided in the forthcoming Table I. We shall describe its content in more detail in the second part of this section, when referring to new proposals of Penrose-type inequalities. In the first part, we will refer to canonical versions.

A. On inequalities (1) and (2)

The mass M_{BH} of the apparent horizon is defined in terms of the area and the quasilocal (Komar-type) angular momentum; see Eq. (14). For such a choice, one has—as discussed above—the relation $M_{\mathrm{C}} \approx M_{\mathrm{BH}}$. This is a kind of virial relation; we shall treat its fullfilment as a test for the self-consistency and correctness of our numerical description. Let us remark that there exists another—exact—virial relation, discussed in Ref. [18], that can be used to check the numerical self-consistency.

Our polytropic matter within a torus satisfies the dominant energy condition. There is no analytic proof, but there exists numerical evidence [18–20] that the asymptotic mass M_{ADM} is not smaller than the quasilocal mass $M_{\mathrm{BH}} = \sqrt{\frac{S}{16\pi} + \frac{4\pi J_{\mathrm{BH}}^2}{S}}$. Thus, Eq. (2) should hold:

$$M_{\mathrm{ADM}} \geq \sqrt{\frac{S}{16\pi} + \frac{4\pi J_{\mathrm{BH}}^2}{S}}. \tag{23}$$

Obviously, the original Penrose inequality [Eq. (1)] also holds true.

The above statements of this subsection should be true whenever there exist numerical solutions. The inspection of Table I confirms this expectation.

B. On new proposals

Inequality (3) uses the size measure $\tilde{R}(S)$ of the apparent horizon [16]. This quantity is difficult to calculate, but we can use a bound that was shown in Ref. [16]—that $\tilde{R}(S)$ is not larger than $\sqrt{10}M_{\mathrm{C}}$—which in turn is approximated by $\sqrt{10}M_{\mathrm{BH}}$. Thus, the necessary condition for the validity of Eq. (3) reads

$$M_{\mathrm{ADM}}^2 \geq \frac{S}{16\pi} + \frac{J_{\mathrm{BH}}^2}{10M_{\mathrm{BH}}^2}. \tag{24}$$

Inequality (24) is valid in all our numerical examples reported in the forthcoming Table I—compare relevant values in the column denoted as M_{ADM} with suitable entries in the last column denoted as I_3. This does not mean, however, that Eq. (3) is confirmed, since Eq. (24) constitutes only the necessary condition.

The inequality

$$M_{\mathrm{BH}}^2 \geq \frac{S}{16\pi} + \frac{J_{\mathrm{BH}}^2}{4M_{\mathrm{BH}}^2} \tag{25}$$

follows directly from the definition of M_{BH}, since $M_{\mathrm{BH}}^2 \geq \frac{S}{16\pi}$. The equality occurs for $a = J_{\mathrm{BH}}/m = 0$. Table I confirms this—compare relevant values in the column denoted as M_{BH} with suitable entries in the last column denoted as I_3.

The quasilocal inequalities (2) and (3) are awkward in a sense, since they require the use of quasilocal measures of the angular momentum. There exists a conserved quantity related with Killing vectors, in stationary and axially symmetric quantities, that gives rise to a distinguished (Komar-type) quasilocal measure of the angular momentum. We used this fact in Sec. II. Unfortunately, there is no such quasilocal measure in general spacetimes. The question arises whether one can replace J_{BH} with its global counterpart J_{ADM}—that is, whether

$$M_{\mathrm{ADM}}^2 \geq \frac{S}{16\pi} + \frac{4\pi J_{\mathrm{ADM}}^2}{S}, \tag{26}$$

or (in a weaker formulation)

$$M_{\mathrm{ADM}}^2 \geq \frac{S}{16\pi} + \frac{J_{\mathrm{ADM}}^2}{4M_{\mathrm{ADM}}^2}, \tag{27}$$

at least for stationary and asymptotically flat spacetimes with compact material systems. This restriction to compact material systems is necessary, since it is easy to envisage a classical mechanical system, with an arbitrarily large angular momentum, so that both inequalities (26) and (27) are broken. In all examples considered below, the circumferential radii of the outermost part of the tori are smaller than 39 M_{ADM}.

Let us mention here the recent work of Kopiński and Tafel [30], in which they consider spacetimes arising from a class of perturbations of the spinless Schwarzschild geometry. These perturbations carry an angular momentum that yields the asymptotic value J_{ADM}. Kopiński and Tafel prove that Eq. (27) is valid for such spacetimes.

When using Table I, in order to test the inequality (26), one should compare entries of the column designated as M_{ADM} with relevant terms in the column denoted as I_1. One can see that Eq. (26) is not valid for systems with heavy toroids, irrespective of the spin of the central black holes—see the cases H1–H15, L3MR, and L5M1–L5M5. Let us remark that a similar conclusion can be drawn from Table II of Ref. [31], but that numerical analysis has used a perturbative approach, and therefore it is not convincing. For lighter disks, the inequality (26) is valid for spins of the black hole that are not too big, but it is broken if a is large enough—see just a few examples corresponding to $a = 0.9$: L5, L10, L15, and L20.

TABLE I. Black hole–torus solutions. Subsequent columns contain (from the left to the right) the solution number, the rotation law parameter δ, the black-hole spin parameter a, the inner radius of the torus r_1, the outer radius of the torus r_2, the total asymptotic mass M_{ADM}, the black-hole mass M_{BH}, the black-hole surface S, the toroid angular momentum J_{T}, the total angular momentum J_{ADM}, and variants of terms in various Penrose inequalities: $I_1 = \sqrt{S/(16\pi) + 4\pi J_{\mathrm{ADM}}^2/S}$, $I_2 = \sqrt{S/(16\pi) + J_{\mathrm{ADM}}^2/(4M_{\mathrm{ADM}}^2)}$, $I_3 = \sqrt{S/(16\pi) + J_{\mathrm{BH}}^2/(4M_{\mathrm{BH}}^2)}$. The solutions were obtained assuming $m = 1$, $\kappa = (1 - 3\delta)/(1 + \delta)$, and $\gamma = 4/3$.

No.	δ	a	r_1	r_2	M_{ADM}	M_{BH}	S	J_{T}	J_{ADM}	I_1	I_2	I_3
L1	−1/3	−0.9	6	41	1.100	1.001	36.19	0.5204	−0.3796	0.8775	0.8659	0.9604
L2	−1/3	−0.5	6	41	1.100	1.003	47.26	0.5008	8.431×10^{-4}	0.9697	0.9697	1.001
L3	−1/3	0	6	41	1.100	1.005	50.79	0.4846	0.4846	1.034	1.029	1.005
L4	−1/3	0.5	6	41	1.100	1.003	47.23	0.4883	0.9883	1.095	1.068	1.001
L5	−1/3	0.9	6	41	1.100	1.000	36.17	0.4947	1.395	1.181	1.059.	0.9601
L6	−1/7	−0.9	6	41	1.100	1.001	36.21	0.4758	−0.4242	0.8848	0.8704	0.9605
L7	−1/7	−0.5	6	41	1.100	1.004	47.32	0.4528	-4.716×10^{-2}	0.9705	0.9704	1.002
L8	−1/7	0	6	41	1.100	1.006	50.87	0.4326	0.4326	1.029	1.025	1.006
L9	−1/7	0.5	6	41	1.100	1.004	47.28	0.4331	0.9331	1.083	1.059	1.001
L10	−1/7	0.9	6	41	1.100	1.001	36.18	0.4371	1.337	1.158	1.0436	0.9602
L11	−1/10	−0.9	6	41	1.100	1.001	36.21	0.4676	−0.4324	0.8862	0.8712	0.9605
L12	−1/10	−0.5	6	41	1.100	1.004	47.32	0.4446	-5.543×10^{-2}	0.9707	0.9706	1.002
L13	−1/10	0	6	41	1.100	1.006	50.87	0.4242	0.4242	1.028	1.024	1.006
L14	−1/10	0.5	6	41	1.100	1.004	47.28	0.4242	0.9242	1.081	1.057	1.001
L15	−1/10	0.9	6	41	1.100	1.001	36.18	0.4277	1.328	1.154	1.041	0.9602
L16	0	−0.9	6	41	1.100	1.001	36.21	0.4477	−0.4525	0.8897	0.8733	0.9605
L17	0	−0.5	6	41	1.100	1.004	47.32	0.4246	-7.535×10^{-2}	0.9710	0.9709	1.002
L18	0	0	6	41	1.100	1.006	50.88	0.4041	0.4041	1.026	1.023	1.006
L19	0	0.5	6	41	1.100	1.004	47.29	0.4030	0.9030	1.076	1.053	1.001
L20	0	0.9	6	41	1.100	1.001	36.18	0.4052	1.305	1.145	1.035	0.9602
H1	0	−0.9	8	20	2.000	1.011	37.88	5.307	4.407	2.683	1.403	0.9756
H2	0	0	8	20	2.000	1.079	58.49	4.879	4.879	2.506	1.628	1.079
H3	0	0.9	8	20	2.000	1.007	37.29	4.942	5.842	3.499	1.695	0.9702
H4	−0.2	−0.9	8	20	2.000	1.011	37.88	5.362	4.462	2.713	1.414	0.9755
H5	−0.2	0	8	20	2.000	1.078	58.46	4.938	4.938	2.531	1.639	1.078
H6	−0.2	0.9	8	20	2.000	1.007	37.28	5.013	5.913	3.539	1.711	0.9702
H7	−0.4	−0.9	8	20	2.000	1.011	37.87	5.406	4.506	2.737	1.422	0.9755
H8	−0.4	0	8	20	2.000	1.078	58.41	4.988	4.988	2.552	1.648	1.078
H9	−0.4	0.9	8	20	2.000	1.007	37.27	5.075	5.975	3.574	1.724	0.9701
H10	−0.6	−0.9	8	20	2.000	1.011	37.87	5.436	4.536	2.753	1.428	0.9754
H11	−0.6	0	8	20	2.000	1.077	58.35	5.032	5.032	2.572	1.656	1.077
H12	−0.6	0.9	8	20	2.000	1.007	37.26	5.134	6.034	3.608	1.737	0.9700
H13	−0.8	−0.9	8	20	2.000	1.011	37.88	5.437	4.537	2.753	1.428	0.9756
H14	−0.8	0	8	20	2.000	1.077	58.28	5.074	5.074	2.591	1.664	1.077
H15	−0.8	0.9	8	20	2.000	1.007	37.24	5.219	6.119	3.657	1.755	0.9698
L3MR	−1/3	0	50	75	2.000	1.016	51.90	9.984	9.984	5.016	2.695	1.016
L5M1	−1/3	0.9	6	41	1.650	1.003	36.66	3.379	4.279	2.647	1.553	0.9646
L5M2	−1/3	0.9	6	41	2.475	1.008	37.46	8.382	9.282	5.445	2.064	0.9717
L5M3	−1/3	0.9	6	41	3.713	1.017	38.79	17.57	18.47	10.55	2.637	0.9836
L5M4	−1/3	0.9	6	41	5.569	1.033	41.17	35.47	36.37	20.11	3.389	1.004
L5M5	−1/3	0.9	6	41	8.353	1.066	46.01	72.76	73.66	38.51	4.512	1.046

We have found only one counterexample to the inequality (27)—see the case L3MR and compare relevant elements in the columns denoted as M_{ADM} and I_2.

V. CONCLUSIONS

We numerically investigate the validity of various versions of the Penrose inequality—in particular, those that include angular momentum. This is done by analyzing a stationary, axially symmetric system consisting of a black hole and a rotating polytropic torus. The original version formulated by Penrose [2] is always true. Formulations, that bound the mass M_{ADM} by quasilocal quantities—the area of the black hole S, its mass M_{BH}, and angular momentum J_{BH}—are also satisfied in our numerical solutions. We have found, however, counterexamples to those versions of the inequality, that are expressed in terms of the asymptotic angular momentum J_{ADM}.

[1] R. Penrose, Gravitational collapse: The role of general relativity, Rivista del Nuovo Cimento 252 (1969).

[2] R. Penrose, Naked singularities, Ann. N.Y. Acad. Sci. **224**, 125 (1973).

[3] J. Karkowski, E. Malec, and Z. Swierczynski, The Penrose inequality in spheroidal geometries, Classical Quant. Grav. **10**, 1361 (1993).

[4] J. Karkowski, P. Koc, and Z. Swierczynski, Penrose inequality for gravitational waves, Classical Quant. Grav. **11**, 1535 (1994).

[5] E. Malec and N. O'Murchadha, Trapped surfaces and the Penrose inequality in spherically symmetric geometries, Phys. Rev. D **49**, 6931 (1994); E. Malec, M. Iriondo, and N. O'Murchadha, Constant mean curvature slices of asymptotically flat spherical spacetimes, Phys. Rev. D **54**, 4792 (1996).

[6] S. Hayward, Inequalities Relating Area, Energy, Surface Gravity, and Charge of Black Holes, Phys. Rev. Lett. **81**, 4557 (1998).

[7] G. Huisken and T. Ilmanen, The Riemannian Penrose inequality, Int. Math. Res. Not. **1997**, 1045 (1997).

[8] G. Huisken and T. Ilmanen, The inverse mean curvature flow and the Riemannian Penrose inequality, J. Diff. Geom. **59**, 353 (2001).

[9] H. Bray, Proof of the Riemannian Penrose inequality using the positive mass theorem, J. Diff. Geom. **59**, 177 (2001).

[10] J. Frauendiener, On the Penrose Inequality, Phys. Rev. Lett. **87**, 101101 (2001).

[11] E. Malec, M. Mars, and W. Simon, On the Penrose Inequality for General Horizons, Phys. Rev. Lett. **88**, 121102 (2002).

[12] J. Karkowski and E. Malec, The general Penrose inequality: Lessons from numerical evidence, Acta Phys. Pol. B **36**, 59 (2005).

[13] M. Mars, Present status of the Penrose inequality, Classical Quant. Grav. **26**, 193001 (2009).

[14] D. Christodoulou, Reversible and Irreversible Transformations in Black-Hole Physics, Phys. Rev. Lett. **25**, 1596 (1970).

[15] S. Dain and M. E. Gabach-Clement, Geometrical inequalities bounding angular momentum and charges in General Relativity, Living Rev. Relativity **21**, 5 (2018).

[16] P. Anglada, Comments on Penrose inequality with angular momentum for outermost apparent horizons, Classical Quant. Grav. **37**, 065023 (2020).

[17] J. Jaracz and M. Khuri, Bekenstein bounds, Penrose inequalities and black hole formation, Phys. Rev. D **97**, 124026 (2018).

[18] M. Shibata, Rotating black hole surrounded by self-gravitating torus in the puncture framework, Phys. Rev. D **76**, 064035 (2007).

[19] J. Karkowski, W. Kulczycki, P. Mach, E. Malec, A. Odrzywołek, and M. Piróg, General-relativistic rotation: Self-gravitating fluid tori in motion around black holes, Phys. Rev. D **97**, 104034 (2018).

[20] J. Karkowski, W. Kulczycki, P. Mach, E. Malec, A. Odrzywołek, and M. Piróg, Self-gravitating axially symmetric disks in general-relativistic rotation, Phys. Rev. D **97**, 104017 (2018).

[21] S. R. Brandt and E. Seidel, Evolution of distorted rotating black holes. I. Methods and tests, Phys. Rev. D **52**, 856 (1995).

[22] L. G. Fishbone and V. Moncrief, Relativistic fluid disks in orbit around Kerr black holes, Astrophys. J. **207**, 962 (1976).

[23] E. Butterworth and I. Ipser, Rapidly rotating fluid bodies in general relativity, Astrophys. J. **200**, L103 (1975).

[24] J. M. Bardeen, A variational principle for rotating stars in General Relativity, Astrophys. J. **162**, 71 (1970).

[25] W. Kulczycki and E. Malec, General-relativistic rotation laws in fluid tori around spinning black holes, Phys. Rev. D **101**, 084016 (2020).

[26] P. Mach and E. Malec, General-relativistic rotation laws in rotating fluid bodies, Phys. Rev. D **91**, 124053 (2015).

[27] J. Knopik, P. Mach, and E. Malec, General-relativistic rotation laws in rotating fluid bodies: Constant linear velocity, Acta Phys. Pol. B **46**, 2451 (2015).

[28] O. Schenk and K. Gärtner, Solving unsymmetric sparse systems of linear equations with PARDISO, Future Gener. Comput. Syst. **20**, 475 (2004).

[29] E. Anderson *et al.*, *LAPACK Users' Guide* (SIAM, Philadelphia, 1999).

[30] J. Kopiński and J. Tafel, The Penrose inequality for nonmaximal perturbations of the Schwarzschild initial data, Classical Quant. Grav. **37**, 105006 (2020).

[31] J. Karkowski, P. Mach, E. Malec, M. Piróg, and N. Xie, Rotating systems, universal features in dragging and antidragging effects, and bounds of angular momentum, Phys. Rev. D **94**, 124041 (2016).