Three-body effective potential in general relativity at second post-Minkowskian order and resulting post-Newtonian contributions

Florian Loebbert[●],[*] Jan Plefka[●],[†] Canxin Shi,[‡] and Tianheng Wang[§]

Institut für Physik und IRIS Adlershof, Humboldt-Universiät zu Berlin,
Zum Großen Windkanal 6, 12489 Berlin, Germany

We study the post-Minkowskian (PM) and post-Newtonian (PN) expansions of the gravitational three-body effective potential. At order 2PM a formal result is given in terms of a differential operator acting on the maximal generalized cut of the one-loop triangle integral. We compute the integral in all kinematic regions and show that the leading terms in the PN expansion are reproduced. We then perform the PN expansion unambiguously at the level of the integrand. Finding agreement with the 2PN three-body potential after integration, we explicitly present new $G^2 v^4$-contributions at order 3PN and outline the generalization to $G^2 v^{2n}$. The integrals that represent the essential input for these results are obtained by applying the recent Yangian bootstrap directly to their ϵ-expansion around three dimensions. The coordinate space Yangian generator that we employ to obtain these integrals can be understood as a special conformal symmetry in a dual momentum space.

I. INTRODUCTION

The three-body problem in Newtonian gravity has been a source of inspiration in mathematics and physics since the time of Newton himself. Families of special solutions are known and tied to names such as Euler, Lagrange and Poincaré [1]. This system of nonintegrable differential equations poses a challenge to the theory of nonlinear systems and numerical approaches to date. They are of clear importance for celestial mechanics and space-flight, and have even been inspirational for science fiction [2]. In general relativity the problem is more challenging, as there are now genuine *N-body interactions* going beyond the Newtonian 2-body potential. As observations indicate that many galaxies, including our own, contain supermassive black holes in their core, these *N*-body interactions might be important for the dynamics of multiple-star systems in their vicinity [3]. With the advent of gravitational wave astronomy [4–6] the gravitational radiation emitted by mergers of compact *binaries* is now observable. It is an interesting question whether genuine three-body systems,

such as hierarchical systems where a black-hole binary is traversed by a third lighter compact object, will be observable in the future as well [7–11].

In the nonrelativistic (post-Newtonian) limit of general relativity the leading three-body interactions are due to Einstein, Infeld and Hoffmann [12,13] and arise from the effective potential [14]

$$V_{3-\text{body}}^{1\text{PN}} = -\sum_{i=1}^{3} \sum_{\substack{j=1 \\ j \neq i}}^{3} \sum_{\substack{k=1 \\ k \neq i}}^{3} \frac{G^2}{2} \frac{m_i m_j m_k}{r_{ij} r_{ik}}, \quad (1)$$

with r_{ij} denoting the spatial distance of the two massive bodies m_i and m_j, G is Newton's constant and we set $c = 1$. In the nomenclature of the two-body problem this is the first post-Newtonian (1PN) contribution to the effective potential in which the velocity squared v^2 and the coupling term Gm/r are of the same order due to the virial theorem. The velocity dependent 1PN terms contributing to the potential beyond (1) are pure two-body interactions of order Gv^2. Numerical simulations of the relativistic three-body problem to date have mostly incorporated general relativity by restricting to the pure two-body PN terms to various orders [7,15–18], as the three-body interactions (1) and beyond are computationally costly, yet relevant [3,8]. In fact, a number of numerical studies incorporating the three-body interactions up to the presently known 2.5PN order exist [8,19–22] demonstrating their relevance for the dynamics. Simulations of three black holes in full numerical relativity [23–25] are challenging.

[*]florian.loebbert@physik.hu-berlin.de
[†]jan.plefka@hu-berlin.de
[‡]canxin.shi@physik.hu-berlin.de
[§]tianheng.wang@physik.hu-berlin.de

The two-body conservative potential for spin-less compact binaries is known up to 4PN level for the potential [26–37], including parts of 5PN [38–40] and 6PN [41–46]. The situation for the N-body problem is considerably more open. For three bodies the effective potential is known to 2PN order [8,19,47–49], generalizing (1) by three-body terms of order $G^3 m^4/r^3$ as well as $v^2 G^2 m^3/r^2$ which entered the above-mentioned numerical studies [8,19,20,22]. The complexity of the three-body 2PN potential already increases considerably, cp. Eq. (52). For $N \geq 4$ the effective potential is in fact *unknown* at 2PN in an analytical form due to an unsolved two-loop spatial integral. The *unintegrated* N-body conservative potential at 2PN was presented in [50].

Turning to the weak gravitational field but arbitrary velocity limit—known as the post-Minkowskian (PM) limit, where one only expands in Newton's coupling but leaves the velocity inert—a lot of progress has been made on the two-body problem recently. Using methods of scattering amplitudes for perturbative quantum gravity, the 2PM [51,52] and 3PM [42,53,54] (including radiation reaction effects [55,56]) results for the effective potential have been established. A worldline effective field theory formalism for the PM expansion was recently formulated [57] and has now been successfully applied to order 3PM [58]. Earlier worldline-based PM calculations can be found in Refs. [59–63] for the conservative sector. The relation between the world-line quantum field theory and the scattering amplitude approach was recently clarified in [64]. Despite this progress, for the N-body problem nothing is known beyond 1PM order at which there are no genuine higher body interactions [61].

It is the aim of this paper to improve on this and to construct the 2PM effective potential in the three-body case (the essential 2PM formulae straightforwardly generalize to N bodies). This in turn may be employed to determine all the velocity dependent contributions to the potential at order G^2 in the post-Newtonian expansion, i.e., the terms of order $v^{2n} G^2 m^3/r^2$. As the complexity of these contributions grows dramatically we shall explicitly provide only the so far unknown $v^4 G^2 m^3/r^2$ terms which contribute to the 3PN terms in the potential in Sec. VII. The general tools to determine the higher velocity terms will be provided.

We employ the PM worldline effective quantum field theory formalism based on [57,64], generalizing the non-relativistic (PN) effective field theory approach of [65] to general relativity. The three-body 2PM potential essentially follows from a single Feynman diagram connecting the three-graviton vertices with the world-lines resulting in a one-loop three-point integral with coordinate space Green's functions [59]. This integral features a Yangian level-one symmetry [66,67] and is related to a generalized cut of the four-point box integral, which has recently been obtained from Yangian bootstrap [68,69]. Generalizing the calculation of [59], we explicitly show that our three-point integral is indeed proportional to one of the four Yangian invariants found in [68]. We then demonstrate that the PN expansion is most efficiently performed at the integrand level, which results in a family of three-point integrals in three dimensions with half integer propagator powers. Again, this family of divergent integrals is invariant under a Yangian level-one generator, which allows to bootstrap their expansion in the dimensional regularization parameter ϵ. This level-one symmetry can alternatively be interpreted as a special conformal symmetry in a dual momentum space, cf. [70–72] for the connection between the two symmetries. We explicitly perform the PN expansion to NNLO yielding the previously unknown $v^4 G^2 m^3/r^2$ terms at the 3PN level and illustrate the generalization to $v^{2n} G^2 m^3/r^2$.

This paper is organized as follows: after a general discussion of the worldline effective field theory in the Polyakov formulation in Sec. II we construct the 2PM potential in Sec. III. The computation of the emerging three-point key integral in various kinematical regions is relegated to Appendix A. Section IV discusses the 1PN limit of the 2PM potential recovering the Einstein–Infeld–Hofmann Lagrangian. In Sec. V we lay out our general approach to integrate the 2PM potential in the nonrelativistic PN expansion at the level of the integrand making use of a level-one Yangian symmetry for the emerging master integrals. As concrete applications of this procedure we then recover the known 2PN three-body potential up to the static term in Sec. VI, and in Sec. VII provide all three-body terms at the 3PN order that scale quartically in velocities and show that they reproduce the known results in the two-body limit.

II. EFFECTIVE FIELD THEORY

Consider three massive spinless point particles coupled to Einstein gravity via the action

$$S = S_{\text{EH}} + S_{\text{gf}} + S_{\text{pp}}. \tag{2}$$

Here we have defined

$$S_{\text{EH}} = -\frac{2}{\kappa^2} \int d^4x \sqrt{-g} R + (\text{GHY term})$$

$$= -\frac{2}{\kappa^2} \int d^4x \sqrt{-g} [g^{\mu\nu} (\Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\rho} - \Gamma^\rho_{\mu\nu} \Gamma^\lambda_{\rho\lambda})], \tag{3}$$

with $\kappa^2 = 32\pi G$, the gravitational coupling and a Gibbons–Hawking–York (GHY) boundary term [73,74]. For the point particles we start out with the action

$$S'_{\text{pp}} = -\sum_i m_i \int d\tau_i \sqrt{g_{\mu\nu}(x(\tau_i)) u_i^\mu(\tau_i) u_i^\nu(\tau_i)}, \tag{4}$$

with the 4-velocities $u_i^\mu = \frac{dx_i^\mu}{d\tau_i}$ integrated along their worldlines. It turns out to be more advantageous to work with the

Polyakov formulation of the point-particle action. Upon introducing the einbein $e_i = e(x(\tau_i))$ this action reads

$$S_{\text{pp}} = -\sum_{i=1}^{3} \frac{m_i}{2} \int d\tau_i e_i \left(g_{\mu\nu} u_i^\mu(\tau_i) u_i^\nu(\tau_i) + \frac{1}{e_i^2} \right), \quad (5)$$

which preserves reparametrization invariance by the transformation rule for e_i. Solving the algebraic equations of motion for the inverse einbein yields $e_i^{-1} = \sqrt{g_{\mu\nu} u_i^\mu(\tau_i) u_i^\nu(\tau_i)}$ and plugging this back into the action recovers the original action (4).

In the weak field expansion of the metric we take $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, using the mostly minus convention. We choose the standard de Donder gauge fixing term $S_{\text{gf}} = \int d^4 x f_\mu f^\mu$ with $f^\mu = \partial^\nu h^\mu{}_\nu - \frac{1}{2} \partial^\mu h^\nu{}_\nu$. This yields the graviton Feynman propagator

$$\approx\!\!\!\!\approx = \frac{i}{2} \frac{P^{\mu\nu\rho\sigma}}{k^2 + i\varepsilon}, \quad (6)$$

with $P^{\mu\nu\rho\sigma} = \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma}$. The advantage of the Polyakov formulation (5) is that it only gives rise to a single graviton worldline interaction:

$$\vdash\!\!\!\approx\!\!\!\approx = -i\kappa\, e(\tau) u^\mu(\tau) u^\nu(\tau). \quad (7)$$

In the bulk we will only need the three-graviton vertex, which may be found, e.g., in [75]. An important aspect in the construction of the classical effective action is the causality structure of the propagator as was recently stressed in [55]. The Fourier transform to coordinate space of the graviton Feynman propagator reads ($x_{ij}^\mu = x_i^\mu - x_j^\mu$)

$$\bar{D}_{ij} = \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 + i\varepsilon} e^{ik \cdot x_{ij}}$$
$$= \frac{1}{4\pi^2} \frac{i}{x_{ij}^2 - i\varepsilon} = -\frac{1}{4\pi} \delta(x_{ij}^2) + \frac{i}{4\pi^2 x_{ij}^2}, \quad (8)$$

where the famous distributional identity

$$\lim_{\epsilon \to 0^+} \frac{1}{y \pm i\epsilon} = \text{pv} \frac{1}{y} \mp i\pi \delta(y) \quad (9)$$

was used in the last step, thereby dropping the principal value label. In order to construct the classical, conservative action for PM gravity, one should restrict to the *real* part D_{ij} defined as:

$$D_{ij} = \text{Re}(\bar{D}_{ij}) = -\frac{1}{4\pi} \delta(x_{ij}^2). \quad (10)$$

This propagator obeys the Green's function identity

$$\Box D_{ij} = -\delta^{(4)}(x_{ij}). \quad (11)$$

It may also be expressed as

$$\delta(x^2) = \frac{\delta(ct - r)}{2r} + \frac{\delta(ct + r)}{2r}, \quad (12)$$

where $r = |\mathbf{x}|$, making manifest that the sum of the retarded and advanced propagator, i.e., the time symmetric propagator, is the real part of the Feynman propagator. The conservative effective action S_{eff} may then be obtained upon integrating out the graviton fluctuations.

III. 2PM POTENTIAL

Up to order 2PM the effective action is expanded as

$$S_{\text{eff}} = S^{\text{free}} + \kappa^2 S^{\text{1PM}} + \kappa^4 S^{\text{2PM}} + \mathcal{O}(\kappa^6), \quad (13)$$

where the free contribution takes the form of (5) with $g_{\mu\nu}$ replaced by $\eta_{\mu\nu}$. Using the Feynman rule (7), it is straightforward to compute the 1PM order. It follows from a single graviton exchange between each pair of point masses

$$\kappa^2 S^{\text{1PM}} = \sum_i \sum_{j \neq i} \underset{i \quad j}{\approx\!\!\!\approx}$$
$$= \sum_i \sum_{j \neq i} \int d\hat{\tau}_i d\hat{\tau}_j \frac{\kappa^2 m_i m_j}{32\pi} \left[u_{ij}^2 - \frac{1}{2} u_i^2 u_j^2 \right] \delta(x_{ij}^2), \quad (14)$$

with $d\hat{\tau}_i := e_i d\tau_i$, and $u_{jk} := u_j \cdot u_k$. There are no three-body interactions at this order. Moving on to 2PM, we find the first genuine three-body interaction in the theory, which arises from a single Feynman diagram. In coordinate space it reads

$$\approx\!\!\!\approx = \kappa^4 \int \frac{d^3\hat{\tau}}{(4\pi)^3} P(x_1(\tau_1), x_2(\tau_2), x_3(\tau_3)), \quad (15)$$

where we have defined $d^3\hat{\tau} = d\hat{\tau}_1 d\hat{\tau}_2 d\hat{\tau}_3$ as well as

$$8(m_1 m_2 m_3)^{-1} P(x_i(\tau_i))$$
$$:= \pi(4u_{12}^2 u_3^2 - 4u_{12} u_{13} u_{23} - u_1^2 u_2^2 u_3^2) \delta(x_{12}^2) \delta(x_{13}^2)$$
$$+ \left(u_{12}^2 u_3^\mu u_3^\nu - \frac{1}{2} u_1^2 u_2^2 u_3^\mu u_3^\nu + 2u_{13} u_{23} u_2^\mu u_1^\nu \right) \partial_{x_1,\mu} \partial_{x_2,\nu} I_{3\delta}$$
$$+ (\text{cyclic}), \quad (16)$$

with the integral $I_{3\delta}$ further discussed below. Note that we have discarded all terms proportional to $u_i \cdot \partial_{x_i}$, which can be written as derivatives $d/d\tau_i$. Modulo integration by

parts, the τ_i-derivatives act on the u_i and the einbein e_j, which also solely depends on the u_j after solving the equations of motion, see (22). In the end, the τ_i-derivatives yield terms involving accelerations, which, by proper field redefinition of x_i, can be replaced by lower order equations of motion that lift these terms to the next order in κ^2, cf. [76]. Note that we will employ this mechanism at several points of the paper.

To complete the three-body action we need to include the two-body interactions at 2PM. These can be obtained from (15) by identifying two of the three world-lines and multiplying with a symmetry factor $1/2$. The full 2PM three-body action thus becomes

$$S^{2PM} = \frac{1}{6}\int\frac{d^3\hat{\tau}}{(4\pi)^3}{\sum_{i,j,k}}' P(x_i(\tau_1), x_j(\tau_2), x_k(\tau_3)), \quad (17)$$

where the sum $\sum'_{i,j,k}$ runs over $i, j, k = 1, 2, 3$ but excludes $i = j = k$. Moreover, propagators that have both ends on the same worldline vanish in dimensional regularization. We note that in fact this becomes the N-body 2PM action if we allow i, j, k to run from 1 to N.

The central ingredient in the above formula (16) for the three-body contribution to the effective potential is the integral

$$I_{3\delta} := \int d^4x_0\,\delta(x_{01}^2)\delta(x_{02}^2)\delta(x_{03}^2) = \quad$$

$$(18)$$

which is interesting for various reasons. In the present paper it arises as the one-loop three-point integral in coordinate space (black solid diagram). Alternatively, we can interpret it as the generalized maximal cut of the momentum space triangle integral where all propagators are put on shell (green dashed diagram), expressed in terms of region momenta x_j, which map to the dual momenta R_j via

$$R_j^\mu := x_{j+1}^\mu - x_{j-1}^\mu. \quad (19)$$

Moreover, $I_{3\delta}$ is related to a generalized cut of the four-point (box) integral, in the limit where one point is sent to infinity. The box integral is invariant under a Yangian algebra, an extension of its well known conformal symmetry [66,77]. As such, in the region $R_j^2 < 0$ the integral is given by the minimal transcendentally solution of the Yangian constraints found in [68] (modulo a piecewise constant):

$$I_{3\delta} = \frac{C}{\sigma}, \qquad \sigma^2 := (R_2 \cdot R_3)^2 - R_2^2 R_3^2. \quad (20)$$

Note that due to $R_1 + R_2 + R_3 = 0$ this representation is not unique and one may pick any two R_i's to define σ^2.

To obtain $I_{3\delta}$, it is useful to generalize the steps of Westpfahl [59], who evaluated the integral for the retarded propagator. This generalization performed in Appendix A shows that the value of the integral depends on the sign of σ^2. In fact, for $R_j^2 < 0$ with $j = 1, 2, 3$ the expression (20) can be compared with the result of [59] which shows that $C(\sigma^2 > 0, R_j^2 < 0) = \pi/4$ in the above expression. However, more care is needed to obtain C for generic kinematics. The explicit calculation given in Appendix A shows that for $\sigma^2 > 0$ we have

$$I_{3\delta} = \frac{\pi}{4\sigma}\Theta(-R_1^2 R_2^2 R_3^2). \quad (21)$$

Here Θ denotes the Heaviside-function as defined in (A7). For $\sigma^2 < 0$ the integral diverges and for $\sigma^2 = 0$ it is proportional to $\sum_i \delta(R_j^2)$, see Appendix A.

IV. THE 1PN EXPANSION

In this section we want to provide a first test of the above expression for the full 2PM effective action against known results for the three-body potential at 1PN order. For this we first solve the equation of motion $\delta S/\delta e_i = 0$ for e_i perturbatively up to order κ^2:

$$e_i = \frac{1}{\sqrt{u_i^2}} + \sum_{j\neq i}\int d\tau_j \frac{\kappa^2 m_j}{16\pi\sqrt{u_i^6 u_j^2}}\left(u_{ij}^2 - \frac{1}{2}u_i^2 u_j^2\right) + \mathcal{O}(\kappa^4).$$

$$(22)$$

Plugging this solution back into (13) and expanding to order κ^4 yields the 2PM effective action free of the einbein. We then consider its nonrelativistic limit, choosing the convenient gauge $\tau_i = t_i$. Reintroducing the speed of light c such that

$$u_i^\mu = \left(1, \frac{\mathbf{v}_i}{c}\right), \qquad \frac{\partial}{\partial x_i^\mu} = \left(\frac{\partial}{c\partial t_i}, \frac{\partial}{\partial \mathbf{x}_i}\right), \qquad \kappa \to \frac{\kappa}{c}, \quad (23)$$

we see that in $P(x_i)$ of (16) only the second line contributes at leading order in c^{-1}:

$${\sum_{i,j,k}}' P(x_i) = -\frac{3\pi m_1 m_2 m_3}{8}\sum_i\sum_{\substack{j\neq i\\k\neq i}}\delta(x_{ij}^2)\delta(x_{ik}^2) + \mathcal{O}(c^{-2}).$$

$$(24)$$

Note that we have rewritten the sum by discarding propagators with both ends on the same worldline. Using the nonrelativistic expansion of the propagator (10)

$$\delta(x_{ij}^2) = \frac{\delta(t_i - t_j)}{r_{ij}} - \frac{r_{ij}}{2c^2} \partial_{t_i} \partial_{t_j} \delta(t_i - t_j)$$
$$+ \frac{r_{ij}^3}{24c^4} \partial_{t_i}^2 \partial_{t_j}^2 \delta(t_i - t_j) + \mathcal{O}(c^{-4}), \quad (25)$$

where $r_{ij} = |\mathbf{r}_{ij}|$ with $\mathbf{r}_{ij} = \mathbf{x}_i - \mathbf{x}_j$, yields a localized time integration in the effective action (17). After some rearrangements, we find the 1PN three-body effective action[1]

$$S = \sum_i \int dt \left[-m_i + \frac{1}{c^2} \left(\frac{m_i \mathbf{v}_i^2}{2} + \sum_{j \neq i} \frac{Gm_i m_j}{2r_{ij}} \right) \right.$$
$$+ \frac{1}{c^4} \left(\frac{m_i \mathbf{v}_i^4}{8} + \sum_{j \neq i} \frac{Gm_i m_j}{4r_{ij}} (6\mathbf{v}_i^2 - (\mathbf{n}_{ij} \cdot \mathbf{v}_i)(\mathbf{n}_{ij} \cdot \mathbf{v}_j) \right.$$
$$\left. \left. - 7\mathbf{v}_i \cdot \mathbf{v}_j) - \sum_{j \neq i} \sum_{k \neq i} \frac{G^2 m_i m_j m_k}{2r_{ij} r_{ik}} \right) \right], \quad (26)$$

where we abbreviate $\mathbf{n}_{ij} := \mathbf{r}_{ij}/r_{ij}$ and $G = \kappa^2/32\pi$. This result agrees with the well known 1PN expression [14].

V. POST-NEWTONIAN EXPANSION AND INTEGRAL BOOTSTRAP

The 1PN expansion obtained in the previous section merely tests the second line of the three-body contribution (16) to the effective potential. In order to obtain the expansion at 2PN order, also the third line in (16) has to be taken into account. This includes second derivatives of the three-delta integral, $\partial_j^\mu \partial_k^\nu I_{3\delta}$, cf. the Θ-function in (21). As outlined in detail in Appendix B, taking these derivatives leads to lengthy expressions in terms of delta functions and their derivatives which are hard to control. In fact, it is simpler to perform the nonrelativistic expansion directly on the level of the integrand of $I_{3\delta}$ as we will demonstrate in the following. For convenience of the reader we briefly summarize the below strategy: First, we will show that expanding the integrand of $I_{3\delta}$ leads to the family of key integrals given in (31). We will then use the Yangian level-one symmetry of these integrals, i.e., invariance under the differential operator (33), to obtain the differential equations (36). Finally, we explicitly demonstrate how these equations are solved in the form of (39), which results in the expressions for the dimensionally regularized integrals that enter into the effective potential. To start we consider the nonrelativistic expansion of the propagator of (8), generalized to D spatial dimensions in the so-called potential region $\omega := k^0 \ll |\mathbf{k}|$, writing

$$\frac{1}{k^2} = \frac{1}{\omega^2 - \mathbf{k}^2} = -\sum_{\alpha=1}^\infty \frac{\omega^{2\alpha-2}}{(\mathbf{k}^2)^\alpha}. \quad (27)$$

Inserting this expansion into the Fourier transformed expression for the time-symmetric propagator yields the common PN-expanded propagator

$$\delta(x_{0i}^2) = 4\pi \int \frac{d^D k}{(2\pi)^D} e^{i\mathbf{k} \cdot \mathbf{x}_{0i}} \sum_{\alpha=0}^\infty \frac{(-1)^\alpha \partial_{t_i}^{2\alpha} \delta(t_{0i})}{c^{2\alpha}(\mathbf{k}^2)^{\alpha+1}}, \quad (28)$$

having performed the energy (ω) integral. Hence, with the expression for the D-dimensional Fourier transform of the momentum space propagator,

$$\int \frac{d^D k}{(2\pi)^D} \frac{e^{i\mathbf{k} \cdot \mathbf{x}}}{(\mathbf{k}^2)^\alpha} = \frac{1}{4^\alpha \pi^{D/2}} \frac{\Gamma_{D/2-\alpha}}{\Gamma_\alpha} r^{2\alpha-D}, \quad (29)$$

we can write the key integral $I_{3\delta}$ in the PN-expansion for general spatial D as

$$I_{3\delta} = \sum_{\alpha,\beta,\gamma=0}^\infty \frac{(-1)^{\alpha+\beta+\gamma}}{(2c)^{2(\alpha+\beta+\gamma)} \pi^{3(D/2-1)}} \frac{\Gamma_{\hat{\alpha}} \Gamma_{\hat{\beta}} \Gamma_{\hat{\gamma}}}{\Gamma_{\alpha+1} \Gamma_{\beta+1} \Gamma_{\gamma+1}}$$
$$\times \int dt_0 \partial_{t_1}^{2\alpha} \delta(t_{01}) \partial_{t_2}^{2\beta} \delta(t_{02}) \partial_{t_3}^{2\gamma} \delta(t_{03}) I_3^D[\hat{\alpha}, \hat{\beta}, \hat{\gamma}]. \quad (30)$$

Here, $\Gamma_\alpha = \Gamma(\alpha)$ denotes the Gamma-function, we use the shorthand $\hat{\alpha} = D/2 - \alpha - 1$ and we have introduced the following family of (Euclidean) integrals:

$$I_3^D[a_1, a_2, a_3] := \int \frac{d^D \mathbf{x}_0}{(\mathbf{x}_{01}^2)^{a_1} (\mathbf{x}_{02}^2)^{a_2} (\mathbf{x}_{03}^2)^{a_3}}. \quad (31)$$

These integrals represent the central nontrivial input for the above expansion (30) and we will now discuss how to compute them. Notably, in [78] the integrals $I_3^D[a_1, a_2, a_3]$ for generic propagator powers a_j and spacetime dimension D have been expressed in terms of Appell hypergeometric functions F_4, which converge for small values of the effective ratio variables r_{12}/r_{13} and r_{23}/r_{13}. In the present situation we would like to avoid making assumptions on these ratios, which would imply a limited validity of the resulting effective potential. Moreover, note that here we are merely interested in the special case of half integer propagator powers a_j in three dimensions, which satisfy the condition

$$a_1 + a_2 + a_3 \leq \frac{D}{2}. \quad (32)$$

In particular, this condition implies that the integrals of interest are divergent in strictly three dimensions and we thus consider their ϵ-expansion around $D = 3$ in dimensional regularization. Importantly, these integrals are accessible via a bootstrap approach, cf. [68,72]: they feature a nonlocal Yangian level-one symmetry, i.e., they are annihilated by the differential operator

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$$\hat{P}^\mu := \frac{i}{2} \sum_{k=1}^{3} \sum_{j=1}^{k-1} (P_j^\mu D_k + P_{j\nu} L_k^{\mu\nu} - (j \leftrightarrow k)) + \sum_{j=1}^{3} s_j P_j^\mu,$$

(33)

where we have used the following representation of the momentum, Lorentz and dilatation generator of the conformal algebra:

$$P_j^\mu = -i\partial_{x_j}^\mu,$$
$$L_j^{\mu\nu} = ix_j^\mu \partial_{x_j}^\nu - ix_j^\nu \partial_{x_j}^\mu,$$
$$D_j = -ix_{j\mu}\partial_{x_j}^\mu - i.$$

(34)

The so-called evaluation parameters s_j entering the definition of the level-one generator $\hat{P}^\mu$ in (33) take values [67]

$$\{s_j\} = \frac{1}{2}\{a_2 + a_3, a_3 - a_1, -a_1 - a_2\}.$$

(35)

Notably, in a dual momentum space, introduced via the transformation (19), i.e., $R_j = x_{j+1} - x_{j-1}$, the level-one generator $\hat{P}$ translates into a representation of the special conformal generator [72]. Invariance under $\hat{P}^\mu$ implies two independent partial differential equations (cf. [68] for the PDEs in terms of ratio variables)

$$A_1 I_3 = 0, \qquad A_2 I_3 = 0,$$

(36)

with the second order differential operators

$$A_1 = +r_{12}(\bar{w}_D - 2a_2)\partial_{r_{13}} - 2r_{12}r_{23}\partial_{r_{13}}\partial_{r_{23}} - r_{12}r_{13}\partial_{r_{13}}^2$$
$$+ r_{13}(\bar{w}_D + 2a_3)\partial_{r_{12}} - 2r_{12}^2\partial_{r_{12}}\partial_{r_{13}} - r_{12}r_{13}\partial_{r_{12}}^2,$$
$$A_2 = +r_{12}(\bar{w}_D + 2a_1)\partial_{r_{23}} - r_{12}r_{23}\partial_{r_{23}}^2$$
$$- r_{23}(\bar{w}_D + 2a_3)\partial_{r_{12}} + r_{12}r_{23}\partial_{r_{12}}^2.$$

(37)

Here, for the conformal weight of the integrals (31), we have introduced the abbreviation

$$w_D = D - 2(a_1 + a_2 + a_3),$$

(38)

and $\bar{w}_D = w_D - 1$. For $D = 3 - 2\epsilon$ we make the following ansatz for the ϵ-expansion of the integral I_3, which is inspired by [79]:

$$\mu^{-2\epsilon} I_3^{3-2\epsilon} = \frac{A}{2\epsilon} + B + C \log\left(\frac{r_{12} + r_{13} + r_{23}}{\mu}\right) + \mathcal{O}(\epsilon).$$

(39)

Here μ denotes some mass scale and A, B, C represent polynomials whose form is constrained by the scaling of the integral:

$$X = \sum_{j=0}^{w_3} \sum_{k=0}^{w_3-j} f_{jk}^{(X)} r_{12}^j r_{13}^k r_{23}^{w_3-j-k}.$$

(40)

For $X \in \{A, B, C\}$ the constant coefficients of the polynomial are denoted by $f_{jk}^{(X)}$. We note that the polynomial B can always be shifted by a term proportional to C via a modification of the mass scale μ. The below results are thus to be understood modulo such a shift. As the coefficients of $1/\epsilon$ and $\log\mu$ are correlated in the ϵ-expansion of (39), we must have $A = -C$ which we also find from the bootstrap arguments.

The solution of the homogeneous differential equations (36) will depend on some undetermined constants. In general, these can for instance be fixed by comparing a coincident point limit of the solution with the following well known expression for the two-point integral, cf. e.g., [80]:

$$\int \frac{d^D x_0}{x_{01}^{2a_1} x_{02}^{2a_2}} = \pi^{\frac{D}{2}} \frac{\Gamma_{a_1+a_2-\frac{D}{2}}\Gamma_{\frac{D}{2}-a_1}\Gamma_{\frac{D}{2}-a_2}}{\Gamma_{a_1}\Gamma_{a_2}\Gamma_{D-a_1-a_2}} r_{12}^{D-2a_1-2a_2}.$$

(41)

However, for the lower propagator powers considered below, some of the arguments of the Gamma-functions will actually be zero. It is thus useful to note that the Laplacian acting on leg 1 of the integral generates a recursive structure on the above integrals, e.g.,

$$\Delta_1 I_3[a_1, a_2, a_3] = 2a_1(2a_1 + 2 - D)I_3[a_1 + 1, a_2, a_3],$$

(42)

and similar for legs 2 and 3. This equation can alternatively be used to relate the undetermined coefficients for integrals with negative propagator powers to the leading-order "seed" integral $I_3[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$.

In the following we bootstrap the integrals contributing to the leading terms of the nonrelativistic expansion (30) using the level-one Yangian PDEs (37). We have compared the expansion of the below results for small ratios r_{12}/r_{13} and r_{23}/r_{13} to the expressions in terms of Appell hypergeometric functions given in [78] finding full agreement, see also [72] for our conventions. The following integrals serve as input for the three-body effective potential via (30) and (16).

A. Order c^0: $I_3[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$

The leading order contribution to the expansion (30) is given by propagator powers $a_j = 1/2$ for $j = 1, 2, 3$. Using the ansatz (39) it is straightforward to solve the PDEs (37) in the ϵ-expansion around $D = 3$, which yields

$$\mu^{-2\epsilon} I_3\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right] = \frac{b_1}{2\epsilon} - b_2 \log\left(\frac{r_{12} + r_{13} + r_{23}}{\mu}\right) + \mathcal{O}(\epsilon),$$

(43)

for some undetermined constants b_1, b_2. The parameters b_1, b_2 are fixed by comparison with the two-point integral (41) to $b_1 = b_2 = 4\pi$. Note that we do not display an additional constant that can be shifted by modification of the mass scale μ. The above logarithmic result for this integral is already contained in [79].

B. Order c^{-2}: $I_3[\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}]$

In complete analogy to the above, we find at next-to-leading order the following ϵ-expansion of the single contributing integral:

$$
\mu^{-2\epsilon} I_3\left[\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right]
$$
$$
= -\frac{2\pi}{3}\left[\frac{r_{12}^2 - r_{13}^2 - r_{23}^2}{2\epsilon} - (r_{12} - r_{13})(r_{12} - r_{23})\right.
$$
$$
\left. - (r_{12}^2 - r_{13}^2 - r_{23}^2)\log\left(\frac{r_{12} + r_{13} + r_{23}}{\mu}\right)\right] + \mathcal{O}(\epsilon).
$$
$$(44)$$

Below we will employ this result to obtain new contributions to the three-body effective potential at 3PN, which scale as $v^4 G^2 m^3 / c^8 r^2$.

C. Order c^{-4}: $I_3[-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}]$ and $I_3[\frac{1}{2}, \frac{1}{2}, -\frac{3}{2}]$

To demonstrate that the above bootstrap approach easily generalizes to higher orders, let us also consider the next order. Note, however, that due to its length we will not evaluate the resulting contribution to the effective potential in this paper, see Sec. VII for the previous order. At order c^{-4} of the nonrelativistic expansion (30) two integrals contribute. With the ansatz (39) we can again solve the above partial differential equations to find solutions of the form

$$
\mu^{-2\epsilon} I_3^{3-2\epsilon} = b\left[\frac{A}{2\epsilon} - B - A\log\left(\frac{r_{12} + r_{13} + r_{23}}{\mu}\right)\right] + \mathcal{O}(\epsilon).
$$
$$(45)$$

Note again that the polynomial B is only defined modulo a shift by A due to the arbitrariness of the mass scale μ. Here we have

$$
A\left[-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right] = -\frac{1}{3}r_{12}^4 + (r_{13}^2 - r_{23}^2)^2 - \frac{2}{3}r_{12}^2(r_{13}^2 + r_{23}^2),
$$

$$
B\left[-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right] = \frac{r_{12}^4}{9} - r_{12}(r_{13} - r_{23})^2(r_{13} + r_{23})
$$
$$
+ \frac{1}{3}r_{12}^3(r_{13} + r_{23}) - r_{13}r_{23}(r_{13} - r_{23})^2
$$
$$
+ \frac{1}{9}r_{12}^2(5r_{13}^2 - 3r_{13}r_{23} + 5r_{23}^2), \quad (46)
$$

as well as

$$
A\left[\frac{1}{2}, \frac{1}{2}, -\frac{3}{2}\right] = -r_{12}^4 - r_{13}^4 - r_{23}^4
$$
$$
+ 2r_{12}^2(r_{13}^2 + r_{23}^2) - \frac{2}{3}r_{13}^2r_{23}^2,
$$

$$
B\left[\frac{1}{2}, \frac{1}{2}, -\frac{3}{2}\right] = -\frac{4r_{12}^4}{3} + r_{13}r_{23}^3 - \frac{4}{9}r_{13}^2r_{23}^2 + r_{13}^3r_{23}
$$
$$
+ r_{12}^3(r_{13} + r_{23}) + r_{12}^2(2r_{13}^2 - r_{23}r_{13} + 2r_{23}^2)
$$
$$
- \frac{1}{3}r_{12}(5r_{13}^3 + 3r_{23}r_{13}^2 + 3r_{23}^2r_{13} + 5r_{23}^3).
$$
$$(47)$$

The overall constants b in (45) are fixed by relating them to the coefficients for the seed integral (43) via the recursion (42):

$$
b\left[-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right] = -\frac{\pi}{10}, \quad b\left[\frac{1}{2}, \frac{1}{2}, -\frac{3}{2}\right] = -\frac{3\pi}{10}. \quad (48)
$$

This provides all the necessary information to generate the 4PN order G^2 contributions to the effective potential.

VI. THE 2PN EXPANSION

We now proceed to compute the 2PN expansion of the effective action as introduced in Sec. IV. This also serves as a test for the integral (43). For the third line of (16), we decompose the sum as

$$
\sum_{i,j,k}' \to \sum_i \sum_{j\neq i} \sum_{k\neq i,j} + \left(\sum_i \sum_j\bigg|_{k=i} + \text{(cyclic)}\right). \quad (49)
$$

Here we refer to the first term on the right-hand side as the three-body interaction and to the remaining terms as the two-body interactions. When identifying two of the three indices, we encounter a divergence $1/r_{ij}|_{j=i}$ and an indefinite unit vector $\mathbf{n}_{ij}|_{j=i}$. In light of the vanishing of propagators with both ends on the same worldline, we propose to regularize the divergences as $1/r_{ij}|_{j=i} \to 0$. Terms of odd order in $\mathbf{n}_{ij}|_{j=i}$ also vanish due to the antisymmetry in the indices. For the quadratic terms in $\mathbf{n}_{ij}|_{j=i}$ of the 2PN result we adopt the following limiting prescription:

$$
\mathbf{n}_{ij} \cdot \mathbf{v}_\alpha \mathbf{n}_{ij} \cdot \mathbf{v}_\beta|_{j=i} \to \mathbf{v}_\alpha \cdot \mathbf{v}_\beta. \quad (50)
$$

That is, whenever the identification of two points yields an expression as given on the left-hand side, we replace it by the right-hand side. This prescription is natural from the perspective of dimensional analysis and symmetry

considerations, and it reproduces the correct results as given in the literature. With regard to the 3PN result to be discussed in Sec. VII we already give the rule

$$\mathbf{n}_{ij} \cdot \mathbf{v}_\alpha \mathbf{n}_{ij} \cdot \mathbf{v}_\beta \mathbf{n}_{ij} \cdot \mathbf{v}_\rho \mathbf{n}_{ij} \cdot \mathbf{v}_\sigma|_{j=i}$$

$$\rightarrow \mathbf{v}_\alpha \cdot \mathbf{v}_\beta \mathbf{v}_\rho \cdot \mathbf{v}_\sigma + \mathbf{v}_\alpha \cdot \mathbf{v}_\rho \mathbf{v}_\sigma \cdot \mathbf{v}_\beta + \mathbf{v}_\alpha \cdot \mathbf{v}_\sigma \mathbf{v}_\beta \cdot \mathbf{v}_\rho. \quad (51)$$

$$S^{2\text{PN}} = \sum_i \int \frac{dt}{c^6} \left\{ \frac{m_i \mathbf{v}_i^6}{16} + \sum_{j \neq i} \frac{G m_i m_j}{16 r_{ij}} [3(\mathbf{n}_{ij} \cdot \mathbf{v}_i)^2 (\mathbf{n}_{ij} \cdot \mathbf{v}_j)^2 - 6\mathbf{n}_{ij} \cdot \mathbf{v}_i \mathbf{n}_{ij} \cdot \mathbf{v}_j \mathbf{v}_{ij}^2 - 2(\mathbf{n}_{ij} \cdot \mathbf{v}_j)^2 \mathbf{v}_i^2 \right.$$

$$+ 3\mathbf{v}_i^2 \mathbf{v}_j^2 + 2(\mathbf{v}_i \cdot \mathbf{v}_j)^2 - 20\mathbf{v}_i^2 \mathbf{v}_i \cdot \mathbf{v}_j + 14\mathbf{v}_i^4] + \sum_{j \neq i} \frac{G^2 m_i m_j^2}{2 r_{ij}^2} [33(\mathbf{n}_{ij} \cdot \mathbf{v}_{ij})^2 - 17 \mathbf{v}_{ij}^2]$$

$$+ \sum_{j \neq i} \sum_{k \neq i} \frac{G^2 m_i m_j m_k}{8} \left[\frac{1}{r_{ij} r_{ik}} (4(\mathbf{n}_{ij} \cdot \mathbf{v}_j)^2 + 18\mathbf{v}_i^2 - 16\mathbf{v}_j^2 - 32\mathbf{v}_i \cdot \mathbf{v}_j + 32\mathbf{v}_j \cdot \mathbf{v}_k) \right.$$

$$+ \frac{1}{r_{ij}^2} (14\mathbf{n}_{ik} \cdot \mathbf{v}_k \mathbf{n}_{ij} \cdot \mathbf{v}_k - 12\mathbf{n}_{ij} \cdot \mathbf{v}_i \mathbf{n}_{ik} \cdot \mathbf{v}_k + \mathbf{n}_{ij} \cdot \mathbf{n}_{ik} (\mathbf{n}_{ik} \cdot \mathbf{v}_k)^2 - \mathbf{n}_{ij} \cdot \mathbf{n}_{ik} \mathbf{v}_k^2) \right]$$

$$+ \sum_{j \neq i} \sum_{k \neq i,j} G^2 m_i m_j m_k \left[\frac{2(\mathbf{n}_{ij} - \mathbf{n}_{jk}) \cdot \mathbf{v}_{ij}}{(r_{ij} + r_{ik} + r_{jk})^2} (4(\mathbf{n}_{ij} + \mathbf{n}_{ik}) \cdot \mathbf{v}_{ij} + (\mathbf{n}_{ik} + \mathbf{n}_{jk}) \cdot \mathbf{v}_{ik}) \right.$$

$$\left. \left. + \frac{9(\mathbf{n}_{ij} \cdot \mathbf{v}_{ij})^2 - 9\mathbf{v}_{ij}^2 + 2(\mathbf{n}_{ij} \cdot \mathbf{v}_{ik})^2 - 2\mathbf{v}_{ik}^2}{r_{ij}(r_{ij} + r_{ik} + r_{jk})} \right] \right\} + G^3 \times [\text{static term}], \quad (52)$$

where we define $\mathbf{v}_{ij} := \mathbf{v}_i - \mathbf{v}_j$. Here we have performed a field redefinition to push terms that involve accelerations to higher orders in G. We have checked that our result agrees with the literature [47,48] up to a total derivative. Note that we do not have access to the static (velocity independent) term at $\mathcal{O}(G^3)$ in our approach as it stems from a 3PM computation.

VII. NEW CONTRIBUTIONS AT 3PN

In this section we explicitly evaluate the contributions to the 3PN three-body effective potential. Limiting the number of point masses to two gives the two-body 3PN effective action, which we checked to agree with [27] up to a total derivative. Next to the novel three-point $G^2 v^4$ terms, the below expression contains terms that scale as $G v^6$, as well as two-point terms of order $G^2 v^4$ which have been known before. The full 3PN action can be written in the form

We note that the $1/\epsilon$-term in (43) naturally drops out in the final expression for the action due to the derivatives that have to be applied. Moreover, we expect this property to hold to all orders in the PN expansion. This is explicitly shown to be true in the 3PN calculation of Sec. VII. The 2PN effective action reads

$$S^{3\text{PN}} = \sum_i \int \frac{dt}{c^8} \left\{ \frac{5}{128} m_i \mathbf{v}_i^8 + L_{(A)}^{3\text{PN}} + L_{(B)}^{3\text{PN}} \right.$$

$$\left. + L_{(C)}^{3\text{PN}} + L_{(D)}^{3\text{PN}} \right\} + \mathcal{O}(G^3). \quad (53)$$

Note that the terms at order G^3 are not given here and require two yet unknown four-point integrals at one and two loops. Moreover, there are additional G^4 contributions at 3PN. In (53) we have ordered the various terms, which are explicitly given in the following, by their power of G and the structure of summations. Terms from perturbative solutions of the equations of motion for the einbein, cf. (22), contribute at various places. Explicit expressions for the terms in (53) are also provided in an ancillary file to this paper [81]. The term $L_{(A)}^{3\text{PN}}$ originates from the 1PM action and reads

$$L_{(A)}^{3\text{PN}} = \sum_{j \neq i} \frac{G m_i m_j}{32 r_{ij}} [-5(\mathbf{n}_{ij} \cdot \mathbf{v}_i)^3 (\mathbf{n}_{ij} \cdot \mathbf{v}_j)^3 + 3\mathbf{n}_{ij} \cdot \mathbf{v}_i (\mathbf{n}_{ij} \cdot \mathbf{v}_j)^2 (2\mathbf{v}_i^2 \mathbf{n}_{ij} \cdot \mathbf{v}_j + 6\mathbf{v}_i^2 \mathbf{n}_{ij} \cdot \mathbf{v}_i - 5\mathbf{v}_i \cdot \mathbf{v}_j \mathbf{n}_{ij} \cdot \mathbf{v}_i)$$

$$+ \mathbf{n}_{ij} \cdot \mathbf{v}_i \mathbf{n}_{ij} \cdot \mathbf{v}_j (10(\mathbf{v}_i \cdot \mathbf{v}_j)^2 + 8\mathbf{v}_i^2 \mathbf{v}_i \cdot \mathbf{v}_j - 5\mathbf{v}_i^2 \mathbf{v}_j^2 - 14\mathbf{v}_i^4) + 2(\mathbf{n}_{ij} \cdot \mathbf{v}_j)^2 \mathbf{v}_i^2 (5\mathbf{v}_i \cdot \mathbf{v}_j - 3\mathbf{v}_i^2)$$

$$- 6\mathbf{v}_i^2 \mathbf{v}_j^2 (\mathbf{n}_{ij} \cdot \mathbf{v}_i)^2 + 16\mathbf{v}_i^4 \mathbf{v}_j^2 + 2(\mathbf{v}_i \cdot \mathbf{v}_j)^3 + 12\mathbf{v}_i^2 (\mathbf{v}_i \cdot \mathbf{v}_j)^2 - 19\mathbf{v}_i^2 \mathbf{v}_j^2 \mathbf{v}_i \cdot \mathbf{v}_j - 34\mathbf{v}_i^4 \mathbf{v}_i \cdot \mathbf{v}_j + 22\mathbf{v}_i^6]. \quad (54)$$

Here we have added a total derivative as given in (C1) in Appendix C. Again, accelerations have been pushed to the next order in G by means of field redefinitions. The next term $L_{(B)}^{3\text{PN}}$ stems from the two-body interactions of the third line of (16) and reads

$$L^{3PN}_{(B)} = \sum_{j \neq i} \frac{G^2 m_i m_j^2}{4 r_{ij}^2} [(-200 \mathbf{v}_i \cdot \mathbf{v}_j + 167 \mathbf{v}_i^2 + 66 \mathbf{v}_j^2)(\mathbf{n}_{ij} \cdot \mathbf{v}_i)^2 - 2(99 \mathbf{v}_i^2 + 64 \mathbf{v}_j^2 - 130 \mathbf{v}_i \cdot \mathbf{v}_j) \mathbf{n}_{ij} \cdot \mathbf{v}_j \mathbf{n}_{ij} \cdot \mathbf{v}_i$$

$$- 44(\mathbf{n}_{ij} \cdot \mathbf{v}_i - \mathbf{n}_{ij} \cdot \mathbf{v}_j)^2 (2(\mathbf{n}_{ij} \cdot \mathbf{v}_i)^2 + (\mathbf{n}_{ij} \cdot \mathbf{v}_j)^2) + (65 \mathbf{v}_i^2 + 96 \mathbf{v}_j^2 - 128 \mathbf{v}_i \cdot \mathbf{v}_j)(\mathbf{n}_{ij} \cdot \mathbf{v}_j)^2$$

$$- 98(\mathbf{v}_i \cdot \mathbf{v}_j)^2 + 96 \mathbf{v}_j^2 \mathbf{v}_i \cdot \mathbf{v}_j + \mathbf{v}_i^2 (134 \mathbf{v}_i \cdot \mathbf{v}_j - 49 \mathbf{v}_j^2) - 51 \mathbf{v}_i^4 - 32 \mathbf{v}_j^4]. \tag{55}$$

Moreover, the term $L^{3PN}_{(C)}$ to (53) receives contributions from the second line of (16) as well as from field redefinitions and total derivatives that we use to remove terms that involve accelerations:

$$L^{3PN}_{(C)} = \sum_{j \neq i} \sum_{k \neq i} \frac{G^2 m_i m_j m_k}{16} \left\{ \frac{1}{r_{ij} r_{ik}} [2(\mathbf{n}_{ij} \cdot \mathbf{v}_j)^2 (16 \mathbf{v}_i \cdot \mathbf{v}_j - 18 \mathbf{v}_i^2 - 32 \mathbf{v}_j \cdot \mathbf{v}_k + 12 \mathbf{v}_j^2 - (\mathbf{n}_{ik} \cdot \mathbf{v}_k)^2) \right.$$

$$+ 64 \mathbf{v}_i \cdot \mathbf{v}_j (2(\mathbf{n}_{ik} \cdot \mathbf{v}_k)^2 + \mathbf{v}_i \cdot \mathbf{v}_k - 2 \mathbf{v}_j \cdot \mathbf{v}_k - \mathbf{v}_j^2) + 16 \mathbf{v}_j^2 (8 \mathbf{v}_j \cdot \mathbf{v}_k - 2 \mathbf{v}_j^2 - 2 \mathbf{v}_k^2 - (\mathbf{n}_{ik} \cdot \mathbf{v}_k)^2)$$

$$+ 16 \mathbf{v}_i^2 (3 \mathbf{v}_j^2 + 2 \mathbf{v}_j \cdot \mathbf{v}_k - 10 \mathbf{v}_i \cdot \mathbf{v}_j) - 6(\mathbf{n}_{ij} \cdot \mathbf{v}_j)^4 + 96(\mathbf{v}_i \cdot \mathbf{v}_j)^2 + 49 \mathbf{v}_i^4] + \frac{1}{3 r_{ij}^2} [20(\mathbf{n}_{ik} \cdot \mathbf{v}_k)^3 (\mathbf{n}_{ij} \cdot \mathbf{v}_i - \mathbf{n}_{ij} \cdot \mathbf{v}_k)$$

$$- 3 \mathbf{n}_{ij} \cdot \mathbf{n}_{ik} ((\mathbf{n}_{ik} \cdot \mathbf{v}_k)^2 - \mathbf{v}_k^2)(3(\mathbf{n}_{ij} \cdot \mathbf{v}_j)^2 + 8 \mathbf{v}_i \cdot \mathbf{v}_j + (\mathbf{n}_{ik} \cdot \mathbf{v}_k)^2 + 2 \mathbf{n}_{ik} \cdot \mathbf{v}_i \mathbf{n}_{ik} \cdot \mathbf{v}_k + 6 \mathbf{v}_i \cdot \mathbf{v}_k - 5 \mathbf{v}_i^2 - 4 \mathbf{v}_j^2 - 4 \mathbf{v}_k^2)$$

$$+ 6((\mathbf{n}_{ik} \cdot \mathbf{v}_k)^2 - \mathbf{v}_k^2)(3 \mathbf{n}_{ij} \cdot \mathbf{v}_j \mathbf{n}_{ik} \cdot \mathbf{v}_i - 3 \mathbf{n}_{ij} \cdot \mathbf{v}_j \mathbf{n}_{ik} \cdot \mathbf{v}_j - 3 \mathbf{n}_{ik} \cdot \mathbf{v}_i \mathbf{n}_{ij} \cdot \mathbf{v}_k + 4 \mathbf{n}_{ij} \cdot \mathbf{v}_i \mathbf{n}_{ik} \cdot \mathbf{v}_j - \mathbf{n}_{ij} \cdot \mathbf{v}_i \mathbf{n}_{ik} \cdot \mathbf{v}_i)$$

$$+ 6 \mathbf{n}_{ik} \cdot \mathbf{v}_k [\mathbf{n}_{ij} \cdot \mathbf{v}_k (19 \mathbf{v}_i^2 + 28 \mathbf{v}_j^2 - 56 \mathbf{v}_i \cdot \mathbf{v}_j - 2 \mathbf{v}_i \cdot \mathbf{v}_k - 21(\mathbf{n}_{ij} \cdot \mathbf{v}_j)^2) + 6 \mathbf{n}_{ij} \cdot \mathbf{v}_j (6 \mathbf{v}_i \cdot \mathbf{v}_j + 7 \mathbf{v}_i \cdot \mathbf{v}_k - 6 \mathbf{v}_i^2 - 7 \mathbf{v}_j \cdot \mathbf{v}_k)$$

$$\left. + 2 \mathbf{n}_{ij} \cdot \mathbf{v}_i (11 \mathbf{v}_i^2 - 12 \mathbf{v}_j^2 - 23 \mathbf{v}_i \cdot \mathbf{v}_k + 28 \mathbf{v}_j \cdot \mathbf{v}_k + 9(\mathbf{n}_{ij} \cdot \mathbf{v}_j)^2)] + 18 \mathbf{v}_k^2 \mathbf{n}_{ik} \cdot \mathbf{v}_k (5 \mathbf{n}_{ij} \cdot \mathbf{v}_k - 4 \mathbf{n}_{ij} \cdot \mathbf{v}_i)] \right\}. \tag{56}$$

The term $L^{3PN}_{(D)}$ contributing to the above action originates from the three-body parts of the third line of (16) and can be expressed in terms of derivatives that act on the integrals $I_3[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$ and $I_3[\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}]$ as given in (43) and (44) of Sec. V:

$$L^{3PN}_{(D)} = \sum_{j \neq i} \sum_{k \neq i,j} \frac{G^2 m_i m_j m_k}{4\pi} \left\{ [(6 \mathbf{v}_i^2 + \mathbf{v}_k^2 - 8 \mathbf{v}_i \cdot \mathbf{v}_j)(\mathbf{v}_{ki} \cdot \partial_{x_i})(\mathbf{v}_{kj} \cdot \partial_{x_j}) + (8 \mathbf{v}_{ik}^2 - 4 \mathbf{v}_k^2)(\mathbf{v}_{ji} \cdot \partial_{x_i})(\mathbf{v}_{ij} \cdot \partial_{x_j})] I_3 \left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right] \right.$$

$$+ (\mathbf{v}_k \cdot \partial_{x_k})^2 [(\mathbf{v}_{ki} \cdot \partial_{x_i})(\mathbf{v}_{kj} \cdot \partial_{x_j}) + 2(\mathbf{v}_{ik} \cdot \partial_{x_k})(\mathbf{v}_{ij} \cdot \partial_{x_j}) + 4(\mathbf{v}_{ji} \cdot \partial_{x_i})(\mathbf{v}_{ij} \cdot \partial_{x_j})$$

$$\left. + 8(\mathbf{v}_{jk} \cdot \partial_{x_k})(\mathbf{v}_{kj} \cdot \partial_{x_j})] I_3 \left[\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right] \right\}. \tag{57}$$

Here the integrals I_3 depend on the three external points i, j, k as opposed to Sec. V, where the labels 1,2,3 were used. For convenience we display again the expressions (43) and (44):

$$\mu^{-2\epsilon} I_3 \left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right] = \frac{2\pi}{\epsilon} - 4\pi \log \left(\frac{r_{ij} + r_{ik} + r_{jk}}{\mu} \right) + \mathcal{O}(\epsilon), \tag{58}$$

$$\mu^{-2\epsilon} I_3 \left[\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right] = -\frac{2\pi}{3} \left[\frac{r_{ij}^2 - r_{ik}^2 - r_{jk}^2}{2\epsilon} - (r_{ij} - r_{ik})(r_{ij} - r_{jk}) - (r_{ij}^2 - r_{ik}^2 - r_{jk}^2) \log \left(\frac{r_{ij} + r_{ik} + r_{jk}}{\mu} \right) \right] + \mathcal{O}(\epsilon). \tag{59}$$

Note again that by identifying two indices in the above $L^{3PN}_{(D)}$ and using the prescriptions given in (50) and (51) we obtain the corresponding two-body contributions (55). The $1/\epsilon$-poles and the mass scale μ in the expressions for the integrals I_3 drop out after taking the derivatives in (57). This property persists at least to the 4PN order.

The structure of the result for $L^{3PN}_{(D)}$ after evaluating these derivatives is displayed in Appendix C.

Note that the contribution from the second line of (16), which is already contained in the above expression (56) for $L^{3PN}_{(C)}$, can also explicitly be written in the form before taking derivatives

$$L_{(\text{second})}^{\text{3PN}} = \sum_{j \neq i} \sum_{k \neq i} G^2 m_i m_j m_k \left\{ \frac{1}{8 r_{ij} r_{ik}} [\mathbf{v}_i^2 (\mathbf{v}_i^2 - 20 \mathbf{v}_j^2 + 16 \mathbf{v}_j \cdot \mathbf{v}_k) + 2 \mathbf{v}_j^2 (32 \mathbf{v}_i \cdot \mathbf{v}_k + 16 \mathbf{v}_j \cdot \mathbf{v}_k - 7 \mathbf{v}_j^2 - 9 \mathbf{v}_k^2) \right.$$

$$+ 32 \mathbf{v}_i \cdot \mathbf{v}_j (\mathbf{v}_i \cdot \mathbf{v}_j - \mathbf{v}_i \cdot \mathbf{v}_k - 2 \mathbf{v}_j \cdot \mathbf{v}_k)] + (\mathbf{v}_i^2 - 3 \mathbf{v}_j^2 - 3 \mathbf{v}_k^2 + 8 \mathbf{v}_j \cdot \mathbf{v}_k)(\mathbf{v}_k \cdot \partial_{x_k})^2 \frac{r_{ik}}{2 r_{ij}}$$

$$\left. - \left[(\mathbf{v}_k \cdot \partial_{x_k})^2 (\mathbf{v}_j \cdot \partial_{x_j})^2 \frac{r_{ij} r_{ik}}{4} + (\mathbf{v}_k \cdot \partial_{x_k})^4 \frac{r_{ik}^3}{12 r_{ij}} \right] \right\}. \tag{60}$$

This completes the details describing the final result (53) for the G^2 contributions to the 3PN effective action. We note again that in principle we could proceed with the same method to compute higher order contributions to the effective potential of order $G^2 v^{2n}$. The two new integrals that contribute to the next order of the expansion were already given in Sec. V. However, due to the length of the above terms at $G^2 v^4$, see also the expanded result in Appendix C, we refrain here from explicitly evaluating the contributions at the next order $G^2 v^6$.

VIII. CONCLUSIONS AND OUTLOOK

In the present paper we have extended the known results for the three-body effective potential in general relativity as follows:

(i) At order 2PM the potential is given by (17) expressed via a differential operator that acts on the three-point integral $I_{3\delta}$ evaluated in Appendix A.

(ii) At 3PN, new $G^2 v^4$-contributions were obtained in Sec. VII and are explicitly provided in an ancillary file to this paper [81].

(iii) The key integrals contributing to the effective potential at $G^2 v^{2n}$ can be obtained by the bootstrap approach described in Sec. V. Due to their length, we here refrain from evaluating the resulting expressions for the effective potential.

There are a number of interesting directions that should be further explored. First, it would be important to establish the connection between the above PN results and the direct nonrelativistic expansion of the (integrated) expression (17) at 2PM. Approaching this problem one faces the lengthy distributional expressions given in Appendix B, whose PN expansion appears to require some regularization of diverging contributions. For this reason we have performed the PN expansion at the level of the 2PM integrand, and then evaluated the integrals. Still, rederiving the PN expansion from the final 2PM expression would represent an important cross check of the result. This should also be useful to understand the interplay of the different kinematical regions for the integral discussed in Appendix A.

With regard to the higher order PN contributions which scale as $G^2 v^{2n}$, it would be interesting to bootstrap a closed formula for the ϵ-expansion of the family of integrals (31) around three dimensions. Exploiting the Yangian level-one

symmetry discussed above, this should be feasible along the lines of [68]. Here it would be great to prove that the observed mechanism which makes the divergent contributions drop out in the final expression for the potential persists to all orders. Similarly, it should be explored how far these bootstrap methods reach in obtaining integrals at higher orders of the PM or PN expansion.

The Yangian level-one symmetry that we employed can also be understood as a special conformal symmetry in a dual momentum space. Here the dualized momenta relate to the above position space variables via (19) (not via Fourier transform). It would be interesting to investigate the employed symmetry in Fourier space and to see if there is a relation to the curious conformal symmetry of graviton amplitudes observed in [82].

Finally, one should see if one can feed the above contributions to the effective potential into numerical simulations updating the studies of [8,19,20,22]. Eventually it would be fascinating if the effect of the three-body interactions obtained here could be observed in the future.

APPENDIX A: THE 3δ INTEGRAL

Here we present a detailed calculation of the 3δ-integral given by

$$I_{3\delta} = \int d^4 x_0 \delta(x_{01}^2) \delta(x_{02}^2) \delta(x_{03}^2). \tag{A1}$$

Recall $\sigma^2 = (R_2 \cdot R_3)^2 - R_2^2 R_3^2$ as defined in (20). Importantly, the quantity $-\sigma^2$ may be seen as the square of the area of the parallelogram spanned by x_1, x_2, x_3 and

thus characterizes the space $\mathcal{M}$ spanned by these three points:

$$\sigma^2 > 0, \quad \mathcal{M} \text{ is 2D Minkowskian,}$$
$$\sigma^2 = 0, \quad \mathcal{M} \text{ is a 1D straight line,}$$
$$\sigma^2 < 0, \quad \mathcal{M} \text{ is 2D Euclidean.}$$

We now explicitly evaluate the above integral for these three cases, generalizing the computation of Westpfahl [59] for the three-point integral with retarded propagators.

1. The case $\sigma^2 > 0$

We choose four basis vectors $R_2^\mu, R_3^\mu, \xi_1^\mu, \xi_2^\mu$ such that we can express the integration vector as

$$x_{01}^\mu = \tau R_2^\mu + \bar{\tau} R_3^\mu + r(\cos\varphi \xi_1^\mu + \sin\varphi \xi_2^\mu), \quad r \geq 0. \quad (A2)$$

Here $\{\xi_1, \xi_2\}$ denotes the (orthogonal) unit basis of the perpendicular complement of $\mathcal{M}$:

$$\xi_i \cdot R_j = 0, \qquad \xi_i \cdot \xi_j = -\delta_{ij}, \quad \text{for } i, j = 1, 2. \quad (A3)$$

In these coordinates the integration measure reads

$$d^4 x_0 = \frac{1}{2} \sigma d\tau d\bar{\tau} dr^2 d\varphi, \quad (A4)$$

and the integral simplifies to

$$\int \frac{d^4 x_0}{4\sigma^2} \delta\left(r^2 + \frac{R_1^2 R_2^2 R_3^2}{4\sigma^2}\right) \delta\left(\tau - \frac{R_3^2 R_1 \cdot R_2}{2\sigma^2}\right)$$
$$\times \delta\left(\bar{\tau} + \frac{R_2^2 R_1 \cdot R_3}{2\sigma^2}\right). \quad (A5)$$

This straightforwardly yields

$$I_{3\delta}(\sigma^2 > 0) = \frac{\pi}{4\sigma} \Theta(-R_1^2 R_2^2 R_3^2). \quad (A6)$$

Here Θ denotes the Heaviside step function defined as

$$\Theta(x) = \begin{cases} 1, & \text{for } x > 0, \\ \frac{1}{2}, & \text{for } x = 0, \\ 0, & \text{for } x < 0. \end{cases} \quad (A7)$$

Hence, we conclude that in the region $\sigma^2 > 0$ the piecewise constant in (20) is given by

$$C(\sigma^2 > 0) = \frac{\pi}{4} \Theta(-R_1^2 R_2^2 R_3^2). \quad (A8)$$

Note that since $\delta(x_{01}^2)$ is the Green's function of the d'Alembertian, see (11), the above integral $I_{3\delta}$ satisfies

$$\partial_1^2 I_{3\delta} = 4\pi\delta(x_{12}^2)\delta(x_{13}^2). \quad (A9)$$

In the region $\sigma^2 > 0$ this is guaranteed by the Heaviside function in (A6); dropping the Θ-function in (A6) would yield a vanishing result as $\partial_1^2 \sigma^{-1} = 0$.

2. The case $\sigma^2 < 0$

In the region where $\sigma^2 < 0$, we can span x_{01} as

$$x_{01}^\mu = tT^\mu + \tau R_2^\mu + \bar{\tau} R_3^\mu + r\xi^\mu. \quad (A10)$$

Here T^μ and ξ^μ denote again unit vectors that are orthogonal to each other and to R_2^μ, R_3^μ, with T^μ being timelike and ξ^μ spacelike. The volume element in this coordinate system is

$$d^4 x_0 = \sqrt{-\sigma^2} dt dr d\tau d\bar{\tau}, \quad (A11)$$

and the integral becomes

$$\begin{aligned} I_{3\delta} &= \int d^4 x_0 \delta(t^2 + (\tau R_2^\mu + \bar{\tau} R_3^\mu)^2 - r^2) \\ &\quad \times \delta(2\tau R_2 \cdot R_3 + (2\bar{\tau} + 1)R_3^2) \\ &\quad \times \delta(2\bar{\tau} R_2 \cdot R_3 + (2\tau - 1)R_2^2) \\ &= \frac{\sqrt{-\sigma^2}}{-4\sigma^2} \int dt dr \delta\left(t^2 - r^2 - \frac{R_1^2 R_2^2 R_3^2}{4\sigma^2}\right) \\ &= \frac{1}{4\sqrt{-\sigma^2}} \int_{-\infty}^{+\infty} \frac{dr}{\sqrt{r^2 + 1}} \to \infty. \quad (A12) \end{aligned}$$

Hence, for $\sigma^2 < 0$ the integral diverges.

3. The case $\sigma^2 = 0$

Finally, for $\sigma^2 = 0$ the surface spanned by the vectors connecting x_1, x_2 and x_3 degenerates into a line. We define the unit vector on this line as R_u^μ, and we set $R_i^\mu = \omega_i R_u^\mu$. Depending on the nature of this line one finds different expressions as follows. For the line being timelike we have

$$\begin{aligned} x_{01}^\mu &= \tau R_u^\mu + r(\xi_1^\mu \cos\theta + \xi_2^\mu \sin\theta \cos\phi + \xi_3^\mu \sin\theta \sin\phi), \\ d^4 x_0 &= r^2 \sin\theta d\tau dr d\theta d\phi, \quad (A13) \end{aligned}$$

and thus

$$\begin{aligned} I_{3\delta} &= \int d^4 x_0 \delta(\tau^2 - r^2)\delta(\omega_3^2 + 2\tau\omega_3)\delta(\omega_2^2 - 2\tau\omega_2) \\ &= \begin{cases} \infty & \omega_1 \omega_2 \omega_3 = 0, \\ 0 & \text{otherwise.} \end{cases} \quad (A14) \end{aligned}$$

For a spacelike line and with $T \cdot R_u = 0$, we have

$$x_{01}^\mu = tT^\mu + \tau R_u^\mu + r(\xi_1^\mu \cos\theta + \xi_2^\mu \sin\theta),$$

$$d^4 x_0 = r dt d\tau dr d\theta, \tag{A15}$$

which implies

$$I_{3\delta} = \int d^4 x_0 \delta(t^2 + \tau^2 - r^2)\delta(\omega_3^2 + 2\tau\omega_3)\delta(\omega_2^2 - 2\tau\omega_2)$$

$$= \begin{cases} \infty & \omega_1\omega_2\omega_3 = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{A16}$$

And finally for a lightlike line with $T \cdot R_u \neq 0$, we obtain

$$x_{01}^\mu = tT^\mu + \tau R^\mu + r(\xi_1^\mu \cos\theta + \xi_2^\mu \sin\theta),$$

$$d^4 x_0 = \sqrt{(T \cdot R_u)^2 - T^2 R_u^2}\, r dt d\tau dr d\theta, \tag{A17}$$

such that

$$I_{3\delta} = \int d^4 x_0 \delta(t^2 + 2t\tau T \cdot R_u - r^2)$$

$$\times \delta(2t\omega_3 T \cdot R_u)\delta(2t\omega_2 T \cdot R_u)$$

$$\sim \int d\tau \delta(0) = \infty. \tag{A18}$$

Hence, the result for $\sigma^2 = 0$ may be summarized as

$$I_{3\delta}(\sigma^2 = 0) \sim \delta(R_1^2) + \delta(R_2^2) + \delta(R_3^2). \tag{A19}$$

In total we thus conclude that the 3δ-integral can be expressed as

$$I_{3\delta} = \begin{cases} \frac{\pi}{4\sigma}\Theta(-R_1^2 R_2^2 R_3^2), & \sigma^2 > 0, \\ \sim\delta(R_1^2) + \delta(R_2^2) + \delta(R_3^2), & \sigma^2 = 0, \\ \infty, & \sigma^2 < 0. \end{cases} \tag{A20}$$

We note that when using the result for $\sigma^2 > 0$ it can be useful to expand the theta-function according to

$$\Theta(-R_1^2 R_2^2 R_3^2) = +\Theta(-R_1^2)\Theta(-R_2^2)\Theta(-R_3^2)$$

$$+ \Theta(-R_1^2)\Theta(+R_2^2)\Theta(+R_3^2)$$

$$+ \Theta(+R_1^2)\Theta(-R_2^2)\Theta(+R_3^2)$$

$$+ \Theta(+R_1^2)\Theta(+R_2^2)\Theta(-R_3^2). \tag{A21}$$

APPENDIX B: DERIVATIVES OF THE 3δ INTEGRAL

In this Appendix we explicitly evaluate the expressions for the second order derivatives of the triple-delta integral $I_{3\delta}$ for $\sigma^2 > 0$, cf. (A20). These enter into the three-body effective potential via (16). A priori we find four terms

$$\partial_1^\mu \partial_2^\nu I_{3\delta}(\sigma^2 > 0) = +\left(\partial_1^\mu \partial_2^\nu \frac{\pi}{4\sigma}\right)\Theta(-R_1^2 R_2^2 R_3^2) + \left(\partial_1^\mu \frac{\pi}{4\sigma}\right)\partial_2^\nu\Theta(-R_1^2 R_2^2 R_3^2)$$

$$+ \left(\partial_2^\nu \frac{\pi}{4\sigma}\right)\partial_1^\mu\Theta(-R_1^2 R_2^2 R_3^2) + \frac{\pi}{4\sigma}\partial_1^\mu \partial_2^\nu \Theta(-R_1^2 R_2^2 R_3^2), \tag{B1}$$

which evaluate to

$$\Theta(-R_1^2 R_2^2 R_3^2)\partial_1^\mu \partial_2^\nu \frac{\pi}{4\sigma} = \frac{\pi\Theta(-R_1^2 R_2^2 R_3^2)}{4\sigma^5}[3(R_1 \cdot R_2)(R_1^\mu R_1^\nu R_2^2 + R_2^\mu R_2^\nu R_1^2 - R_1^\mu R_2^\nu R_1 \cdot R_2 - R_2^\mu R_1^\nu R_1 \cdot R_2)$$

$$+ \sigma^2(\eta^{\mu\nu} R_1 \cdot R_2 + R_1^\mu R_2^\nu + R_2^\mu R_1^\nu)], \tag{B2}$$

$$\frac{\pi}{4\sigma}\partial_1^\mu \partial_2^\nu \Theta(-R_1^2 R_2^2 R_3^2) = \frac{\pi}{4\sigma}[2\eta^{\mu\nu}\delta(R_3^2) + 4R_3^\mu R_3^\nu \delta'(R_3^2)]\text{sgn}(R_1^2 R_2^2) + \frac{8\pi}{4\sigma}[R_2^\mu R_1^\nu \delta(R_1^2)\delta(R_2^2)\text{sgn}(R_3^2)$$

$$- R_2^\mu R_3^\nu \delta(R_2^2)\delta(R_3^2)\text{sgn}(R_1^2) - R_3^\mu R_1^\nu \delta(R_1^2)\delta(R_3^2)\text{sgn}(R_2^2)], \tag{B3}$$

$$\left(\partial_1^\mu \frac{\pi}{4\sigma}\right)\partial_2^\nu\Theta(-R_1^2 R_2^2 R_3^2) = \frac{\pi}{4\sigma}\left[\frac{4R_1^\mu R_1^\nu}{R_2^2 - R_3^2}\delta(R_1^2)\text{sgn}(R_2^2 R_3^2)\right.$$

$$\left. - \frac{4R_2^\mu R_3^\nu}{R_1^2 - R_2^2}\delta(R_3^2)\text{sgn}(R_1^2 R_2^2) + R_3^\mu R_3^\nu \frac{R_1^2 + R_2^2}{\sigma^2}\delta(R_3^2)\text{sgn}(R_1^2 R_2^2)\right]. \tag{B4}$$

Here the last line also enters into (B1) with the labels 1 and 2 interchanged. Note the appearance of the derivative of the delta function in the first line of (B3) that one could resolve using $\delta'(R_3^2) = -\delta(R_3^2)/R_3^2$. Putting these terms together, Eq. (B1) then becomes (ordered by the number of delta functions)

$$\partial_1^\mu \partial_2^\nu I_{3\delta} = \frac{\pi \Theta(-R_1^2 R_2^2 R_3^2)}{4\sigma^5} [3(R_1 \cdot R_2)(R_1^\mu R_1^\nu R_2^2 + R_2^\mu R_2^\nu R_1^2 - R_1^\mu R_2^\nu R_1 \cdot R_2 - R_2^\mu R_1^\nu R_1 \cdot R_2) + \sigma^2(\eta^{\mu\nu} R_1 \cdot R_2 + R_1^\mu R_2^\nu + R_2^\mu R_1^\nu)]$$

$$+ \frac{\pi}{4\sigma}\left[+ \frac{4R_1^\mu R_1^\nu}{R_2^2 - R_3^2}\delta(R_1^2)\mathrm{sgn}(R_2^2 R_3^2) + \frac{4R_2^\mu R_2^\nu}{R_1^2 - R_3^2}\delta(R_2^2)\mathrm{sgn}(R_1^2 R_3^2) \right]$$

$$+ \frac{\pi}{4\sigma}\left[-\frac{4R_1^\mu R_3^\nu}{R_2^2 - R_1^2}\delta(R_3^2) - \frac{4R_2^\mu R_3^\nu}{R_1^2 - R_2^2}\delta(R_3^2) + 2R_3^\mu R_3^\nu \frac{R_1^2 + R_2^2}{\sigma^2}\delta(R_3^2) + 2\eta^{\mu\nu}\delta(R_3^2) + 4R_3^\mu R_3^\nu \delta'(R_3^2) \right]\mathrm{sgn}(R_1^2 R_2^2)$$

$$+ \frac{8\pi}{4\sigma}[+R_2^\mu R_1^\nu \delta(R_1^2)\delta(R_2^2)\mathrm{sgn}(R_3^2) - R_2^\mu R_3^\nu \delta(R_2^2)\delta(R_3^2)\mathrm{sgn}(R_1^2) - R_3^\mu R_1^\nu \delta(R_1^2)\delta(R_3^2)\mathrm{sgn}(R_2^2)]. \quad (B5)$$

Performing the PN expansion starting from this expression seems (also conceptually) much harder than working on the level of the integrand of $I_{3\delta}$ in (16). The latter is demonstrated in Sec. V.

APPENDIX C: DETAILS ON 3PN

In the computation of the 3PN potential, we added the following total derivative to remove the dependence on the derivative of accelerations and possible spurious poles for $r_{ij} \to \infty$:

$$L^{\mathrm{td}} = \sum_{j \neq i} \frac{Gm_i m_j}{48c^8} \frac{\mathrm{d}}{\mathrm{d}t} [r_{ij}(21\mathbf{a}_i \cdot \mathbf{v}_j - 18\mathbf{a}_i \cdot \mathbf{v}_i)((\mathbf{n}_{ij} \cdot \mathbf{v}_j)^2 + \mathbf{v}_j^2) + r_{ij}\mathbf{n}_{ij} \cdot \mathbf{a}_i \mathbf{n}_{ij} \cdot \mathbf{v}_j((\mathbf{n}_{ij} \cdot \mathbf{v}_j)^2 - 3\mathbf{v}_j^2)]. \quad (C1)$$

Due to its length, here we display only an excerpt of the genuine three-body contribution to the 3PN effective potential from the third line of (16). The full result is given in an ancillary file [81]. The expression below is organized according to the rational functions of the spatial distances, where each function is multiplied by a sum of numerator structures that scale as v^4. Note that some numerator structures begin with the same terms but they do not agree. Evaluating the derivatives in (57) yields the expression

$$L_{(D)}^{\mathrm{3PN}} = \sum_{j \neq i} \sum_{k \neq i,j} G^2 m_i m_j m_k$$

$$\times \left\{ \frac{1}{(r_{ij} + r_{jk} + r_{ik})^2}\left((\mathbf{n}_{ik} \cdot \mathbf{v}_i)\left(\frac{16}{3}(\mathbf{n}_{ij} \cdot \mathbf{v}_i)^3 - 12(\mathbf{n}_{ij} \cdot \mathbf{v}_i)^2(\mathbf{n}_{ij} \cdot \mathbf{v}_j) + \frac{20}{3}(\mathbf{n}_{ij} \cdot \mathbf{v}_i)(\mathbf{n}_{ij} \cdot \mathbf{v}_j)^2 \right) + 245 \text{ terms} \right) \right.$$

$$+ \frac{1}{r_{ij}(r_{ij} + r_{jk} + r_{ik})}\left(\mathbf{v}_i^2\left(\frac{16}{3}(\mathbf{n}_{ij} \cdot \mathbf{v}_i)^2 - 11(\mathbf{n}_{ij} \cdot \mathbf{v}_i)(\mathbf{n}_{ij} \cdot \mathbf{v}_j) + \frac{16}{3}(\mathbf{n}_{ij} \cdot \mathbf{v}_j)^2 \right) + 45 \text{ terms} \right)$$

$$- \frac{r_{ij}}{(r_{ij} + r_{jk} + r_{ik})^3}\left(\frac{8}{3}(\mathbf{n}_{ij} \cdot \mathbf{v}_i)^4 - 6(\mathbf{n}_{ij} \cdot \mathbf{v}_i)^3(\mathbf{n}_{ij} \cdot \mathbf{v}_j) + \frac{8}{3}(\mathbf{n}_{ij} \cdot \mathbf{v}_i)^2(\mathbf{n}_{ij} \cdot \mathbf{v}_j)^2 + 286 \text{ terms} \right)$$

$$- \frac{r_{ik}r_{jk}}{r_{ij}(r_{ij} + r_{jk} + r_{ik})^3}((\mathbf{n}_{ik} \cdot \mathbf{v}_k)^2(16(\mathbf{n}_{ij} \cdot \mathbf{v}_i)^2 - 36(\mathbf{n}_{ij} \cdot \mathbf{v}_i)(\mathbf{n}_{ij} \cdot \mathbf{v}_j) + 16(\mathbf{n}_{ij} \cdot \mathbf{v}_j)^2) + 69 \text{ terms})$$

$$- \frac{r_{ij}^2}{(r_{ij} + r_{jk} + r_{ik})^4}(8(\mathbf{n}_{ij} \cdot \mathbf{v}_i)^4 - 18(\mathbf{n}_{ij} \cdot \mathbf{v}_i)^3(\mathbf{n}_{ij} \cdot \mathbf{v}_j) + 16(\mathbf{n}_{ij} \cdot \mathbf{v}_i)^2(\mathbf{n}_{ij} \cdot \mathbf{v}_j)^2 + 143 \text{ terms})$$

$$+ \frac{r_{ik}r_{jk}}{(r_{ij} + r_{jk} + r_{ik})^4}((\mathbf{n}_{ik} \cdot \mathbf{v}_k)^2(16(\mathbf{n}_{ij} \cdot \mathbf{v}_i)^2 - 36(\mathbf{n}_{ij} \cdot \mathbf{v}_i)(\mathbf{n}_{ij} \cdot \mathbf{v}_j) + 16(\mathbf{n}_{ij} \cdot \mathbf{v}_j)^2) + 114 \text{ terms})$$

$$- \left[\frac{1}{r_{ik}(r_{ij} + r_{jk} + r_{ik})}\left(\frac{4}{3}\mathbf{v}_i^4 - \frac{4}{3}((\mathbf{n}_{ik} \cdot \mathbf{v}_i)^2 - (\mathbf{n}_{ik} \cdot \mathbf{v}_k)^2)\mathbf{v}_i^2 + 20 \text{ terms} \right) + (i \leftrightarrow j) \right]$$

$$- \left[\frac{r_{ik}}{r_{jk}(r_{ij} + r_{jk} + r_{ik})^2}\left((\mathbf{n}_{jk} \cdot \mathbf{v}_k)^2\left(\frac{8}{3}(\mathbf{n}_{ij} \cdot \mathbf{v}_i)^2 - 6(\mathbf{n}_{ij} \cdot \mathbf{v}_i)(\mathbf{n}_{ij} \cdot \mathbf{v}_j) + \frac{8}{3}(\mathbf{n}_{ij} \cdot \mathbf{v}_j)^2 \right) + 23 \text{ terms} \right) + (i \leftrightarrow j) \right]$$

$$+ \left. \left[\frac{r_{ij}}{r_{ik}(r_{ij} + r_{jk} + r_{ik})^2}\left((\mathbf{n}_{jk} \cdot \mathbf{v}_k)^2\left(\frac{8}{3}(\mathbf{n}_{ij} \cdot \mathbf{v}_i)^2 - 6(\mathbf{n}_{ij} \cdot \mathbf{v}_i)(\mathbf{n}_{ij} \cdot \mathbf{v}_j) + \frac{8}{3}(\mathbf{n}_{ij} \cdot \mathbf{v}_j)^2 \right) + 47 \text{ terms} \right) + (i \leftrightarrow j) \right] \right.$$

$$-\left[\frac{r_{ik}}{r_{ij}(r_{ij}+r_{jk}+r_{ik})^2}(8(\mathbf{n}_{ij}\cdot\mathbf{v}_i)^4-18(\mathbf{n}_{ij}\cdot\mathbf{v}_i)^3(\mathbf{n}_{ij}\cdot\mathbf{v}_j)+12(\mathbf{n}_{ij}\cdot\mathbf{v}_i)^2(\mathbf{n}_{ij}\cdot\mathbf{v}_j)^2+93\,\text{terms})+(i\leftrightarrow j)\right]$$

$$-\left[\frac{r_{ik}}{(r_{ij}+r_{jk}+r_{ik})^3}(16(\mathbf{n}_{ij}\cdot\mathbf{v}_i)^4-36(\mathbf{n}_{ij}\cdot\mathbf{v}_i)^3(\mathbf{n}_{ij}\cdot\mathbf{v}_j)+24(\mathbf{n}_{ij}\cdot\mathbf{v}_i)^2(\mathbf{n}_{ij}\cdot\mathbf{v}_j)^2+285\,\text{terms})+(i\leftrightarrow j)\right]$$

$$+\left[\frac{r_{ik}^2}{r_{jk}(r_{ij}+r_{jk}+r_{ik})^3}\left((\mathbf{n}_{jk}\cdot\mathbf{v}_k)^2\left(\frac{16}{3}(\mathbf{n}_{ij}\cdot\mathbf{v}_i)^2-12(\mathbf{n}_{ij}\cdot\mathbf{v}_i)(\mathbf{n}_{ij}\cdot\mathbf{v}_j)+\frac{16}{3}(\mathbf{n}_{ij}\cdot\mathbf{v}_j)^2\right)+46\,\text{terms}\right)+(i\leftrightarrow j)\right]$$

$$-\left[\frac{r_{ij}^2}{r_{ik}(r_{ij}+r_{jk}+r_{ik})^3}\left((\mathbf{n}_{ik}\cdot\mathbf{v}_i)^2\left(\frac{8}{3}(\mathbf{n}_{ij}\cdot\mathbf{v}_i)^2-6(\mathbf{n}_{ij}\cdot\mathbf{v}_i)(\mathbf{n}_{ij}\cdot\mathbf{v}_j)+\frac{8}{3}(\mathbf{n}_{ij}\cdot\mathbf{v}_j)^2\right)+58\,\text{terms}\right)+(i\leftrightarrow j)\right]$$

$$-\left[\frac{r_{ij}r_{jk}}{r_{ik}(r_{ij}+r_{jk}+r_{ik})^3}\left((\mathbf{n}_{ik}\cdot\mathbf{v}_i)^2\left(\frac{8}{3}(\mathbf{n}_{ij}\cdot\mathbf{v}_i)^2-6(\mathbf{n}_{ij}\cdot\mathbf{v}_i)(\mathbf{n}_{ij}\cdot\mathbf{v}_j)+\frac{8}{3}(\mathbf{n}_{ij}\cdot\mathbf{v}_j)^2\right)+72\,\text{terms}\right)+(i\leftrightarrow j)\right]$$

$$-\left[\frac{r_{ik}^2}{r_{ij}(r_{ij}+r_{jk}+r_{ik})^3}\left((\mathbf{n}_{ik}\cdot\mathbf{v}_k)^2\left(\frac{40}{3}(\mathbf{n}_{ik}\cdot\mathbf{v}_i)^2-30(\mathbf{n}_{ik}\cdot\mathbf{v}_i)(\mathbf{n}_{ik}\cdot\mathbf{v}_j)+\frac{40}{3}(\mathbf{n}_{ik}\cdot\mathbf{v}_j)^2\right)+75\,\text{terms}\right)+(i\leftrightarrow j)\right]$$

$$+\left[\frac{r_{ik}^2}{(r_{ij}+r_{jk}+r_{ik})^4}((\mathbf{n}_{ik}\cdot\mathbf{v}_k)^2(16(\mathbf{n}_{ij}\cdot\mathbf{v}_i)^2-36(\mathbf{n}_{ij}\cdot\mathbf{v}_i)(\mathbf{n}_{ij}\cdot\mathbf{v}_j)+16(\mathbf{n}_{ij}\cdot\mathbf{v}_j)^2)+109\,\text{terms})+(i\leftrightarrow j)\right]$$

$$-\left[\frac{r_{ij}r_{ik}}{(r_{ij}+r_{jk}+r_{ik})^4}(16(\mathbf{n}_{ij}\cdot\mathbf{v}_i)^4-36(\mathbf{n}_{ij}\cdot\mathbf{v}_i)^3(\mathbf{n}_{ij}\cdot\mathbf{v}_j)+24(\mathbf{n}_{ij}\cdot\mathbf{v}_i)^2(\mathbf{n}_{ij}\cdot\mathbf{v}_j)^2+174\,\text{terms})+(i\leftrightarrow j)\right]$$

$$-\left[\frac{r_{ik}^2(r_{ij}+r_{jk})}{r_{ij}r_{jk}(r_{ij}+r_{jk}+r_{ik})^3}\left((\mathbf{n}_{jk}\cdot\mathbf{v}_k)^2\left(\frac{8}{3}(\mathbf{n}_{ij}\cdot\mathbf{v}_i)^2-6(\mathbf{n}_{ij}\cdot\mathbf{v}_i)(\mathbf{n}_{ij}\cdot\mathbf{v}_j)+\frac{8}{3}(\mathbf{n}_{ij}\cdot\mathbf{v}_j)^2\right)+21\,\text{terms}\right)+(i\leftrightarrow j)\right]$$

$$+\left[\frac{r_{ik}^2(r_{ik}+2r_{jk})}{r_{ij}r_{jk}(r_{ij}+r_{jk}+r_{ik})^4}((\mathbf{n}_{ik}\cdot\mathbf{v}_k)^2(8(\mathbf{n}_{ij}\cdot\mathbf{v}_i)^2-36(\mathbf{n}_{ij}\cdot\mathbf{v}_i)(\mathbf{n}_{ij}\cdot\mathbf{v}_j)+8(\mathbf{n}_{ij}\cdot\mathbf{v}_j)^2)+33\,\text{terms})+(i\leftrightarrow j)\right]\Bigg\}.$$

(C2)

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