Monopole fluctuation of the CMB and its gauge invariance

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The standard theoretical description $\Theta(\hat{n})$ of the observed cosmic microwave background (CMB) temperature anisotropies is gauge dependent. It is, however, well known that the gauge mode is limited to the monopole and that the higher angular multipoles Θ_l ($l \geq 1$) are gauge invariant. Several attempts have been made in the past to properly define the monopole fluctuation, but the resulting values of the monopole power C_0 are infinite due to the infrared divergences. The infrared divergences arise from the contribution of the uniform gravitational potential to the monopole fluctuation, in violation of the equivalence principle. Here we present the gauge-invariant theoretical description of the observed CMB temperature anisotropies and compute the monopole power $C_0 = 1.66 \times 10^{-9}$ in a ΛCDM model. While the gauge dependence in the standard calculations originates from the ambiguity in defining the hypersurface for the background CMB temperature $\bar{T}$ today, it is in fact well defined and one of the fundamental cosmological parameters. We argue that once the cosmological parameters are chosen, the monopole fluctuation can be unambiguously inferred from the angle average of the observed CMB temperature, making it a model-dependent "observable." Adopting simple approximations for the anisotropy formation, we derive a gauge-invariant analytical expression for the observed CMB temperature anisotropies to study the CMB monopole fluctuation and the cancellation of the uniform gravitational potential contributions on large scales.

I. INTRODUCTION

Two years after the discovery of the cosmic microwave background (CMB) radiation by Penzias and Wilson in 1965 [1], Sachs and Wolfe published their pioneering work [2] about the formation of the CMB temperature anisotropies. Since then, the theoretical description of the CMB anisotropies has been extensively studied in many works (see, e.g., [3–8]). The first detection of the CMB anisotropies by the Cosmic Background Explorer (COBE) satellite was announced in 1992 [9], and a variety of experiments (ground, balloon, and space based) have been carried out since then. The polarization of the CMB was discovered in 2002 by the Degree Angular Scale Interferometer telescope [10]. Launched in 2001, the Wilkinson Microwave Anisotropy Probe satellite collected data for nine years, and the final data were released in 2012 [11]. Its successor, the Planck satellite, was launched in 2009, and the final data was released in a series of papers in 2018 (see, e.g., [12]). Measurements on small angular scales were provided by the Atacama Cosmology Telescope [13] and the South Pole Telescope [14]. In addition to observational data, accurate numerical computations of the CMB temperature anisotropies are provided by different versions of the Boltzmann codes (e.g., CMBFAST [15], CAMB [16], CLASS [17]).

Given these recent developments, we revisit the standard theoretical description of the CMB temperature anisotropies, in particular, focusing on the monopole fluctuation. Once the Universe expands and cools enough, the CMB photons decouple from free electrons and propagate toward the observer, so the theoretical description of the observed CMB temperature naturally involves the physical quantities at the decoupling point, along the photon path, and at the observer position. At the linear order in perturbations, the gravitational redshift, the Doppler effect, and the intrinsic temperature fluctuation form the contributions to the CMB temperature anisotropies at the decoupling position, while the CMB temperature is further affected during the propagation of the CMB photons through the inhomogeneous Universe, known as the integrated Sachs-Wolfe effect [2]. At the observer position, the observer motion is the dominant contribution to the CMB temperature anisotropies, but there exist other relativistic contributions.

The standard theoretical description of the observed CMB temperature anisotropies is the temperature fluctuation at the observer position, and it is gauge dependent, just like any perturbation quantities. However, since the gauge transformation of the temperature fluctuation at the

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observer position is isotropic, only the theoretical description of the CMB monopole anisotropy is affected, and the theoretical descriptions of all the other higher-order angular multipoles are gauge invariant. The gauge dependence of the standard expression originates from the ambiguity in defining the hypersurface for the background CMB temperature today: being a function of time, the background temperature in the standard background-perturbation split of the observed photon temperature depends on the coordinate system chosen to describe the time coordinate of the observer. However, it was shown [18] that the background CMB temperature today is in fact well defined and there exists no further gauge ambiguity.

The gauge issues associated with the theoretical description of the CMB temperature anisotropies was extensively discussed in the comprehensive work [19] by Zibin and Scott. An analytical expression for the CMB temperature anisotropies was derived, and the resulting monopole and the dipole transfer functions were discussed in detail. It was stated [19] that the choice of the observer hypersurface (or the time coordinate of the observer) is not uniquely fixed by any physical prescription and this ambiguity affects the monopole of the CMB temperature anisotropy. By specifying the observer hypersurface to be one of the uniform energy density, their expression for the observed CMB temperature anisotropy is gauge invariant, but the resulting monopole power C_0 is divergent. Other attempts have been made in the past to derive a gauge-invariant expression, though the focus was not on the monopole fluctuation (see, e.g., [20]). All the predictions for the monopole power are divergent as well.

In this work we derive a gauge-invariant analytical expression for the observed CMB temperature anisotropies $\hat{\Theta}(\hat{n})$ with particular emphasis on the CMB monopole anisotropy and its gauge invariance. We find that the resulting monopole power is devoid of any divergences, as the uniform gravitational potential contributions to the monopole fluctuation on large scales are canceled in accordance with the equivalence principle. Our numerical computation shows for the first time that the (finite) monopole power is $C_0 = 1.66 \times 10^{-9}$ in a ΛCDM universe. We investigate the validity of our expression by comparing to the Boltzmann codes and compare our results to the previous work.

It is often said that the monopole fluctuation is not directly observable, as it contributes to the angle average of the observed CMB temperature together with the arbitrary background temperature. We argue that the background CMB temperature today in Eq. (7) is unambiguously defined in a given model and its cosmological parameters, and in fact it is one of the fundamental cosmological parameters [18]. Once a choice of the cosmological parameters is made, the monopole fluctuation *can* be inferred from the angle-average of the observed CMB temperature. Therefore, the CMB monopole fluctuation is

an observable in a sense similar to the case in galaxy clustering, where the observed galaxy number density and its power spectrum are observables, once a choice of cosmological parameters is made.

The organization of this paper is as follows: in Sec. II A, we investigate the coordinate dependence of the background photon temperature. Our main result, a gauge-invariant analytical expression for the observed CMB temperature anisotropies, is derived in Sec. II B. We clarify some issues associated with both the sky average of the observed photon temperature and the observability of the monopole fluctuation in Sec. II C and decompose our analytical expression for the temperature anisotropies in terms of observed angle in Sec. II D. We focus on the monopole and the dipole in Secs. III A and III B, respectively, and numerically compute their power. Their large-scale limits are investigated in more detail in Sec. III C. We compare our result to previous work in Sec. III D and conclude with a discussion in Sec. IV.

II. OBSERVED CMB TEMPERATURE

Here we present our theoretical description of the observed CMB temperature. In Sec. II A we discuss the coordinate dependence of the background CMB temperature and the issues associated with it. A gauge-invariant expression for the observed CMB temperature anisotropy is then derived in Sec. II B. We discuss the sky average of the observed temperature and clarify some ambiguities concerning the observability of the monopole fluctuation in Sec. II C and derive an expression for the multipole coefficients of the observed CMB temperature anisotropy in Sec. II D.

A. Background CMB temperature $\bar{T}$

In perturbation analysis, it is convenient to split any quantities into a background and a perturbation. For CMB photons in a thermal equilibrium, the CMB temperature T at a given spacetime position x^μ can be split as

$$T(x^\mu) = \bar{T}(\eta)[1 + \Theta(x^\mu)], \tag{1}$$

where η is the conformal time coordinate. The background temperature $\bar{T}(\eta)$ represents the photon temperature in a homogeneous Universe, and the remaining part defines the perturbation or the (dimensionless) temperature fluctuation $\Theta := T(x^\mu)/\bar{T}(\eta_o) - 1$. The homogeneity and isotropy in the background Universe renders the background temperature $\bar{T}(\eta)$ only a function of time and independent of spatial position. This implies that the value of the background CMB temperature $\bar{T}(\eta)$ at a given spacetime position depends on our choice of coordinate system, or the time coordinate η for the given position x^μ. In particular, the background temperature $\bar{T}(\eta_o)$ for the observer today

will depend on the time coordinate η_o of the observer, where the subscript o indicates the observer position.

Given the diffeomorphism symmetry in general relativity, we can choose any coordinate system to describe physics, and we consider a general coordinate transformation

$$x^\mu \mapsto \tilde{x}^\mu = x^\mu + \xi^\mu, \qquad \xi^\mu := (\xi, L^{,\alpha}), \qquad (2)$$

where $\alpha, \beta, \cdots$ represent the spatial indices, while $\mu, \nu, \cdots$ represent the spacetime indices. Under the coordinate transformation, the observer position today is then described by two different coordinates $x_o^\mu := (\eta_o, x_o^\alpha)$ and $\tilde{x}_o^\mu := (\tilde{\eta}_o, \tilde{x}_o^\alpha)$ with the relation

$$\tilde{\eta}_o = \eta_o + \xi_o, \qquad \tilde{x}_o^\alpha = x_o^\alpha + L_o^{,\alpha}. \qquad (3)$$

It is now apparent that the background CMB temperature $\bar{T}(\eta_o)$ at the observer position in two different coordinate systems is different by

$$\bar{T}(\tilde{\eta}_o) = \bar{T}(\eta_o) + \bar{T}'(\eta_o)\xi_o, \qquad (4)$$

where the prime denotes the derivative with respect to the conformal time and we expanded to the linear order in perturbations. After recombination, the CMB photons free stream, and the observer at x_o^μ today measures the black-body temperature $T(\hat{n})$ of the CMB photons along the observed direction $\hat{n}$. The fact that the observed CMB temperature $T(\hat{n}) = \bar{T}(\eta_o)(1 + \Theta)$ arriving at the observer position x_o^μ should be independent of our coordinate choice provides the consistency relation for the temperature fluctuation Θ under the coordinate transformation:

$$\tilde{\Theta}(\tilde{x}_o^\mu) = \Theta(x_o^\mu) + \mathcal{H}_o\xi_o, \qquad (5)$$

and it is indeed consistent with the gauge transformation property of Θ at a given coordinate x^μ, where $\mathcal{H}$ is the conformal Hubble parameter, $\bar{T} \propto 1/a$, and we suppressed the dependence of the temperature fluctuation on the observed direction $\hat{n}$.

While the observed CMB temperature $T(\hat{n})$ along the direction $\hat{n}$ is independent of coordinate system, both the background $\bar{T}(\eta_o)$ and the perturbation Θ separately depend on our choice of coordinate system. We remove this coordinate dependence of the background temperature $\bar{T}(\eta_o)$ by introducing a fixed reference time $\bar{\eta}_o$ (or $\bar{t}_o$), independent of our coordinate choice:

$$\bar{\eta}_o := \int_0^\infty \frac{dz}{H(z)}, \qquad \bar{t}_o := \int_0^\infty \frac{dz}{H(z)(1+z)}, \qquad (6)$$

where $H(z) = (1+z)\mathcal{H}$ is the Hubble parameter. The coordinate-independent reference time $\bar{t}_o$ is known as the age of the (homogeneous) Universe, and $\bar{\eta}_o$ is the conformal time, corresponding to the proper time $\bar{t}_o$. Note that

the exact value of our reference $\bar{\eta}_o$ (or its theoretical prediction) depends on our choice of cosmological parameters, but it is independent of our choice of coordinate system to describe the observed Universe. Furthermore, it was noted [18,21] that the background CMB temperature $\bar{T}(\bar{\eta}_o)$, not $\bar{T}(\eta_o)$, is really the CMB temperature today in a homogeneous Universe, corresponding to the cosmological parameter ω_γ, or the radiation density parameter. For later convenience we define

$$\bar{T} := \bar{T}(\bar{\eta}_o). \qquad (7)$$

Having defined the reference time $\bar{\eta}_o$ in Eq. (6), we express the observer position in terms of $\bar{\eta}_o$ to take advantage of its coordinate independence as

$$\eta_o := \bar{\eta}_o + \delta\eta_o, \qquad (8)$$

where the coordinate (time) lapse $\delta\eta_o$ represents the difference of the time coordinate η_o of the observer in a given coordinate system, compared to the reference time $\bar{\eta}_o$. Mind that Eq. (8) is not the usual background-perturbation split as in Eq. (1), rather it is a way to describe the observer time coordinate η_o in terms of a coordinate-independent reference $\bar{\eta}_o$. Under the coordinate transformation in Eq. (2), we can derive

$$\delta\eta_o \mapsto \widetilde{\delta\eta}_o = \delta\eta_o + \xi_o, \qquad (9)$$

based on the coordinate independence of $\bar{\eta}_o$ in Eq. (6). A similar relation can be derived for the spatial coordinate shift with respect to the reference position [22], but for our purposes of the linear-order analysis only the coordinate lapse $\delta\eta_o$ will be used.

Using the reference time $\bar{\eta}_o$, the background temperature $\bar{T}(\eta_o)$ at the observer position today is expressed as

$$\bar{T}(\eta_o) = \bar{T}(\bar{\eta}_o)(1 - \mathcal{H}_o\delta\eta_o), \qquad (10)$$

and hence the observed CMB temperature along $\hat{n}$ is now

$$T(\hat{n}) = \bar{T}(\bar{\eta}_o)(1 - \mathcal{H}_o\delta\eta_o + \Theta_o), \qquad (11)$$

where $\Theta_o := \Theta(x_o^\mu)$ is the temperature fluctuation at the observer position. The coordinate independence of $T(\hat{n})$ can be readily verified by using the gauge transformation properties in Eqs. (5) and (9), and the combination

$$\hat{\Theta}(\hat{n}) := \Theta_o(\hat{n}) - \mathcal{H}_o\delta\eta_o, \qquad (12)$$

is the linear-order gauge-invariant expression for the observed CMB temperature anisotropies. The coordinate lapse $\delta\eta_o$ is independent of the observed direction $\hat{n}$, as it is associated with the observer motion, rather than observations of CMB photons. Note, however, that we have not

specified the observer and the gauge-invariant expression in Eq. (12) is general.

The coordinate lapse $\delta\eta_o$ of the observer can be derived simply by integrating the observer four velocity over the observer path [23]. In a homogeneous Universe, all the observers are stationary, and the four velocity is $\bar{u}^\mu = (1, \vec{0})/a$, where a is the scale factor. The integration of the time component of the observer four velocity yields $\bar{\eta}_o$ in Eq. (6). Due to the inhomogeneity in the Universe, the observer four velocity $u^\mu = (1 - \alpha, U^\alpha)/a$ deviates from the stationary motion in a homogeneous Universe, introducing the coordinate lapse $\delta\eta_o$, where U^α is the peculiar velocity and α is the metric perturbation in the time component (see Appendix A for our notation convention). At the linear order in perturbations, the coordinate lapse is derived in [23] as

$$\delta\eta_o = -\frac{1}{a_o} \int_0^{\bar{t}_o} dt\, \alpha, \tag{13}$$

where the integral is in fact along the motion of the observer but equivalent to the integral along the time coordinate at the linear order. This expression indeed satisfies the transformation relation in Eq. (9).

Ignoring the vector perturbation, the spatial part of the observer four-velocity can be expressed in terms of a scalar velocity potential v, i.e., $u_\alpha =: -av_{,\alpha}$. If we assume that the observer follows the geodesic motion $0 = u^\nu u^\mu_{;\nu}$, Eq. (13) for the coordinate lapse can be solved to yield

$$\delta\eta_o = -v_o, \tag{14}$$

where the semicolon represents the covariant derivative with respect to $g_{\mu\nu}$. As discussed, the coordinate lapse $\delta\eta_o$ in Eq. (13) is generic for all observers at the linear order, and Eq. (14) is also generic for all observers on a geodesic motion. However, it depends on spatial position and the geodesic path.

B. Gauge-invariant expression for the observed CMB temperature anisotropies

Equation (12) is the gauge-invariant description of the observed CMB temperature anisotropies given the gauge-dependent expression of the CMB temperature fluctuation Θ at the observer position, which can be obtained by evolving the Einstein-Boltzmann equation. Instead, we derive a simple analytic expression for the CMB temperature anisotropies $\hat{\Theta}(\hat{n})$ in Eq. (12). Here we assume that (1) the baryon-photon fluid is tightly coupled until the recombination, (2) the recombination takes place instantaneously as soon as the temperature of the baryon-photon fluid reaches T_* set by atomic physics, (3) the baryon-photon fluid simultaneously decouples, and (4) no further interaction occurs for the free-streaming photons. This approximation has been adopted in literature to gain

intuitive understanding of CMB physics (see, e.g., [2,5,6,19]).

As the Universe expands, the temperature of the baryon-photon fluid cools down, and it becomes T_* at some spacetime position denoted as x_*^μ, at which the CMB photons decouple from the baryons. The equilibrium temperature T_* is again split into a background $\bar{T}$ and a perturbation Θ as

$$T_* := T(x_*^\mu) = \bar{T}(\eta_*)[1 + \Theta_*], \tag{15}$$

where $\Theta_* := \Theta(x_*^\mu)$ is the temperature fluctuation at x_*^μ. Under the tight coupling approximation, the baryon-photon fluid is fully described by the density and the velocity and is devoid of any higher-order moments in the photon distribution such as the anisotropic pressure. Note that while T_* is, under our approximation, a unique number set by atomic physics, the individual components $\bar{T}(\eta_*)$ and Θ_* depend on our choice of coordinates x_*^μ to parametrize the physical spacetime position at which the baryon-photon fluid decouples.

Once the CMB photons decouple, they free stream and the observer measures the CMB temperature $T(\hat{n})$ in the rest frame along the observed direction $\hat{n}$. With information that the observed CMB photons originate from the initial temperature T_*, the observed CMB temperature $T(\hat{n})$ can be used to define the observed redshift z_{obs} of the position x_*^μ, at which the photons started to free stream toward the observer:

$$1 + z_{\text{obs}}(\hat{n}) := \frac{T_*}{T(\hat{n})}. \tag{16}$$

The CMB photons follow the Planck distribution, and Wien's displacement law states that the wavelength at the peak of the distribution is inversely proportional to the temperature. The advantage in expressing $T(\hat{n})$ in terms of $z_{\text{obs}}(\hat{n})$ is that we can utilize the well-known expression for the observed redshift z_{obs} and understand how the observed CMB temperature $T(\hat{n})$ is affected throughout the photon propagation, instead of solving the complicated Boltzmann equation. Note that no quantities in Eq. (16) depend on our coordinate choice. In fact, this approach to describing the observed CMB temperature anisotropies was first developed in [2], but the modern approach in literature focuses on the Boltzmann equation as it simplifies the calculations of the angular multipoles.

The observed redshift is also split into a background and a perturbation as

$$1 + z_{\text{obs}}(\hat{n}) := (1 + z_*)(1 + \delta z_*), \tag{17}$$

where the background redshift z_* of the position x_*^μ is literally the expression of the time coordinate η_*:

$$1 + z_* = \frac{a(\bar{\eta}_o)}{a(\eta_*)} = \frac{\bar{T}(\eta_*)}{\bar{T}(\bar{\eta}_o)}. \tag{18}$$

It is convention to set $a(\bar{\eta}_o) \equiv 1$, while it is noted that the scale factor at the observer position is

$$a(\eta_o) = a(\bar{\eta}_o) + a'(\bar{\eta}_o)\delta\eta_o \neq 1. \tag{19}$$

Consequently, the time coordinate z_* of the decoupling position differs in two different coordinates in Eq. (2) as

$$1 + \tilde{z}_* = (1 + z_*)(1 - \mathcal{H}_*\xi_*). \tag{20}$$

The perturbation δz_* in the observed redshift can be derived by solving the geodesic equation (see, e.g., [24]) as

$$\delta z_*(\hat{n}) = -H_*\chi_* + (\mathcal{H}\delta\eta + H\chi)_o - [v_{\chi,a}n^\alpha + \alpha_\chi]_o^* \\ - \int_0^{\bar{r}_*} d\bar{r}(\alpha_\chi - \varphi_\chi)', \tag{21}$$

where n^α is the α component of the unit directional vector $\hat{n}$ and $\bar{r}$ is the line-of-sight distance. The script $*$ indicates that the quantities are evaluated at the decoupling point with x_*^μ. α_χ and φ_χ are the two gauge-invariant potentials corresponding to the Bardeen variables Φ_A and Φ_H in [25], and χ is the scalar shear of the normal observer. The velocity potential v is combined with the scalar shear χ to form the gauge-invariant variable v_χ (see Appendix A). The term $\alpha_\chi|_o^*$ (the Sachs-Wolfe effect [2]) accounts for the gravitational redshift induced by the difference in the gravitational potential at departure x_*^μ and arrival x_o^μ, and the integral term is the integrated Sachs-Wolfe effect that takes into account the variation in time of the scalar metric perturbations α_χ and φ_χ. The individual terms in Eq. (21) are expressed in terms of gauge-invariant variables (such as α_χ) and the gauge-invariant combination $(\mathcal{H}\delta\eta + H\chi)_o$, except the first term $H_*\chi_*$. Therefore, the perturbation δz_* is gauge dependent and transforms as

$$\widetilde{\delta z}_* = \delta z_* + \mathcal{H}_*\xi_*. \tag{22}$$

Note, however, that together with Eq. (20), the combination for the observed redshift in Eq. (17) remains unchanged, as the coordinate transformation in Eq. (2) describes the same physical spacetime point of the baryon-photon decoupling in two different coordinates and the physical observables should be independent of our coordinate choice.

Using Eqs. (15)–(18) and noting $\bar{T} := \bar{T}(\bar{\eta}_o)$, the observed CMB temperature can be written as

$$T(\hat{n}) = \bar{T}[1 + \hat{\Theta}(\hat{n})], \tag{23}$$

and we arrive at one of our main results, or the gauge-invariant expression for the CMB temperature anisotropies $\hat{\Theta}(\hat{n})$ at the linear order in perturbation:

$$\hat{\Theta}(\hat{n}) := \Theta_o(\hat{n}) - \mathcal{H}_o\delta\eta_o = \Theta_* - \delta z_*(\hat{n}). \tag{24}$$

The first equation states that the observed CMB temperature anisotropies are not described by the gauge-dependent Θ_o, but the gauge-invariant $\hat{\Theta}$, which includes the correction $\mathcal{H}_o\delta\eta_o$ due to the observer position. The second equation provides a physical description for the observed CMB temperature anisotropies. Owing to the fluid approximation, the temperature fluctuation Θ_* at decoupling is isotropic and does not depend on the observed direction $\hat{n}$. However, it is evaluated at the point of decoupling x_*^μ, which is a function of the observed direction.

C. Sky average of the observed CMB temperature and the observed monopole fluctuation

The observed CMB temperature $T(\hat{n})$ can be averaged over the sky to yield the mean temperature:

$$\langle T \rangle_\Omega := \int \frac{d^2\hat{n}}{4\pi} T(\hat{n}) = \bar{T}[1 + \hat{\Theta}_0], \tag{25}$$

where $\hat{\Theta}_0$ is the angle-averaged anisotropy (or monopole fluctuation)

$$\hat{\Theta}_0 := \int \frac{d^2\hat{n}}{4\pi} \hat{\Theta}(\hat{n}), \tag{26}$$

and it should not be confused with the gauge-dependent temperature fluctuation $\Theta_o(\hat{n})$ at the observer position (mind the difference in the two subscripts 0 and o). In comparison, the ensemble average of the observed CMB temperature is

$$\langle T(\hat{n}) \rangle = \bar{T}, \qquad \langle \hat{\Theta}(\hat{n}) \rangle = 0. \tag{27}$$

We want to emphasize that $\bar{T}$, not $\langle T \rangle_\Omega$ [or a coordinate-dependent $\bar{T}(\eta_o)$], is the CMB temperature today in a homogeneous Universe, corresponding to a cosmological parameter, while $\langle T \rangle_\Omega$ is the observed CMB temperature today upon angle average.

Equations (25) and (27) make it clear that the observed mean temperature $\langle T \rangle_\Omega$, for instance, from the COBE Far Infrared Absolute Spectrometer (FIRAS) [26] differs from the background CMB temperature $\bar{T}$, or the ensemble average, as it includes the monopole fluctuation $\hat{\Theta}_0$ at our position. It was pointed out [18,27] that the ensemble average is equivalent to the Euclidean average, including not only the angle average over the sky but also the spatial average over different observer positions. Consequently, the mean temperature $\langle T \rangle_\Omega$ today (or the angle average) depends on the spatial position of the observation due to the monopole fluctuation, and its value alone cannot determine the cosmological parameter ω_γ (or $\bar{T}$). This implies that if one takes $\langle T \rangle_\Omega$ as the "ensemble average," there is *no*

observed monopole fluctuation by construction. In practice, given the rms fluctuation amplitude $\sim 10^{-5}$ of the monopole computed in Sec. III A, the difference between $\langle T \rangle_\Omega$ and $\bar{T}$ is negligible.

In addition, Eq. (25) makes it clear that once the cosmological parameter ω_γ (or $\bar{T}$) is chosen, the observed mean temperature $\langle T \rangle_\Omega$ directly translates into the "observed" monopole fluctuation $\hat{\Theta}_0$. Although model dependent, the monopole fluctuation *can* be inferred once a cosmological model is chosen. The resulting value is model dependent, but independent of our choice of coordinate system. On the other hand, the observed photon temperature can be split according to Eq. (1) into the two gauge-dependent quantities $\bar{T}(\eta_o)$ and $\Theta(x_o^\mu)$, and the angle-averaged CMB temperature takes the form

$$\langle T \rangle_\Omega := \int \frac{d^2\hat{n}}{4\pi} T(\hat{n}) = \bar{T}(\eta_o)[1 + \Theta_0(x_o^\mu)], \quad (28)$$

where $\Theta_0(x_o^\mu)$ is the angle average (or monopole) of the temperature fluctuation $\Theta(x_o^\mu)$ at the observer position. The background temperature $\bar{T}(\eta_o)$ is ambiguous as it depends on the time coordinate of the observer and therefore on the coordinate system chosen. Accordingly, $\Theta_0(x_o^\mu)$ is ambiguous as well. This ambiguity in the background-perturbation split is the reason why the monopole fluctuation is often referred to be unobservable. However, the CMB temperature today in a homogeneous Universe is correctly described by $\bar{T}(\bar{\eta}_o)$ and there is no ambiguity in its definition in a given model and its cosmological parameters.

Any measurements of cosmological observables in practice involve measurement uncertainties, and hence the cosmological parameters in a given model have uncertainties in their best-fit values, which result in uncertainties in the theoretical predictions. However, this aspect is rather independent from the goal of this work. These uncertainties in the predictions for the cosmological observables are solely due to the uncertainties in our estimates of the cosmological parameters, not due to the ambiguities in the theoretical predictions. The primary goal in this current investigation is to have a unique prediction for the monopole power C_0, given a model and its assumed cosmological parameters.

D. Multipole expansion

The observed CMB temperature anisotropy $\hat{\Theta}(\hat{n})$ is traditionally decomposed in terms of spherical harmonics $Y_{lm}(\hat{n})$ as

$$\hat{\Theta}(\hat{n}) = \sum_{lm} \hat{a}_{lm} Y_{lm}(\hat{n}), \quad (29)$$

and the multipole coefficients are

$$\hat{a}_{lm} = \int d^2\hat{n} Y_{lm}^*(\hat{n}) \hat{\Theta}(\hat{n}). \quad (30)$$

Defining the standard multipole coefficients a_{lm} (without hat) in the same way for the gauge-dependent temperature fluctuation $\Theta_o(\hat{n})$ at the observer position, we derive the relation between the two different multipole coefficients

$$\hat{a}_{lm} = a_{lm} - \sqrt{4\pi} \mathcal{H}_o \delta\eta_o \delta_{l0} \delta_{m0}. \quad (31)$$

With no angular dependence for $\delta\eta_o$, the difference resides only at the monopole with $l = 0$, reaffirming that all the multipole coefficients a_{lm} derived in literature are gauge invariant for $l \geq 1$. The gauge dependence of the (standard) monopole a_{00} is well known and also evident in Eq. (5). However, we emphasize that the correct monopole coefficient $\hat{a}_{00}$ is indeed gauge invariant and well defined.

While the difference is limited to the monopole coefficient in theory, *all* the multipole coefficients are indeed affected in reality, though the impact is rather negligible in practice due to the small rms fluctuation amplitude of the monopole. The standard Boltzmann codes such as CAMB [16] and CLASS [17] provide the multipole coefficients a_{lm} and their angular power spectra $C_l := \langle |a_{lm}|^2 \rangle$ with $l \geq 2$, and the comparison to observations determines the best-fit cosmological parameters. However, in the standard data analysis of CMB measurements, the background CMB temperature is set equal to the observed mean temperature $\bar{T} \equiv \langle T \rangle_\Omega$ by hand, and this formally incorrect procedure results in two problems [18]: (1) the background dynamics in our model predictions differs from the background evolution in our Universe, unless the monopole $\hat{\Theta}_0$ at our position happens to be zero by accident. (2) By using the observed mean $\langle T \rangle_\Omega$ as the background temperature $\bar{T}$, the angular multipole coefficients obtained from the observations correspond to

$$\hat{a}_{lm}^{\text{obs}} = \frac{\hat{a}_{lm}}{1 + \hat{\Theta}_0}. \quad (32)$$

While these two issues are shown [18] to have negligible impact on our current cosmological parameter analysis due to the small rms fluctuation of the monopole, these systematic errors in the standard data analysis are always present, which may become a significant component in the systematic errors in future surveys (see, e.g., [28] for recent discussion).

Combining Eqs. (21) and (24), we expressed the observed CMB temperature anisotropies as

$$\hat{\Theta} = \Theta_{\chi*} + [v_{\chi,\alpha} n^\alpha + \alpha_\chi]_o^* + H_o v_{\chi o} + \int_0^{\bar{r}_*} d\bar{r}(\alpha_\chi - \varphi_\chi)', \quad (33)$$

where the temperature fluctuation Θ_* at decoupling in Eq. (24) is combined with the gauge-dependent term $H_* \chi_*$

in Eq. (21) to form a gauge-invariant temperature fluctuation Θ_χ in the conformal Newtonian gauge

$$\Theta_\chi := \Theta + H\chi, \qquad (34)$$

and we assumed that the observer motion is geodesic in Eq. (14) to form a gauge-invariant variable for the scalar velocity potential v_χ at the observer position. Note that the decoupling position x_*^μ is a function of the observed direction $\hat{n}$. To derive the expressions for the multipole coefficients $\hat{a}_{lm}$, we first define the transfer functions $\mathbb{T}(\eta, k)$ for the gauge-invariant variables in Eq. (33) in terms of the primordial fluctuation $\zeta(\mathbf{k})$ set at the initial condition. For instance, the transfer function for α_χ is then

$$\alpha_\chi(\eta, \mathbf{k}) =: \mathbb{T}_{\alpha_\chi}(\eta, k)\zeta(\mathbf{k}), \qquad (35)$$

providing the relation of $\alpha_\chi(\eta, \mathbf{k})$ in Fourier space at any conformal time η to the initial condition $\zeta(\mathbf{k})$ of the comoving gauge curvature $\zeta := \varphi - \mathcal{H}v$, where the initial power spectrum is set as $\Delta_\zeta^2 := k^3 P_\zeta/2\pi^2 = A_s(k/k_\circ)^{n_s-1}$ in terms of the primordial fluctuation amplitude A_s at pivot scale $k_\circ$ and the spectral index n_s (see, e.g., [12]). It is well known that on large scales $k \to 0$ the comoving gauge curvature is conserved in time, and the gauge-invariant variable is then related as

$$\alpha_\chi(\eta_{\mathrm{mde}}) = -\frac{3}{5}\zeta, \qquad (36)$$

where we suppressed the scale dependence and considered the conformal time in the matter-dominated era (mde). This implies that the transfer function has the limit

$$\lim_{k\to 0}\mathbb{T}_{\alpha_\chi}(\eta_{\mathrm{mde}}, k) = -\frac{3}{5}. \qquad (37)$$

Using the plane-wave expansion

$$e^{i\mathbf{k}\cdot\mathbf{x}} = 4\pi\sum_{l,m}i^l j_l(kr)Y_{lm}(\hat{\mathbf{k}})Y_{lm}^*(\hat{\mathbf{x}}), \qquad (38)$$

the multipole coefficients $\hat{a}_{lm}$ of the CMB temperature anisotropies can be derived according to Eq. (30) as

$$\begin{aligned}
\hat{a}_{lm} = 4\pi i^l \int \frac{dkk^2}{2\pi^2}&\left\{(\mathbb{T}_{\Theta_\chi} + \mathbb{T}_{\alpha_\chi})_* j_l(k\bar{r}_*) \right.\\
&+ k\mathbb{T}_{v_\chi*}j_l'(k\bar{r}_*) - (\mathbb{T}_{\alpha_\chi} - H\mathbb{T}_{v_\chi})_o\delta_{l0} - \frac{k}{3}\mathbb{T}_{v_\chi o}\delta_{l1}\\
&\left.+ \int_0^{\bar{r}_*}d\bar{r}(\mathbb{T}_{\alpha_\chi}' - \mathbb{T}_{\varphi_\chi}')j_l(k\bar{r})\right\}\int\frac{d\Omega_k}{4\pi}Y_{lm}^*(\hat{k})\zeta(\mathbf{k}),
\end{aligned} \qquad (39)$$

where $j_l(x)$ are the spherical Bessel functions, $\delta_{ll'}$ is the Kronecker delta, and we suppressed the k dependence of

the transfer functions, while the time dependence is indicated in terms of subscripts. In the matter dominated Universe, the gravitational potential is constant, and the integrated Sachs-Wolfe contribution vanishes. So, the dominant contributions to the CMB temperature anisotropies today at higher angular multipoles ($l \geq 2$) are the temperature fluctuation with the gravitational potential contribution at the source plus the Doppler effect (see, e.g., [5,6,29])

$$\hat{a}_{lm} \propto (\mathbb{T}_{\Theta_\chi} + \mathbb{T}_{\alpha_\chi})_* j_l(k\bar{r}_*) + k\mathbb{T}_{v_\chi*}j_l'(k\bar{r}_*). \qquad (40)$$

III. MONOPOLE AND DIPOLE OF THE CMB TEMPERATURE ANISOTROPIES

The multipole coefficients $\hat{a}_{lm}$ in observations can be summed over m to yield an estimate of the angular power spectrum C_l. Its theoretical prediction can be obtained by taking the ensemble average of the multipole coefficients as

$$C_l = \langle|\hat{a}_{lm}|^2\rangle = 4\pi\int d\ln k\Delta_\zeta^2(k)|\mathcal{T}_l(k)|^2, \qquad (41)$$

where the stochasticity in $\zeta(\mathbf{k})$ is averaged out to yield the power spectrum by using

$$\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\rangle = \frac{(2\pi)^3}{k_1^2}P_\zeta(k_1)\delta^D(k_1 - k_2)\delta^D(\Omega_{k_1} - \Omega_{k_2}), \qquad (42)$$

and the various effects in the curly bracket in Eq. (39) are lumped into the transfer function $\mathcal{T}_l(k)$ for the angular power spectrum

$$\begin{aligned}
\mathcal{T}_l(k) :=& (\mathbb{T}_{\Theta_\chi} + \mathbb{T}_{\alpha_\chi})_* j_l(k\bar{r}_*) + k\mathbb{T}_{v_\chi*}j_l'(k\bar{r}_*) - \frac{k}{3}\mathbb{T}_{v_\chi o}\delta_{l1}\\
&- (\mathbb{T}_{\alpha_\chi} - H\mathbb{T}_{v_\chi})_o\delta_{l0} + \int_0^{\bar{r}_*}d\bar{r}(\mathbb{T}_{\alpha_\chi}' - \mathbb{T}_{\varphi_\chi}')j_l(k\bar{r}).
\end{aligned} \qquad (43)$$

Since the primordial power spectrum Δ_ζ^2 is nearly scale invariant, the shape of the transfer function $\mathcal{T}_l(k)$ contains all the information about the angular power spectrum C_l. Our main focus in this section is on the monopole and the dipole transfer functions, which can be readily obtained from Eq. (43) with $l = 0$ and $l = 1$.

Our analytical expression describes the simple situation in which the CMB photons in thermal equilibrium with T_* are emitted at the decoupling point x_*^μ and these photons are measured by the observer at x_o^μ. Though both the temperature and the radiation energy density are observer-dependent quantities, the dependence arises at the second order from the Lorentz boost, and no further specification of the source and

the observer frames is necessary. However, the observed redshift z_{obs} and its perturbation δz are affected by the motion of the source and the observer via the linear-order Doppler effect. At the decoupling point x^μ_*, the source is described as the baryon-photon fluid, so the velocity at x^μ_* is the velocity of the baryon-photon fluid. For the observer motion, we assume that the observer is moving together with matter (baryons and dark matter), $v_o \equiv v_m(\eta_o, k)$. It is important to note that the velocity terms in Eq. (21) are those specifying the rest frames at the emission (or decoupling) and the observation of the photons; they do not have to be the velocity potential of the same fluid at both emission and observer points.

In Secs. III A and III B we present our numerical computation of the monopole transfer function $\mathcal{T}_0(k)$ and the dipole transfer function $\mathcal{T}_1(k)$. Section III C takes a closer look at the large-scale limit of the two transfer functions $\mathcal{T}_0(k)$ and $\mathcal{T}_1(k)$. We then compare our results to previous work on $\mathcal{T}_0(k)$ and $\mathcal{T}_1(k)$ in Sec. III D. We choose the conformal Newtonian gauge ($\chi \equiv 0$) for the numerical calculations of our gauge-invariant expression in Eq. (43).

To calculate the transfer functions of the individual scalar perturbation variables, we use the Cosmic Linear Anisotropy Solving System (CLASS) [17] and the Code for Anisotropies in the Microwave Background (CAMB) [16]. We find that the difference between two Boltzmann code solvers is negligible in our calculations. Here we adopt the ΛCDM model with the cosmological parameters consistent with the *Planck* 2018 results [12]: dimensionless Hubble parameter $h = 0.6732$, the baryon density parameter $\Omega_b h^2 = 0.02299$, the (cold) dark matter density parameter $\Omega_{\mathrm{cdm}} h^2 = 0.12011$, the reionization optical depth $\tau = 0.0543$, the scalar spectral index $n_s = 0.96605$, and the primordial amplitude $\ln(10^{10} A_s) = 3.0448$ at $k_\circ = 0.05$ Mpc^{-1}.

A. Monopole

The transfer function of the monopole in the conformal Newtonian gauge is

$$
\begin{aligned}
\mathcal{T}_0(k) &= [\mathbb{T}_\Theta(\eta_*, k) + \mathbb{T}_\psi(\eta_*, k)] j_0(k\bar{r}_*) \\
&\quad + k\mathbb{T}_{v_\gamma}(\eta_*, k) j_0'(k\bar{r}_*) - \mathbb{T}_\psi(\eta_o, k) + H_o \mathbb{T}_{v_m}(\eta_o, k) \\
&\quad + \int_0^{\bar{r}_*} d\bar{r} [\mathbb{T}_\psi'(\eta, k) - \mathbb{T}_\phi'(\eta, k)] j_0(k\bar{r}),
\end{aligned} \tag{44}
$$

where $\psi := \alpha$ and $\phi := \varphi$ in the conformal Newtonian gauge. Though $\psi \approx -\phi$ already at the decoupling, we use the exact transfer functions for ψ and ϕ to compute the monopole transfer function $\mathcal{T}_0(k)$. The monopole fluctuation is composed of the photon temperature fluctuation Θ_*, the gravitational redshift ψ_*, and the baryon-photon velocity $v_{\gamma*}$ at the decoupling point, the gravitational redshift ψ_o and the observer velocity potential v_m at the observer position, and finally the integrated Sachs-Wolfe effect (ISW). Note that the velocity potential
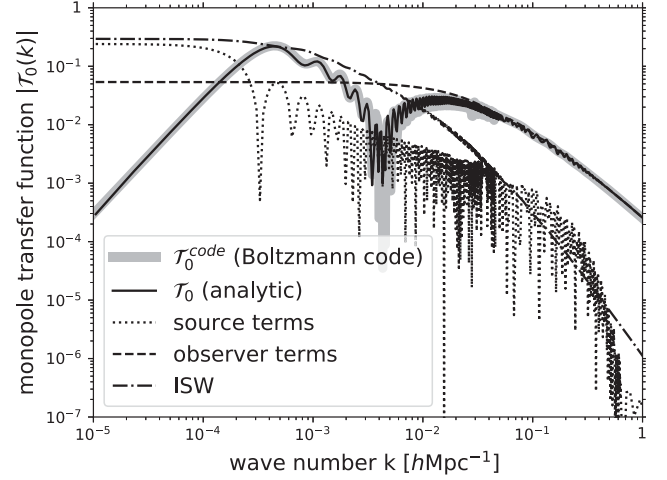

FIG. 1. Monopole transfer function $\mathcal{T}_0$ and individual contributions as a function of the wave number k. The analytic expression (44) derived in this paper is represented by the black solid line, while the contributions from terms evaluated at decoupling (denoted as source terms) and at the observer position are indicated by the dotted and the dashed line, respectively. The dot-dashed line corresponds to the ISW contribution. The thick gray line shows the transfer function from the Boltzmann codes serving as a reference for our analytical expression.

v_m at the observer position arises from the coordinate lapse $\delta\eta_o$ in Eq. (14), while the Doppler effect by the observer velocity is absent in the monopole transfer function. For later convenience, we refer to those contributions at the decoupling x^μ_* as the source terms, while those at the observer position as the observer terms. Although they are not individually observables, the decomposition into the individual components helps understand the monopole transfer function intuitively.

Figure 1 describes the monopole transfer function $\mathcal{T}_0(k)$ and its individual contributions. To facilitate the comparison, we plot the absolute values of the individual transfer functions. The ISW effect (dot-dashed) and the contributions at the decoupling (source terms; dotted) are the dominant contribution to the monopole transfer function on large scales, while the contributions at the observer position (observer terms; dashed) are dominant on small scales, as the source terms are suppressed due to the spherical Bessel function. The source terms oscillate rapidly largely due to the baryon-photon velocity v_γ, and because of the Silk damping [30] the baryon-photon fluctuations decay fast on small scales. On large scales, all three contributions (source terms, observer terms, and ISW) are constant, and these individual contributions on superhorizon scales ($k \ll H_0 = 3.3 \times 10^{-4} h$ Mpc^{-1}) result in infrared divergences for the monopole power in Eq. (41), respectively. However, as evident in Fig. 1, the sum of all the individual contributions to $\mathcal{T}_0$ (solid) falls as k^2 on large scales due to the cancellation of the individually diverging contributions, and the monopole power C_0 is finite and
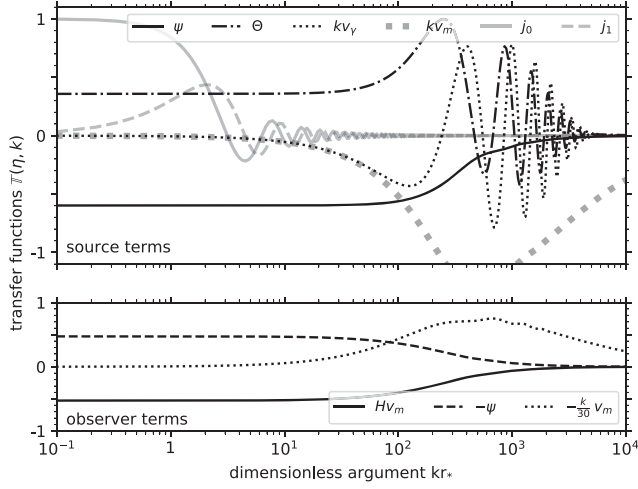
FIG. 2. Individual contributions to the monopole and dipole transfer functions as a function of dimensionless argument kr_*, where r_* is the comoving distance to the last scattering surface. The range of the x axis is approximately the same as in Figs. 1 and 3. The top panel shows terms evaluated at decoupling (i.e., source terms), in addition to the two spherical Bessel functions $j_0(x)$ and $j_1(x)$. The bottom panel shows terms evaluated at the observer position. Note that the dotted curve shows $kv_m/30$ instead of $kv_m/3$, with another factor of 10 to fit in the plot.

devoid of any divergences. To put it differently, the theoretical prediction for the observed CMB temperature today upon angle average in Eq. (25) is finite and independent of the fluctuations on very large scales beyond our horizon (see also [19]).

To better understand the subtle cancellation on large scales, we show the individual contributions of the source terms and the observer terms in Fig. 2 in linear scale as a function of dimensionless argument kr_* in the Bessel function, where the distance to the decoupling is $r_* = 13.87$ Gpc (or $9.34\,h^{-1}$ Gpc) at $z_* = 1088$ with our fiducial cosmological parameters, so that the x range in Fig. 2 is approximately equivalent to the range of the wave number in Fig. 1. The upper panel shows the contributions at the decoupling or the source terms that are the photon temperature fluctuation Θ (dot-dashed), the gravitational redshift ψ (solid) and the Doppler effect v_γ (dotted). Since these quantities contribute to the observed monopole fluctuation via photon propagation from the decoupling point toward the observer position, all three terms are multiplied by the spherical Bessel function $j_0(x)$ (gray solid) or its derivative $j_0'(x)$, and hence are suppressed by $1/x$ (or $1/x^2$ for the Doppler term) on small scales ($x \gg 10$). On large scales ($j_0 \simeq 1$), the first two contributions Θ and ψ are dominant and in fact constant in k but with opposite signs, giving rise to cancellation between the two contributions. The Doppler effect v_γ is again negligible on large scales, as it is a gradient ($\propto k$) and $j_0' \simeq 0$.

The bottom panel shows the contributions at the observer position or the observer terms that are made of the

gravitational redshift ψ (dashed) and the coordinate lapse Hv_m (solid). Since the observer terms are without the spherical Bessel function, their contributions are relatively larger than the source terms on small scales, though the transfer functions still decay on small scales. The two contributions on large scales are also constant but with opposite signs. Their amplitude is almost identical, resulting in near cancellation, so the contribution of the observer terms is smaller than the source terms on large scales, apparent in Fig. 1. These two contributions (the source and the observer terms) add up to nearly cancel the positive contribution from the line-of-sight integration or the ISW term. Mind that while the gravitational potentials (ψ and ϕ) decay in time, the line-of-sight direction increases backward in time to yield the positive contribution of the ISW term.

To check the validity of our analytical expression, we plot the monopole transfer function $\mathcal{T}_0^{\text{code}}(k)$ (gray solid) in Fig. 1 from the Boltzmann codes CLASS and CAMB. Numerical computation in these Boltzmann codes is performed in the synchronous gauge, where $\alpha = \beta = 0$. Using the residual gauge freedom in the synchronous gauge (see, e.g., [23]), an extra gauge condition is imposed in the Boltzmann codes, in which $v_m = 0$. So, the resulting gauge condition is indeed (dark-matter) comoving-synchronous gauge. The observed CMB temperature anisotropies in this gauge condition are then

$$\hat{\Theta}(\hat{n}) = \Theta_o^{\text{sync}}(\hat{n}), \tag{45}$$

from Eq. (24), where the coordinate lapse is vanishing with $v_m = 0$. Therefore, we can use the transfer function for the photon temperature fluctuation today from the Boltzmann codes for the monopole transfer function (gray solid) in Fig. 1:

$$\mathcal{T}_0^{\text{code}}(k) = \mathbb{T}_\Theta^{\text{sync}}(\eta_o, k) \tag{46}$$

(see Appendix B for more details). Note that the transfer function $\mathbb{T}_\Theta^{\text{sync}}$ in the Boltzmann codes (hence $\mathcal{T}_0^{\text{code}}$) is obtained by numerically solving the full Boltzmann equation to the present day without any approximations we adopted for our analytical calculations, while our monopole transfer function $\mathcal{T}_0$ (solid curve) is obtained by using the analytic expression in Eq. (44).

Figure 1 shows an astonishing agreement for the monopole transfer function (solid and gray curves) on all scales, which strongly supports that our approximations for the analytical expression capture the essential physics of the CMB anisotropy formation, at least for the monopole. Our approximations neglect collisions against free electrons after the recombination and assume a sharp transition from tight coupling to complete decoupling, none of which matters on large scales. Though we do expect the breakdown of our approximations on small scales, the monopole transfer function on small scales is dominated by the

contribution at the observer position, independent of the validity of our approximations.

Using the monopole transfer function, we numerically compute the monopole power in Eq. (41)

$$C_0 = \langle |\hat{a}_{00}|^2 \rangle = 1.66 \times 10^{-9}, \qquad (47)$$

and the rms fluctuation of the angle-averaged anisotropy is

$$\sqrt{\langle \hat{\Theta}_0^2 \rangle} = \sqrt{\frac{C_0}{4\pi}} = 1.15 \times 10^{-5}. \qquad (48)$$

We emphasize that this is the *first time* to correctly compute these values (see Sec. III D). It turns out that the rms fluctuation in our ΛCDM model is very small, and in particular a lot smaller than the measurement uncertainty in the COBE FIRAS observation [26,31],

$$\langle T \rangle_\Omega = 2.7255 \pm 5.7 \times 10^{-4} \text{ K.} \qquad (49)$$

This explains why the systematic errors in the standard cosmological parameter estimation are in practice negligible despite the formally incorrect assumption of setting $\bar{T} \equiv \langle T \rangle_\Omega$ (see [18]).

We want to emphasize again that the monopole fluctuation $\hat{\Theta}_0$ *is* directly "observable" once a choice of cosmological parameters is made, as the latter uniquely fixes the value of the background temperature $\bar{T}$. The monopole fluctuation can then be inferred from the observed mean temperature $\langle T \rangle_\Omega$ via Eq. (25), although the resulting value is of course model dependent. This is discussed in more detail in Sec. II C.

In the past analytical calculations have been performed in particular with the choice of the conformal Newtonian gauge. However, the gauge invariance of the expression was not verified, and the coordinate lapse $\delta\eta_o$ at the observer position is neglected. To be fair, the focus of previous analytic calculation (see, e.g., [5,6]) is on the higher angular multipoles rather than the monopole and the dipole. Apparent from Figs. 1 and 2, the monopole transfer function would be constant in k on large scales, if $\delta\eta_o$ is neglected, and the resulting monopole power C_0 would be infinite. Comparison to previous work will be presented in Sec. III D.

Figure 2 also shows the velocity potential v_m (gray dotted) of the dark matter at the decoupling point. On large scales, it is identical to the photon velocity v_γ (and hence the baryon velocity $v_b = v_\gamma$), but it deviates significantly on small scales, as dark matter decoupled in the early Universe and evolved separately from the baryon-photon plasma. However, the impact of using v_m instead of v_γ on the monopole transfer function $\mathcal{T}_0(k)$ is negligible.

B. Dipole

The transfer function of the dipole in the conformal Newtonian gauge is

$$\begin{aligned}
\mathcal{T}_1(k) = &[\mathbb{T}_\Theta(\eta_*, k) + \mathbb{T}_\psi(\eta_*, k)] j_1(k\bar{r}_*) \\
&+ k\mathbb{T}_{v_\gamma}(\eta_*, k) j_1'(k\bar{r}_*) - \frac{k}{3} \mathbb{T}_{v_m}(\eta_o, k) \\
&+ \int_0^{\bar{r}_*} d\bar{r} [\mathbb{T}_\psi'(\eta, k) - \mathbb{T}_\phi'(\eta, k)] j_1(k\bar{r}).
\end{aligned} \qquad (50)$$

Similar to the monopole, the dipole transfer function has the same contributions of the source terms (the photon temperature fluctuation Θ_*, the gravitational redshift ψ_*, and the baryon-photon velocity $v_{\gamma*}$) and the integrated Sachs-Wolfe effect. These contributions are already shown in Fig. 2, but they are multiplied with the spherical Bessel function for the dipole ($l = 1$: gray dashed). The key difference compared to the monopole transfer function arises from the observer terms. The observer velocity v_m contributes to the dipole via the Doppler effect, whereas the gravitational redshift ψ_o and the coordinate lapse $\delta\eta_o$ drop out in the dipole transfer function.

Figure 3 plots the dipole transfer function $\mathcal{T}_1(k)$ and its individual contributions. The observer velocity (dashed) is the dominant contribution on all scales, and this contribution $(-kv_m/3)$ is positive as shown in Fig. 2 (but note that the dotted curve is $kv_m/30$ instead of $kv_m/3$ with another factor ten to fit in the plot). On large scales $k < 10^{-3}h^{-1}$ Mpc, the source terms (dotted) and the integrated Sachs-Wolfe term (dot-dashed) become comparable to the observer velocity contribution (dashed), and these contributions cancel each other to yield the dipole transfer function (solid) in proportion to k^3. As evident in Fig. 2, the individual transfer functions such as $\mathbb{T}_\Theta$, $\mathbb{T}_\psi$, and so on become constant on large scales,
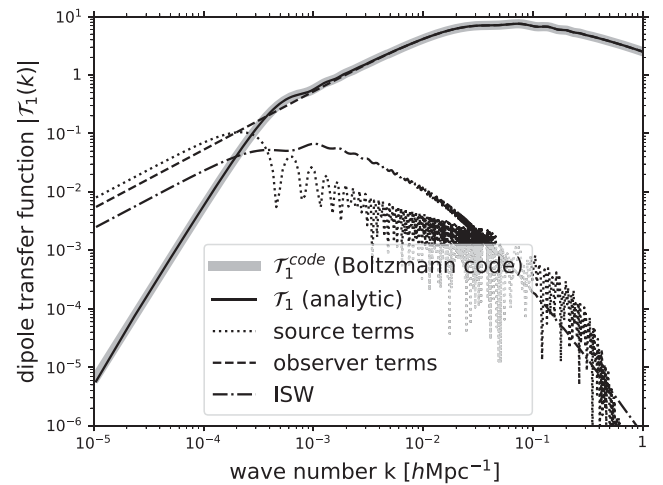

FIG. 3. Dipole transfer function $\mathcal{T}_1$ and individual contributions as a function of the wave number k, in the same format as in Fig. 1.

and hence they all fall as k due to the spherical Bessel function $j_1(x)$ (or due to the extra k factor for $\mathbb{T}_{v_\gamma}$ and $\mathbb{T}_{v_m}$). Upon cancellation of these contributions on large scales, the dipole transfer function picks up extra k^2 (as in the monopole transfer function) to scale with k^3.

Again to check the validity of our analytical expression of the dipole transfer function, we plot the dipole transfer function $\mathcal{T}_1^{\mathrm{code}}(k)$ (gray solid) from the Boltzmann codes. The numerical computation in the Boltzmann codes is performed in the (dark-matter) comoving-synchronous gauge, where the matter velocity is zero ($v_m = 0$). Since the dipole is the spatial energy flux of the CMB photon distribution measured by the observer, the dipole transfer function is literally the relative velocity between the observer and the CMB photon fluid (see Appendix B; this point was also emphasized in [19]). Therefore, in the comoving-synchronous gauge, the dipole transfer function can be obtained as

$$\mathcal{T}_1^{\mathrm{code}}(k) = \frac{k}{3}\mathbb{T}_{v_\gamma}^{\mathrm{sync}}(\eta_o, k).\tag{51}$$

As in the case of the monopole transfer function, our analytical expression provides an accurate description of the dipole transfer function, in particular on large scales, where the cancellation of the individual contributions takes place. The dipole transfer function is again dominated by the contribution at the observer position on small scales, where our approximation is expected to be less accurate.

Using the dipole transfer function, we numerically compute the dipole power in Eq. (41)

$$C_1 = \frac{1}{3}\sum_m \langle |\hat{a}_{1m}|^2 \rangle = 4.51 \times 10^{-6},\tag{52}$$

and the rms fluctuation of the relative velocity

$$\sigma_{v_r}^2 = \langle \vec{v}_r \cdot \vec{v}_r \rangle = \int d\ln k\, \Delta_\zeta^2(k)|k\mathbb{T}_{v_r}(\eta_o, k)|^2,$$

$$= \frac{9C_1}{4\pi} = (540\ \mathrm{km/s})^2.\tag{53}$$

The Planck measurement [32] of the CMB dipole anisotropy yields that our rest frame is moving 369.82 ± 0.11 km/s with respect to the CMB rest frame, which is consistent with our theoretical expectation of the one-dimensional rms relative velocity $\sigma_{v_r}/\sqrt{3} = 311$ km/s. Note that our scalar velocity potential v_γ is related to the variable θ_γ in the convention of [33] through $\theta_\gamma = k^2 v_\gamma$.

It should be emphasized that the dipole is a measure of the relative velocity, not the absolute velocity. The velocity potential v gauge transforms as $\tilde{v} = v - \xi$, in the same way for all species, such that the relative velocity is gauge invariant. In relativity, the absolute velocity has no physical meaning and only the relative velocity at the same

spacetime position has physical significance. At different positions, not only the observer velocity but also the photon velocity vary from those at our position, invalidating the notion that the CMB rest frame provides an absolute frame for all observers in the Universe.

C. Large-scale limit of the monopole and the dipole transfer functions

Having presented our numerical calculations of the monopole and the dipole transfer functions in Eqs. (44) and (50), here we investigate their large-scale behavior analytically by taking the limit $k \to 0$. For a small argument $x = kr_* \ll 1$, the spherical Bessel function can be approximated as

$$j_l(x) = \frac{2^l l!}{(2l+1)!}x^l + \mathcal{O}(x^{l+2}),\tag{54}$$

and for the monopole and the dipole, they become

$$j_0(x) \simeq 1 + \mathcal{O}(x^2), \qquad j_1(x) \simeq \frac{1}{3}x + \mathcal{O}(x^3).\tag{55}$$

In the limit $k \to 0$, the monopole transfer function in Eq. (44) is approximated as

$$\begin{aligned}\mathcal{T}_0(k) &\simeq \mathbb{T}_\Theta(\eta_*, k) + \mathbb{T}_\psi(\eta_*, k) - \mathbb{T}_\psi(\eta_o, k)\\ &\quad + H_o\mathbb{T}_{v_m}(\eta_o, k) + \int_0^{\bar{r}_*} d\bar{r}[\mathbb{T}'_\psi(\eta, k) - \mathbb{T}'_\phi(\eta, k)],\end{aligned}\tag{56}$$

where the baryon-photon velocity $v_{\gamma *}$ term is dropped due to its k factor and the spherical Bessel function $j'_0(x)$. Figure 2 shows all these contributions are constant in k on large scales. In this large-scale limit, the integrated Sachs-Wolfe term can be analytically integrated by part to yield

$$\mathcal{T}_0(k) \simeq \mathbb{T}_\Theta(\eta_*, k) + \mathbb{T}_\phi(\eta_*, k) + H_o\mathbb{T}_{v_m}(\eta_o, k) - \mathbb{T}_\phi(\eta_o, k),\tag{57}$$

where $d\bar{r}$ is the line-of-sight integration ($d/d\bar{r} = \partial_r - \partial_\eta$) and the two gravitational redshift terms ψ are canceled. The conservation equation of the photon energy-density takes the form

$$\dot{\Theta} + \dot{\phi} = 0,\tag{58}$$

on large scales (see, e.g., [33]), yielding

$$\Theta(\eta_*) + \phi(\eta_*) = \Theta(\eta_o) + \phi(\eta_o) = \mathfrak{C},\tag{59}$$

where $\mathfrak{C}$ is a constant and we suppressed the scale dependence as these quantities are taken in the limit

$k \to 0$. This further simplifies the monopole transfer function as

$$\mathcal{T}_0(k) \simeq \mathbb{T}_\Theta(\eta_o, k) + H_o \mathbb{T}_{v_m}(\eta_o, k). \quad (60)$$

Combining the time-time and the time-space components of the Einstein equation, we derive the relation between the velocity potential and the matter density on large scales:

$$v_m = -\frac{\delta\rho_m}{3aH\bar{\rho}_m} = \frac{\delta\rho_m}{a\dot{\bar{\rho}}_m}, \quad (61)$$

where we used the background conservation equation for the matter density. Further assuming the adiabaticity for each species at $k = 0$

$$\frac{\delta\rho_m}{\dot{\bar{\rho}}_m} = \frac{\delta\rho_\gamma}{\dot{\bar{\rho}}_\gamma}, \quad (62)$$

the two contributions to the monopole transfer function $\mathcal{T}_0(k)$ cancel in the large-scale limit. The next-leading order contribution ($\propto k$) could come from the same terms in Eq. (56), as the next-leading order in $j_0(x)$ is proportional to k^2. Each term in Eq. (56) can be expanded as a power series in k, and the subsequent derivations are exactly the same for terms in proportion to k, since the conservation Eq. (58) is valid up to k^2. While we chose the observer moving together with matter for computing $\delta\eta_o$, the coordinate lapse $\delta\eta_o$ is in fact independent of this choice, as discussed in Sec. II A.

The absence of contributions that are independent of scales or in proportion to k is the consequence of the equivalence principle, which states the equality of the gravitational and the inertial mass. Consequently, a local observer cannot tell the existence of a uniform gravitational force, as the reference frame and the apparatus of the local observer are affected altogether in the same way. For our calculations, a uniform gravitational potential corresponds to the constant contributions in individual transfer functions such as $\mathbb{T}_\psi$, while a uniform gravitational acceleration corresponds to the contributions in proportion to k (or the gradient of the potential contributions). It was shown [34–37] that the theoretical descriptions of the luminosity distance and galaxy clustering are devoid of such contributions.

Applying the equivalence principle to the CMB monopole transfer function $\mathcal{T}_0(k)$, we find that the gravitational potential fluctuations or the gravitational accelerations of wavelength larger than $\bar{r}_*$ act as uniform fields and they have no impact on our local physical observables such as the observed CMB temperature (or the monopole fluctuation). This argument is borne out by the cancellation of individually diverging contributions to the monopole power at low k in our numerical and analytical calculations of the monopole transfer function $\mathcal{T}_0(k)$. Keep in mind that if any

of the potential contributions such as the coordinate lapse $\delta\eta_o$ at the observer position is ignored, the cancellation of each contribution at low k would not take place, and the monopole transfer function $\mathcal{T}_0(k)$ is nonvanishing even in the limit $k \to 0$, leading to an infinite monopole power C_0 (hence the observed CMB temperature) or the monopole power highly sensitive to the lower cutoff scale in the integral of Eq. (41). This would work against the equivalence principle, as the superhorizon scale fluctuations dictate our local observables or put it differently we can infer the existence of such superhorizon scale fluctuations based on our local observables.

In the limit $k \to 0$, the dipole transfer function in Eq. (50) is approximated as

$$\mathcal{T}_1(k) \simeq [\mathbb{T}_\Theta(\eta_*, k) + \mathbb{T}_\psi(\eta_*, k)]\frac{k\bar{r}_*}{3} + \frac{k}{3}\mathbb{T}_{v_\gamma}(\eta_*, k)$$
$$- \frac{k}{3}\mathbb{T}_{v_m}(\eta_o, k) + \int_0^{\bar{r}_*} d\bar{r}[\mathbb{T}'_\psi(\eta, k) - \mathbb{T}'_\phi(\eta, k)]\frac{k\bar{r}}{3}. \quad (63)$$

Given that the individual transfer functions are constant at low k, it is apparent that the dipole transfer function goes at least as k. Using the conservation equation (58), we first replace the term ϕ' with Θ' in the integrated Sachs-Wolfe term, and by performing the integration by part, the dipole transfer function can be expressed as

$$\frac{3}{k}\mathcal{T}_1(k) \simeq \mathbb{T}_{v_\gamma}(\eta_*, k) - \mathbb{T}_{v_m}(\eta_o, k)$$
$$+ \int_0^{\bar{r}_*} d\bar{r}[\mathbb{T}_\psi(\eta, k) + \mathbb{T}_\Theta(\eta, k)]. \quad (64)$$

Similarly to the monopole, we make use of the conservation equation for the photon energy momentum on large scales

$$\dot{v}_\gamma = \frac{1}{a}\psi + \frac{1}{a(\bar{\rho}_\gamma + \bar{p}_\gamma)}\delta p_\gamma = \frac{1}{a}(\psi + \Theta), \quad (65)$$

and integrate along the line of sight to obtain

$$\mathbb{T}_{v_\gamma}(\eta_*) - \mathbb{T}_{v_\gamma}(\eta_o) = -\int_0^{\bar{r}_*} d\bar{r}(\mathbb{T}_\psi + \mathbb{T}_\Theta), \quad (66)$$

where we suppressed the scale dependence. Using the adiabaticity condition in Eq. (62) and the velocity potential for matter in Eq. (61), we find that the dipole transfer function vanishes on large scales. Again, the large-scale limit of the conservation equation is valid up to k^2, and the leading term in the dipole transfer function comes in proportion to k^3.

D. Comparison to previous work

1. Zibin and Scott 2008

In the comprehensive work by Zibin and Scott [19], an analytic expression of the observed CMB temperature anisotropies was derived with close attention to the gauge issues associated with the monopole and the dipole transfer functions. Despite the apparent difference in their approach and the notation convention, the theoretical description in their work is based on the same assumptions adopted in this work—CMB photons decouple abruptly from the local plasma at the temperature T_E (E: emission) in the rest frame of the baryon-photon plasma with four-velocity u_E^μ, and they freely propagate to the observer, moving together with matter, where the photons are received with temperature T_R (R: reception) along the observed direction n^α.

Compared to our expression in Eq. (33), the key difference lies in the observer position today. Here we briefly compare how the difference arises in the work [19]. The main calculation is to derive the exact description of the temperature ratio in their Eq. (24) and its linearized equation (29):

$$\int_R^E dt\, H_n N = \int_{\bar{t}_R}^{\bar{t}_E} dt\, H_n N + (\bar{H}\delta t_D)_R^E, \qquad (67)$$

where H_n is the derivative of the photon frequency along the line-of-sight, N is the time lapse in the Arnowitt-Deser-Misner (ADM) formalism [38], and δt_D describes the deviation of the exact positions at emission and reception from the background coordinate $\bar{t}$. The left-hand side is essentially the line-of-sight integration but replaced with the coordinate integration to yield the frequency ratio of photons emitted at the position E and received at the position R. Since the computation of the line-of-sight integration is performed for an ADM normal observer with u_{ADM}^μ, they perform Lorentz boosts δt_B both at the emission and the reception to match the physical frames of the baryon-photon plasma and the observer, of which the four-velocity is then $u^\mu = u_{\mathrm{ADM}}^\mu + \delta t_B{}^{\cdot\mu}$. This yields their main equation (35) for the observed CMB temperature anisotropies:

$$\frac{\delta T(n^\mu)}{\bar{T}_R} = \int_{\bar{t}_R}^{\bar{t}_E} dt\, \delta(H_n N) + (\bar{H}\delta t_D + n^\mu \delta t_{B,\mu})_R^E, \quad (68)$$

where the background temperature at reception is defined as

$$\bar{T}_R := T_E \exp\left[\int_{\bar{t}_R}^{\bar{t}_E} dt\, \bar{H}\right], \qquad (69)$$

and their photon propagation direction n^μ in a coordinate is the opposite of our observed direction n^α in addition to the overall scale factor a:

$$n^\mu = -\frac{1}{a}(0, n^\alpha) + \mathcal{O}(1). \qquad (70)$$

This formula describes the observed anisotropies $\hat{\Theta}(\hat{n})$, corresponding to our Eq. (24), and the derivation above is equivalent to our Eq. (16) for the observed redshift with T_* replaced by T_E. In more detail, the line-of-sight integration was computed in the conformal Newtonian gauge in their Eq. (41) as

$$\int_{\bar{t}_R}^{\bar{t}_E} dt\,\delta(H_n N) \to \int_0^{\bar{r}_*} d\bar{r}(\psi - \phi)' + \psi\Big|_o^*, \qquad (71)$$

where we used our notation convention in the right-hand side. The boost parameters δt_B are set to transform an ADM normal observer to an observer without any spatial energy flux in their Eq. (43):

$$\delta t_B = -\frac{H\psi - \dot{\phi}}{4\pi G(\rho + p)}. \qquad (72)$$

Using the Einstein equation, we derive that the boost parameter is indeed

$$\delta t_B \to -a v_N, \qquad (73)$$

corresponding to an observer velocity, moving together with the total matter component, where the subscript N denotes that the quantity is computed in the conformal Newtonian gauge in addition to the two gravitational potentials ψ and ϕ. Since the radiation energy density is already smaller at the decoupling, this velocity at emission would correspond to the matter component. Note that the velocity at emission in our formula is the one for the baryon-photon fluid, regardless of the validity of the approximation in the analytical expression, as the rest frame of the photon emission is specified by the baryon-photon plasma. However, we showed in Fig. 2 that since the matter velocity is the same as the baryon-photon velocity on large scales, this difference has no impact on the monopole transfer function. At reception, this boost parameter gives the matter velocity, as in our formula.

Now we come to the temporal displacement terms δt_D. Compared to our Eq. (24) for $\hat{\Theta}(\hat{n})$, the absence of the temperature fluctuation Θ_* at decoupling is apparent in Eq. (68), as their equations are derived specifically for the hypersurface in which the photon density is uniform, i.e., $\Theta_* \equiv 0$. This corresponds to a gauge choice, set by δt_D at emission. The time coordinate of the emission point is $\bar{t}_E$, corresponding to our η_*, and the CMB photon temperature has no fluctuation, i.e.,

$$T_E =: \bar{T}(t_E) = \bar{T}(\bar{t}_E)(1 - H\delta t_D). \qquad (74)$$

By identifying to Eq. (15), we find that the temporal displacement δt_D at emission plays a role of restoring the broken gauge symmetry:

$$-H\delta t_D \to \Theta_*. \tag{75}$$

Indeed, the temporal displacement terms δt_D were computed at both positions by gauge transforming to the uniform energy density gauge in their Eq. (42)

$$\delta t_D = -\frac{\delta\rho_N}{\dot{\bar{\rho}}} = -\frac{3H(H\psi - \dot{\phi}) + \frac{A}{a^2}\phi}{12\pi GH(\rho + p)}, \tag{76}$$

where the matter or the photon density fluctuations are also vanishing with the adiabaticity assumption. Using the Einstein equation, we find that the temporal displacements are

$$\delta t_D \to \frac{\delta_N}{3H}, \tag{77}$$

where $\delta_N := \delta - \chi(\dot{\bar{\rho}}/\bar{\rho})$ is the gauge-invariant expression for the density fluctuation in the conformal Newtonian gauge. While this transformation δt_D correctly captures the emission point in Eq. (75), the observer position today is not at the hypersurface of the uniform energy density or the matter density. In fact, it was argued [19] that δt_D at reception is *not uniquely* fixed by any physical prescription, the choice of δt_D at reception only affects the monopole, and they chose it to be the gauge transformation to the uniform energy density gauge as above.

We argued in Sec. II A that the observer position is uniquely fixed, once we assume that the observer is moving together with the matter, as its trajectory in time determines the current position in Eqs. (8) and (13). In fact, the coordinate lapse $\delta\eta_o$ in Eq. (13) is generic for all observers at the linear order. By further assuming that the observer motion is geodesic, the expression for the observer position, or the coordinate (time) lapse is simplified as $\delta\eta_o = -v_o$ in Eq. (14). The temporal displacement δt_D at reception in [19] is then expressed as

$$\delta t_D \to \delta\eta_o - \left.\frac{\Delta\phi}{12\pi Ga^2H(\rho + p)}\right|_o = \delta\eta_o + \frac{\delta_v}{3H_o}, \tag{78}$$

where $\delta_v := \delta_N - av_N(\dot{\bar{\rho}}/\bar{\rho})$ is the gauge-invariant expression for the matter density in the comoving gauge (or the usual matter density from the Boltzmann codes) and we used the Einstein equation in the last equality.

One subtlety in [19] is that the "background" observed temperature $\bar{T}_R$ in Eq. (69) or their Eq. (34) needs further clarification. In order to set $\bar{T}_R$ equal to our coordinate-independent background CMB temperature $\bar{T} := \bar{T}(\bar{\eta}_o)$, the explicit definition of $\bar{t}_R$ and $\bar{t}_E$ in a coordinate-independent way would be needed. Upon setting $\bar{T}_R \to \bar{T}$, we derive the relation of the observed CMB temperature anisotropies in [19] in Eq. (68) to our expression as
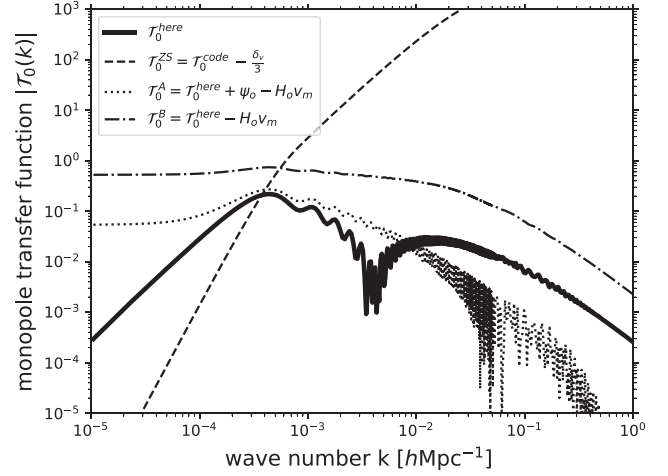

FIG. 4. Comparison of the analytic expression of the monopole transfer function derived in this paper (solid) and the one derived in [19] (dashed), the one without the observer terms (dotted), and the one without the coordinate lapse (dot-dashed). All the monopole transfer functions except $\mathcal{T}_0^{\text{here}}$ lead to an infinite monopole power.

$$\hat{\Theta}_{ZS} = \hat{\Theta}_{\text{here}} - \frac{1}{3}\delta_v. \tag{79}$$

We emphasize again that the difference arises due to the observer position today and it only affects the monopole transfer function. Figure 4 compares the monopole transfer function in this work (solid) and in [19] (dashed). Given that the matter density fluctuation in the comoving gauge $\delta_v \propto k^2\phi$, the monopole transfer function is expected to behave as $\mathcal{T}_0^{ZS}(k) \propto k^2$ on large scales. However, the adiabatic condition in the comoving gauge imposes

$$\Theta = \frac{1}{4}\delta_\gamma = \frac{1}{3}\delta_m, \tag{80}$$

leading to another cancellation on large scales, and the resulting behavior is $\mathcal{T}_0^{ZS}(k) \propto k^4$ and the monopole power C_0^{ZS} is devoid of infrared divergences. Note that the coordinate lapse vanishes in the comoving gauge and our gauge-invariant expression $\hat{\Theta}$ coincides with Θ in the comoving gauge. However, as the density fluctuation δ_v becomes dominant on small scales, the monopole transfer function $\mathcal{T}_0^{ZS}(k) \propto \delta_v(k)$ (dashed) increases with $\simeq k^{0.15}$, leading to the UV divergence in the monopole power C_0. It was concluded in [19] that "this divergence makes it impossible to quantify the total power C_0." The dipole and other multipoles are the same as derived in this work. Note that the transfer function in [19] is defined with an extra factor five: $\mathcal{T}_l^{ZS} = 5\mathcal{T}_l^{\text{here}}$.

2. Case A: Without the observer terms

There exist few work that focus on the gauge invariance of the full observed CMB temperature anisotropies, rather than the expressions for the higher angular multipoles. In Hwang and Noh [20] the gauge invariance of the observed CMB temperature anisotropies was investigated, though the main focus is not the monopole anisotropy. Starting with the same assumption that the observed CMB temperature anisotropies at one direction originate from the photons emitted at a single point, they derive the expression for the observed CMB temperature anisotropies in their Eq. (14):

$$\frac{T(\hat{n})}{\bar{T}(\eta_o)} - 1 = (\Theta + H\chi)_* + (v_{\chi,\alpha}n^\alpha + \alpha_\chi)_* - (v_{\chi,\alpha}n^\alpha)_o$$
$$- H_o\chi_o - \alpha_{\chi o} + \int_0^{\bar{r}_*} d\bar{r}(\alpha_\chi - \varphi_\chi)', \quad (81)$$

where we used our notation convention to express the right-hand side of the equation. They noted that the first two terms at the source is the gauge-invariant combination $(\Theta + H\chi)_* \to \Theta_{\chi*}$. More importantly, they identified the gauge dependence of the expression for the observed CMB temperature anisotropies due to the term $H_o\chi_o$. Since it is independent of the angular direction (also the term $\alpha_{\chi o}$), they argued that it will be absorbed into the background CMB temperature or the angle average $\langle T \rangle_\Omega$ in Eq. (25). So they arrived at the gauge-invariant expression for the observed CMB temperature anisotropies in their Eq. (15),

$$\hat{\Theta}_A = \Theta_{\chi*} + (v_{\chi,\alpha}n^\alpha + \alpha_\chi)_* - (v_{\chi,\alpha}n^\alpha)_o + \int_0^{\bar{r}_*} d\bar{r}(\alpha_\chi - \varphi_\chi)', \quad (82)$$

after removing the two terms at the observer position without the angular dependence. Despite the procedure to subtract the contribution to $\langle T \rangle_\Omega$, Eq. (82) still has the monopole fluctuation.

In comparison to our gauge-invariant expression in Eq. (33), it is clear that the gauge dependence in Eq. (81) arises from the background temperature at the observer position $\bar{T}(\eta_o)$ in the left-hand side. As shown in Eq. (10), the value of $\bar{T}(\eta_o)$ depends on our choice of coordinate, and it is composed of the background temperature $\bar{T}$ in a homogeneous Universe and the coordinate lapse $\delta\eta_o$. This yields extra terms in Eq. (81) that are not included in $\hat{\Theta}_A$:

$$-H_o\delta\eta_o - H_o\chi_o - \alpha_{\chi o}. \quad (83)$$

Consequently, we derive the relation of the observed CMB temperature anisotropies to our expression as

$$\hat{\Theta}_A = \hat{\Theta}_{\text{here}} + \alpha_{\chi o} - H_o v_{\chi o}. \quad (84)$$

Figure 4 shows the monopole transfer function for $\hat{\Theta}_A$ (dotted). With two extra potential terms missing, no cancellation occurs on large scales, and the transfer function does not vanish in the infrared. Therefore, despite the absence of divergences in the UV, the monopole power is also infinite $C_0^A = \infty$.

3. Case B: Without the coordinate lapse

Another common case in literature is to neglect the coordinate lapse. Though calculations are done properly (up to $\delta\eta_o$), less attention is paid to the gauge invariance of the resulting expression. In the pioneering work [2], the expression for the observed CMB temperature anisotropies was derived (see also [39,40]) by using the relation of the observed CMB temperature to the observed redshift in Eq. (16). Perturbing all the quantities in Eq. (16), we obtain their key equation for the observed CMB temperature anisotropies

$$\frac{\Delta T(\hat{n})}{T(\hat{n})} = \frac{\Delta T_*}{T_*} - \frac{\Delta z}{1 + z_{\text{obs}}}, \quad (85)$$

which is similar to our Eq. (24). By defining the last-scattering surface as a hypersurface of the uniform energy density $\Delta T_* \equiv 0$, they computed the perturbation to the observed redshift. However, the devils are again in details, and the ambiguities arise from identifying the correct background quantities. At the linear order in perturbations, Δz can be safely equated to $(1 + z_*)\delta z$ in our notation convention, but without the coordinate lapse $\delta\eta_o$. Again, the missing lapse term is owing to the lack of proper consideration of the observer position today. The final expression for the observed CMB temperature anisotropies is then

$$\hat{\Theta}_B = \hat{\Theta}_{\text{here}} - H_o v_{\chi o}. \quad (86)$$

Figure 4 shows the monopole transfer function for $\hat{\Theta}_B$ (dot-dashed line). Similar to $\hat{\Theta}_A$, the monopole power is plagued with infrared divergences, but without UV divergences. The monopole power C_0^B is again infinite.

IV. DISCUSSION

We have derived an analytic expression for the observed CMB temperature anisotropies $\hat{\Theta}(\hat{n})$ in Eqs. (24) and (33) and investigated the gauge issues in comparison to previous work. It is well known that a general coordinate transformation in Eq. (2) induces a gauge transformation for the temperature fluctuation Θ at the observer position x_o^μ:

$$\tilde{\Theta}(\tilde{x}_o^\mu) = \Theta(x_o^\mu) + \mathcal{H}_o\xi_o, \quad (87)$$

as the background CMB temperature depends only on the time coordinate

$$\bar{T}(\tilde{\eta}_o) = \bar{T}(\eta_o) + \bar{T}'(\eta_o)\xi_o. \qquad (88)$$

Consequently, the temperature fluctuation Θ at the observer position is gauge dependent (in fact at any position). However, since the gauge mode is independent of observed angular direction $\hat{n}$, only the monopole fluctuation Θ_0 is gauge dependent, and the other angular multipoles Θ_l with $l \geq 1$ are gauge invariant when decomposed in terms of observed angle $\hat{n}$. This statement is correct, but the gauge dependence of the temperature fluctuation Θ at the observer position indicates that it cannot be the correct description of the observed CMB temperature anisotropies. To put it differently, the theoretical prediction for the observed values of the CMB temperature anisotropies should be independent of our choice of gauge condition.

With a few exceptions [19,20], relatively little attention has been paid in literature to this flaw in the theoretical description of the observed CMB temperature anisotropies, largely because the angular multipoles a_{lm} of the temperature fluctuation $\Theta(\hat{n})$ with $l \geq 1$ are gauge invariant and they contain most of the cosmological information, and also because the angle-averaged CMB temperature, or the combination of the background and the fluctuation

$$\langle T \rangle_\Omega := \int \frac{d^2\hat{n}}{4\pi} T(\hat{n}) = \bar{T}(\eta_o)[1 + \Theta_0(x_o^\mu)], \qquad (89)$$

is well measured and gauge invariant, where $\Theta_0(x_o^\mu)$ is the angle average (or monopole) of the temperature fluctuation $\Theta(x_o^\mu)$ at the observer position and it gauge transforms as in Eq. (87). However, we showed that the monopole power C_0 computed by using the gauge-dependent description of $\Theta(\hat{n})$ is *infinite*, logarithmically diverging in the infrared.

The angle-averaged CMB temperature is expected to fluctuate around the background temperature from place to place, but its variance C_0 cannot be infinite. Figures 1 and 2 show that the infrared divergence in the monopole power originates from the gravitational potential contributions on very large scales or low k. Those contributions act as a uniform gravitational potential on scales smaller than their wavelength, and they should have *no* impact on any local measurements, as any test particles and the measurement apparatus would move together, according to the equivalence principle. In fact, our investigation of the monopole fluctuation in the large-scale limit in Sec. III C proves that our gauge-invariant expression for the observed CMB temperature anisotropies $\hat{\Theta}$ in Eqs. (24) and (33) (as opposed to the gauge-dependent expression Θ) indeed contains numerous components that act as a uniform gravitational potential on very large scales, but their large-scale contributions cancel to yield the leading-order contribution in proportion to k^2. The infrared divergence of the monopole power arises due to the use of the gauge-dependent expression of the CMB temperature fluctuation, which neglects one or a few contributions, breaking the

gauge invariance and the subtle balance for the cancellation. In fact, the cancellation of such contributions is stronger, as the equivalence principle states that a uniform gravitational acceleration cannot be measured locally, preventing any gradient contributions of the gravitational potentials on large scales.

In Sec. III D, we have compared our gauge-invariant expression for the observed CMB temperature anisotropies to previous work. The major physical process of the CMB temperature anisotropy formation was fully identified in the pioneering work by Sachs and Wolfe in 1967 [2]—once the baryon-photon plasma cools to decouple, the CMB photons propagate freely in space, and they are measured by the observer. This physical process naturally involves the physical quantities along the photon path as well as those at the decoupling position and the observer position. The first contribution is referred to as the integrated Sachs-Wolfe effect, and the contributions at the source position are made of the gravitational redshift (or Sachs-Wolfe effect), the Doppler effect, and the (intrinsic) temperature fluctuation, while the contributions at the observer position were often neglected in literature. In the comprehensive work by Zibin and Scott [19], this physical process was carefully examined in close attention to the gauge invariance of the theoretical description of the observed CMB temperature anisotropies. The observer should be moving together with baryon and matter components, at least in the linear-order description, and this observer motion contributes to the dipole anisotropy. According to [19], however, there still remains one ambiguity, which is the choice of the observer hypersurface, or the time coordinate of the observer position. This is evident in Eqs. (87) and (88), and such ambiguity in defining the observer position is one source for the gauge dependence of the analytical expressions in literature. A hypersurface of the uniform energy density was chosen in [19], though it was also noted that this choice is not unique. By specifying the hypersurface, their expression for the observed CMB temperature is gauge invariant, but the resulting monopole power is UV divergent.

In this work we have shown that there is *no* ambiguity in describing the observed CMB temperature anisotropies, as expected for any physical observations; the observer position today is uniquely determined, once a physical choice of the observer is made. The observer is moving together with the matter component, and its time coordinate η_o today can be computed by following the motion of the observer in time. Compared to the reference time coordinate $\bar{\eta}_o$ of the observer in the background in Eq. (6), the observer time coordinate today deviates from $\bar{\eta}_o$ by the coordinate lapse $\delta\eta_o$, defined as in Eq. (8): $\eta_o := \bar{\eta}_o + \delta\eta_o$. The coordinate lapse $\delta\eta_o$, of course, gauge transforms, as the time coordinate of the observer depends on the choice of coordinate system. However, it is this extra gauge-dependent term that compensates for the gauge dependence

in the temperature fluctuation Θ at the observer position, as expressed in Eq. (24). Due to the absence of the angular dependence, the expressions for the higher-order multipoles are gauge invariant.

Using our gauge-invariant expression for the observed CMB temperature anisotropies in Eq. (24) and the Boltzmann codes CLASS and CAMB, we have numerically computed the monopole power for the first time

$$C_0 = 1.66 \times 10^{-9}, \tag{90}$$

corresponding to the rms monopole fluctuation

$$\sqrt{\langle \hat{\Theta}_0^2 \rangle} = \sqrt{\frac{C_0}{4\pi}} = 1.15 \times 10^{-5}. \tag{91}$$

The largest contribution to the monopole power is the gravitational potential at the decoupling point, which is also the main source for the Sachs-Wolfe plateau at low angular multipoles. A finite value of the monopole power indicates that the angle average of the CMB temperature fluctuates in space around the background CMB temperature $\bar{T}(\bar{\eta}_o)$ evaluated at the reference time coordinate $\bar{\eta}_o$, and the COBE FIRAS measurement $\langle T \rangle_\Omega$ is *not* the background temperature $\bar{T}(\bar{\eta}_o)$, but one with the monopole fluctuation $\hat{\Theta}_0$ at our position. Given that the current measurement uncertainty is ~ 10 times larger than the rms fluctuation, the impact of properly accounting for the difference between $\bar{T}(\bar{\eta}_o)$ and $\langle T \rangle_\Omega$ is negligible [18] for the CMB power spectrum analysis. With $\bar{T} := \bar{T}(\bar{\eta}_o)$ being one of the fundamental cosmological parameters, the monopole fluctuation $\hat{\Theta}_0$ (or the angle-averaged anisotropy) can be inferred from the measurement of $\langle T \rangle_\Omega$ once a cosmological model is chosen. This is in contrast to the coordinate-dependent $\bar{T}(\eta_o)$ and $\Theta_0(x_o^\mu)$ in Eq. (89), which are not separable from the measurement of $\langle T \rangle_\Omega$ due to their ambiguous definition. Hence the monopole fluctuation $\hat{\Theta}_0$ is a model-dependent, but coordinate-independent, "observable."

ACKNOWLEDGMENTS

APPENDIX A: METRIC CONVENTION AND GAUGE TRANSFORMATION PROPERTIES OF THE PERTURBATION VARIABLES

In this appendix we introduce the notation convention used in this paper. The background Universe is described by a Robertson-Walker metric:

$$ds^2 = -a^2(\eta)d\eta^2 + a^2(\eta)\bar{g}_{\alpha\beta}dx^\alpha dx^\beta,$$

with conformal time η and scale factor $a(\eta)$. To account for the inhomogeneities in the Universe, we introduce scalar perturbations to the metric tensor with the following convention:

$$g_{00} := -a^2(1 + 2\alpha), \qquad g_{0\alpha} := -a^2\beta_{,\alpha},$$
$$g_{\alpha\beta} := a^2[(1 + 2\varphi)\bar{g}_{\alpha\beta} + 2\gamma_{,\alpha|\beta}]. \tag{A1}$$

The comma represents the coordinate derivative, while the vertical bar represents the covariant derivative with respect to the three metric $\bar{g}_{\alpha\beta}$. We do not consider vector and tensor perturbations in this paper. The timelike four velocity vector is introduced as

$$u^\mu = \frac{1}{a}(1 - \alpha, U^\alpha), \qquad -1 = u_\mu u^\mu, \tag{A2}$$

and we define the scalar velocity potentials as

$$U^\alpha := -U^{,\alpha}, \qquad v := U + \beta. \tag{A3}$$

Under the general coordinate transformation in Eq. (2), the metric tensor gauge transforms as

$$\delta_\xi g_{\mu\nu} = \tilde{g}_{\mu\nu}(x) - g_{\mu\nu}(x) = -\pounds_\xi g_{\mu\nu}$$

at linear order. The Lie derivative $\pounds$ of a rank 2 tensor is given by

$$\pounds_\xi g_{\mu\nu} = g_{\mu\nu,\rho}\xi^\rho + g_{\rho\nu}\xi^\rho{}_{,\mu} + g_{\mu\rho}\xi^\rho{}_{,\nu}.$$

For the time-time component, this leads to

$$\tilde{\alpha}(x) = \alpha(x) - \frac{1}{a}(a\xi)',$$

where the prime denotes the partial derivative with respect to conformal time η. The time-space component and the space-space component of the metric reveal the transformation properties of the other scalar perturbation variables introduced in Eq. (A1):

$$\tilde{\beta} = \beta + L' - \xi, \qquad \tilde{\varphi} = \varphi - \mathcal{H}\xi, \qquad \tilde{\gamma} = \gamma - L.$$

Since theoretical descriptions of observables have to be gauge invariant, it is most convenient to work with gauge-invariant variables [25]:

$$\alpha_\chi := \alpha - \frac{1}{a}\chi', \qquad \varphi_\chi := \varphi - H\chi,$$

with the scalar shear of the normal observer

$$\chi := a(\beta + \gamma'),$$

introduced in [41]. It gauge transforms as $\tilde{\chi} = \chi - a\xi$. The two potentials correspond to the Bardeen variables $\alpha_\chi \rightarrow \Phi_A$ and $\varphi_\chi \rightarrow \Phi_H$ in [25]. The energy-momentum tensor transforms in the same way as the metric tensor. The time-time and the time-space components reveal the transformation properties of the density fluctuation and the scalar velocity potential:

$$\tilde{\delta} = \delta - \frac{\bar{\rho}'}{\bar{\rho}}\xi, \qquad \tilde{v} = v - \xi.$$

Therefore, we construct two gauge-invariant quantities

$$\Theta_\chi := \frac{1}{4}\delta_\chi^\gamma = \frac{1}{4}\delta_\gamma + H\chi, \qquad v_\chi := v - \frac{\chi}{a}.$$

APPENDIX B: NUMERICAL COMPUTATION OF THE MONOPOLE AND DIPOLE TRANSFER FUNCTIONS

For the black-body radiation, the temperature anisotropy Θ is related to the fluctuation in the photon distribution function f at the linear order as

$$f(q, \hat{n}) = -\frac{d\bar{f}}{d\ln q}\Theta(\hat{n}), \tag{B1}$$

where q is the comoving momentum and the photon distribution function in the background is

$$\bar{f}(q) = \left[\exp\left(\frac{q}{a\bar{T}(\eta)}\right) - 1\right]^{-1}. \tag{B2}$$

Noting that the temperature anisotropy is independent of q, we can integrate over q to derive

$$\Theta(\hat{n}) = \frac{2\pi}{a^4\bar{\rho}_\gamma}\int dq\, q^3 f(q, \hat{n}), \tag{B3}$$

where the photon energy density is

$$\bar{\rho}_\gamma(\eta) = \frac{2}{a^4}\int d^3q\, q\bar{f}(q). \tag{B4}$$

The temperature anisotropy can be decomposed in terms of observed angle, and the multipole coefficients in Eq. (30) are

$$a_{lm} = \frac{2\pi}{a^4\bar{\rho}_\gamma}\int d^2\hat{n}\, Y_{lm}^*(\hat{n})\int dq\, q^3 f(q, \hat{n}). \tag{B5}$$

The monopole coefficient is, therefore, related to the photon density fluctuation δ_γ as

$$a_{00} = \sqrt{4\pi}\times\frac{1}{4}\delta_\gamma^{\text{sync}} = \sqrt{4\pi}\Theta^{\text{sync}}, \tag{B6}$$

and using Eq. (39) the monopole transfer function is obtained as

$$\mathcal{T}_0(k) = \mathbb{T}_\Theta^{\text{sync}}(k), \tag{B7}$$

where $\delta\rho_\gamma = \bar{\rho}_\gamma\delta_\gamma$ is related to the distribution function $f(q, \hat{n})$ as in Eq. (B4). For the monopole transfer function, we need to use the gauge-invariant expression $\hat{\Theta}(\hat{n})$, rather than the gauge-dependent expression $\Theta(\hat{n})$. In the comoving-synchronous gauge, where the coordinate lapse vanishes, both values are identical.

The dipole coefficient is

$$a_{10} = \sqrt{\frac{3\pi}{4}}\frac{2}{a^4\bar{\rho}_\gamma}\int d^3q\, q\cos\theta f(q, \hat{n}) = \sqrt{\frac{3\pi}{4}}\frac{s_z}{\bar{\rho}_\gamma}, \tag{B8}$$

where the spatial energy flux of the photon distribution is defined as

$$s^i := \frac{2}{a^4}\int d^3q\, q^i f(q, \hat{n}). \tag{B9}$$

Using the fluid description for the photon energy-momentum tensor, the spatial energy flux of the photon distribution can be expressed as

$$s^i = (\bar{\rho} + \bar{p})_\gamma u_\gamma^i \tag{B10}$$

in terms of the photon velocity u_γ^i measured in the observer rest frame. The photon velocity

$$u_\gamma^i = [e_i]_\mu^{\text{obs}} u_\gamma^\mu = (v_{\text{obs}} - v_\gamma)^{,i} \tag{B11}$$

is literally the relative velocity between the observer and the photon fluid at the linear order, and it is evidently gauge invariant, where $[e_i]_{\text{obs}}^\mu$ is a spatial directional vector (or a spatial tetrad) of the observer (see, e.g., [24,42]). Finally, by using Eq. (39), the dipole transfer function is obtained as

$$\mathcal{T}_1(k) = \frac{k}{3}(\mathbb{T}_{v_\gamma} - \mathbb{T}_{v_{\text{obs}}})(k). \tag{B12}$$

In the comoving-synchronous gauge, where the observer velocity vanishes, the dipole transfer function becomes

$$\mathcal{T}_1(k) = \frac{k}{3}\mathbb{T}_{v_\gamma}^{\text{sync}}(k). \tag{B13}$$

The monopole and the dipole transfer functions $\mathcal{T}_0(k)$ and $\mathcal{T}_1(k)$ are obtained by using the above expressions and numerically evaluating $\mathbb{T}_\Theta^{\text{sync}}(k)$ and $\mathbb{T}_{v_\gamma}^{\text{sync}}(k)$ at present day from the Boltzmann codes CLASS and CAMB.

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