Triple-X and beyond: Hadronic systems of three and more $X(3872)$

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The $X(3872)$ resonance has been conjectured to be a $J^{PC} = 1^{++}$ charm meson-antimeson two-body molecule. Meanwhile, there is no experimental evidence for larger, few-body compounds of multiple charm meson-antimeson pairs which would resemble larger molecules or nuclei. Here, we investigate such multimeson states to the extent of what can be deduced theoretically from essentials of the interaction between uncharged D^0 and D^{*0} mesons. From a molecular $X(3872)$, we predict a $4X$ (4^{++}) octamer with a binding energy $B_{4X} > 2.08$ MeV, assuming a $D^{*0}\bar{D}^0$ system close to the unitary limit [as suggested by the mass of the $X(3872)$]. If we consider heavy-quark spin symmetry explicitly, the $D^{*0}\bar{D}^{*0}$ (2^{++}) system is close to unitarity, too. In this case, we predict a bound $3X$ (3^{++}) hexamer with $B_{3X} > 2.29$ MeV and a more deeply bound $4X$ octamer with $B_{4X} > 11.21$ MeV. These results exemplify with hadronic molecules a more general phenomenon of equal-mass two-species Bose systems composed of equal number of either type: the emergence of unbound four- and six-boson clusters in the limit of a short-range two-body interaction which acts only between bosons of different species. Finally, we also study the conditions under which a $2X$ (2^{++}) tetramer might form.

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I. INTRODUCTION

Systems of particles with a two-body scattering length a significantly larger than the interaction range R ($a \gg R$) share a series of common/universal properties, which encompass a multitude of phenomena in atomic, nuclear, and particle physics [1]. This invariance with respect to a continuous scale transformation, however, holds strictly only in the two-body sector. In the few-body spectrum, this *continuous* scale invariance survives only partially in a *discrete* version. An example of this is the Efimov effect [2], i.e., the appearance of a geometric bound-state spectrum of three-boson systems in the unitary limit ($a/R \to \infty$). This effect was found for the first time a decade ago in experiments with caesium atoms [3], and it is now known to extend to systems of nonidentical particles [4] as well as systems of more than three particles [5,6]. In nuclear physics, the Efimov effect plays a role in the description of the triton [7,8] and ^{4}He [9], halo nuclei [10–14], and the Hoyle state [15,16]. In bosonic systems with more than three particles, the same effect realizes stable clusters (see e.g., [17] where up to 60 bosons where analyzed).

Compared with atoms and nucleons, it is more difficult to find instances of universality in hadronic physics where the $X(3872)$ resonance [18] might qualify as a hadronic system close to the unitary limit. The X has been conjectured to be a hadronic molecule [19,20], more precisely, a relatively shallow bound state of two hadrons because of its proximity to the $D^{*0}\bar{D}^0$ threshold (~0.01 MeV) and its narrow width. This shallowness, in particular, is a signature of universal behavior [21]. To explore the consequences of universality, we will describe X and multi-X systems with a contact-range theory [21] and use D ($\bar{D}$) mesons (antimesons) as fundamental degrees of freedom. With this approach, we aim to expose characteristic features of composite systems with the minimal assumptions and data on the constituents. Alternative descriptions may improve on accuracy if the coupling to other channels, meson exchanges, etc., [22–26] is considered. The bulk properties of the systems we analyze below, however, will not be affected by these refinements.

The identification of universal properties in systems composed of more than two charm mesons is an intriguing open question because charm meson-antimeson interactions produce qualitatively new features that are absent

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in systems of identical particles. For instance, both three-body systems $D^0D^0\bar{D}^{*0}$ and $D^{*0}D^{*0}\bar{D}^0$ do neither form trimers nor do they display the Efimov effect [27]. Along with heavy-quark spin symmetry (HQSS) [28,29] and the associated, more tightly constrained charm meson-antimeson potential enters new features. In the two-body sector, we expect from HQSS the interaction in the $D^{*0}\bar{D}^0$ ($J^{PC} = 1^{++}$) X-channel identical to the one in the $D^{*0}\bar{D}^{*0}$ (2^{++}) channel, suggesting the existence of a partner molecule of the X [30,31]. Like the $X(3872)$, this partner is expected to be shallow but its survival as a bound or virtual state, or as a resonance depends on a number of uncertainties [31–33]. HQSS challenges the original expectation of an unbound $D^{*0}D^{*0}\bar{D}^0$ $J = 2$ three-body system and could thus facilitate the Efimov effect [34].

In four-meson systems and beyond, we expect to find new universal phenomena different from the ones known to emerge in atomic and nuclear composites [17,35]. We will consider, in particular, systems of $N = 2, 3, 4$ $D^0\bar{D}^{*0}$ pairs with maximum spin, i.e., $J = 2, 3, 4$, respectively. Bound "polymers" of this kind exhibit a characteristic scaling inversely proportional to the square of the interaction range, i.e., $B_{2N} \propto 1/R^2$. We infer from this scaling the Thomas collapse [36] of these systems along with the implied Efimov effect. This collapse is eventually avoided owing to short-range effects, e.g., the finite interaction range. Conversely, the Efimov effect is impaired by long-range deviations from unitarity, i.e., a finite scattering length. Specifically, $D^{*0}\bar{D}^{*0}$ pairs can decay strongly to $D^{(*)0}\bar{D}^0/D^{(*)+}\bar{D}^-$ via a short-range D-wave operator [37] inducing such a finite interaction width. Using data on the X in support of the assumption of an infinite $D^{*0}\bar{D}^0$ scattering length (zero binding energy of the X molecule) and disregarding HQSS, we predict a bound state of four X's: an octamer. As the tetramer and the hexamer are unbound under these circumstances, this octamer resembles a *so-called* Brunnian [38,39] state: a generalization of a Borromean structure. Finally, we predict that a $D^{*0}\bar{D}^{*0}$ interaction close to the unitary limit, will stabilize the hexamer and thus induce the transition from a Brunnian to a Borromean system (a still unbound tetramer with a hexamer resembling a Borromean bound state of X's).

II. THEORY AND CALCULATION METHOD

We treat the above-mentioned polymers as a nonrelativistic few-body problem. The charm meson and antimesons comprising these polymers have a ground ($D/\bar{D}$) and excited ($D^*/\bar{D}^*$) state. Their isospin $I = 1/2$ discriminates between neutral and charged states. Because of their mass difference, we will only consider the neutral mesons which dominate the X wave function tail. In the unitary limit, we have to consider only *resonant*[1] two-body interactions. Mass and quantum numbers of the X from the molecular perspective hint toward such a resonant behavior in the $D^{*0}\bar{D}^0$ (1^{++}) channel (i.e., the X channel) [20,21], while HQSS lets us expect the $D^{*0}\bar{D}^{*0}$ (2^{++}) channel to be resonant, too [30,31]. All other combinations are assumed to be nonresonant and set to zero. Nonresonant interaction pairs would increase the total binding of the systems slightly. Thus, they do not alter our conclusions.

To describe the resonant pairs, we employ a contact two-body potential

$$V(\boldsymbol{r}; R_c) = C(R_c)\delta^{(3)}(\boldsymbol{r}; R_c), \tag{1}$$

regularized with the Gaussian cutoff function

$$\delta^{(3)}(\boldsymbol{r}; R_c) = \frac{e^{-(r/R_c)^2}}{\pi^{3/2}R_c^3}.$$

The cutoff radius R_c and coupling constant $C(R_c)$ are calibrated (renormalized) to the mass of the $X(3872)$,

$$M_X = m(D^0) + m(D^{*0}) - B_X. \tag{2}$$

Here, $m(D^0)$, $m(D^{*0})$ denote the masses of D^0 and D^{*0} mesons, respectively, and the binding energy B_X is positive for a stable state. As we are interested in the universal properties of charmed meson clusters, we will set $B_X = 0$ in accordance with the current experimental value $B_X = -0.01 \pm 0.18$ MeV putting it slightly above threshold [40]. The potential underlying this threshold state is attractive, i.e., $C(R_c) < 0$, and would bind the system if increased by an infinitesimal amount. The contact-range potential is visualized in Fig. 1 for $R_c = 1.0, 1.5, 2.0$ fm, where the relation between an interaction of shorter range with a larger coupling (i.e., a deeper potential), as to ensure that the $D^{*0}\bar{D}^0$ bound state is always at threshold, is apparent. When solving the Schrödinger equation with this potential (1), we will consider all the mesons to have an identical mass of $m = 1933.29$ MeV, i.e., twice the reduced mass of the $D^{*0}\bar{D}^0$ pair within the X because it is the most important resonant interaction in the multi-X systems [see derivation of (4)–(12) below]. Corrections are deemed to be subleading and without impact on the qualitative description of the many-particle states.

Renormalized predictions, in principle, require that observables are cutoff independent. We will show below that hexamer and octamer binding energies do not conform with this demand as they Thomas collapse if $R_c \to 0$. In few-body systems, this type of divergence can be renormalized via an additional three-boson datum [7,8] which, as of now, is unavailable in the few-X sector.

Despite this obstacle, we can obtain information about the existence of bound states and estimates of their binding energies. To this end, we choose a cutoff range near the theory's breakdown scale which is determined by the longest omitted short-range component of the interaction.

[1]Two particles interact resonantly if their S-wave scattering length diverges and a two-body bound state with zero energy at threshold exists.

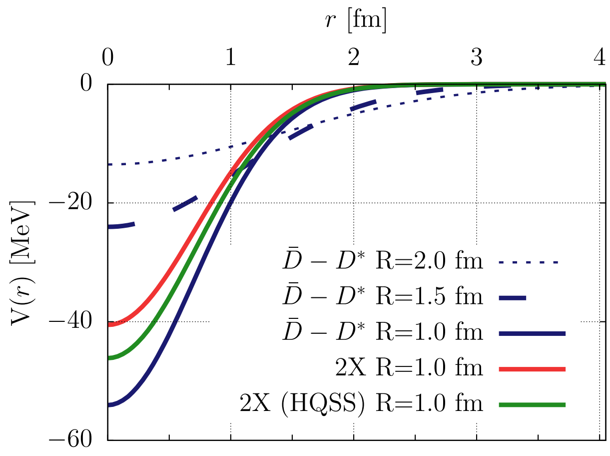

FIG. 1. Regularized contact-range $\bar{D} - D^*$ potential (1) with strengths $C(R_c)$ renormalized to a X molecule at threshold for cutoff radii $R_c = 1.0, 1.5, 2.0$ fm (solid, dashed, dotted blue lines), i.e., the depth is determined by the coupling strength (1) and changes with R_c in order to generate the same diverging two-body scattering length independent of R_c. The effective meson-antimeson interactions for $R_c = 1.0$ fm in the four-body $2X$ system [see (10)] are shown in red for comparison. The enhancement of the potential due to HQSS is shown in green [see (15)]. In these two cases, the effective interaction is a factor of 0.75 and 0.854 weaker to that of the $\bar{D} - D^*$ potential owing to the existence of noninteracting meson-antimeson pairs in the multimeson wave functions [see (6)–(15)].

This missing component is in case of the X the charged channel,[2] i.e., the $D^{*+}D^-$ component of the X wave function [23,24]. The characteristic momentum scale of the charged channel is $M_{\rm ch} \simeq 125$ MeV. It is sensible to expect a cutoff in the vicinity of $M_{\rm ch} R_c \sim 1$ for which the three-body counterterm vanishes, and that it remains numerically small within some interval around it. This smallness suffices to foresee that their inclusion would have no effect on the character of a state: bound will remain bound, resonance will remain resonance, etc. Hence, a bound state found within a cutoff range around $M_{\rm ch} R_c \sim 1$ (specifically, we chose $R_c = 1.0$–2.0 fm) can reliably be considered a renormalized prediction which will not change character even with the proper calibration of a collapse-preventing counterterm.

III. INTERACTION BETWEEN MESON PAIRS

We exemplify the few-X calculations with a detailed discussion of the four-body, i.e., two-X problem. First, we treat the X as a pure $D^{*0}\bar{D}^0$ two-body system. This approximation disregards the shorter-range $D^{*+}\bar{D}^-$ component and assumes the X wave function to be

$$\Psi_X = \phi_X(\boldsymbol{r}) \frac{1}{\sqrt{2}} [|D^0\bar{D}^{*0}\rangle + |D^{*0}\bar{D}^0\rangle], \tag{3}$$

with the spatial two-body wave function $\phi_X(\boldsymbol{r})$. The charm meson-antimeson potential in the X channel is defined for the linear combination $D^{*0}\bar{D}^0 + D^0\bar{D}^{*0}$ (the positive C-parity combination). It is practical to use a Fock representation of the potential,

$$\begin{aligned} V_X(\boldsymbol{r}; R_c) = \frac{V_D(\boldsymbol{r}; R_c)}{2} &[|D^0\bar{D}^{*0}\rangle\langle D^0\bar{D}^{*0}| \\ &+ |D^{*0}\bar{D}^0\rangle\langle D^{*0}\bar{D}^0|] \\ &+ \frac{V_E(\boldsymbol{r}; R_c)}{2} [|D^0\bar{D}^{*0}\rangle\langle D^{*0}\bar{D}^0| \\ &+ |D^{*0}\bar{D}^0\rangle\langle D^0\bar{D}^{*0}|], \end{aligned} \tag{4}$$

with a direct (V_D) and an exchange term (V_E) which combine to the potential in the X channel, $V_X = V_D + V_E$. As no negative C-parity partner of the X has been found yet, we assume $|V_D + V_E| \gg |V_D - V_E|$. Moreover, the isospin-breaking decays of the X [23,24] allow access to V_E and corroborate this inequality [42]. Hence, we use the approximation $V_D = V_E = \frac{1}{2}V$ and

$$V_X = \frac{V}{2} [|D^0\bar{D}^{*0}\rangle + |D^{*0}\bar{D}^0\rangle][\langle D^0\bar{D}^{*0}| + \langle D^{*0}\bar{D}^0|], \tag{5}$$

where the $(\vec{r}, R_c)$ dependence of the potential has been dropped to improve readability.

The two-X tetramer contains in principle the six possible permutations of the $|D^0D^0\bar{D}^{*0}\bar{D}^{*0}\rangle$ state that result from exchanging ground- and excited-state mesons (we assume the spins of all the $D^{*0}/\bar{D}^{*0}$ mesons/antimesons to point in the same direction). However, these permutations are further constrained by symmetries, as we require (i) positive C-parity (i.e., invariance with respect to the exchange of particles and antiparticles) and (ii) D^0 and D^{*0} to obey Bose statistics which we realize with symmetric internal and spatial wave function components as they are expected to provide the majority of the attraction (i.e., symmetric combinations of D^0D^{*0} and $D^{*0}D^0$[3]). This reduces the number of relevant states from six to two,

[2]Pion effects are naïvely expected to enter perturbatively at subleading orders [41], suggesting leading-order predictions which are indistinguishable in pionfull and pionless treatments. Analogous to the conjectured effect of the charged components of the D's, the inclusion of pions is expected to change the breakdown scale of the theory and generates a finite-range interaction which adds to the attraction in larger clusters instead of their disintegration.

[3]Antisymmetric combinations—the nuclear analog are proton-proton or neutron-neutron spin-1 contributions to, e.g., ^{4}He—demand an odd angular momentum with a perturbatively small effect in the leading-order framework employed in this work.

$$|1\rangle = \frac{|D^0 D^{*0}\bar{D}^{*0}\bar{D}^0\rangle + |D^{*0}D^0\bar{D}^0\bar{D}^{*0}\rangle}{\sqrt{2}}, \tag{6}$$

$$|2\rangle = \frac{|D^0 D^{*0}\rangle + |D^{*0}D^0\rangle}{\sqrt{2}} \frac{|\bar{D}^{*0}\bar{D}^0\rangle + |\bar{D}^0\bar{D}^{*0}\rangle}{\sqrt{2}}. \tag{7}$$

In this basis, the potential reads [insert (6) and (7) in (5)]

$$\sum_{ij} V_X(\boldsymbol{r}_{ij}; R_c) \begin{pmatrix} |1\rangle \\ |2\rangle \end{pmatrix} = \begin{pmatrix} 2\bar{V} & \sqrt{2}\bar{V} \\ \sqrt{2}\bar{V} & \bar{V} \end{pmatrix} \begin{pmatrix} |1\rangle \\ |2\rangle \end{pmatrix}, \tag{8}$$

where $\bar{V}$ represents the average of the potential for all resonant pairs. Considering that V_X involves particle-antiparticle interactions only and the same ordering as in $|1\rangle$ and $|2\rangle$ (i.e., indexing particles before antiparticles),

$$\bar{V} = \frac{1}{4}[V(\boldsymbol{r}_{13}) + V(\boldsymbol{r}_{14}) + V(\boldsymbol{r}_{23}) + V(\boldsymbol{r}_{24})]. \tag{9}$$

The diagonalization of (8) yields

$$\sum_{ij} V_X(\boldsymbol{r}_{ij}; R_c)|X_2\rangle = 3\bar{V}|X_2\rangle, \tag{10}$$

as the most attractive configuration, with $|X_2\rangle = \sqrt{\frac{2}{3}}|1\rangle + \sqrt{\frac{1}{3}}|2\rangle$ being a four-meson eigenstate of $\sum V_X$. The original coupled-channel problem has thereby been recast into a single-channel form.

The steps detailed above for the tetramer can be straightforwardly applied to the hexamer and octamer. The six-body case comprises, in principle, 20 possible permutations of the $|D^0D^0D^0\bar{D}^{*0}\bar{D}^{*0}\bar{D}^{*0}\rangle$ state, which are reduced to two states by symmetry constraints. In the eight-body case, there are 70 possible permutations of the $|D^0D^0D^0D^0\bar{D}^{*0}\bar{D}^{*0}\bar{D}^{*0}\bar{D}^{*0}\rangle$ state, which are reduced to three symmetric ones. The potential can be diagonalized, as before in the four-body case, leading to a series of eigenvalues and eigenvectors of which the most attractive configurations are

$$\sum_{ij} V_X(\boldsymbol{r}; R_c)|X_3\rangle = 6\bar{V}|X_3\rangle, \tag{11}$$

$$\sum_{ij} V_X(\boldsymbol{r}; R_c)|X_4\rangle = 10\bar{V}|X_4\rangle. \tag{12}$$

Again, $\bar{V}$ represents the average of the potential experienced by the interacting pairs, while $|X_3\rangle$, $|X_4\rangle$ are the eigenvectors in the internal space of the interaction that correspond to the most attractive configuration.

In order to analyze the effect of HQSS on our predictions, we modify the above-derived interaction. First, we infer from HQSS a potential in the $D^{*0}\bar{D}^{*0}$ (2^{++}) channel identical to that in the X channel. Note the approximate character of this symmetry and the resulting hypothetical nature of the 2^{++} partner of the X. The HQSS extension of the two-body potential V_X of (5) is

$$V_X^{\mathrm{HQSS}} = V_X + V|D^{*0}\bar{D}^{*0}\rangle\langle D^{*0}\bar{D}^{*0}|. \tag{13}$$

Coupling between the 1^{++} and the 2^{++} channels is precluded in the two-body sector but, nevertheless, these transitions become possible in the few-X sector, where these states appear as intermediate structures in the wave function.

We use the four-body case once again to exemplify how the additional interaction term leads to more attraction than expected earlier. In the basis (6), (7), the potential (13) now reads

$$\sum_{ij} V_X^{\mathrm{HQSS}}(\boldsymbol{r}; R_c) \begin{pmatrix} |1\rangle \\ |2\rangle \end{pmatrix} = \begin{pmatrix} 2\bar{V} & \sqrt{2}\bar{V} \\ \sqrt{2}\bar{V} & 2\bar{V} \end{pmatrix} \begin{pmatrix} |1\rangle \\ |2\rangle \end{pmatrix}, \tag{14}$$

whose diagonalization gives

$$\sum_{ij} V_X^{\mathrm{HQSS}}(\boldsymbol{r}; R_c)|X_2'\rangle = (2+\sqrt{2})\bar{V}|X_2'\rangle, \tag{15}$$

with the more attractive eigenvalue $(2+\sqrt{2})\bar{V}$ and eigenvector $|X_2'\rangle = (|1\rangle + |2\rangle)/\sqrt{2}$. For the six- and eight-body systems, the same assumptions and symmetries result in

$$\sum_{ij} V_X^{\mathrm{HQSS}}(\boldsymbol{r}; R_c)|X_3'\rangle = \frac{1}{2}(11+\sqrt{13})\bar{V}|X_3'\rangle, \tag{16}$$

$$\sum_{ij} V_X^{\mathrm{HQSS}}(\boldsymbol{r}; R_c)|X_4'\rangle = (8+\sqrt{22})\bar{V}|X_4'\rangle, \tag{17}$$

with eigenvectors $|X_3'\rangle$ and $|X_4'\rangle$ corresponding to configurations in which the potential is most attractive.

In both cases, with and without HQSS, the spectrum of a system composed of $N/2$ X particles is given by the Schrödinger equation

$$\left(-\sum_{i<j}^{N} \frac{\hbar^2}{2m}(\boldsymbol{\nabla}_{ij})^2 + \eta \sum_{\substack{1\le i\le N/2 \\ (N/2+1)\le j\le N}} V(\boldsymbol{r}_{ij})\right)\phi_N(\boldsymbol{r}) = E\phi_N(\boldsymbol{r}), \tag{18}$$

with $\phi_N(\boldsymbol{r}) = \langle \boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N|(N/2)X\rangle$, η the eigenvalues of (8) ($\eta = \{3/4, 6/9, 10/16\}$, respectively, for $2X$, $3X$, and $4X$), $\boldsymbol{r}_{ij} = \boldsymbol{r}_i - \boldsymbol{r}_j$ with the index $i(j)$ representing a charm meson (antimeson), where we have indexed the particles first and then the antiparticles, and with m being twice the reduced mass of the D^0D^{*0} system, i.e., $m \approx 1933$ MeV. Finally, E refers to the energy of the N-body system, where we are interested in bound states ($B = -E > 0$).

In practice, we solve the Schrödinger equation with the Stochastic-Variational Method [43]. In our implementation, this method expands the wave function in correlated Gaussian functions $[(N-1)\times(N-1)$ relative Jacobi coordinates], with a nonzero interaction between the relevant pairs (meson-antimeson). We abstain from an explicit symmetrization of the spatial wave function, i.e., we do not project onto $L=0$ and assume that the central and parity-preserving character of the potential will produce the energetically favorable symmetric ground states in the course of the variational optimization.

IV. RESULTS AND CONCLUSIONS

Assuming the charm meson-antimeson interaction in the X-channel to dominate, i.e., with the average interactions (10)–(12), we find solutions to (18) of the four-body ($2X$) and six-body ($3X$) systems to be unbound. Adding another X, we predict the eight-body ($4X$) system bound with $B_{4X}>2.08$ MeV. Including the attraction induced by HQSS, i.e., the interactions in (15)–(17), the eight-body binding energy increases to $B_{4X}^{\mathrm{HQSS}}>11.21$ MeV. Furthermore, the six-body system becomes bound with $B_{3X}>2.29$ MeV. These results represent sensible lower bounds for the binding energies of the respective systems obtained at a regularization scale of about 2 fm, a value deemed soft enough for an attractive three-body counterterm. Furthermore, any attraction from the nonresonant mesonic interactions (set to zero in our calculations) is expected to increase binding energies.

As alluded to in the Introduction, this appearance of few-body clusters bound by a few MeV as a result of a resonant two-body system with close-to zero binding energy is not unprecedented. In comparison with those universal $A=4$, 5, 6 bosonic clusters found [44,45] attached to $A=3$ Efimov states, the constraint of a resonant interaction in the X-channel, only, amounts to an N-body problem with each particle interacting only with $N/2$ particles. Furthermore, the strength of this interaction is reduced and is no longer resonant. To be explicit, instead of 6(28) resonant interaction pairs in a system of 4(8) bosons, the limitation to resonant interaction only in the 1^{++} channel yields a 4(8) equal-mass boson problem with 4(16) nonresonant interaction pairs. Multi-X boundstates are thus not expected to expose molecular-X behavior. Therefore, they should be approached, as done in this study, as multi-$D/\bar{D}$ systems. We also notice that three X bosons do not show an Efimov spectrum.

In Fig. 2, we show the regulator dependence of the binding energies as a signature of the Thomas collapse. Originally, this collapse is expected for one-channel systems of identical particles in the zero-range limit [36]. Here, we demonstrate the occurrence of the collapse for a more complex system with more than one channel and where a certain number of interaction pairs have been removed. A range of cutoffs over which the effect of the unenforced renormalization condition (e.g., the canonical three-body counterterm) is expected to vanish is marked in the figure (gray area).

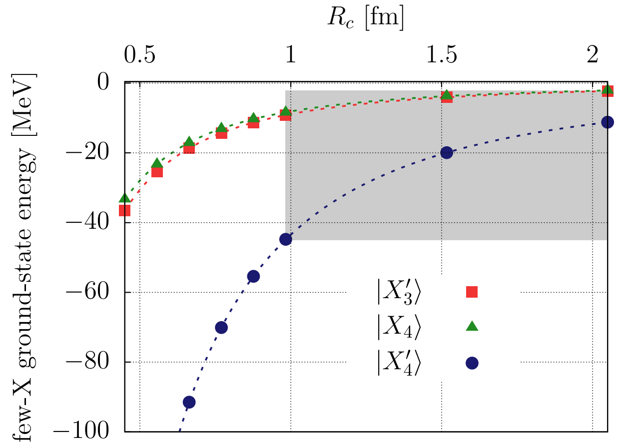

FIG. 2. Cutoff-radius dependence of the ground-state binding energies of few-X systems. With a resonant meson-antimeson interaction in the X channel and the $J^{PC}=2^{++}$ partner channel, $3X'$—a 6-meson state—(red square, dotted) and $4X'$—an eight-meson state—(blue square, dotted) clusters are bound. Solely with a resonant X-channel interaction, only the $4X$ (blue circle, dashed) is bound. The binding energies are proportional to $1/R_c^2$ (dashed/dotted lines) and indicate a Thomas collapse of the systems. The ensuing counterterm(s) are expected to vanish within the gray shaded area, while the total R_c range spans from the typical hadron size up to a scale set by the expected charged components of the X.

Another effect of the reduction of resonantly interacting pairs found here is the cutoff-independent ratio between $4X$ and $3X$ energies, $B_4/B_3\sim 4.9$. Compared with the ratio found in [46,47], $B_4/B_3\sim 4.6$, we conclude that reducing the number of interacting pairs widens the gap between the energy of N- and $(N+1)$-boson systems. However, a single counterterm should still suffice to renormalize both systems. Interpreted more generally, this study hints toward new universal systems in which part of the resonant interactions are amplified by the presence of more two-body channels or totally removed by symmetry effects. Consequences of deviations from universality and the effect of multiple open two-body channels on universal ratios are problems beyond the scope of this work.

Finally, we revisit the X_2 and assess the conditions for its binding. The heavy-quark content of the X_2 is $cc\bar{c}\bar{c}$, a double charm-anticharm content which was first observed when detecting Ξ_{cc}^{++} [48] and more recently the fully charm tetraquark [49]. These measurements indicate a possible X_2 discovery in the near future. In this regard, it is interesting to notice that the $D^0\bar{D}^0$ interaction has been theorized to be attractive and strong enough as to even support a bound state [31,50,51]. As this interaction is not connected to the V_X potential via HQSS, its strength is unknown. Its structure, however, can be included it our framework by refining (13) as follows:

$$\tilde{V}_X^{\mathrm{HQSS}}=V_X^{\mathrm{HQSS}}+\lambda V|D^0\bar{D}^0\rangle\langle D^0\bar{D}^0|. \tag{19}$$

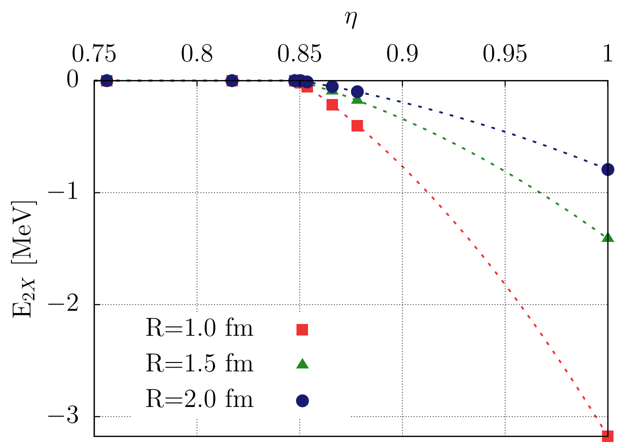

FIG. 3. Ground-state energy of the four-body $2X$ system as a function of the dimensionless coupling strength η [see (18)]. A bound state forms for $\eta \geq \eta^* = 0.876(1)$. If only the X-channel interaction [i.e., the $D^{*0}\bar{D}^0$ (1^{++}) system] is considered to be resonant then $\eta = 0.75$, which is insufficient for the formation of a $2X$ tetramer. If the $D^{*0}\bar{D}^{*0}$ (2^{++}) channel is also considered [see (20)], then $\eta = 0.854$, pretty close to η^* Finally, for an attractive $D^0\bar{D}^0$ (0^{++}) interaction with at least 17.4(8)% of the X channel strength, η surpasses the critical value and the $2X$ binds. The dependence of the bound-state energy on η fits a parabola for all considered cutoffs (dotted lines).

Here, λ parametrizes the relative strength of the interaction in the $D^0\bar{D}^0$ (0^{++}) channel with respect to the X. The diagonalization of this modified potential [see (10)] is

$$\sum_{ij} \tilde{V}_X^{\mathrm{HQSS}}(\boldsymbol{r};R_c)|X_2''\rangle = \left(2 + \frac{\lambda}{2} + \sqrt{2 + \frac{\lambda^2}{4}}\right)\bar{V}|X_2''\rangle = 4\eta(\lambda)\bar{V}|X_2''\rangle. \tag{20}$$

Compared with (18), the strength of the potential η is now a function of the additional attraction parameter λ. We find numerically that if $\eta \geq \eta^* = 0.876(1)$ the four-body $(2X)$ system binds (see Fig. 3). This critical strength is approximately constant for all considered cutoffs $R_c = 1.0, 1.5, 2.0$ fm. An analysis and explanation of this phenomenon remain beyond this paper. The condition $\eta \geq \eta^*$ is equivalent to $\lambda \geq \lambda^* = 0.174(8)$, i.e., the $2X$ will bind if an additional $(>20\%)$ $D^0\bar{D}^0$ attraction is provided. As bound states of the $D^0\bar{D}^0$ system are a conceivable scenario [31,50–53], the enhanced attraction and the ensuing bound $2X$ tetramer are plausible.

In summary, we have shown how the substructure of a unitary dimer—the X—affects the spectrum of its cluster states. This spectrum differs from the one predicted for pointlike bosons in the unitary limit [47] in an intriguing aspect. Namely, under certain assumptions about the meson-antimeson interaction, the X cluster states realize a novel generalization of Borromean/Brunnian systems. Regardless of the enormous practical difficulties which hamper an experimental (or numerical, in the lattice) verification of our conjectures (double charm-anticharm production has only been recently achieved [48,49]), we deem the exposition of the mechanism which "delays" the formation of bound structures—the onset of binding with $4X$ and $3X$, but not necessarily with $2X$ under the assumptions we made—as a noteworthy result of the above. Yet, in this later case, the $2X$ tetramer will bind provided we include a weakly attractive $D^0\bar{D}^0$ interaction in our calculations of about 20% the strength of the X-channel potential. In view of previous conjectures about a possible $D^0\bar{D}^0$ molecule [31,50,51], this condition might very well be met in the real world. If this is to be the case, the $2X$ tetramer will be experimentally accessible in the near future.

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[1] E. Braaten and H. W. Hammer, Phys. Rep. **428**, 259 (2006).
[2] V. Efimov, Phys. Lett. **33B**, 563 (1970).
[3] T. Kraemer, M. Mark, P. Waldburger, J. G. Danzl, C. Chin, B. Engeser, A. D. Lange, K. Pilch, A. Jaakkola, H.-C. Nägerl *et al.*, Nature (London) **440**, 315 (2006).
[4] K. Helfrich, H. W. Hammer, and D. S. Petrov, Phys. Rev. A **81**, 042715 (2010).
[5] Y. Castin, C. Mora, and L. Pricoupenko, Phys. Rev. Lett. **105**, 223201 (2010).
[6] B. Bazak and D. S. Petrov, Phys. Rev. Lett. **118**, 083002 (2017).

[7] P. F. Bedaque, H. W. Hammer, and U. van Kolck, Phys. Rev. Lett. **82**, 463 (1999).
[8] P. F. Bedaque, H. W. Hammer, and U. van Kolck, Nucl. Phys. **A676**, 357 (2000).
[9] S. König, H. W. Grießhammer, H. W. Hammer, and U. van Kolck, Phys. Rev. Lett. **118**, 202501 (2017).
[10] D. V. Federov, A. S. Jensen, and K. Riisager, Phys. Rev. Lett. **73**, 2817 (1994).
[11] W. Horiuchi and Y. Suzuki, Phys. Rev. C **74**, 034311 (2006).
[12] D. L. Canham and H. W. Hammer, Eur. Phys. J. A **37**, 367 (2008).
[13] B. Acharya, C. Ji, and D. R. Phillips, Phys. Lett. B **723**, 196 (2013).
[14] C. Ji, C. Elster, and D. R. Phillips, Phys. Rev. C **90**, 044004 (2014).
[15] H. W. Hammer and R. Higa, Eur. Phys. J. A **37**, 193 (2008).
[16] R. Higa, H. W. Hammer, and U. van Kolck, Nucl. Phys. **A809**, 171 (2008).
[17] J. Carlson, S. Gandolfi, U. van Kolck, and S. A. Vitiello, Phys. Rev. Lett. **119**, 223002 (2017).
[18] S. K. Choi *et al.* (Belle Collaboration), Phys. Rev. Lett. **91**, 262001 (2003).
[19] N. A. Tornqvist, arXiv:hep-ph/0308277.
[20] M. Voloshin, Phys. Lett. B **579**, 316 (2004).
[21] E. Braaten and M. Kusunoki, Phys. Rev. D **69**, 074005 (2004).
[22] E. S. Swanson, Phys. Lett. B **588**, 189 (2004).
[23] D. Gamermann and E. Oset, Phys. Rev. D **80**, 014003 (2009).
[24] D. Gamermann, J. Nieves, E. Oset, and E. Ruiz Arriola, Phys. Rev. D **81**, 014029 (2010).
[25] I. W. Lee, A. Faessler, T. Gutsche, and V. E. Lyubovitskij, Phys. Rev. D **80**, 094005 (2009).
[26] V. Baru, A. Filin, C. Hanhart, Y. Kalashnikova, A. Kudryavtsev, and A. Nefediev, Phys. Rev. D **84**, 074029 (2011).
[27] D. L. Canham, H. W. Hammer, and R. P. Springer, Phys. Rev. D **80**, 014009 (2009).
[28] N. Isgur and M. B. Wise, Phys. Lett. B **237**, 527 (1990).
[29] N. Isgur and M. B. Wise, Phys. Lett. B **232**, 113 (1989).
[30] M. P. Valderrama, Phys. Rev. D **85**, 114037 (2012).
[31] J. Nieves and M. P. Valderrama, Phys. Rev. D **86**, 056004 (2012).
[32] V. Baru, E. Epelbaum, A. A. Filin, C. Hanhart, U.-G. Meißner, and A. V. Nefediev, Phys. Lett. B **763**, 20 (2016).
[33] E. Cincioglu, J. Nieves, A. Ozpineci, and A. U. Yilmazer, Eur. Phys. J. C **76**, 576 (2016).
[34] M. P. Valderrama, Phys. Rev. D **98**, 034017 (2018).
[35] J. A. Tjon, Phys. Lett. **56B**, 217 (1975).
[36] L. H. Thomas, Phys. Rev. **47**, 903 (1935).
[37] M. Albaladejo, F. K. Guo, C. Hidalgo-Duque, J. Nieves, and M. P. Valderrama, Eur. Phys. J. C **75**, 547 (2015).
[38] N. A. Baas, D. V. Fedorov, A. S. Jensen, K. Riisager, A. G. Volosniev, and N. T. Zinner, Phys. At. Nucl. **77**, 336 (2014).
[39] H. Brunn, Sitz. Bayer. Akad. Wiss. Math-Phys. Klasse **22**, 77 (1892).
[40] P. Zyla *et al.* (Particle Data Group), Prog. Theor. Exp. Phys. (**2020**), 083C01.
[41] S. Fleming, M. Kusunoki, T. Mehen, and U. van Kolck, Phys. Rev. D **76**, 034006 (2007).
[42] C. Hidalgo-Duque, J. Nieves, and M. P. Valderrama, Phys. Rev. D **87**, 076006 (2013).
[43] Y. Suzuki and K. Varga, *Stochastic Variational Approach to Quantum-Mechanical Few-Body Problems*, Lecture Notes in Physics Monographs (Springer, Berlin, 1998).
[44] J. von Stecher, Phys. Rev. Lett. **107**, 200402 (2011).
[45] B. Bazak, M. Eliyahu, and U. van Kolck, Phys. Rev. A **94**, 052502 (2016).
[46] A. Deltuva, Phys. Rev. A **82**, 040701 (2010).
[47] J. Carlson, S. Gandolfi, U. van Kolck, and S. Vitiello, Phys. Rev. Lett. **119**, 223002 (2017).
[48] R. Aaij *et al.* (LHCb Collaboration), Phys. Rev. Lett. **119**, 112001 (2017).
[49] R. Aaij *et al.* (LHCb Collaboration), Sci. Bull. **65**, 1983 (2020).
[50] Y.-J. Zhang, H.-C. Chiang, P.-N. Shen, and B.-S. Zou, Phys. Rev. D **74**, 014013 (2006).
[51] D. Gamermann, E. Oset, D. Strottman, and M. Vicente Vacas, Phys. Rev. D **76**, 074016 (2007).
[52] S. Prelovsek, S. Collins, D. Mohler, M. Padmanath, and S. Piemonte, arXiv:2011.02542.
[53] X.-K. Dong, F.-K. Guo, and B.-S. Zou, arXiv:2101.01021.
[54] http://esnt.cea.fr.