New approach to the study of nonrelativistic bosonic string in flat spacetime

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A new approach to the study of nonrelativistic bosonic string in flat spacetime is introduced, based on a holistic Hamiltonian analysis of the minimal action for the string. This leads to a structurally new form of the action which is, however, equivalent to the known results since, under appropriate limits, it interpolates between the minimal action (Nambu-Goto type) where the string metric is taken to be that induced by the embedding and the Polyakov type of action where the world sheet metric components are independent fields. The equivalence among different actions is established by a detailed study of symmetries using constraint analysis. Various vexing issues in the existing literature are clarified. The interpolating action mooted here is shown to reveal the geometry of the string and may be useful in analyzing nonrelativistic string coupled with curved background.

I. INTRODUCTION

String theory has been developed as an approach towards quantum gravity [1]. Though many interesting results have been deduced (including the theory of Einstein's gravity itself), there are many difficulties also [2]. It is not our purpose to go in to the details of the issues. We only mention that string moving in a nonrelativistic background is an interesting subject on its own that has many welcome features and low energy stringy phenomena can be investigated with more confidence. For instance, such field theories have been proved to be unitary and ultraviolet free [3]. Nonrelativistic string theories (NRST) are useful in the study of nonrelativistic holography, which have found applications in the strongly correlated systems in condensed matter physics [4]. The literature of NRST is quite rich and expanding [3,5]. However, for obvious reasons, there are many issues left in the studies of nonrelativistic strings, some of which will be discussed here.

The feasibility of Galilean invariant closed string was first demonstrated in [6]. They considered an open string without a brane in the presence of the $NS - NS$ two-form field. In the limit when the background field goes critical, under a very low energy excitation limit, a nonrelativistic closed string action emerges. Note that in the limiting procedure the interaction field plays a crucial role. Also notable is the occurrence of the additional fields which are very difficult to explain. The action of the interacting string

obtained in [6] involves the metric. However, the metric is considered to be induced by the embedding. This particular type of action was originally proposed by Nambu and Goto for relativistic string. This action is the minimal action containing just the number of fields equal to the number of coordinates. Nonrelativistic strings with this type of action will be called the Nambu-Goto (NG) (type) nonrelativistic string. However, there is a caveat. The action contains two extra fields other than the coordinates. This discrepancy is of fundamental significance. It expands the phase space. *A priori*, it is unclear that these extra degrees of freedom do not hamper the consistency of the string theory. Such a formulation is therefore not satisfactory [6]. One asks what relation these new fields have with the string geometry? Note also that in later works the issue is rarely discussed.

The nonrelativistic string is, perhaps, technically more difficult than the relativistic string. A string necessarily is a relativistic structure. So the world sheet metric is Lorentzian. The bulk transverse to the world sheet has Galilean geometry. The world sheet is not just any arbitrary pullback from the spacetime manifold as assumed in the existing literature. The situation is exactly comparable with what happens in GR during Hamitonoian formuulation, albeit in the reverse order. For nonrelativistic string, Polyakov (type) theory was mooted in [7] where the two-dimensional nonlinear sigma model in the presence of a Kalb-Ramond two-form field and a dilaton is extended by any pullback from the background. This is bound to be entangled in contradictions. For instance, the number of degrees of freedom with that of [6] do not match.[1]

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[1]This is because of gauge fixing.

In this paper we have provided a new approach to the classical nonrelativistic string evolving freely. It consists in viewing the string action as a generalization of the covariant particle model. We assume that the NG form of nonrelativistic action will be obtained as the $c \to \infty$ limit of the relativistic NG string. Note that the string world sheet is necessarily relativistic. So the limit must be judiciously taken. From this nonrelativistic limit the corresponding string action is obtained. No *ad hoc* variables are introduced. In a specific gauge the standard action for a stretched string vibrating transversally is reproduced with the correct tension and velocity. The next step is a detailed constraint analysis of the action which leads to the first order form of the Hamiltonian. Reducing the momenta in terms of the fields and back substitution gives us an equivalent action.[2] It is called the interpolating action because it can be cast in the Polyakov form. It may be recalled that a similar approach for relativistic strings was proposed earlier [8].

Before ending the introductory section it will be appropriate to give an account of the organization of the paper. In Sec. II we have derived the nonrelativistic action of the string in Nambu-Goto form from its relativistic counterpart. Several authors discussed this model [5,6], where interaction appears to play a role. We followed the instance of the nonrelativstic particle [9] and took a simple $c \to \infty$ limit in a way that exploited the structure of the system. In Sec. III a detailed canonical analysis of the model is provided. Symmetries in the canonical level are retained by working in the gauge independent approach. Particularly remarkable is the identification of the gauge generator with the diffeomorphisms of the string world sheet. The dynamics in phase space is then analyzed by fixing the gauge. In Sec. IV, the interpolating action for the free nonrelativistic string is obtained. We have demonstrated that this action may be reduced to the previous (Nambu-Goto) form. More important is the evolution of the Polyakov type action from the interpolating action *including the two extra fields*. This section also provides a canonical analysis of the interpolating action. The connections between the Hamiltonian analysis of different actions have been analyzed to show the equivalence of the new action with the Nambu-Goto action from the canonical point of view. In the Sec. V the geometrical connection was pushed further. Finally, we conclude in Sec. VI.

II. THE NAMBU-GOTO ACTION FOR THE BOSONIC STRING

In the Introduction, we have indicated that the nonrelativistic string was obtained in [6] by considering a relativistic Nambu-Gote bosonic string action coupled to a

background $NS - NS$ two-form field B in a certain limit, where the scale of internal energy of the string is much less than the energy scale of motion about the direction of B. This enables an expansion in a power series and we can derive an action to the first order which is Galilean invariant. Again, the Polyakov type action used in [7] depends crucially on the background gauge field. In this paper we will consider a free string. Just as the relativistic point particle in the $c \to \infty$ limit goes to the nonrelativistic particle, the NG action of nonrelativistic string may be deduced from the relativistic string in the same limit, although there are certain technicalities.

Unlike the particle which is represented by a point, the string is an one-dimensional object which is described by a parameter σ. So during its evolution it traces a two-dimensional world sheet. This surface is mapped by two coordinates, τ and σ, where τ is timelike and σ is spacelike. The world sheet is embedded in a background spacetime. If the background is Poincare symmetric we obtain a flat relativistic string. On the other hand, a flat nonrelativisitc (NR) string is defined if the background symmetry is Galilean.

The relativistic Nembu-Goto action of the bosonic string is given by

$$S_{\text{NG}} = -N \int d\sigma d\tau \sqrt{-\det h_{\alpha\beta}} \qquad (1)$$

where N is a dimensionful constant. The metric $h_{\alpha\beta}$ is induced by the target space and given by

$$h_{\alpha\beta} = \eta_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu. \qquad (2)$$

Here $X^\mu = X^\mu(\tau, \sigma)$ corresponds to the coordinates of a point on the string with τ labeling timelike and σ labeling a spacelike direction on the world sheet[3] while μ represents the coordinates of the background spacetime in which the world sheet traced by the string is embedded. So, in this formalism the metrics are not independent fields. Note that $\eta_{\mu\nu} = \text{diag} - 1, +1, +1\ldots$ is the Lorentzian metric in the target space. Now, expanding the determinant we get

$$S_{\text{NG}} = -N \int d\sigma d\tau$$
$$\times \left[\left(\frac{\partial X^\mu}{\partial \tau} \frac{\partial X_\mu}{\partial \sigma} \right)^2 - \left(\frac{\partial X^\mu}{\partial \tau} \frac{\partial X_\mu}{\partial \tau} \right) \left(\frac{\partial X^\mu}{\partial \sigma} \frac{\partial X_\mu}{\partial \sigma} \right) \right]^{\frac{1}{2}}. \qquad (3)$$

If we consider the low energy phenomenology of the relativistic string, then the effects in the target space are nonrelativistic. Let the dimension of the embedding space be $(D + 2)$. At a given time the string intersects the embedding space along a line. We take this line as the X^1 coordinate line. Then X^0 and X^1 are longitudinal to

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[2]Similar technique has been employed in [6] but in a different context.

[3]Collectively, they will be represented by σ^α, $\alpha = 1, 2$.

the string and the rest is transverse. Note that we could take any of $X^2, ..., X^D$ in place of X^1. So no particular gauge choice is associated with this prescription.

There are different methods of obtaining the NR approximation of (3) which are variants of the Inonu-Wigner contraction method. Here, one or more of the coordinates X^μ are scaled and finally a limit of the scaling parameter is taken. While this may be algebraically tenable [5], it is not physically motivated. Indeed the most natural prescription would be to reintroduce "c" in the timelike variable as $X^0 = ct$ and then take the $c \to \infty$ limit. This is adopted here.

Simplification of (3) using $X^0 = ct$ gives

$$S_{NG} = -N \int d\sigma d\tau \left[c^2(\dot{t}X'^1 - \dot{X}^1 t')^2 + \sum_k c^2(\dot{t}X'^k - \dot{X}^k t')^2 \right.$$
$$- \sum_k (\dot{X}^1 X'^k - \dot{X}^k X'^1)^2$$
$$\left. + \sum_{k,l} (\dot{X}^k X'^k \dot{X}^l X'^l - \dot{X}^k \dot{X}^k X'^l X'^l) \right]^{\frac{1}{2}} \quad (4)$$

where a dot over a symbol denotes derivative with respect to τ while a prime as a superscript implies differentiation with respect to σ. Also c is made explicit.

So far our result is relativistic. Now, remember that the X^1 axis is longitudinal to the string and the rest ($k = 2, 3, ..., D$) are in the transverse section. So $\dot{X}^1 \ll c$ and $dX^k \ll dX^1$ since in the low energy scenario the slope of the transverse vibration of the string is very small. Thus the nonrelativistic limit of the action (4) is

$$S_{NG} = -N \int d\sigma d\tau (c(\dot{t}X'^1 - \dot{X}^1 t')) \left[1 + \frac{\sum_k c^2(\dot{t}X'^k - \dot{X}^k t')^2}{c^2(\dot{t}X'^1 - \dot{X}^1 t')^2} \right.$$
$$\left. - \frac{\sum_k (\dot{X}^1 X'^k - \dot{X}^k X'^1)^2}{c^2(\dot{t}X'^1 - \dot{X}^1 t')^2} \right]^{\frac{1}{2}} \quad (5)$$

where we have neglected the last term of (4), as it is of higher order of smallness. Now $ct = X^0$ and taking the leading term of the small quantities (also making sum over k implicit), we get

$$\mathcal{L}_{NG} = -N \left[\frac{(\dot{X}^0 X'^k - \dot{X}^k X'^0)^2}{2(\dot{X}^0 X'^1 - \dot{X}^1 X'^0)} - \frac{(\dot{X}^1 X'^k - \dot{X}^k X'^1)^2}{2(\dot{X}^0 X'^1 - \dot{X}^1 X'^0)} \right]. \quad (6)$$

Note that we have dropped the first term within the square brackets as it is a total boundary,

$$(\dot{X}^0 X'^1 - \dot{X}^1 X'^0) = \frac{\partial}{\partial \tau}(X^0 X'^1) - \frac{\partial}{\partial \sigma}(X^0 \dot{X}^1). \quad (7)$$

Equation (6) is the NR Nambu-Goto form of the Lagrangian for a bosonic string. The derivation is based

on the usual $c \to \infty$ limit along with certain physical inputs. This result was obtained in [5], using a variant of the Inonu-Wigner contraction referred to earlier which somewhat obscures the physical origin inherent in our derivation. Another aspect of the construction is the impact of the relativistic nature of the string which enforces that the metric in the 0–1 plane is Lorentzian, even in our example of nonrelativistic phenomena. We denote this metric by $\eta_{\mu\nu}; \mu, \nu = 0, 1 = \text{diag}(1, -1)$. We can now rewrite the Lagrangian (6) in a less clumsy form,

$$\mathcal{L}_{NG} = -N(2\epsilon_{\mu\nu}\dot{X}^\mu X'^\nu)^{-1}(\dot{X}^\mu X^{k\prime} - \dot{X}^k X^{\mu\prime})^2, \quad (8)$$

by using the covariant notation.

One can wonder in what sense the action

$$S_{NG} = \int d\sigma d\tau \mathcal{L}_{NG} \quad (9)$$

with $\mathcal{L}_{NG}$ given by (8), is Galilean invariant. That the string is essentially relativistic makes the question nontrivial. Let us now discuss the issue. We have already mentioned that we are considering the string to be in motion in a nonrelativistic background. So one would expect that physics in the background remains unaltered under the Galilean transformations. But not all elements of the group can be included here. The values of τ and σ, which specify a point on the string world sheet, should not change.[4] Consider the Galilean transformations in the usual way,

$$\delta X^0 = -\epsilon,$$
$$\delta X^1 = \epsilon^1 + \omega^1{}_j X^j - v^1 X^0,$$
$$\delta X^k = \epsilon^k + \omega^k{}_l X^l - v^k X^0. \quad (10)$$

We now calculate the change of the Lagrangian (8) under (10). The result is

$$\delta \mathcal{L}_{NG} = N\omega^{1i}(\dot{X}^0 X'^i - X'^0 \dot{X}^i)$$
$$\times [(\epsilon_{\mu\nu}\dot{X}^\mu X'^\nu)^{-2}(\dot{X}^\mu X^{k\prime} - \dot{X}^k X^{\mu\prime})^2]. \quad (11)$$

Now the X^1 axis is assumed to be lying along the $\tau = $ constant direction. So a nonzero ω^{1i} would mean the change of the world sheet parameters, which is contrary to the concept of the global coordinate transformations in the target space. Hence $\omega^{1i} = 0$. So the Galilean transformations under which the theory (8) is symmetric are (10), supplemented by this condition. For ready reference we write the modified symmetry transformations as

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[4]Just as the value of proper time locating a particle on its world line is insensitive to the Galilean transformations of the background.

$$\delta X^0 = -\epsilon,$$

$$\delta X^1 = \epsilon^1 - v^1 X^0,$$

$$\delta X^k = \epsilon^k + \omega^k{}_l X^l - v^k X^0; k, l > 1. \qquad (12)$$

Another demand we would like to place on (8) is that it should reduce to the Lagrangian of the classical vibrating string in the appropriate limit. Let us take a string stretched between $x = 0$ to $x = a$ along the x axis, vibrating transversely along the y axis. From elementary analysis we get the Lagrangian as

$$L = \left[\frac{1}{2}\mu^0\left(\frac{\partial y}{\partial t}\right)^2 - \frac{1}{2}T^0\left(\frac{\partial y}{\partial x}\right)^2\right] \qquad (13)$$

where μ^0 is the linear mass density and T^0 is the tension of the string. Now let us see in what way we may reproduce this result from (8). Putting $X^0 = c\tau = ct$ and $X^1 = \sigma$ we get[5]

$$\mathcal{L}_{\mathrm{NG}} = \left[\frac{1}{2}\frac{N}{c}\left(\frac{\partial X^k}{\partial t}\right)^2 - \frac{1}{2}Nc\left(\frac{\partial X^k}{\partial X^1}\right)^2\right] \qquad (14)$$

These are k copies of the transversally vibrating classical string if we identify $\frac{N}{c} = \mu^0$ and $Nc = T^0$. Thus the constant N in (8) is related to the tension of the string. Expectedly, the velocity is given by $c^2 = \frac{T^0}{\mu^0}$, as happens classically.

In the work of [5] the NR limit of the relativistic Nambu-Goto action is taken in two distinct ways, motivated by group contraction techniques, so that one either gets a stringy (vibrating) NR string or a particlelike (nonvibrating) string. In our case we generalize the limiting prescription of the relativistic particle to that of the relativistic string with appropriate technicalities that account for the characteristics of a string. Thus ours is a unique result which is similar to the stringy limit of [5]. The fact that it reduces to the usual classical string (14) is a consequence of a special choice of coordinates that breaks the reparametrization symmetry or, equivalently, the gauge symmetry, since the two here are identifiable, as shown in the next section.

III. CANONICAL ANALYSIS OF THE MODEL

We have now established our Lagrangian. The next step is a Hamiltonian analysis of the model. Now, being a reparametrization invariant theory, it is already covariant [10]. The model is thus an interesting example of a constrained system, and because of its NR nature, more involved than its relativistic counterpart. As we have already mentioned, these studies are entirely new as the

Hamiltonian analysis available in the literature [11] does not provide a faithful Dirac treatment of the model.

A constrained system with first class constraints necessarily possesses gauge degrees of freedom [12]. General symmetries of such systems can be derived without fixing the gauges. In fact we will see that this gauge independent approach will lead to the derivation of a new action. The importance of canonical analysis can thus be hardly overestimated. On the contrary, this aspect has been less emphasized in the literature. We will therefore try to give a holistic account of the topic. For clarity of presentation we divide our results in a number of subsections.

A. Phase space structure

Here the fields are $X^0(\tau, \sigma)$, $X^1(\tau, \sigma)$, $X^k(\tau, \sigma)$ (where $k = 2, 3, \ldots, D$).

The canonical momenta corresponding to X^0 is[6]

$$\Pi_0 = \frac{\partial \mathcal{L}}{\partial \dot{X}^0}$$

$$= \left[X'^k(\epsilon_{\mu\nu}\dot{X}^\mu X'^\nu)^{-1}(\dot{X}^0 X'^k - \dot{X}^k X'^0)\right.$$

$$\left. - \frac{X'^1}{2}(\epsilon_{\mu\nu}\dot{X}^\mu X'^\nu)^{-2}(\dot{X}^\mu X'^{k'} - \dot{X}^k X'^{\mu'})^2\right]. \qquad (15)$$

Similarly that for X^1 is

$$\Pi_1 = \frac{\partial \mathcal{L}}{\partial \dot{X}^1}$$

$$= \left[-X'^k(\epsilon_{\mu\nu}\dot{X}^\mu X'^\nu)^{-1}(\dot{X}^1 X'^k - \dot{X}^k X'^1)\right.$$

$$\left. + \frac{X'^0}{2}(\epsilon_{\mu\nu}\dot{X}^\mu X'^\nu)^{-2}(\dot{X}^\mu X'^{k'} - \dot{X}^k X'^{\mu'})^2\right]. \qquad (16)$$

Also for X_k,

$$\Pi^k = \frac{\partial \mathcal{L}}{\partial \dot{X}^k} = (-\epsilon_{\mu\nu}\dot{X}^\mu X'^\nu)^{-1}[X'^\mu(\dot{X}_\mu X'^k - \dot{X}^k X'_\mu)]. \qquad (17)$$

Using these in the definition of the canonical Hamiltonian we get

$$H_c(\tau) = \int d\sigma(\Pi^\mu \dot{X}_\mu - \mathcal{L}).$$

A straightforward calculation gives $H_c(\tau) = 0$. This is a characteristic of the already covariant theories. From an inspection of the expressions of the momenta, the following primary constraints emerge:

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[5]That such a choice is possible is justified in Sec. III.4 through a detailed Hamiltonian analysis.

[6]We rename the NG Lagrangian as $\mathcal{L}_{\mathrm{NG}} \to \mathcal{L}$ and set $N = 1$.

$$\Omega_1 = \Pi^\mu X'_\mu \approx 0,$$
$$2\Omega_2 = \mathbf{\Pi}^2 + \mathbf{X}'^2 - 2\sigma_{\alpha\beta}\Pi^\alpha X'^\beta \approx 0, \qquad (18)$$

where $\sigma_{\alpha\beta}$ is a second Pauli matrix. The fundamental Poisson brackets of the theory are given by

$$\{X^\mu(\tau,\sigma), \Pi_\nu(\tau,\sigma')\} = \eta^\mu_\nu \delta(\sigma - \sigma') \qquad (19)$$

while the others vanish. Using these Poisson brackets it is easy to work out the algebra of the constraints,

$$\{\Omega_1(\sigma), \Omega_2(\sigma')\} = (\Omega_2(\sigma) + \Omega_2(\sigma'))\partial_\sigma \delta(\sigma - \sigma'),$$
$$\{\Omega_1(\sigma), \Omega_1(\sigma')\} = (\Omega_1(\sigma) + \Omega_1(\sigma'))\partial_\sigma \delta(\sigma - \sigma'),$$
$$\{\Omega_2(\sigma), \Omega_2(\sigma')\} = (\Omega_1(\sigma) + \Omega_1(\sigma'))\partial_\sigma \delta(\sigma - \sigma'). \qquad (20)$$

Clearly, the Poisson brackets between the constraints (18) are weakly involutive. So the set (18) is first class.

The total Hamiltonian is

$$H_T = \int d\sigma(\rho\Omega_1 + \lambda\Omega_2) \qquad (21)$$

where ρ and λ are Lagrange multipliers and Ω_1 and Ω_2 are shown in (18).

Conserving the primary constraints, no new secondary constraints emerge since the constraint algebra simply closes. The total set of constraints of the NG theory is then given by the first class system (18).

The Nambu-Goto string is a constrained system. So its description is redundant. If it is embedded in a $D + 1$ dimensional spacetime, the number of fields in the configuration space is $D + 1$. The corresponding number of variables in the phase space is $2(D + 1)$. Then the number of degrees of freedom in the configuration space is given by

$$n = \frac{1}{2}(2(D+1) - 4) = D - 1 \qquad (22)$$

This result is consistent with our understanding about the nonrelativistic excitation taking place in the transverse direction and we identify the $(D - 1)$ variables X^k as the physical set. Also, we understand from another angle why $\omega^{1i} = 0$ should hold [recall the second equation of (12)].

B. Studies of local symmetries

The string world sheet is a two-dimensional manifold which is charted by the parameters τ and σ. The physical theory should not depend on any particular parametrization. In other words we should have invariance under reparametrization (a mapping of the manifold on itself i.e. a diffeomorphism)

$$\tau' = \tau'(\tau, \sigma),$$
$$\sigma' = \sigma'(\tau, \sigma), \qquad (23)$$

which becomes for infinitesimal diffeomorphism

$$\tau' = \tau + \delta\tau,$$
$$\sigma' = \sigma + \delta\sigma. \qquad (24)$$

The increments $\delta\sigma$ and $\delta\tau$ both are functions of σ and τ. In the Lagrangian level the diffieomorphismn invariance is conceptually clear. The Lagrangian is a world sheet scalar. Its form variation under (24) is given by $\delta\mathcal{L} = \delta\sigma_a \frac{\partial\mathcal{L}}{\partial\sigma^a}$ where $\sigma^0 = \sigma$ and $\sigma^1 = \tau$.

The Jacobian of the transformation is $(1 + \partial_a\delta\sigma_a)$. Direct substitution in the action gives

$$\delta S = \int d\sigma d\tau \frac{\partial}{\partial\sigma_a}(\mathcal{L}\delta\sigma_a) = 0 \qquad (25)$$

since the variations vanish at the boundary. So the theory (8) is invariant under (24).

Looking from the Hamiltonian point of view such action level symmetries should correspond to the gauge symmetries of the model. The gauge redundancy accounts for the diffeomorphism invariances and vice versa. So one should be able to establish an exact mapping between gauge and reparametrization parameters. We will derive the explicit form of the mapping now.

C. Mapping between gauge and reparametrization symmetries

According to the Dirac conjecture all the first class constraints generate gauge transformations. But the gauge parameters associated with these transformations are not independent. It is known that the number of independent primary first class constraints equals the number of independent gauge parameters [13,14]. Since in the present example two primary first class constraints form the set of constraints, the gauge generator can be written down immediately,

$$G(\tau) = \int d\sigma(\omega_1(\sigma)\Omega_1 + \omega_2(\sigma)\Omega_2). \qquad (26)$$

The corresponding gauge variations are

$$\delta_G X^0 = [X^0, G]_{PB} = \omega_1(\sigma)X^{0\prime} - \omega_2 X^{1\prime},$$
$$\delta_G X^1 = [X^1, G]_{PB} = \omega_1(\sigma)X^{1\prime} - \omega_2(\sigma)X^{0\prime},$$
$$\delta_G X^k = [X^k, G]_{PB} = \omega_1(\sigma)X^{k\prime} + \omega_2(\sigma)\Pi^{k\prime} \qquad (27)$$

with the exact number of independent parameters.

Now, the variations due to diffeomorphism (24) are

$$\delta_D X^\mu = \xi_1 X^{\mu\prime} + \xi_2 \dot{X}^\mu \qquad (28)$$

where $\xi_1 = -\delta\sigma$ and $\xi_2 = -\delta\tau$ are defined in (24). We see "velocities" appearing in the expression of variations (28). To exhibit the one-to-one correspondence we have to substitute $\dot{X}^\mu$ in (28) from its equation of motion, $\dot{X}^\mu = [X^\mu, H_T]$, where H_T is the total Hamiltonian given by (21). This calculation gives us for $\mu = 0$

$$\dot{X}^0 = \rho X^{\prime 0} - \lambda X^{\prime 1}. \qquad (29)$$

Substituting this in (28) we get

$$\delta_D X^0 = (\xi_1 + \rho\xi_2)X^{\prime 0} - \xi_2\lambda X^{\prime 1}. \qquad (30)$$

Comparison with $\delta_G X^0$ gives us the desired mapping,

$$\omega_1 = (\xi_1 + \xi_2\rho),$$
$$\omega_2 = \xi_2\lambda. \qquad (31)$$

The same results will be obtained for any component of μ. Thus the mapping (31) exhibits the complete equivalence of the gauge symmetries with the diffeomorhism invariances of the model.

D. Gauge fixed analysis

We have seen that the analysis of the symmetries is best done in the gauge independent approach. However, for the study of dynamics in the phase space one must eradicate the gauge redundancy by a gauge choice. A gauge is a condition in phase space which makes a first class constraint second class, thereby reducing two degrees of freedom. While this is a necessary condition, it is not sufficient. In order to specify the physical set, one has to properly obtain the canonical variables. These variables can always be obtained (this is the content of the Maskawa-Nakajima theorem [15]) such that the Dirac brackets among these variables is the same as the Poisson brackets. Then one can proceed with the quantization by replacing these brackets by suitable commutators.

We assume the standard gauges,

$$\Omega_3 = X^1 - \sigma \approx 0, \qquad \Omega_4 = X^0 + c\tau \approx 0. \qquad (32)$$

Note that if we include the gauge conditions as constraints with the existing set $\{\Omega_1 \approx 0, \Omega_1 \approx 0\}$, then all of the constraints become second class. The phase space can now be reduced by implementing these constraints strongly. The canonical structure is now optimum, described by the phase space variables $\{X^k, \Pi_k\}$, and the symplectic structure is given by the Dirac brackets [10,12]. The structure of the Dirac brackets between the coordinates is rich with physical significance and worthy to be studied carefully.

The Dirac brackets between two phase space variables is defined by [12]

$$[A(\sigma), B(\sigma')]_{DB}$$
$$= [A(\sigma), B(\sigma')]_{PB}$$
$$- \int d\sigma_1 d\sigma_2 [A(\sigma), \Omega_i(\sigma_1)]_{PB} \Delta^{ij}(\sigma_1, \sigma_2)[\Omega_j(\sigma_2), B(\sigma')]_{PB} \qquad (33)$$

where the matrix Δ^{ij} is the inverse of the matrix Δ_{ij} formed by the constraint algebra,

$$\Delta_{ij} = [\Omega_i, \Omega_j]_{PB}; i = 1, 2, 3, 4 \qquad (34)$$

which is necessarily nonsingular and admits an inverse. Using these definitions and the Poisson brackets between the second class constraints (18) and (32), we get

$$\Delta_{ij} = \begin{bmatrix} 0 & 0 & -\delta(\sigma-\sigma') & 0 \\ 0 & 0 & 0 & \delta(\sigma-\sigma') \\ \delta(\sigma-\sigma') & 0 & 0 & 0 \\ 0 & -\delta(\sigma-\sigma') & 0 & 0 \end{bmatrix}.$$

The inverse is easily found,

$$\Delta^{ij} = \begin{bmatrix} 0 & 0 & \delta(\sigma-\sigma') & 0 \\ 0 & 0 & 0 & -\delta(\sigma-\sigma') \\ -\delta(\sigma-\sigma') & 0 & 0 & 0 \\ 0 & \delta(\sigma-\sigma') & 0 & 0 \end{bmatrix}.$$

It is then straightforward to calculate the Dirac brackets between the previous canonically conjugate variables in the gauge independent analysis,

$$[X^0(\sigma,\tau), \Pi_0(\sigma',\tau)]_{DB} = 0,$$
$$[X^1(\sigma,\tau), \Pi_1(\sigma',\tau)]_{DB} = 0,$$
$$[X^i(\sigma,\tau), \Pi_j(\sigma',\tau)]_{DB} = \delta^i_j \delta(\sigma-\sigma'). \qquad (35)$$

We will provide a detailed derivation of the first equation. Starting from the definition of the Dirac brackets we get

$$[X^0(\sigma,\tau), \Pi_0(\sigma',\tau)]_{DB}$$
$$= [X^0(\sigma,\tau), \Pi_0(\sigma',\tau)]_{PB}$$
$$- \int d\sigma_1 d\sigma_2 [X^0(\sigma), \Omega_2(\sigma_1)]\Delta^{24}((\sigma_1,\sigma_2)[\Omega_4(\sigma_2), \Pi_0]$$
$$= \delta(\sigma-\sigma') - \int d\sigma_1 d\sigma_2 \delta(\sigma-\sigma_1)X^{1'}\delta(\sigma_1-\sigma_2)\delta(\sigma_2-\sigma')$$
$$= 0 \qquad (36)$$

where we have imposed the constraint Ω_3 strongly[7] so that $X^{1'} = 1$. Similarly we can derive the other two equations of the set (35). Incidentally the last relation of (35) is the only nonzero bracket.

Thus X^0, X^1 (and their conjugate momenta) are out of the dynamics and (X^i, Π^i), $i = 2, 3, \ldots$ are the canonical pairs. Also note that the total Hamiltonian vanishes when the constraints are strongly implemented. In this situation we have to identify a Hamiltonian in the reduced phase space which will generate the equations of motion with respect to the Dirac brackets.

Remember that canonically the Hamiltonian may be considered as the conjugate to the time parameter. So we identify the new Hamiltonian as

$$H = c \int d\sigma [\Pi^0]_g \qquad (37)$$

where the subscript g denotes that gauge fixed value. Using the gauge conditions (32) which now allow us to put $\dot{X}^0 = -c$ and $X^1 = \sigma$ we find by substitution from (15)

$$H = \frac{c}{2} \int d\sigma \left[(\partial_\sigma X^k))^2 + \frac{1}{c^2} (\partial_\tau X^k)^2 \right]. \qquad (38)$$

We see that the proposed Hamiltonian is positive definite. Remarkably, this Hamiltonian is a sum of harmonic oscillator terms. Use the definition of Π^k and the gauge fixing conditions to get

$$\Pi^k = \frac{1}{c} \partial_\tau X^k. \qquad (39)$$

Then from (38) we can write

$$H = \frac{c}{2} \int [(\partial_\sigma X^k)^2 + (\Pi^k)^2]. \qquad (40)$$

This is however the transversely vibrating string Hamiltonian. The corresponding Lagrangian, obtained by an inverse Legendre transformation, reproduces (14) with $N = 1$.

As stated earlier we have shown the passage to the usual classical string, clearly validating the special choice of the coordinates as a good choice of gauge. This demonstration is a nice application of our formalism.

IV. THE INTERPOLATING LAGRANGIAN

In this section the canonical analysis of the previous section will be used from an inverse approach to develop a new Lagrangian, which will be shown to have the remarkable property of interpolating between NG and Polyakov

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[7]This is permitted as all the Poisson brackets have already been evaluated.

Lagrangians. In the case of relativistic string one can easily deduce the NG string from the Polyakov string on shell, by substituting the metric from its equation of motion in the original Lagrangian under definite conditions. But for the nonrelativitic string, the Polyakov action requires to be supplemented by two world sheet fields, the origin of which is hard to trace [3]. Thus the equivalence of the two actions becomes problematic. In the following discussion we will see that the Hamiltonian analysis can be used to identify the source of the additional fields in the interpolating action which eventually permeates to the Polyakov form. Also, this action is remarkable due to its connection with the geometry and may be useful in coupling the string with a curved background.

The Lagrangian corresponding to the Hamiltonian H_T (21) is

$$\mathcal{L}_I = \Pi_\mu \dot{X}^\mu - \rho\Omega_1 - \lambda\Omega_2 \qquad (41)$$

where the multipliers ρ and λ are given the status of independent fields. The Lagrange equations corresponding to Π^0, Π^1, and Π^k are now respectively,

$$\dot{X}^0 - \rho X'^0 + \lambda X'^1 = 0,$$
$$\dot{X}^1 - \rho X'^1 + \lambda X'^0 = 0,$$
$$\dot{X}^k - \rho X'^k - \lambda\Pi^k = 0. \qquad (42)$$

Solving ρ and λ from the set (42), we get

$$\rho = \frac{\dot{X}^1 X'^1 - \dot{X}^0 X'^0}{(X'^1)^2 - X'^{02}},$$

$$\lambda = \frac{-\dot{X}^0 X'^1 + \dot{X}^1 X'^0}{(X'^1)^2 - (X'^0)^2}. \qquad (43)$$

It is possible to eliminate all the momenta from (41) using (42). Now, reinterpret ρ and λ as independent fields. Also, Eqs. (43) will now be promoted to Lagrangian constraints. Doing all these steps the following Lagrangian is obtained:

$$\mathcal{L}_I = \frac{1}{2\lambda} [(\dot{X}^k)^2 - 2\rho\dot{X}^k X'^k + (\rho^2 - \lambda^2)(X'^k)^2]$$

$$+ \beta\left(\rho - \frac{\dot{X}^1 X'^1 - \dot{X}^0 X'^0}{X'^{12} - X'^{02}}\right)$$

$$+ \alpha\left(\lambda + \frac{\dot{X}^0 X'^1 - \dot{X}^1 X'^0}{X'^{12} - X'^{02}}\right). \qquad (44)$$

This is the appearance of the interpolating Lagrangian announced earlier. We will presently show that we can derive both the Nambu-Goto and Polyakov forms of the nonrelativistic string action from (44). This is the reason for dubbing (44) as the interpolating Lagrangian.

The above Lagrangian is a new result that emerges on exploiting the constraint structure of the Nambu-Goto type

action. The equivalence of the two can thus be shown easily, as done below. But its equivalence with the Polyakov type form is quite nontrivial, which has also been demonstrated. This reveals that the interpolating action is not merely of academic interest since it is crucial in showing the equivalence of the Nambu-Goto and Polyakov type forms. Such a clear-cut and conceptually clean derivation is lacking. We may also recall the relativistic case where an interpolating action was presented to show a similar equivalence [8]. However, in that case, this could also be established directly by simply opening the square root of the Nambu-Goto action and it was not essential to go through this intermediate step of interpolating Lagrangian.

A. Passage to the Nambu-Goto type

The derivation of the Nambu-Goto action (4) from (44) is trivial. The multipliers simply enforce the solutions (43). Putting it back in (44) reproduces the expected result.

B. Passage to the Polyakov type form

This sounds really interesting for the Polyakov form brings in an independent metric on the world sheet but in the interpolating Lagrangian there is no explicit reference to such a metric. Also, the world sheet fields that are invoked in the Lagrangian look like a hurdle. The startling observation in the following analysis is the solution at one stroke, where aspects of Riemannian geometry of the world sheet converge with the canonical structure of the nonrelativistic string. From this confluence the Polyakov type action emerges including the extra fields.

Let us first observe that with the help of the fields ρ and λ we can construct a 2×2 matrix

$$h^{ij} = (-h)^{-\frac{1}{2}} \begin{pmatrix} \frac{1}{\lambda} & -\frac{\rho}{\lambda} \\ -\frac{\rho}{\lambda} & \frac{\rho^2 - \lambda^2}{\lambda} \end{pmatrix} \qquad (45)$$

where h is the determinant of the inverse matrix h_{ij}. The consistency of the construction can be verified by the computation of $\det h^{ij}$, which yields

$$\det h^{ij} = \frac{1}{h} \qquad (46)$$

showing that the matrix h_{ij} is a 2×2 real symmetric matrix. Also the inverse to h_{ij} is h^{ij}. Note that obeying the constraints, h may assume any nonzero value. We propose h_{ij} as the metric on the world sheet and write the interpolating Lagrangian, (44), in terms of the elements of h^{ij} as

$$\mathcal{L}_{\mathcal{I}} = \frac{1}{2}\sqrt{-h}\,h^{ij}\partial_i X^k \partial_j X^k + \mathcal{L}_e. \qquad (47)$$

The first part is formally the same as the relativistic bosonic string, though here only transverse degrees of freedom are dynamical. The second part is due to the constraints (43), enforced in the Lagrangian, again by multipliers,

$$\mathcal{L}_e = \beta\left(-\frac{h^{01}}{h^{00}} - \frac{\dot{X}^1 X'^1 - \dot{X}^0 X'^0}{(X'^1)^2 - X'^{02}}\right)$$
$$+ \alpha\left(\frac{1}{\sqrt{-h}h^{00}} + \frac{\dot{X}^0 X'^1 - \dot{X}^1 X'^0}{X'^{12} - X'^{02}}\right) \qquad (48)$$

Note that from the identification of the metric we can show that

$$\lambda = \frac{1}{\sqrt{-h}h^{00}}; \qquad \rho = -\frac{h^{01}}{h^{00}} \qquad (49)$$

which has been used in writing (48). The Lagrangian (47), with $\mathcal{L}_e$ given by (48), is the string action given in the Polyakov form. We see that the matrix h^{ij} now represents independent fields. Indeed, h^{ij} resembles the metric on the world sheet. We can compare the action with the corresponding relativistic action. The first term is of the same form but only transverse degrees of freedom are involved. This is consistent with the previous analysis given here. The fields α and β are the two extra fields included in the Polyakov form. For nonrelativistic string action considered in previous studies [3] such fields are included. However, their appearance was not explained. We have seen they follow from the canonical analysis. The issue will be considered in more detail in the following section.

C. The interpolating Lagrangian—canonical analysis

The last section has established that the interpolating action (44) is a versatile tool to study nonrelativistic strings. We have shown that under appropriate conditions the model interpolates between the Nambu-Goto and Polyakov forms of the string action. This equivalence will further be elucidated by the canonical analysis.

We have already shown the action level reduction of the interpolating action to the Nambu-Goto string. But that does not necessarily imply that the two theories have the same physical content. The canonical structure of the interpolating theory determines its energy spectrum. Also, the first class constraints determine the gauge symmetry of the models. In the following discussion a canonical analysis of the new action is provided which further elucidates its equivalence with the NG form.

The interpolating Lagrangian will be the starting point of the comparison. The momenta corresponding to the basic fields in (44) lead to one genuine momentum Π_k,

$$\Pi_k = \frac{\dot{X}^k}{\lambda} - \frac{\rho}{\lambda} X'^k \qquad (50)$$

while the rest are constraints, two of which are first class defined as

$$\pi_\rho \approx 0; \qquad \pi_\lambda \approx 0 \qquad (51)$$

while the remaining are second class,

$$\Sigma_1 = \pi_\alpha \approx 0; \qquad \Sigma_2 = \pi_\beta \approx 0,$$

$$\Sigma_3 = \Pi_0 - \frac{\alpha X'^1 + \beta X'^0}{(X'^{12} - X'^{02})} \approx 0,$$

$$\Sigma_4 = \Pi_1 + \frac{\alpha X'^0 + \beta X'^1}{(X'^{12} - X'^{02})} \approx 0. \qquad (52)$$

The canonical Hamiltonian is

$$H_c = \int d\sigma \left[\lambda \left(\frac{\Pi_k^2 + X'^{k2}}{2} - \alpha \right) + \rho(\Pi^k X^{k'} - \beta) \right]. \qquad (53)$$

The second class constraints are strongly implemented by using Dirac brackets instead of the Poisson. The total Hamiltonian is then defined as

$$H_T = H_c + \int d\sigma(\nu_1 \pi_\lambda + \nu_2 \pi_\rho). \qquad (54)$$

From conserving the first class constraints we get two more constraints. Note that now we have to use, instead of the Poisson brackets, the relevant Dirac brackets. However it is easy to see that the Dirac brackets and the Poisson brackets are identical for the variables that are involved in the iterative computation of constraints. The new constraints are

$$[\pi_\rho, H_T] \approx 0 \rightarrow \Pi^k X'^k - \beta = \Phi_1 \approx 0,$$

$$[\pi_\lambda, H_T] \approx 0 \rightarrow \frac{1}{2}(\Pi^{k2} + X'^{k2}) - \alpha = \Phi_2 \approx 0. \qquad (55)$$

The constraints are now all first class. Their algebra is strongly involutive, except for the pair (Φ_1, Φ_2) which satisfies an algebra identical to (20),

$$\{\Phi_1(\sigma), \Phi_2(\sigma')\} = (\Phi_2(\sigma) + \Phi_2(\sigma'))\partial_\sigma \delta(\sigma - \sigma'),$$

$$\{\Phi_1(\sigma), \Phi_1(\sigma')\} = (\Phi_1(\sigma) + \Phi_1(\sigma'))\partial_\sigma \delta(\sigma - \sigma'),$$

$$\{\Phi_2(\sigma), \Phi_2(\sigma')\} = (\Phi_1(\sigma) + \Phi_1(\sigma'))\partial_\sigma \delta(\sigma - \sigma'). \qquad (56)$$

Let us next perform a degree of freedom count. The total number of phase space degrees of freedom is $2(6 + k)$. There are four second class constraints and four first class constraints. Hence the number of independent phase space degrees of freedom is

$$n = 2(6 + k) - 4 - 2 \times 4 = 2k. \qquad (57)$$

Hence the independent number of configuration space degrees of freedom is k, which we take to be the X^k variables. This precisely matches with our earlier counting and identification.

Since the constraints in (52) are strongly implemented, it is possible to solve for α and β. We get

$$\alpha = \Pi^0 X'^1 + \Pi_1 X'^0,$$

$$\beta = -\Pi^0 X'^0 - \pi_1 X'^1. \qquad (58)$$

Substituting these in (55) we obtain

$$\Phi_1 = \Omega_1; \qquad \Phi_2 = \Omega_2 \qquad (59)$$

which are identical to the two first class constraints (18) of the original Nambu-Goto model.

This shows that the interpolating model is perfectly viable, allowing for a systematic analysis of constraints that is needed to show its equivalence with other types of actions. This Hamltonian analysis complements the Lagrangian formulation presented previously.

V. CONNECTION WITH GEOMETRY

In the above we have introduced a new action for the nonrelativistic bosonic string which has the merit of interpolating between the Nambu-Goto form on one side and the Polyakov form of the action on the other. Thus both types of actions can be related in one go. In the non-relativistic variety this task is not simple, as evidenced in the literature [3,5]. First of all, metric components in the transverse directions only appear in the Polyakov type action and there is a clear compartmentalization of the directions horizontal to the string and transverse to the string in the target space. The low energy excitations are entirely transverse. The horizontal components of the metric are not dynamical. Probably due to this proviso two new fields are included in the action which are devoid of any dynamics [5]. How these fields are related with the geometry (i.e. their connection with the metric) is not known. Clearly the action level correspondence between the different forms of the action, so transparent in the relativistic formulations, appear to be missing. The equivalence could be established by an arduous path [7]. The new action found here not only was shown to bridge the different forms, it also generated the additional fields. Moreover it has connected the geometric elements with the multipliers in the Hamiltonian. In the following we will further investigate the connection and one would appreciate that this connection is not accidental.

So far, the metric induced on the world sheet was not discussed because the discussion could proceed without reference to the metric. A metric on the world sheet was derived as the metric induced by the embedding [see

Eq. (1)]. But the Polyakov form gives an independent metric. So geometry of the world sheet is now interwoven with the dynamics. In deriving the Polyakov type action from the Nambu-Goto through the interpolating action, we have connected the form of the metric (45) composed with the Hamiltonian constructs ρ and λ, with the world sheet geometry. This evolution of the metric as dynamical fields is certainly a new input in the existing literature.

Our construction (45) is reminiscent of the Arnowitt-Deser-Misner (ADM) decomposition of general relativity [16]. In the ADM representation the metric of the four-dimensional Riemannian spacetime $^{(4)}\gamma^{\mu\nu}$ is split as[8]

$$\begin{pmatrix} ^{(4)}\gamma^{00} & ^{(4)}\gamma^{0m} \\ ^{(4)}\gamma^{0k} & ^{(4)}\gamma^{km} \end{pmatrix} = \begin{pmatrix} -\dfrac{1}{(N)^2} & \dfrac{(N^m)}{(N)^2} \\ \dfrac{(N^k)}{(N)^2} & \left(\gamma^{km} - \dfrac{(N^k)(N^m)}{(N)^2}\right) \end{pmatrix}. \quad (60)$$

Here, k, m take the values 1, 2, 3. γ^{km} is the metric on a three-dimensional hypersurface embedded in the four-dimensional spacetime. N and N^k are the arbitrary lapse and shift variables which are nothing but the Lagrange multipliers of the theory. From (60), a similar structure for $d = 2$ assumes the following form:

$$\begin{pmatrix} ^{(2)}\gamma^{00} & ^{(2)}\gamma^{01} \\ ^{(2)}\gamma^{01} & ^{(2)}\gamma^{11} \end{pmatrix} = \begin{pmatrix} -\dfrac{1}{(N)^2} & \dfrac{(N^1)}{(N)^2} \\ \dfrac{(N^1)}{(N)^2} & \left(\gamma^{11} - \dfrac{(N^1)^2}{(N)^2}\right) \end{pmatrix}. \quad (61)$$

Comparing this with (45) we can easily establish

$$N^1 \mapsto \rho \quad \text{and} \quad (N)^2 \mapsto -\lambda\sqrt{-g} \quad \text{and} \quad \gamma \mapsto g. \quad (62)$$

Thus the identification (45) is the same as the ADM foliation of the world sheet. The fields λ and ρ are manifestations of the arbitrariness along the string (the lapse N) and an arbitrariness transverse to the string i.e. the relative time direction (the shift N^1), both on the world sheet.

For further insight and comparison with existing results [3], we rewrite the interpolating Lagrangian in the light-cone coordinates,

$$X = X^0 + X^1, \qquad \bar{X} = X^0 - X^1. \quad (63)$$

The extra piece can now be written as

$$\mathcal{L}_e = \alpha\left(\frac{1}{\sqrt{-h}h^{00}} + \frac{\epsilon^{\alpha\beta}\partial_\alpha X \partial_\beta \bar{X}}{2\bar{X}'X'}\right) + \beta\left(-\frac{h^{01}}{h^{00}} - \frac{\sigma^{\alpha\beta}\partial_\alpha X \partial_\beta \bar{X}}{2X'\bar{X}'}\right) \quad (64)$$

where

$$\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (65)$$

and $\sigma^{\alpha\beta}$ is

$$\sigma^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (66)$$

Combining (47) and (64) we get the interpolating action in light cone coordinates

$$\mathcal{L}_\mathcal{I} = \frac{1}{2}\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^k \partial_\beta X^k + \alpha\left(\frac{1}{\sqrt{-h}h^{00}} + \frac{\epsilon^{\alpha\beta}\partial_\alpha X \partial_\beta \bar{X}}{2\bar{X}'X'}\right) + \beta\left(-\frac{h^{01}}{h^{00}} - \frac{\sigma^{\alpha\beta}\partial_\alpha X \partial_\beta \bar{X}}{2X'\bar{X}'}\right). \quad (67)$$

Now note that the coordinates X^a are defined in a Lorentz plane with metric $\eta_{ab} = (1, -1)$. The string world sheet in its ground state is parallel with the Lorentz plane. When the world sheet metric is an independent field, one has to introduce the tangent space at every point on the world sheet. The tangent space is locally Lorentzian. The coordinates X^0 and X^1 are referred to these coordinates. Let e_α and e_a be the bases at a point on the world sheet and the tangent space at that point, respectively. The vierbein $\Lambda^\alpha{}_a$ and its inverse connect the two bases,

$$e_\alpha = \Lambda_\alpha{}^a e_a. \quad (68)$$

The inverse of $\Lambda_\alpha{}^a$ will be denoted by $\Lambda^\alpha{}_a$.

The vierbeins may be used to factorize the metric,

$$h^{\alpha\beta} = \Lambda^\alpha{}_a \Lambda^\beta{}_b \delta^{ab}. \quad (69)$$

We now give special attention to the part of the Lagrangian where the vierbeins explicitly appear in the theory. It is that part of the Lagrangian which is special to the nonrelativistic theory having no relativistic analog. Now, both X and $\bar{X}$ are world sheet scalars. So we can easily derive the following relations:

$$\partial_\alpha X = \Lambda_\alpha{}^a \partial_a X = \Lambda_\alpha{}^a(\delta_a{}^0 + \delta_a^1) = e_\alpha,$$
$$\partial_\alpha \bar{X} = \Lambda_\alpha{}^a \partial_a \bar{X} = \Lambda_\alpha{}^a(\delta_a{}^0 - \delta_a^1) = \bar{e}_\alpha, \quad (70)$$

where

$$e_\alpha = \Lambda_\alpha^0 + \Lambda_\alpha^1,$$
$$\bar{e}_\alpha = \Lambda_\alpha^0 - \Lambda_\alpha^1. \quad (71)$$

Now we have all the intermediate quantities. We can then write the expression in terms of the basis vectors,

$$\mathcal{L}_{\mathcal{I}} = \frac{1}{2}\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^k \partial_\beta X^k + \alpha\left(\frac{1}{\sqrt{-h}h^{00}} + \frac{\epsilon^{\alpha\beta}e_\alpha \bar{e}_\beta}{2\bar{X}'X'}\right)$$
$$+ \beta\left(-\frac{h^{01}}{h^{00}} - \frac{\sigma^{\alpha\beta}e_\alpha \bar{e}_\beta}{2X'\bar{X}'}\right). \tag{72}$$

The string action in the form (72) clearly reveals the connection with the world sheet geometry. The deduction of this from the canonical analysis of the model (8) is indeed remarkable.

VI. CONCLUSION

Nonrelativistic string theories have recently come to prominence in the literature [3,5–7]. Just as recent studies of nonrelativistic field theories have emphasized the role of geometry to tackle the issue of coupling nonrelativistic field theories with gravity, studies in string theories have raised new questions about geometry. But nay, the NRSTs are interesting even in flat space. Similar to their relativistic counterpart, different actions have been proposed. Broadly we can divide these in two classes—minimal action which contains the world sheet area swept by the string as the Lagrangian [5] and more redundant form of action where the metric elements on the world sheet [3] are considered as independent fields. The former is comparable with the Nambu-Goto type and the latter with the Polyakov type in relativistic strings. But there is one significant difference. In NRSTs in the second type there are two extra fields on the world sheet. In relativistic field theories of Polyakov type one can easily substitute the metric components from their equations of motion to get the Nambu-Goto string. This action level correspondence is not apparent in case of the NRST primarily due to the presence of the extra fields.

We have developed an approach where the starting point was the Nambu-Goto type of action. This was obtained by taking a nonrelativistic $(c \to \infty)$ limit of the relativistic Nambu-Goto action. A unique form for the action was found. By eliminating its gauge freedom it was shown to reduce to the expected form for the nonrelativistic stretched string vibrating transversally. The choice of gauge was justified by a Hamiltonian analysis.

This Hamiltonian analysis was pushed in a big way by constructing from the Nambu-Goto type action an intermediate action that interpolated between the Nambu-Goto and Polyakov type forms. While the connection with the Nambu-Goto form was easily shown, that with the Polyakov type was much more involved. It was possible to identify a metric structure using the fields introduced in the Hamiltonian formulation. This enabled us to connect the string world sheet geometry with the geometry of the embedding which can be identified with that appearing in the Polyakov type action. Thereby, we regain the action level correspondence in NRSTs. This shows the utility of our Hamiltonian approach and explains the problems of showing the equivalence by strictly confining to the Lagrangian approach, as done for the relativistic example.

Another remarkable aspect is the role of Hamiltonian analysis in the formulation of the interpolating action. We have started from the minimal action. To facilitate the introduction of geometry on the world sheet of the string, we have carried out a comprehensive canonical analysis of the model. The action is then enriched by the introduction of the constraints in the Lagrangian by the Lagrange multiplier technique and lifting the status of the multipliers to independent fields. Eliminating the phase space variables by the inverse Legendre procedure, the desired action is obtained. A surprising connection of the new fields with the Arnowit-Deser-Misner construction in general relativity emerged, whereby geometry was introduced in the theory. It was then an easy journey towards the Polyakov type action. Remarkably, the two extra fields appeared spontaneously in the process. We have provided a detailed canonical analysis of the new action. Its phase space structure has been studied. Throughout the paper symmetries of the different actions have been investigated from the canonical point of view and the interpolating Lagrangian is no exception. This analysis has been used to deduce further geometrical connections.

The interplay of canonical analysis and geometry, as evidenced here, brings out a clear picture of the connection of canonical analysis with the metric of the world sheet. Thus the possible use of the action obtained here in case of coupling with gravity appears to be feasible. One has to find ways to relate the background curvature. Our approach is amenable to generalization and can be extended to membranes and higher branes. This is because the analysis has a close parallel with the relativistic case where the equivalence of higher brane actions was demonstrated [17] by generalizing the results for the string case done in [8]. This could be useful since the equivalence of actions for nonrelativistic theories is obviously much more complicated than its relativistic counterpart. These and other issues may yield fresh insights and open up further research.

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