Turnaround physics beyond spherical symmetry

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The concept of turnaround surface in an accelerating universe is generalized to arbitrarily large deviations from spherical symmetry, to close the gap between the idealized theoretical literature and the real world observed by astronomers. As an analytical application, the characterization of turnaround surface is applied to small deviations from spherical symmetry, recovering a previous result while extending it to scalar-tensor gravity.

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I. INTRODUCTION

Thanks to the study of type Ia supernovae [1,2], it has been known since 1998 that the present expansion of the Universe is accelerated. In the context of general relativity, this acceleration is attributed to the presence of a mysterious form of dark energy permeating the Universe and responsible for approximately 70% of its energy content [3]. However, since this dark energy is introduced completely *ad hoc*, there has been much activity in explaining the cosmic acceleration by modifying gravity at large scales and dispensing with dark energy [4,5] (see Refs. [6–10] for reviews). Whatever the explanation for the cosmic acceleration, there are physical phenomena peculiar to an accelerated universe. One of them is the *turnaround radius* of cosmic structures [11–14], which has attracted much attention recently [15–29] because of its potential to test the Λ-cold dark matter model and/or modified gravity.

Consider an accelerating Friedmann-Lemaître-Robertson-Walker (FLRW) universe and superpose a spherical matter condensation acting as a perturbation of the FLRW metric. The local gravitational attraction due to this overdensity tends to make it collapse, while the cosmic expansion tends to disperse it (if this structure is sufficiently large to feel the effect of the cosmic expansion appreciably). The turnaround radius is the minimum scale at which a spherical shell of test particles cannot collapse because of the accelerated cosmic expansion (or, vice versa, the upper limit to the radius of spherical bound structures in an accelerated universe). At the turnaround radius, the local attraction balances the cosmic expansion.

Thus far, with the exception of [29,30], the literature on turnaround physics has been restricted to spherically symmetric situations. Reference [30] studies analytically small deviations from spherical symmetry (Ref. [29], instead, examines larger deviations numerically). However, this idealized situation is still far from being realistic and can easily induce large errors (cf. Refs. [25–28]). Currently, the only reliable method to actually measure the turnaround point consists of using a pancake detection, as proposed in Ref. [31], and then solving for zero velocity (see the discussion of Ref. [25]).

In the presence of spherical symmetry, the turnaround radius trivially defines the "turnaround surface," i.e., the sphere of radius equal to the turnaround radius, but the generalization of this turnaround surface to nonspherical situations of astrophysical interest has not been discussed in the theoretical literature. As a consequence, astronomers attempting to determine the turnaround surface and deduce cosmological information have to grapple with ill-defined theory and basic concepts that are unclear, in addition to major observational challenges.

Here we identify the salient features of the turnaround surface in spherical symmetry and characterize it with a definition suitable for geometries with arbitrarily large deviations from spherical symmetry (however, the deviations from the FLRW metric remain always small). The key idea is to identify the turnaround surface with an equipotential surface of the (local) metric perturbation potential with the special property that, if test particles initially sit on this surface with zero velocity with respect to it, they remain on this surface at later times while it evolves. They must remain at rest on this surface and cannot leave it to collapse because of the self-gravity of the perturbation, nor disperse because of the cosmic expansion.

There is only one critical surface $\mathcal{S}^*$ on which these two opposing forces balance, as in the spherically symmetric case. Any other closed surface $\mathcal{S}^*$ nearby will not have this property: particles will collapse (if $\mathcal{S}^*$ is contained inside the critical surface) or will disperse, expanding faster than those at rest on the critical turnaround surface, if $\mathcal{S}^*$ lies

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outside of it. This characterization captures the essence of turnaround sphere in spherical symmetry and generalizes this concept, while shifting the emphasis from the *size* of this surface (the turnaround radius) to the surface itself.

The study of the relevant equations for specific cosmic structures (observed or hypothetical) requires, in general, a numerical implementation. We can, however, apply our definition to an analytical discussion of small nonsphericities and test our characterization in this situation, which has already been studied in Ref. [30] with a completely different method, based on the splitting of the Hawking-Hayward quasilocal energy contained in the turnaround surface into local and cosmological parts. We recover the results of [30] in our new, general description.

In Sec. II we calculate the timelike geodesics needed in the rest of this paper, while Sec. III provides the general definition of turnaround surface. The application to small nonsphericities is detailed in Sec. IV, while Sec. V extends this result to scalar-tensor gravity. Section VI contains a discussion and the conclusions. We follow the notation of Ref. [32].

II. TIMELIKE GEODESICS IN THE PERTURBED FLRW UNIVERSE

The definition of turnaround surface requires one to consider test particles lying on this surface. They follow timelike geodesics in spacetime; therefore, we first discuss these special curves traced by test particles and clouds (or shells) of dust in the perturbed FLRW spacetime.

The spacetime metric in the conformal Newtonian gauge is

$$\begin{aligned} \mathrm{d}s^2 &= g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta \\ &= a^2(\eta)\{-(1+2\Phi)\mathrm{d}\eta^2 + (1-2\Phi) \\ &\quad \times [\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2)]\}, \end{aligned} \tag{2.1}$$

where η is the conformal time of the unperturbed FLRW universe and $\Phi(x^i)$ describes the Newtonian perturbation. Since we will consider only structures of size much smaller than the Hubble radius H_0^{-1}, the time dependence of Φ can be safely neglected.

Timelike geodesics parametrized by the proper time τ have four-tangents $u^\mu = dx^\mu/d\tau$ that satisfy the geodesic equation

$$\frac{du^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta = 0. \tag{2.2}$$

The only nonvanishing Christoffel symbols of the perturbed FLRW universe (2.1) are given in the Appendix, and they are used to compute the components of the timelike geodesic equation

$$\begin{aligned} \frac{\mathrm{d}u^0}{\mathrm{d}\tau} &= \frac{1}{a(2\Phi+1)}\{a_{,\eta}[2(r^2(u^3)^2\sin^2\theta + r^2(u^2)^2 + (u^1)^2 - (u^0)^2)\Phi - r^2(u^3)^2\sin^2\theta - r^2(u^2)^2 - (u^1)^2 - (u^0)^2] \\ &\quad - 2u^0 a(u^3\Phi_{,\varphi} + u^2\Phi_{,\theta} + u^1\Phi_{,r})\}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} \frac{\mathrm{d}u^1}{\mathrm{d}\tau} &= \frac{1}{a(2\Phi-1)}\{2u^1u^0a_{,\eta}(1-2\Phi) + a[(r^2(u^2)^2 + r^2(u^3)^2\sin^2\theta - (u^1)^2 + (u^0)^2)\Phi_{,r} \\ &\quad - 2u^3u^1\Phi_{,\varphi} - 2u^2u^1\Phi_{,\theta} + 2r((u^3)^2\sin^2\theta + (u^2)^2)\Phi - r(u^3)^2\sin^2\theta - r(u^2)^2]\}, \end{aligned} \tag{2.4}$$

$$\begin{aligned} \frac{\mathrm{d}u^2}{\mathrm{d}\tau} &= \frac{1}{r^2a(2\Phi-1)}\{2r^2u^2u^0a_{,\eta}(1-2\Phi) + a[-2r^2u^2u^3\Phi_{,\varphi} + (-r^2(u^2)^2 + r^2(u^3)^2\sin^2\theta + (u^1)^2 + (u^0)^2)\Phi_{,\theta} \\ &\quad - 2r^2u^1u^2\Phi_{,r} + r(r(u^3)^2\sin(2\theta) - 4u^1u^2)\Phi - r^2(u^3)^2\sin\theta\cos\theta + 2ru^1u^2]\}, \end{aligned} \tag{2.5}$$

$$\begin{aligned} \frac{\mathrm{d}u^3}{\mathrm{d}\tau} &= \frac{1}{r^2a(2\Phi-1)}\{2r^2u^3u^0a_{,\eta}(1-2\Phi) + a[((r^2(u^2)^2 + (u^1)^2 + (u^0)^2)\csc^2\theta - r^2(u^3)^2)\Phi_{,\varphi} \\ &\quad + 2ru^3(-ru^1\Phi_{,r} - ru^2\Phi_{,\theta} + ru^2\cot\theta + u^1) - 4ru^3(ru^2\cot\theta + u^1)\Phi]\}. \end{aligned} \tag{2.6}$$

In an unperturbed FLRW spacetime, the 4-velocity of a comoving observer reads

$$u^\mu_{(0)} = (u^0_{(0)}, \mathbf{0}) = \left(\frac{1}{a}, \mathbf{0}\right) \tag{2.7}$$

in coordinates $(\eta, r, \theta, \varphi)$. Adding a perturbation as in Eq. (2.1), the four-tangent to a timelike geodesic becomes

$$u^\mu = u^\mu_{(0)} + \delta u^\mu = (u^0_{(0)} + \delta u^0, \boldsymbol{\delta u}) = \left(\frac{1}{a} + \delta u^0, \boldsymbol{\delta u}\right); \tag{2.8}$$

the normalization $u^\mu u_\mu = -1$ then yields

$$\delta u^0 = -\frac{\Phi}{a} \tag{2.9}$$

to first order in δu^μ and Φ. Substituting this expression of δu^0 in the normalization of u^μ, one finds

$$\begin{aligned} -1 &= -a^2(1+2\Phi)\left(\frac{1-\Phi}{a}\right)^2 + g_{ij}\delta u^i \delta u^j \\ &= -1 + g_{ij}\delta u^i \delta u^j + \mathcal{O}(\Phi^2). \end{aligned} \tag{2.10}$$

If one then assumes that $\mathcal{O}(\delta u^1) = \mathcal{O}(\delta u^2) = \mathcal{O}(\delta u^3)$, then the latter implies

$$\mathcal{O}(u^i) = \mathcal{O}(\delta u^i) = \mathcal{O}(\Phi), \quad i = 1, 2, 3. \tag{2.11}$$

We can now use these results to study the geodesic equations to first order in the perturbation. In detail, it is easy to show that Eq. (2.3) reduces to an identity to $\mathcal{O}(\Phi)$, whereas for the spatial components one finds

$$\frac{d(\delta u^i)}{d\tau} + \frac{2a_{,\eta}}{a^2}\delta u^i + g^{ij}\partial_j \Phi = 0. \tag{2.12}$$

III. GENERAL DEFINITION OF TURNAROUND SURFACE

Here we generalize the definition of turnaround surface to the case in which deviations from spherical symmetry can be arbitrarily large. An examination of the salient features of the turnaround sphere in spherical symmetry shows that this surface is an equipotential surface of the perturbation potential Φ. This is a necessary property, but it is not sufficient to identify the turnaround surface. We require the extra property that, if test particles initially lay on this surface and have zero initial velocity *with respect to it*, they remain on this surface at later times as the latter evolves. These dust particles, and the surface with respect to which they are at rest, are not comoving with the background FLRW universe. They would be comoving only if the cosmic fluid of this FLRW background universe was dust, but this cannot be true because this universe accelerates. Therefore, these dust particles and the surface they lie on necessarily do not comove with the background. Furthermore, these particles are slowed down by the attraction of the local gravity due to the mass contained inside the turnaround surface; therefore, they expand more slowly than the FLRW background.

This property is still not sufficient to identify the turnaround surface because many timelike geodesics cross the turnaround surface, but we further restrict to timelike geodesics that initially have zero velocity with respect to this surface. Since they satisfy the timelike geodesic equation, which is of second order, assigning their initial position (on Σ_{t_0}) and initial velocity (vanishing with respect to Σ_{t_0}) specifies them completely. In other words, these massive test particles stay on Σ_{t_0} initially and at all later times and expand more slowly than the accelerating FLRW background. Finally, to complete the identification of the initial surface Σ_{t_0}, we require that, on this surface, the attraction due to the Newtonian perturbation balances exactly the cosmic expansion, so that the acceleration of dust particles on this initial surface vanishes.

We require the *turnaround surface* Σ_t at (comoving) time t to be a two-dimensional, closed, simply connected surface that, at all times t, is an equipotential surface of the perturbation Φ such that:

(i) The time evolution of the surface is such that the three-dimensional components of the tangent to the timelike geodesics crossing Σ_t are *locally* proportional to the gradient $\boldsymbol{\nabla}\Phi$ (and therefore perpendicular to Σ_t in the three-dimensional sense):

$$u^i|_{\Sigma_t} = \sigma(t) g^{ij}\partial_j \Phi|_{\Sigma_t}. \tag{3.1}$$

(ii) A dust particle initially comoving with the surface remains on this surface.[1] In other words, if a dust particle is initially comoving *with* Σ_{t_0} (not with the FLRW background), namely

$$u^i|_{\Sigma_{t_0}} = \sigma(t_0) g^{ij}\partial_j \Phi|_{\Sigma_{t_0}}, \tag{3.2}$$

then at a time $t > t_0$ its 3-velocity will satisfy

$$u^i|_{\Sigma_t} = \sigma(t) g^{ij}\partial_j \Phi|_{\Sigma_t}. \tag{3.3}$$

(iii) In an unperturbed FLRW universe, the (purely radial) acceleration of a massive test particle is $\ddot{r} = \ddot{a}r/a$ (this rather intuitive result has been obtained many times in the literature, using various methods [33–70]). In the presence of a spherical perturbation, the turnaround radius is obtained by balancing the attraction of the local inhomogeneity with the cosmic acceleration, or $\frac{M}{r^2} = \frac{\ddot{a}}{a} r$, where M is the mass of the local perturbation. For a general Newtonian perturbation described by the potential Φ, we impose that at every point of the initial surface Σ_{t_0}, assumed to be convex,[2] the acceleration of a

[1] This particle is not comoving with the cosmic fluid because this surface is *not* comoving with the cosmic substratum.

[2] The surface is assumed to be convex to avoid pathological possibilities, such as a mass distribution with two or more centers far away from each other (which, technically, is a mass distribution but has nothing to do with a mass concentration on the verge of collapsing under its own gravity).

massive test particle normal to this surface vanishes because the local attraction $-\boldsymbol{\nabla}\Phi$ balances exactly the force per unit mass due to the cosmic expansion *at that point* $\frac{\ddot{a}(t_0)}{a(t_0)}\boldsymbol{x}_\perp$ (on Σ_{t_0} there is no sideways acceleration due to the local gravity of the perturbation because Σ_{t_0} is an equipotential surface of Φ). In other words, let $\boldsymbol{x}$ denote the position of a point on Σ_{t_0} embedded in the three-dimensional spacelike slice of the spacetime. Thus, denoting by

$$\boldsymbol{n} = \frac{\boldsymbol{\nabla}\Phi}{|\boldsymbol{\nabla}\Phi|}\bigg|_{\Sigma_{t_0}}$$

the normal to Σ_{t_0}, the following condition must hold:

$$-\boldsymbol{\nabla}\Phi = \frac{\ddot{a}(t_0)}{a(t_0)}\boldsymbol{x}_\perp \quad \text{on} \quad \Sigma_{t_0}, \tag{3.4}$$

with $\boldsymbol{x}_\perp \equiv (\boldsymbol{x}\cdot\boldsymbol{n})\boldsymbol{n}$, which implies

$$-|\boldsymbol{\nabla}\Phi|^2 = \frac{\ddot{a}(t_0)}{a(t_0)}\boldsymbol{x}\cdot\boldsymbol{\nabla}\Phi \quad \text{on} \quad \Sigma_{t_0}. \tag{3.5}$$

This condition completes the identification of the initial position of the massive test particles.

To elucidate this definition let us consider some timelike dust comoving with the turnaround surface Σ_{t_0} at some $t = t_0$. Since the dust is initially comoving with Σ_{t_0}, then the velocity of each particle will be such that $u^i|_{\Sigma_{t_0}} \propto g^{ij}\partial_j\Phi|_{\Sigma_{t_0}}$, which implies $\delta u^i|_{\Sigma_{t_0}} = \sigma(t_0)g^{ij}\partial_j\Phi|_{\Sigma_{t_0}}$. If this was not the case, a particle would have a nonvanishing component of $u^i|_{\Sigma_{t_0}}$ tangent to Σ_{t_0}, inducing a tangential movement along the surface; however, we are not interested in this scenario with this definition. Furthermore, Φ is not constant in time: it does not depend explicitly on time, but it has a time dependence through the coordinates on Σ_t, which depend on time, $\Phi = \Phi(x^\alpha(t))$.

Any other shell that does not satisfy precisely the two initial conditions on position and velocity (i) starting on Σ_{t_0}; (ii) having zero initial velocity with respect to Σ_{t_0} at t_0, and (iii) coinciding with the zero acceleration surface at the initial time t_0, will necessarily be forever distinct from Σ_t due to the uniqueness theorem for the solutions of the Cauchy problem associated with the second order geodesic equation. The surface satisfying these two properties simultaneously is unique, and the definition of Σ_t is a true definition.

The turnaround surface deviates from the Hubble flow because of the local attraction due to the mass contained in it, which creates the first order metric perturbation potential. Consistently, in Eq. (3.1), the induced 4-velocity of this shell *relative to the FLRW background* is of first order. The turnaround surface Σ_t evolves to background order (as described by its 4-velocity $u^\mu_{(0)}$ *and* to first order, so that its total 4-velocity is $u^\mu_{(0)} + \delta u^\mu$). The metric perturbation potential Φ calculated on Σ_t depends on the coordinates on it, and this surface evolves; therefore, also $\Phi|_{\Sigma_t}$ *calculated on* Σ_t evolves, although it has no explicit dependence on t in the line element (2.1).

Now, the general result in (2.12) can be recast as

$$\frac{\mathrm{d}(\delta u^i)}{\mathrm{d}t} + 2H\delta u^i + g^{ij}\partial_j\Phi = 0, \tag{3.6}$$

since $u^0_{(0)} = \mathrm{d}\eta/\mathrm{d}\tau = \mathrm{d}\eta/\mathrm{d}t = 1/a$ to order $\mathcal{O}(\Phi^0)$. It is then easy to infer that (3.6) reduces to

$$\frac{1}{a^2}\frac{\mathrm{d}}{\mathrm{d}t}(a^2\delta u^i) = -g^{ij}\partial_j\Phi, \tag{3.7}$$

leading to

$$\delta u^i|_{\Sigma_t} = \frac{a^2(t_0)}{a^2(t)}\delta u^i|_{\Sigma_{t_0}} - \frac{1}{a^2(t)}\int_{t_0}^{t} h^{ij}(x^\alpha(t'))\partial_j\Phi(x^\alpha(t'))\mathrm{d}t', \tag{3.8}$$

given that $h^{ij} \equiv a^2(t)g^{ij} = \mathtt{diag}(1, 1/r^2, 1/(r^2\sin^2\theta))$.

Astronomical observations of the turnaround radius cannot span the entire history of the structures observed since their formation, but only a small redshift interval near the time when the light that is received now was emitted by an object in the sky. Therefore, we linearize the quantities $a(t)$ and the integral to first order in $t - t_0$ (no astronomical observation has a chance to go beyond first order).

From the requirement of locality in (i), i.e., $\epsilon = t - t_0$ very small in comparison with $(H(t_0))^{-1}$, and expanding $a(t) = a(t_0) + H(t_0)(t - t_0) + \ldots$, one obtains

$$\begin{aligned}\delta u^i|_{\Sigma_t} &= \frac{1}{[1 + H(t_0)(t - t_0) + \cdots]^2}\delta u^i|_{\Sigma_{t_0}} \\ &\quad - \frac{1}{a^2(t_0)[1 + H(t_0)(t - t_0) + \cdots]^2}\int_{t_0}^{t}[h^{ij}(x^\alpha(t_0) + \dot{x}^\alpha(t_0)(t - t_0))\partial_j\Phi(x^\alpha(t_0) + \dot{x}^\alpha(t_0)(t - t_0))]\mathrm{d}t' \\ &= [1 - 2H(t_0)(t - t_0)]\delta u^i|_{\Sigma_0} - \frac{[1 - 2H(t_0)(t - t_0)]}{a^2(t_0)}h^{ij}(x^\alpha(t_0))\partial_j\Phi(x^\alpha(t_0))(t - t_0) + \cdots,\end{aligned} \tag{3.9}$$

and finally

$$\delta u^i|_{\Sigma_t} = \left[(1-2H(t_0)\epsilon)\sigma(t_0) - \frac{\epsilon}{a^2(t_0)}\right] h^{ij}\partial_j\Phi|_{\Sigma_{t_0}} + \mathcal{O}(H^2(t_0)\epsilon^2)$$
$$= \left[\sigma(t_0) - \left(2\sigma(t_0)H(t_0) + \frac{1}{a^2(t_0)}\right)\epsilon\right] h^{ij}\partial_j\Phi|_{\Sigma_{t_0}} + \mathcal{O}(H^2(t_0)\epsilon^2). \tag{3.10}$$

Consider now the spherical case. To first order, the equation of radial timelike geodesics reduces to [recall that $u^0_{(0)} = \mathrm{d}\eta/\mathrm{d}\tau = \mathrm{d}\eta/\mathrm{d}t = 1/a$ to order $\mathcal{O}(\Phi^0)$]

$$\frac{\mathrm{d}\delta u^1}{\mathrm{d}t} + 2H\delta u^1 + \frac{\Phi'}{a^2} = 0, \tag{3.11}$$

where $\delta\boldsymbol{u} = (\delta u^1, 0, 0)$, $\boldsymbol{\nabla}\Phi = (\Phi', 0, 0)$, $\Phi = \Phi(r)$, and a prime denotes differentiation with respect to r. Clearly $\boldsymbol{u}$ and $\boldsymbol{\nabla}\Phi$ are parallel. As discussed above, Eq. (3.11) can be recast as

$$\frac{1}{a^2}\frac{\mathrm{d}}{\mathrm{d}t}(a^2\delta u^i) = -\frac{\Phi'}{a^2}, \tag{3.12}$$

which yields

$$\delta u^1 = \left[\sigma(t_0) - \left(2\sigma(t_0)H(t_0) + \frac{1}{a^2(t_0)}\right)(t-t_0)\right]\Phi'$$
$$\equiv \sigma(t)\Phi'(r), \tag{3.13}$$

up to $\mathcal{O}(H^2(t_0)(t-t_0)^2)$. Thus, we have proved that the only 3-velocity perturbation component δu^1 is proportional to the gradient of Φ on the turnaround sphere.

In the coordinates $(\eta, r, \theta, \varphi)$, this sphere evolves with time; hence, the proportionality constant σ must depend on time (and only on time because this is an equipotential surface of the perturbation potential), as described by Eq. (3.13). Moreover, according to Eqs. (3.8) and (3.13), the velocity perturbation δu^i is negative, which can be easily understood using the spherical case as an example. The dynamics of the critical (turnaround) sphere was discussed in Ref. [71]: in an accelerated universe propelled by a cosmic fluid with equation of state $P = w\rho$, with $w \simeq -1$ according to current observations, the areal radius R_c of the critical turnaround sphere evolves according to [71]

$$\frac{\dot{R}_c}{R_c} = \left(w + \frac{4}{3}\right)H \simeq \frac{H}{3} < H. \tag{3.14}$$

This equation compares the expansion rate of the turnaround sphere with that of the cosmic substratum and tells us that the turnaround sphere expands slower than the Hubble flow. Therefore, a particle at rest on it will slow down with respect to the cosmic substratum and will have a radial velocity perturbation $\delta u^1 < 0$.

IV. SMALL NONSPHERICITIES

Let us apply now the previous considerations to small deviations from spherical symmetry. This situation is studied in [30] with a conceptually different method. Reference [30] is based on the splitting of the Hawking-Hayward quasilocal energy enclosed by a 2-surface Σ into a local and a cosmological part: the local part due to the perturbation Φ dominates inside the turnaround surface, while the cosmological part due to the cosmic mass energy enclosed by Σ dominates outside of the turnaround surface. This surface is defined by the equality of these two contributions (previously, this quasilocal energy method was applied to the spherical case [71,72]). The result of Ref. [30] is that, to first order in the metric perturbations and in a parameter ϵ describing the deviations from spherical symmetry, the nonsphericities do not matter and the turnaround surface is still described by the turnaround radius obtained to zero order in ϵ. Given the very different methods used in the present paper and in [30], one should check that the results obtained coincide and that the two methods are compatible. Indeed, the results of [30] are recovered in our new, general description of Sec. III.

To wit: let us go back to the perturbed geodesic equations

$$\frac{\mathrm{d}(\delta u^i)}{\mathrm{d}\tau} + \frac{2a_{,\eta}}{a^2}\delta u^i + g^{ij}\partial_j\Phi = 0. \tag{4.1}$$

Small nonsphericities are introduced by perturbing the otherwise spherical potential $\Phi_0(r)$ as

$$\Phi(r,\theta,\varphi) = \Phi_0(r) + \epsilon f(r,\theta,\varphi),$$
$$\mathcal{O}(\Phi) = \mathcal{O}(\Phi_0), \qquad 0 < \epsilon \ll 1. \tag{4.2}$$

The nonsphericity leads to a further correction in $\delta\boldsymbol{u}$, specifically

$$\delta\boldsymbol{u} = (\delta u^1_{(0)}, 0, 0) + \epsilon\boldsymbol{\delta} \tag{4.3}$$

with $\delta u^1_{(0)}$ denoting the (radial) velocity perturbation in the unperturbed spherical case. Hence, perturbing the geodesic equations and using the results of the spherical case, one finds the equation satisfied by the nonsphericities

$$\frac{\mathrm{d}\delta^i}{\mathrm{d}t} + 2H\delta^i + g^{ij}\partial_j f = 0 \tag{4.4}$$

to $\mathcal{O}(\epsilon)$, where $H \equiv \dot{a}/a$ is the Hubble function with respect to comoving time. Finally, since the perturbed surface must be close to a sphere at all times, the perturbation function $f(r,\theta,\varphi)$ must be controlled so that it does not blow up. Unless some regulating condition is imposed to this regard, the function f could grow very fast and the modified surface could deviate arbitrarily from a sphere even if the expansion parameter ϵ remains small. Therefore, we impose that[3] $\boldsymbol{\nabla} f = \mathcal{O}(\epsilon)$, then $\boldsymbol{\delta} = \mathcal{O}(\epsilon)$. This means that, for small deviations from sphericity, approximating the nonspherical turnaround surface with the unperturbed (spherical) one still gives the correct result to first order in the parameter ϵ that quantifies the nonsphericity in Eq. (4.2).

V. SCALAR-TENSOR GRAVITY

The turnaround radius has been studied also in scalar-tensor gravity and can, in principle, provide information about the theory of gravity at large scales [18–23,29]. In scalar-tensor gravity there is a gravitational slip and a Newtonian perturbation describing a bound structure is described by two metric potentials Ψ and Φ. The perturbed FLRW line element is now

$$\mathrm{d}s^2 = a^2(\eta)\{-(1+2\Psi)\mathrm{d}\eta^2 + (1-2\Phi) \times [\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2)]\}, \tag{5.1}$$

where it is assumed that the small metric perturbations Ψ and Φ are time independent and of the same order. The general definition of turnaround surface given above can still be applied, provided that this surface is now an equipotential surface of Ψ.

Again, the only nonvanishing Christoffel symbols are given in the Appendix and the equations of timelike geodesics with four-tangents u^μ are now

$$\frac{\mathrm{d}u^0}{\mathrm{d}\tau} = \frac{1}{a(2\Psi+1)}\{a_\eta[2(r^2(u^3)^2\sin^2\theta + r^2(u^2)^2 + (u^1)^2)\Phi - 2(u^0)^2\Psi - r^2(u^3)^2\sin^2\theta - r^2(u^2)^2 - (u^1)^2 - (u^0)^2] - 2u^0 a(u^3\Psi_{,\varphi} + u^2\Psi_{,\theta} + u^1\Psi_{,r})\}, \tag{5.2}$$

$$\frac{\mathrm{d}u^1}{\mathrm{d}\tau} = \frac{1}{a(2\Phi-1)}\{2u^1u^0a_{,\eta}(1-2\Phi) + a[(r^2(u^2)^2 + r^2(u^3)^2\sin^2\theta - (u^1)^2)\Phi_{,r} - 2u^3u^1\Phi_{,\varphi} - 2u^2u^1\Phi_{,\theta} + 2r((u^3)^2\sin^2\theta + (u^2)^2)\Phi + (u^0)^2\Psi_{,r} - r(u^3)^2\sin^2\theta - r(u^2)^2]\}, \tag{5.3}$$

$$\frac{\mathrm{d}u^2}{\mathrm{d}\tau} = \frac{1}{r^2a(2\Phi-1)}\{2r^2u^2u^0a_{,\eta}(1-2\Phi) + a[-2r^2u^2u^3\Phi_{,\varphi} + (-r^2(u^2)^2 + r^2(u^3)^2\sin^2\theta + (u^1)^2)\Phi_{,\theta} - 2r^2u^1u^2\Phi_{,r} + r(r(u^3)^2\sin(2\theta) - 4u^1u^2)\Phi + (u^0)^2\Psi_{,\theta} - r^2(u^3)^2\sin\theta\cos\theta + 2ru^1u^2]\}, \tag{5.4}$$

$$\frac{\mathrm{d}u^3}{\mathrm{d}\tau} = \frac{1}{r^2a(2\Phi-1)}\{2r^2u^3u^0a_{,\eta}(1-2\Phi) + a[((r^2(u^2)^2 + (u^1)^2)\csc^2\theta - r^2(u^3)^2)\Phi_{,\varphi} + 2ru^3(-ru^1\Phi_{,r} - ru^2\Phi_{,\theta} + ru^2\cot\theta + u^1) - 4ru^3(ru^2\cot\theta + u^1)\Phi + (u^0)^2\csc^2\theta\Psi_{,\varphi}]\}. \tag{5.5}$$

The 4-velocities $u^\mu = u^\mu_{(0)} + \delta u^\mu$ are given again by Eq. (2.8). Specifically, to first order in both Ψ and δu^μ, the normalization $u_\mu u^\mu = -1$ gives

$$\delta u^0 = -\frac{\Psi}{a}, \qquad u^\mu = \left(\frac{1-\Psi}{a}, \delta u^1, \delta u^2, \delta u^3\right). \tag{5.6}$$

Besides, plugging the latter again into the normalization of u^μ one finds

$$-1 = -a^2(1+2\Psi)\left(\frac{1-\Psi}{a}\right)^2 + g_{ij}\delta u^i\delta u^j = -1 + g_{ij}\delta u^i\delta u^j + \mathcal{O}(\Psi^2), \tag{5.7}$$

that implies $\delta\boldsymbol{u} = \mathcal{O}(\Psi) = \mathcal{O}(\Phi)$.

Substituting (5.6) into the timelike geodesic equations and proceeding as done for general relativity in the previous sections, one can check that the time component of the geodesic equation is identically satisfied. The spatial components give, to first order,

$$\frac{d(\delta u^1)}{d\tau} + \frac{1}{a}\left(\frac{2a_{,\eta}}{a}\delta u^1 + \frac{\Psi_{,r}}{a}\right) = 0, \tag{5.8}$$

$$\frac{d(\delta u^2)}{d\tau} + \frac{1}{ar^2}\left(\frac{2r^2a_{,\eta}}{a}\delta u^2 + \frac{\Psi_{,\theta}}{a}\right) = 0, \tag{5.9}$$

$$\frac{d(\delta u^3)}{d\tau} + \frac{1}{ar^2}\left(\frac{2r^2a_{,\eta}}{a}\delta u^3 + \frac{\Psi_{,\varphi}}{a\sin^2\theta}\right) = 0. \tag{5.10}$$

[3]This assumption has already been used in Ref. [30].

Now we expand the potentials to describe small deviations from spherical symmetry as

$$\Psi(r,\theta,\varphi) = \Psi_0(r) + \epsilon f(r,\theta,\varphi),$$
$$\Phi(r,\theta,\varphi) = \Phi_0(r) + \epsilon h(r,\theta,\varphi), \tag{5.11}$$

with ϵ a smallness parameter. Again, the 4-velocities become

$$u^\mu = u^\mu_{(0)} + \delta u^\mu = (u^0_{(0)} + \epsilon\delta_0, \delta u^1_{(0)} + \epsilon\delta_1, \epsilon\delta_2, \epsilon\delta_3) \tag{5.12}$$

$$= \left(\frac{1-\Psi_0-\epsilon f}{a}, \delta u^1_{(0)} + \epsilon\delta_1, \epsilon\delta_2, \epsilon\delta_3\right). \tag{5.13}$$

Inserting this expansion into the spatial components of the geodesic equations yields

$$\frac{d\delta_1}{dt} + 2H\delta_1 = -\frac{f_{,r}}{a^2}, \tag{5.14}$$

$$\frac{d\delta_2}{dt} + 2H\delta_1 = -\frac{f_{,\theta}}{a^2 r^2}, \tag{5.15}$$

$$\frac{d\delta_3}{dt} + 2H\delta_3 = -\frac{f_{,\varphi}}{a^2 r^2 \sin^2\theta}. \tag{5.16}$$

Again, one needs to control the behavior of the deviations from sphericity by limiting the growth of the function f. This leads to the same results derived above for general relativity. One can conclude that, also in scalar-tensor gravity, small deviations from sphericity can be neglected in the identification of the turnaround surface.

VI. DISCUSSION AND CONCLUSIONS

In spherical symmetry, the turnaround radius clearly corresponds to a sphere of instability. Test particles that start on this surface with zero initial velocity with respect to it remain on it; analogous test particles inside this critical sphere must collapse, while those outside never form a bound system and disperse. The turnaround sphere corresponds to a delicate balance between the local gravitational attraction, which tends to make a dust shell collapse, and the cosmic expansion that tends to disperse it. On either side of the turnaround surface, one of these two forces prevails and moves a test particle away from it, so the position of (dynamical) equilibrium at the turnaround surface is clearly unstable. An actual measurement of the turnaround point will require the observation a specific galaxy near this surface. Therefore, this galaxy should preferably reside in a cold, coherent flow. It seems that this situation has only been studied in numerical simulations, with the conclusion that the needed galaxies are cold near the turnaround point [73]. The 4-velocity perturbation δu^1 used in our calculation is negative because of the self-gravity of the mass contained inside the turnaround surface: this mass slows down the outward motion of geodesic particles relative to the cosmic substratum. In an unperturbed universe, massive test particles starting out with zero (radial) initial velocity relative to the background would not be pulled back this way; hence, it is always $\delta u^1 < 0$ (assuming, of course, that the mass contained in the turnaround surface to be positive, which is the only physically meaningful option).

The turnaround radius is not a fixed point in the phase space of radial timelike geodesics, unless the background universe is de Sitter, which is locally static [68–70]. In a general FLRW background, the (proper or areal) turnaround radius is not constant but depends on time and the turnaround sphere expands (but the dust particles sitting on it have zero acceleration $\ddot{R} = 0$, initially and at all later times, where R is the areal radius). The turnaround sphere is not comoving.

Since the turnaround sphere is a sphere of unstable equilibrium, it marks the upper bound on the radius of any (spherical) bound structure. Because of the instability, a spherical bound structure with radius equal to the turnaround radius will not occur in nature. The turnaround radius is presented correctly in the literature as marking the upper limit to the largest possible size of a bound spherical structure.

Realistic structures in the sky, however, are not spherical nor approximately spherical. This fact is a challenge for astronomers attempting to identify the turnaround surface from observations of bound cosmic structures. This observational challenge is, of course, more complicated if one does not know what a turnaround surface is from the theoretical point of view. This is the gap addressed in the previous sections.

In the absence of spherical symmetry, the "size" of an asymmetric bound structure, or cluster, may be defined operationally in various ways. Each one of them will have advantages and disadvantages and will be somehow questionable. However, it is more important to focus on the *turnaround surface* and to identify it, rather than discussing its "size." Here we have identified the turnaround surface with an equipotential surface of the metric perturbation potential Φ satisfying a special property: dust particles initially sitting on this (nonspherical) surface with zero velocity with respect to it (and with the property that the gravitational acceleration due to the local mass distribution balances exactly the cosmic acceleration) will remain on it as this surface evolves in time. This definition is completely general and seems to be the correct generalization of turnaround sphere. Then, the turnaround radius no longer exists[4] and the size of the critical turnaround surface ceases to play a primary role in the discussion of turnaround physics.

The application of our characterization of turnaround surface to small deviations from spherical symmetry reproduces the previous result of Ref. [30], which was obtained with a completely different method (the splitting of the Hawking mass contained in the turnaround surface into local and cosmological parts [71]). Further applications to realistic situations will be presented in the future.

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APPENDIX: CHRISTOFFEL SYMBOLS OF THE PERTURBED FLRW UNIVERSE

Here we report the only nonvanishing Christoffel symbols of the perturbed FLRW universe (2.1), which are used to compute the timelike geodesics in the text. They are

$$\Gamma^0_{00} = \frac{a_{,\eta}}{a}, \qquad \Gamma^0_{11} = \frac{a_{,\eta}(1-2\Phi)}{a(1+2\Phi)},$$
$$\Gamma^0_{22} = \frac{a_{,\eta} r^2(1-2\Phi)}{a(1+2\Phi)}, \qquad \Gamma^0_{33} = \frac{a_{,\eta} r^2 \sin^2\theta(1-2\Phi)}{a(1+2\Phi)}, \tag{A1}$$

$$\Gamma^0_{01} = \frac{\Phi_{,r}}{1+2\Phi}, \qquad \Gamma^0_{02} = \frac{\Phi_{,\theta}}{1+2\Phi}, \qquad \Gamma^0_{03} = \frac{\Phi_{,\varphi}}{1+2\Phi}, \tag{A2}$$

$$\Gamma^1_{11} = \frac{\Phi_{,r}}{2\Phi-1}, \qquad \Gamma^1_{21} = \frac{\Phi_{,\theta}}{2\Phi-1},$$
$$\Gamma^1_{22} = -\frac{r(r\Phi_{,r}+2\Phi-1)}{2\Phi-1}, \tag{A3}$$

$$\Gamma^1_{31} = \frac{\Phi_{,\varphi}}{2\Phi-1}, \qquad \Gamma^1_{33} = -\frac{r\sin^2\theta(r\Phi_{,r}+2\Phi-1)}{2\Phi-1},$$
$$\Gamma^1_{01} = \frac{a_{,\eta}}{a}, \qquad \Gamma^1_{00} = \frac{\Phi_{,r}}{1-2\Phi}, \tag{A4}$$

$$\Gamma^2_{11} = \frac{\Phi_{,\theta}}{r^2(1-2\Phi)}, \qquad \Gamma^2_{21} = \frac{-r\Phi_{,r}-2\Phi+1}{r(1-2\Phi)},$$
$$\Gamma^2_{22} = \frac{\Phi_{,\theta}}{2\Phi-1}, \tag{A5}$$

$$\Gamma^2_{32} = \frac{\Phi_{,\varphi}}{2\Phi-1}, \qquad \Gamma^2_{33} = -\frac{\sin\theta[\sin\theta\Phi_{,\theta}+\cos\theta(2\Phi-1)]}{2\Phi-1},$$
$$\Gamma^2_{02} = \frac{a_{,\eta}}{a}, \tag{A6}$$

$$\Gamma^2_{00} = \frac{\Phi_{,\theta}}{r^2(1-2\Phi)}, \tag{A7}$$

$$\Gamma^3_{11} = \frac{\csc^2\theta\Phi_{,\varphi}}{r^2(1-2\Phi)}, \qquad \Gamma^3_{22} = \frac{\csc^2\theta\Phi_{,\varphi}}{1-2\Phi},$$
$$\Gamma^3_{31} = \frac{-r\Phi_{,r}-2\Phi+1}{r(1-2\Phi)}, \tag{A8}$$

$$\Gamma^3_{32} = \frac{\Phi_{,\theta}+\cot\theta(2\Phi-1)}{2\Phi-1}, \qquad \Gamma^3_{33} = \frac{\Phi_{,\varphi}}{2\Phi-1},$$
$$\Gamma^3_{03} = \frac{a_{,\eta}}{a}, \tag{A9}$$

$$\Gamma^3_{00} = \frac{\csc^2\theta\Phi_{,\varphi}}{r^2(1-2\Phi)}. \tag{A10}$$

When scalar-tensor gravity is considered, instead of GR, the line element is given by Eq. (5.1) instead of (2.1). In this case, the corresponding nonvanishing Christoffel symbols are

$$\Gamma^0_{00} = \frac{a_{,\eta}}{a}, \qquad \Gamma^0_{11} = \frac{a_{,\eta}(1-2\Phi)}{a(1+2\Psi)},$$
$$\Gamma^0_{22} = \frac{a_{,\eta} r^2(1-2\Phi)}{a(1+2\Psi)}, \qquad \Gamma^0_{33} = \frac{a_{,\eta} r^2 \sin^2\theta(1-2\Phi)}{a(1+2\Psi)} \tag{A11}$$

$$\Gamma^0_{01} = \frac{\Psi_{,r}}{1+2\Psi}, \qquad \Gamma^0_{02} = \frac{\Psi_{,\theta}}{1+2\Psi}, \qquad \Gamma^0_{03} = \frac{\Psi_{,\varphi}}{1+2\Psi} \tag{A12}$$

$$\Gamma^1_{11} = \frac{\Phi_{,r}}{2\Phi-1}, \qquad \Gamma^1_{21} = \frac{\Phi_{,\theta}}{2\Phi-1},$$
$$\Gamma^1_{22} = -\frac{r(r\Phi_{,r}+2\Phi-1)}{2\Phi-1} \tag{A13}$$

$$\Gamma^1_{31} = \frac{\Phi_{,\varphi}}{2\Phi-1}, \qquad \Gamma^1_{33} = -\frac{r\sin^2\theta(r\Phi_{,r}+2\Phi-1)}{2\Phi-1},$$
$$\Gamma^1_{01} = \frac{a_{,\eta}}{a}, \qquad \Gamma^1_{00} = \frac{\Psi_{,r}}{1-2\Phi} \tag{A14}$$

[4]A "turnaround size" could be defined, for example, as $\ell_T = \sqrt{\mathcal{A}_T}$, where $\mathcal{A}_T$ is the area of the turnaround surface defined above. In scalar-tensor gravity, this definition introduces the second potential Φ (in addition to Ψ) in ℓ_T.

$$\Gamma^2_{11} = \frac{\Phi_{,\theta}}{r^2(1-2\Phi)}, \qquad \Gamma^2_{21} = \frac{-r\Phi_{,r} - 2\Phi + 1}{r(1-2\Phi)},$$
$$\Gamma^2_{22} = \frac{\Phi_{,\theta}}{2\Phi - 1}, \tag{A15}$$

$$\Gamma^2_{32} = \frac{\Phi_{,\varphi}}{2\Phi - 1}, \qquad \Gamma^2_{33} = -\frac{\sin\theta[\sin\theta\Phi_{,\theta} + \cos\theta(2\Phi - 1)]}{2\Phi - 1},$$
$$\Gamma^2_{02} = \frac{a_{,\eta}}{a} \tag{A16}$$

$$\Gamma^2_{00} = \frac{\Psi_{,\theta}}{r^2(1-2\Phi)} \tag{A17}$$

$$\Gamma^3_{11} = \frac{\csc^2\theta\Phi_{,\varphi}}{r^2(1-2\Phi)}, \qquad \Gamma^3_{22} = \frac{\csc^2\theta\Phi_{,\varphi}}{1-2\Phi},$$
$$\Gamma^3_{31} = \frac{-r\Phi_{,r} - 2\Phi + 1}{r(1-2\Phi)} \tag{A18}$$

$$\Gamma^3_{32} = \frac{\Phi_{,\theta} + \cot\theta(2\Phi - 1)}{2\Phi - 1}, \qquad \Gamma^3_{33} = \frac{\Phi_{,\varphi}}{2\Phi - 1},$$
$$\Gamma^3_{03} = \frac{a_{,\eta}}{a} \tag{A19}$$

$$\Gamma^3_{00} = \frac{\csc^2\theta\Psi_{,\varphi}}{r^2(1-2\Phi)}. \tag{A20}$$

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