Gravitational scattering at the seventh order in G: Nonlocal contribution at the sixth post-Newtonian accuracy

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A recently introduced approach to the classical gravitational dynamics of binary systems involves intricate integrals (linked to a combination of nonlocal-in-time interactions with iterated $\frac{1}{r}$-potential scattering) which have so far resisted attempts at their analytical evaluation. By using computing techniques developed for the evaluation of multiloop Feynman integrals (notably harmonic polylogarithms and Mellin transform) we show how to analytically compute all the integrals entering the nonlocal-in-time contribution to the classical scattering angle at the sixth post-Newtonian accuracy, and at the seventh order in Newton's constant, G (corresponding to six-loop graphs in the diagrammatic representation of the classical scattering angle).

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I. INTRODUCTION

The detection of the gravitational wave signals emitted by compact binary systems [1] has opened a new path for investigating the structure of the Universe, and offers a novel tool for studying the gravitational interaction. The full exploitation of this new observational tool poses, however, the theoretical challenge to model with improved accuracy the gravitational wave signals emitted during the last orbits of coalescing black-hole binaries.

The latter theoretical challenge has recently motivated the construction of a new approach [2] to the analytical description of the classical conservative dynamics of binary systems. The latter approach is based on a novel way of combining results from several theoretical formalisms, developed for studying the gravitational potential within classical general relativity (GR): post-Newtonian (PN) expansion, post-Minkowskian (PM) expansion, multipolar-post-Minkowskian expansion, effective-field-theory, gravitational self-force approach, and effective one-body method. Another feature of the approach of Ref. [2] is to combine knowledge from gauge-invariant observables of *bound motions*, and from gauge-invariant observables of *scattering motions*. In view of its characteristic multipronged nature, we henceforth refer to the method of Ref. [2] as the tutti frutti (TF) method.

The TF method has succeeded in pushing the state of the art to the sixth post-Newtonian (6PN) accuracy in the conservative dynamics of binary systems [3–5]. More precisely, the TF method has determined the full structure of two gauge-invariant characterizations of the 6PN-accurate dynamics: the scattering angle χ, and the radial action I_r, both being considered as functions of the total center-of-mass (c.m.) energy, $E = \sqrt{s}$, and of the total c.m. angular momentum, J. Both quantities are given as double expansions in powers of the gravitational constant G (PM expansion), and of the inverse velocity of light $1/c$ (PN expansion), each term of these expansions being a polynomial in the symmetric mass ratio $\nu = m_1 m_2/(m_1 + m_2)^2$. Most of the $O(200)$ coefficients entering the latter gauge-invariant characteristics of the 6PN dynamics have been analytically obtained within the TF method except for six coefficients entering the local-in-time Hamiltonian. In addition, the explicit implementation of the TF method requires the evaluation of a certain number of "scattering integrals," A_{mnk}, arising in the computation of the nonlocal-in-time contribution to the scattering angle. Previous work [5] only succeeded in analytically computing a fraction of the latter scattering integrals: namely the A_{mnk}'s for $m = 0$, 1 and for $(mnk) = (200), (221)$. Some other scattering integrals [namely A_{2nk} for $(nk) = (20)$, $(40), (41), (42)$] were only numerically evaluated (with a modest, eight-digit accuracy).

Many computing techniques [6–22] have been developed for the evaluation of multiloop Feynman integrals. We show here how the use of some of these techniques, notably involving the use of Mellin transforms [8], harmonic polylogarithms (HPL) [9], and expansion of hypergeometric functions about half-integer parameters [13], allows one to derive the *analytical values* of all the scattering coefficients A_{mnk}'s entering the nonlocal-in-time contribution at the seventh order in G, and at the 6PN accuracy (the G^7 order corresponds to the value $m=3$ of the first index m of the scattering integrals A_{mnk}). In particular, the present work will determine the exact, analytical values of the $O(G^6)$ scattering integrals A_{2nk} that were left undetermined in Ref. [5], and which enter the full determination of the 6PN *local-in-time* dynamics, via the combination D, defined as [see Eq. (6.29) of [5]]

$$D=\frac{1}{\pi}\left(\frac{5}{2}A_{221}+\frac{15}{8}A_{200}+A_{242}\right). \tag{1.1}$$

The present work is an extension of Ref. [23] which derived the analytical expressions of the scattering coefficients A_{2nk} entering the nonlocal-in-time contribution at the sixth order in G.

II. SETUP ON THE GR SIDE

The TF method extracts information from various classical GR observables. In particular, one of the crucial gauge-invariant observables used in this approach is the *conservative*[1] classical scattering angle χ during a gravitational encounter, considered as a function of the total c.m. energy, $E=\sqrt{s}$, the total c.m. angular momentum, J, and the symmetric mass ratio ν. We use the notation

$$M\equiv m_1+m_2;\qquad \mu\equiv\frac{m_1m_2}{m_1+m_2};$$
$$\nu\equiv\frac{\mu}{M}=\frac{m_1m_2}{(m_1+m_2)^2}. \tag{2.1}$$

The TF approach decomposes $\chi(E,J;\nu)$ into three separate contributions:

$$\chi(E,J,\nu)=\chi^{\rm loc,f}+\chi^{\rm nonloc,h}+\chi^{\rm f\text{-}h}, \tag{2.2}$$

corresponding to an analogous decomposition of the total Hamiltonian: $H(t)=H^{\rm loc,f}(t)+H^{\rm nonloc,h}(t)+\Delta^{\rm f\text{-}h}H(t)$. Here $\chi^{\rm loc,f}$ is the scattering angle that would be induced by the (f-route) local-in-time piece of the Hamiltonian, $H^{\rm loc,f}(t)$. By contrast, $\chi^{\rm nonloc,h}$ is induced by the (h-route) nonlocal-in-time piece of the Hamiltonian, $H^{\rm nonloc,h}(t)$, while the last contribution, $\chi^{\rm f\text{-}h}$, is induced by the complementary (f-route) term $\Delta^{\rm f\text{-}h}H(t)$, which is algorithmically derived [5] from the ν structure of $\chi^{\rm nonloc,h}$. The present work will focus on $\chi^{\rm nonloc,h}$, which is perturbatively determined as a double expansion in powers of the gravitational constant G (PM expansion), and of the inverse velocity of light $1/c$ (PN expansion). It is convenient to express the combined PM + PN expansion of $\chi^{\rm nonloc,h}$ in terms of the dimensionless variables

$$p_\infty\equiv\sqrt{\gamma^2-1},\quad\text{and}\quad j\equiv\frac{cJ}{Gm_1m_2}, \tag{2.3}$$

where the dimensionless energy parameter γ is defined in terms of the total c.m. energy $E=\sqrt{s}$ by

$$\gamma\equiv\frac{E^2-m_1^2c^4-m_2^2c^4}{2m_1m_2c^4}. \tag{2.4}$$

The variable γ is equal both to the Lorentz factor between the two incoming worldlines, and to the μc^2-rescaled effective energy $\mathcal{E}_{\rm eff}$ entering the effective-one-body description [28] of the binary dynamics.

As $j\propto\frac{c}{G}$, the PM expansion of $\chi^{\rm nonloc,h}$ is equivalent to an expansion in inverse powers of j, and reads

$$\frac{1}{2}\chi^{\rm nonloc,h}(\gamma,j;\nu)=+\nu p_\infty^4\left(\frac{A_0^h(p_\infty,\nu)}{j^4}+\frac{A_1^h(p_\infty,\nu)}{p_\infty j^5}+\frac{A_2^h(p_\infty,\nu)}{p_\infty^2j^6}+\frac{A_3^h(p_\infty,\nu)}{p_\infty^3j^7}+O\left(\frac{1}{j^8}\right)\right). \tag{2.5}$$

The last-written contribution $\propto A_3^h(p_\infty,\nu)/(p_\infty^3j^7)$ belongs to the 7PM approximation, $O(G^7)$. The dimensionless coefficients $A_m^h(p_\infty,\nu)$, $m=0,1,2,3,\ldots$, then admit a PN expansion, i.e., an expansion in powers of $p_\infty=O(\frac{1}{c})$, modulo logarithms of p_∞, say

$$A_m^h(p_\infty,\nu)=\sum_{n\geq0}\left[A_{mn}(\nu)+A_{mn}^{\rm ln}(\nu)\ln\left(\frac{p_\infty}{2}\right)\right]p_\infty^n. \tag{2.6}$$

The coefficient $A_{mn}(\nu)$ is a polynomial in ν of order n and parametrizes a term of order $\frac{p_\infty^{4+n-m}}{j^{4+m}}\sim\frac{G^{4+m}}{c^{8+n}}$ (with $m\geq0$, $n\geq0$) in the combined PM + PN expansion of the nonlocal scattering angle. The leading-order contribution to the nonlocal dynamics is at the combined 4PM and 4PN level, i.e., $\propto G^4/c^8$ [29]. The corresponding nonlocal scattering coefficient, coming from $m=0$ and $n=0$, is $A_0^h(p_\infty,\nu)=\pi[-\frac{37}{5}\ln(\frac{p_\infty}{2})-\frac{63}{4}]+O(p_\infty^2)$ [25]. The higher-order logarithmic coefficients $A_{mn}^{\rm ln}(\nu)$ were analytically determined [3–5] so that we shall henceforth focus on the nonlogarithmic coefficients $A_{mn}(\nu)$. Finally, the numerical scattering coefficient A_{mnk} is defined as the coefficient of the kth power of the symmetric mass ratio ν in $A_{mn}(\nu)$:

$$A_{mn}(\nu)\equiv\sum_{k=0}^{n}A_{mnk}\nu^k, \tag{2.7}$$

with $k=0,1,2,\ldots$.

[1] See Refs. [24–27] for discussions including the radiation-reaction contribution to the scattering angle.

III. CLASSICAL PERTURBATIVE EXPANSION OF THE NONLOCAL-IN-TIME SCATTERING ANGLE

Reference [25] has derived a general link [valid to first order in tail effects, i.e., up to $O[(G^4/c^8)^2] = O[G^8/c^{16}]$] between the nonlocal-in-time contribution $\chi^{\rm nonloc,h}$ to the scattering angle and the integrated nonlocal action. Namely,

$$\chi^{\rm nonloc,h}(E,J,\nu) = \frac{\partial W^{\rm nonloc,h}(E,J,\nu)}{\partial J}, \qquad (3.1)$$

where

$$W^{\rm nonloc,h}(E,J;\nu) \equiv \int_{-\infty}^{+\infty} dt H^{\rm nonloc,h}(t) \qquad (3.2)$$

is the integrated (h-route) nonlocal action. The TF method expresses the latter quantity by the following explicit (regularized) twofold integral [to be evaluated along a hyperbolic-motion solution of the local-in-time Hamiltonian $H^{\rm loc,f}(t)$],

$$W^{\rm nonloc,h} = \alpha {\rm Pf}_{\Delta t^h} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{dtdt'}{|t-t'|} \mathcal{F}^{\rm split}_{\rm GW}(t,t') + O(\alpha^2). \qquad (3.3)$$

Here, $\alpha \equiv GE/c^5 = G\sqrt{s}/c^5$; ${\rm Pf}_h$ denotes the partie-finie regularization of the logarithmically divergent t' integration at $t'=t$ [using the harmonic-coordinate-based time scale $\Delta t^h = 2r^h_{12}(t)/c$]; and $\mathcal{F}^{\rm split}_{\rm GW}(t,t')$ is the time-split version (defined below) of the gravitational-wave energy flux (absorbed and then) emitted by the system.[2] The nonlocal expansion (3.3) is keyed by successive powers of α. The $O(\alpha)$ term is called first-order tail; the $O(\alpha^2)$ is the second-order tail contribution, etc. The effects linked to the second-order tail contribution have been analytically derived in [5], at the combined 6PM and 5.5PN accuracy. [The next term in the PN expansion of the second-order tail contribution is at the 6.5PN level, which is beyond the accuracy sought for in the present work.]

We shall deal first with terms belonging to the $O(\alpha)$, first-order tail contribution explicated above. The time-split version of the gravitational-wave energy flux is given, at the needed accuracy, by

$$\mathcal{F}^{\rm split}_{\rm 2PN}(t,t') = \frac{G}{c^5}\left[\frac{1}{5} I^{(3)}_{ab}(t) I^{(3)}_{ab}(t') + \eta^2\left(\frac{1}{189} I^{(4)}_{abc}(t) I^{(4)}_{abc}(t') + \frac{16}{45} J^{(3)}_{ab}(t) J^{(3)}_{ab}(t')\right) + \eta^4\left(\frac{1}{9072} I^{(5)}_{abcd}(t) I^{(5)}_{abcd}(t') + \frac{1}{84} J^{(4)}_{abc}(t) J^{(4)}_{abc}(t')\right)\right]. \qquad (3.4)$$

Here $\eta \equiv 1/c$ and the superscript in parentheses indicates repeated time derivatives. The multipole moments I_L, J_L denote the values of the canonical moments M_L, S_L entering the PN-matched [29–33] multipolar-post-Minkowskian (MPM) formalism [34], when they are reexpressed as explicit functionals of the instantaneous state of the binary system. These multipole moments parametrize (in a minimal, gauge-fixed way) the exterior gravitational field (and therefore the relevant coupling between the system and a long-wavelength external radiation field).

The subscript 2PN on $\mathcal{F}^{\rm split}_{\rm 2PN}(t,t')$ indicates that the multipole moments must be individually evaluated with the PN accuracy needed for knowing $\mathcal{F}^{\rm split}_{\rm 2PN}(t,t')$, and the corresponding ordinary (non-time-split) gravitational wave flux,

$$\mathcal{F}_{\rm 2PN}(t) = \mathcal{F}^{\rm split}_{\rm 2PN}(t,t), \qquad (3.5)$$

with a *fractional* 2PN accuracy. More explicitly, this means that we need the 2PN-accurate value of the quadrupole moment expressed in terms of the material source [35,36]. The other moments (the electric octupole moment I_{ijk}, the electric hexadecapole moment, I_{ijkl}, the magnetic quadrupole moment, J_{ij}, and the magnetic octupole moment, J_{ijk}) need only to be known at the 1PN fractional accuracy [30,31,37]. Their explicit expressions (in the center-of-mass harmonic coordinate frame) have been recalled in Eq. (3.3) and in Table I of Ref. [5].

Introducing the shorthand notation

$$\langle F\rangle_\infty \equiv \int_{-\infty}^{+\infty} dt F(t), \qquad (3.6)$$

and expressing the partie-finie operation ${\rm Pf}_{\Delta t^h}$ entering Eq. (3.3) in terms of a partie-finie operation ${\rm Pf}_{2s/c}$ involving an intermediate length scale s, we decompose the nonlocal integrated action $W^{\rm nonloc,h}$ into two contributions:

$$W^{\rm nonloc,h}(E,j) = W^{\rm tail,h}_1(E,j) + W^{\rm tail,h}_2(E,j) + O(\alpha^2), \qquad (3.7)$$

where

$$W^{\rm tail,h}_1(E,j) \equiv -\alpha\left\langle {\rm Pf}_{2s/c}\int_{-\infty}^{\infty}\frac{dt'}{|t-t'|}\mathcal{F}^{\rm split}_{\rm 2PN}(t,t')\right\rangle_\infty, \qquad (3.8)$$

[2] We consider the conservative dynamics of a binary system interacting in a time-symmetric way.

and

$$W_2^{\rm tail,h}(E,j) \equiv 2\alpha \left\langle \mathcal{F}_{\rm 2PN}(t) \ln\left(\frac{r_{12}^h(t)}{s}\right)\right\rangle_\infty. \qquad (3.9)$$

The integrated nonlocal action $W^{\rm nonloc,h}(E,j)$, and therefore each partial contribution, Eqs. (3.8) and (3.9), has to be evaluated along a 2PN-accurate hyperbolic motion.

IV. QUASI-KEPLERIAN PARAMETRIZATION OF THE HYPERBOLIC MOTION, AND ITS LARGE-ECCENTRICITY EXPANSION

In view of Eq. (3.1), the PM expansion (2.5) of $\chi^{\rm nonloc,h}$ is equivalent to the following expansion of the integrated nonlocal action $W^{\rm nonloc,h}(E,j)$ in inverse powers of j,

$$\frac{cW^{\rm nonloc,h}(\gamma,j;\nu)}{2Gm_1m_2} = -\nu p_\infty^4\left(\frac{A_0^h(p_\infty,\nu)}{3j^3} + \frac{A_1^h(p_\infty,\nu)}{4p_\infty j^4} + \frac{A_2^h(p_\infty,\nu)}{5p_\infty^2 j^5} + \frac{A_3^h(p_\infty,\nu)}{6p_\infty^3 j^6} + O\left(\frac{1}{j^7}\right)\right). \qquad (4.1)$$

Remembering the proportionality between $j = cJ/(Gm_1m_2)$ and the impact parameter b (via $J = bP_{\rm c.m.}$, where $P_{\rm c.m.}$ is the c.m. linear momentum of each body), we see that the computation of the scattering coefficients $A_m^h(p_\infty,\nu)$ amounts to expanding the integrated nonlocal action in inverse powers of b. An explicit way to compute the large-impact-parameter expansion of $W^{\rm nonloc,h}$ is to use the quasi-Keplerian parametrization [38] of the 2PN-accurate hyperbolic-motion solution [39] of the 2PN dynamics of a binary system in harmonic coordinates [40,41].

The hyperbolic quasi-Keplerian parametrization involves a semi-major-axis-like quantity a_r, together with several eccentricity-like quantities e_t, e_r, e_ϕ. The variable parametrizing the time development is an eccentric-anomaly-like (hyperbolic) angle v varying from $-\infty$ to $+\infty$:

$$\begin{aligned} r &= \bar{a}_r(e_r\cosh v - 1),\\ \ell &= \bar{n}(t-t_P) = e_t\sinh v - v + f_t V(v) + g_t \sin V(v),\\ \bar{\phi} &= \frac{\phi-\phi_P}{K} = V(v) + f_\phi \sin 2V(v) + g_\phi \sin 3V(v). \end{aligned} \qquad (4.2)$$

Here, we use adimensionalized variables (and $c = 1$), notably $r = r^{\rm phys}/(GM)$, $t = t^{\rm phys}/(GM)$, while $V(v)$ is given by

$$V(v) = 2\arctan\left[\Omega_{e_\phi}\tanh\frac{v}{2}\right], \qquad (4.3)$$

where

$$\Omega_{e_\phi} \equiv \sqrt{\frac{e_\phi+1}{e_\phi-1}}. \qquad (4.4)$$

The expressions [as functions of the specific binding energy $\bar{E} \equiv (E_{\rm tot} - Mc^2)/(\mu c^2)$ and of the dimensionless angular momentum $j = cJ/(GM\mu)$] of the orbital parameters $\bar{n}$ (hyperbolic mean motion) and K (hyperbolic periastron precession), as well as $\bar{a}_r, e_t, e_r, e_\phi, f_t, g_t, f_\phi, g_\phi$, can be found in Appendix A of Ref. [5]. Let us only recall here the expressions of $\bar{a}_r$, and e_r in terms of $\bar{E}$ and j:

$$\begin{aligned} \bar{a}_r &= \frac{1}{2\bar{E}}\left[1 - \frac{1}{2}\bar{E}\eta^2(-7+\nu) + \frac{1}{4}\bar{E}^2\eta^4\left(1+\nu^2 - 8\frac{(-4+7\nu)}{\bar{E}j^2}\right)\right],\\ e_r^2 &= 1 + 2\bar{E}j^2 + \bar{E}[5\bar{E}j^2(\nu-3) + 2\nu - 12]\eta^2 + \frac{\bar{E}}{j^2}[(4\nu^2+80-45\nu)\bar{E}^2j^4 + (\nu^2+74\nu+30)\bar{E}j^2 + 56\nu - 32]\eta^4. \end{aligned} \qquad (4.5)$$

When using this quasi-Keplerian parametrization, the combined PM + PN expansion of $W^{\rm nonloc,h}(\gamma,j;\nu)$ can be constructed from the combined large-e_r+ large-a_r expansion of the function $W^{\rm nonloc,h}(e_r,a_r)$. On the one hand, as the tail action starts at the 4PN level, we need to work to the next-to-next-to-leading-order (NNLO) in $\frac{1}{a_r} \sim \frac{p_\infty^2}{c^2}$ in order to reach the 6PN accuracy. On the other hand, as the tail action starts at the 4PM level [$O(G^4)$], we need to work to the next-to-next-to-next-to-leading-order ($\rm N^3LO$) in $\frac{1}{e_r}$ in order to reach the 7PM, $O(G^7)$, accuracy (seventh order in $\frac{1}{b}$).

Without presenting too many technical details, let us illustrate the origin of some of the structures entering the scattering integrals A_{mnk} by explaining how one can compute the large-eccentricity expansion of the crucial nonlocal integral

$$\iint dtdt' \frac{dtdt'}{|t-t'|}\mathcal{F}_{\rm GW}^{\rm split}(t,t') \qquad (4.6)$$

entering $W^{\rm nonloc,h}$. The first step is to introduce the auxiliary time variable $T \in [-1,1]$:

$$T \equiv \tanh\frac{v}{2}. \qquad (4.7)$$

In terms of this variable, the 2PN-accurate functional relation between the original (rescaled) time variable $t \equiv \frac{t^{\rm phys}}{GM}$ and the hyperbolic eccentric anomaly v reads

$$t=\frac{2}{\bar{n}}\left[e_t\frac{T}{(1-T^2)}-\operatorname{arctanh}(T)+f_t\arctan(\Omega_{e_\phi}T)+g_t\frac{\Omega_{e_\phi}T}{1+\Omega_{e_\phi}^2T^2}\right],\tag{4.8}$$

with a corresponding expression for t' vs $T'\equiv\tanh\frac{v'}{2}$. One then forms $|t-t'|$, whose 2PN-accurate large-eccentricity expansion reads

$$|t-t'|=|T-T'|\frac{1+TT'}{(1-T^2)(1-T'^2)}\bar{a}_r^{3/2}e_r\times\left[2-(1+2\nu)\frac{\eta^2}{\bar{a}_r}+\frac{1}{4}(8\nu^2-8\nu-1)\frac{\eta^4}{\bar{a}_r^2}\right]\times\left[1+\frac{1}{e_r}\mathcal{P}_1+\frac{1}{e_r^2}\mathcal{P}_2+\frac{1}{e_r^3}\mathcal{P}_3+O\left(\frac{1}{e_r^4}\right)\right],\tag{4.9}$$

with coefficients $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ of the form

$$\mathcal{P}_1=\mathcal{P}_{10}(T,T')+\mathcal{P}_{12}(T,T')\frac{\eta^2}{\bar{a}_r}+\mathcal{P}_{14}(T,T')\frac{\eta^4}{\bar{a}_r^2},$$
$$\mathcal{P}_2=\mathcal{P}_{24}(T,T')\frac{\eta^4}{\bar{a}_r^2},$$
$$\mathcal{P}_3=\mathcal{P}_{34}(T,T')\frac{\eta^4}{\bar{a}_r^2}.\tag{4.10}$$

Let us illustrate the structure of the coefficients $\mathcal{P}_{nm}(T,T')$ entering the $\mathcal{P}_n$'s by citing the expressions of the first few of them. Introducing the shorthand notation

$$At(T,T')\equiv\arctan(T)-\arctan(T'),$$
$$Ath(T,T')\equiv\operatorname{arctanh}(T)-\operatorname{arctanh}(T'),\tag{4.11}$$

we have

$$\mathcal{P}_{10}(T,T')=-\frac{(1-T'^2)(1-T^2)}{(TT'+1)(T-T')}Ath(T,T'),$$
$$\mathcal{P}_{12}(T,T')=\frac{1}{2}(-8+3\nu)\mathcal{P}_{10}(T,T'),$$
$$\mathcal{P}_{14}(T,T')=\frac{1}{8}\nu(-29+3\nu)\mathcal{P}_{10}(T,T')+\frac{1}{8}\nu(-15+\nu)\frac{(TT'-1)(1-T'^2)(1-T^2)}{(1+T'^2)(1+T^2)(TT'+1)},$$
$$\mathcal{P}_{24}(T,T')=-\frac{3}{2}(-5+2\nu)\frac{(1-T'^2)(1-T^2)}{(TT'+1)(T-T')}At(T,T')+\frac{1}{2}\nu(\nu-15)\frac{TT'(1-T^2)(1-T'^2)}{(1+T^2)^2(1+T'^2)^2}-\frac{1}{8}(16-43\nu+\nu^2)\frac{(1-T^2)^2(1-T'^2)^2}{(1+T^2)^2(1+T'^2)^2}+2(-4+7\nu)\frac{(T^2T'^2+1)(T^2+T'^2)}{(1+T^2)^2(1+T'^2)^2},$$
$$\mathcal{P}_{34}(T,T')=\frac{3}{2}(-4+7\nu)\mathcal{P}_{10}(T,T')-\frac{1}{2}\nu(\nu-15)\frac{TT'(TT'-1)(1-T'^2)(1-T^2)}{(TT'+1)(1+T^2)^2(1+T'^2)^2}+\frac{1}{8}(\nu^2+9\nu-60)\frac{(TT'-1)(1-T'^2)(1-T^2)[(TT'+1)^2-(T-T')^2][(TT'-1)^2-(T+T')^2]}{(TT'+1)(1+T^2)^3(1+T'^2)^3}+6(-5+2\nu)\frac{(TT'-1)(1-T'^2)(1-T^2)[T^2(1+T'^2)^2+T'^2(1+T^2)^2]}{(TT'+1)(1+T^2)^3(1+T'^2)^3}.\tag{4.12}$$

Using the above relations one can compute the large-eccentricity expansion of the measure

$$\frac{dtdt'}{|t-t'|}=\frac{1}{|t(T)-t'(T')|}\frac{dt}{dT}\frac{dt'}{dT'}dTdT'\equiv d\mathcal{M}_{(T,T')}.\tag{4.13}$$

Its schematic 2PN-accurate structure reads

$$d\mathcal{M}_{(T,T')}=2e_r\bar{a}_r^{3/2}\left[1-\frac{1+2\nu}{2\bar{a}_r}\eta^2-\frac{1+8\nu-8\nu^2}{8\bar{a}_r^2}\eta^4\right]\times\frac{(1+T'^2)(1+T^2)dTdT'}{(1-T'^2)(1-T^2)(1+TT')|T-T'|}\times\left(1+\frac{\mathcal{M}_1}{e_r}+\frac{\mathcal{M}_2}{e_r^2}+\frac{\mathcal{M}_3}{e_r^3}+O\left(\frac{1}{e_r^4}\right)\right),\tag{4.14}$$

where we have explicitly shown only the LO contribution in the large-eccentricity expansion. The NLO, NNLO and N^{3}LO contributions [respectively described by the coefficients $\mathcal{M}_1(T,T';\nu,\eta)$, $\mathcal{M}_2(T,T';\nu,\eta)$ and $\mathcal{M}_3(T,T';\nu,\eta)$] have long expressions that we do not explicitly display here. Let us simply note that [recalling the definitions Eq. (4.11)] $\mathcal{M}_1(T,T';\nu,\eta)$ involves the function $Ath(T,T')$ linearly, $\mathcal{M}_2(T,T';\nu,\eta)$ involves $Ath(T,T')$, $Ath^2(T,T')$ and $At(T,T')$, while $\mathcal{M}_3(T,T';\nu,\eta)$ involves $Ath(T,T')$, $Ath^2(T,T')$, $Ath^3(T,T')$ as well as $At(T,T')$ and $Ath(T,T')At(T,T')$.

As illustrated here, apart from rational functions of T and T', the large-eccentricity expansion has a polynomial dependence on the transcendental functions $\arctan(T)$, $\arctan(T')$, $\mathrm{arctanh}(T)$ and $\mathrm{arctanh}(T')$. Using these expansions (as well as corresponding expansions of the various multipole moments), one finally gets explicit integral expressions for the scattering coefficients A_{mnk} of the form

$$A_{mnk} = \int_{-1}^{+1}\int_{-1}^{+1}\frac{dTdT'}{|T-T'|}a_{mnk}(T,T'), \qquad (4.15)$$

with integrands $a_{mnk}(T,T')$ of the form

$$a_{mnk}(T,T') = \sum_{p,q\geq 0} R^{mnk}_{pq}(T,T')Ath(T,T')^p At(T,T')^q, \qquad (4.16)$$

where $R^{mnk}_{pq}(T,T')$ are rational functions of T and T', and where we used the shorthands (4.11). The highest power of $Ath(T,T') \equiv \mathrm{arctanh}(T) - \mathrm{arctanh}(T')$ in this expression is directly equal to the order of expansion in $\frac{1}{e_r}$ (and therefore in G, recalling the leading-order expression $e_r \approx \sqrt{1+2\bar{E}j^2}$) of the relativistic hyperbolic motion.

Reference [5] succeeded in analytically computing (up to the 6PN accuracy) the numerical coefficients A_{mnk} when $m=0$ (G^4 level) and $m=1$ (G^5 level). By contrast, the integrands of Eq. (4.15) become so involved when $m=2$ and $m=3$ (G^6 and G^7 levels) that most of them resisted analytical integration by standard integration methods.

V. MULTIPLE POLYLOGARITHMS AND HARMONIC POLYLOGARITHMS

To determine the analytic expressions of the scattering integrals A_{2nk} we follow one of the strategies used in the realm of multiloop Feynman calculus, namely the reduction to *iterated integrals* [6,7,9–12,14–22]. Given a sequence of univariate functions $g_{a_1}(x), g_{a_2}(x), \ldots, g_{a_n}(x)$, assumed (say) to be regular at $x=0$, iterated integrals are recursively defined by $G(a_1,a_2,\ldots,a_n;x) = \int_0^x dt_1 g_{a_1}(t_1) \times G(a_2,\ldots,a_n;t_1)$, with the starting value $G(\varnothing;x)=1$. The simplest class of iterated integrals are the *multiple polylogarithms* defined by considering a sequence of inverse-linear functions: $g_{a_i}(x) = (x-a_i)^{-1}$. These were introduced by Poincaré [42], and have been the topic of many mathematical studies, e.g., [43–47]. They also came up as important tools for expressing certain multiloop Feynman integrals [17,48]. On the other hand, from the practical point of view, a *subclass* of the multiple logarithms, the harmonic polylogarithms (HPL) [9], has turned out to be sufficient, and very useful, to express many Feynman integrals. They are defined by restricting the singular points a_i entering $G(a_1,a_2,\ldots,a_n;x)$ to taking one of the three values $+1,-1$ or 0, and by normalizing the inverse-linear factors in a slightly different way. Specifically, the HPLs are defined as the recursive integrals,

$$H_{i_1,i_2,\ldots i_n}(x) = \int_0^x dt_1 f_{i_1}(t_1)H_{i_2\ldots i_n}(t_1), \qquad (5.1)$$

with $f_{\pm1}(x) = (1\mp x)^{-1}$, $f_0(x) = 1/x$, and a regularization at $x=0$ such that $H_{0,0,\ldots,0}(x) \equiv \ln^n(x)/n!$.

A crucial feature of the multiple polylogarithms, and therefore of the HPLs, is that they enjoy special algebraic properties, going under the names of shuffle algebra, stuffle algebra, scaling invariance, shuffle-antipode relations, Hölder convolution, integration-by-parts identities, etc. In addition, all these special algebraic properties respect a filtration by the *weight*, i.e., by the number n of singular values, $a_1, a_2, \ldots, a_n$, or the number n of indices on $H_{i_1 i_2\ldots i_n}(x)$. The weight corresponds to the number of iterations appearing in the nested integral representation. For instance, at weight 1 a multiple polylogarithm is a simple logarithm, while at weight 2, it is a linear combination of a dilogarithm and a squared logarithm. The remarkable algebraic properties of multiple polylogarithms (and HPLs) allow one to express them algebraically, at any given weight n, in terms of a minimal subset of them, having weights $n' \leq n$. For instance, at weights $n=2$, 3, and 4 the minimal subsets are formed by 3, 8, and 18 elements, respectively. In addition, their evaluation for special values of their arguments $a_1, a_2, \ldots, a_n; x$ can often be reduced to a relatively small number of transcendental constants. This is particularly the case if, besides 0, the arguments $a_1, a_2, \ldots, a_n; x$ are roots of unity. For introductions to the vast literature on the properties, and evaluation, of multiple polylogarithms and HPLs (including computer-program implementations) see, e.g., [10–12,14,15,17–19,21,22,43,44,49,50].

VI. ANALYTIC EVALUATION OF THE $O(G^6)$ SCATTERING INTEGRALS VIA HARMONIC POLYLOGARITHMS

Let us now sketch how we could analytically compute the $O(G^6)$ scattering integrals, i.e., Eq. (4.15), with $m=2$, by reducing these twofold definite integrals to the evaluation of HPLs, of weight ≤ 4, for the values $x=1, i$ of the HPL variable.

First, using symmetry properties of the integrands $a_{mnk}(T,T')$ entering Eq. (4.16), it is possible to reduce the double integration to the triangle $0<T<1, 0<T'<T$. Let us start by discussing the integration over T' on the interval $0<T'<T$. The crucial information needed for discussing this first integration concerns the structure of the integrands $a_{mnk}(T,T')$, and particularly of the denominators entering the rational coefficients $R^{mnk}_{pq}(T,T')$ in Eq. (4.16), when $m=2$. To be concrete, let us discuss the case $(mnk)=(242)$ and exhibit one representative part of the integrand $a_{242}(T,T')$. It reads

$$\frac{-16(1-T^2)^3(1-T'^2)^3P_2(T,T')}{315(1+T^2)^8(1+T'^2)^8(1+TT')^3(T-T')^3}\times\left\{[\mathrm{arctanh}(T)-\mathrm{arctanh}(T')]^2-\frac{(T-T')^2}{2(1-T^2)^2}-\frac{(T-T')^2}{2(1-T'^2)^2}\right\}, \tag{6.1}$$

where $P_2(T,T')$ is a (symmetric) polynomial in T and T', of order 14 in both variables. By partial fractioning (6.1) with respect to T' (keeping T fixed) one is reduced to evaluating integrals of the type

$$\int dT'\frac{\mathrm{arctanh}^p(T')}{(T'-a)^q}, \tag{6.2}$$

where $p=0$, 1, 2, $1\le q\le 8$ and $a=\pm i, -\frac{1}{T}, T$ or ± 1. Integrating by parts (with respect to T'), one can reduce the power q down to $q=1$. At this stage, remembering that $\mathrm{arctanh}(T)=\frac{1}{2}\ln((1+T)/(1-T))$ [and $\arctan(T)=\mathrm{arctanh}(iT)/i$ for other denominators] are (as explained above) of weight 1, we see that the highest-weight term in the numerator, $\propto\ln^2((1+T')/(1-T'))$, is of weight 2, so that its integration over T' with the additional kernel $(T'-a)^{-1}$ will generate terms of weight 3. The explicit computation of the needed integration over $T'\in[0,T]$, with the values of a listed above, is found to involve at most the trilogarithm $\mathrm{Li}_3(z)$ at the rational arguments $z=-\frac{1+T}{1-T}$ or $z=-(\frac{1+T}{1-T})^2$.

Having so obtained an explicit weight-3 expression for the result of the integration over T', we need to perform the final integration over $T\in[0,1]$. This is done in three steps. The first step is the same that was used for the T' integration. There are now polynomial denominators involving powers of T^2+1, powers of $T\pm 1$, and also powers of T. Partial fractioning, and integrating by parts, one can reduce these powers to the first power. Second, we use the definition of HPLs to express the integrals containing T^{-1} and $(T\pm 1)^{-1}$ in terms of HPLs. Third, we consider the integrals containing $(T\pm i)^{-1}$: these cannot be directly cast in HPL format (which admits poles only at $T=0,\pm 1$). Therefore, we modify the integrands by the insertion of a parameter x, to be later replaced by a suitable value, so as to obtain the original integral back. Following a technique introduced many years ago to analytically evaluate multiloop Feynman integrals [6,7], the integral, now function of x, is reduced to iterated integrals of the type $\int_0^x dx_1(x_1-a_1)^{-1}\int_0^{x_1}dx_2(x_2-a_2)^{-1}\ldots$, by combining repeated differentiations with respect to x with partial fractioning, and integrations by parts, followed by quadratures to get back the original integral.

Let us show an example of this technique: all the A_{2nk} integrals contain, after the T' integration, the same combination of integrals of weight $w=4$,

$$\hat{J}=\int_0^1 dT\frac{-2\ln^3(\frac{1-T}{1+T})-3\mathrm{Li}_3[-(\frac{1-T}{1+T})^2]}{1+T^2}. \tag{6.3}$$

We modify the integral (6.3), to let it acquire a dependence on the new parameter x, i.e., $\hat{J}\to J(x)$, in the following way:

$$J(x)\equiv i\int_0^1 dT(1-x^2)\times\frac{-2\ln^3(\frac{1-T}{1+T})-3\mathrm{Li}_3[(\frac{(1-T)(1-x)}{(1+T)(1+x)})^2]}{2x(T+x)(T+1/x)}. \tag{6.4}$$

It is easily seen that the original integral is recovered at the value $x=i$, that is $\hat{J}=J(i)$, and that $J(1)=0$. By differentiating and reintegrating 3 times over x, on the model of $J(x)=\int_1^x dx(dJ(x)/dx)$, $J(x)$ can be expressed in terms of HPLs at weight $w\le 4$; namely,

$$\begin{aligned}iJ(x)={}&\frac{23}{240}\pi^4-21\ln(2)\zeta(3)+\pi^2\ln^2(2)-\ln^4(2)-24a_4+\frac{21}{2}H_{-1}(x)\zeta(3)-\frac{3}{2}H_0(x)\zeta(3)+\frac{21}{2}H_1(x)\zeta(3)\\&+\frac{1}{2}\pi^2H_{0,-1}(x)+\frac{1}{2}\pi^2H_{0,1}(x)-\frac{3}{2}\pi^2H_{-1,-1}(x)-\frac{3}{2}\pi^2H_{-1,1}(x)-\frac{3}{2}\pi^2H_{1,-1}(x)-\frac{3}{2}\pi^2H_{1,1}(x)\\&+12H_{0,1,-1}(x)\ln(2)+12H_{0,1,1}(x)\ln(2)-12H_{0,-1,-1,-1}(x)+6H_{0,-1,-1,0}(x)-12H_{0,-1,1,-1}(x)\\&+6H_{0,-1,1,0}(x)-12H_{0,1,-1,-1}(x)+6H_{0,1,-1,0}(x)-12H_{0,1,1,-1}(x)+6H_{0,1,1,0}(x)-6H_{-1,-1,-1,0}(x)\\&-6H_{-1,-1,1,0}(x)-6H_{-1,1,-1,0}(x)-6H_{-1,1,1,0}(x)-6H_{1,-1,-1,0}(x)-6H_{1,-1,1,0}(x)-6H_{1,1,-1,0}(x)\\&-6H_{1,1,1,0}(x)+12H_{0,-1,-1}(x)\ln(2)+12H_{0,-1,1}(x)\ln(2).\end{aligned} \tag{6.5}$$

TABLE I. Analytical results for the $O(G^6)$ scattering coefficients A_{2nk}.

Coefficient	Value
$\pi^{-1}A_{200}$	$-\frac{99}{4}-\frac{2079}{8}\zeta(3)$
$\pi^{-1}A_{220}$	$-\frac{41297}{112}-\frac{9216}{7}\ln(2)+\frac{49941}{64}\zeta(3)$
$\pi^{-1}A_{221}$	$\frac{1937}{8}+\frac{3303}{4}\zeta(3)$
$\pi^{-1}A_{240}$	$\frac{1033549}{4536}+\frac{10704}{7}\ln(2)-\frac{40711}{128}\zeta(3)$
$\pi^{-1}A_{241}$	$\frac{8008171}{8064}+\frac{75520}{21}\ln(2)-\frac{660675}{256}\zeta(3)$
$\pi^{-1}A_{242}$	$-\frac{583751}{864}-\frac{100935}{64}\zeta(3)$

This result expresses $\hat{J}=J(i)$ in terms of the values at the fourth root of unity, i, of HPLs of weight $w\le 4$ [together with $a_4\equiv \mathrm{Li}_4(1/2)$, and lower-weight quantities such as π^2 and $\zeta(3)$]. Using [12], we expressed the needed values of the HPLs at $x=i$ in terms of a small subset of irreducible constants of weight $w\le 4$; namely, $\mathrm{K}=\mathrm{Im}\mathrm{Li}_2(i)=\sum_{n=0}^{\infty}(-1)^n/(2n+1)^2$ (Catalan's constant), $\mathrm{Q}_3=\mathrm{Im}H_{0,1,1}(i)$, $\mathrm{Q}_4=\mathrm{Im}H_{0,1,1,1}(i)$, $a_4=\mathrm{Li}_4(1/2)$ and $\beta(4)=\mathrm{Im}\mathrm{Li}_4(i)$. The irreducible weight-4 constants are found to cancel when evaluating $\hat{J}=J(i)$ by means of Eq. (6.5) to yield

$$\hat{J}=J(i)=-\frac{1}{2}\pi^2\mathrm{K}+\frac{9}{2}\pi\zeta(3). \tag{6.6}$$

Applying our technique to all the scattering integrals A_{2nk}, we found that they could all be expressed in terms of the values of HPLs of weight $w\le 4$ at the arguments $x=1$ or $x=i$. Similarly to what happens for $\hat{J}=J(i)$, the irreducible weight-4 constants are found to cancel in the evaluation of all the scattering integrals A_{2nk}. Actually, the final results for the A_{2nk}'s are found to factorize as the product of π with constants of weight ≤ 3. For instance, we found

$$A_{242}=-\pi\left(\frac{583751}{864}+\frac{100935}{64}\zeta(3)\right). \tag{6.7}$$

Our complete analytical results for the A_{2nk}'s are listed in Table I. We give below the relations between such coefficients and those used in Ref. [5] to parametrize the (nonlogarithmic) part of the scattering angle [see Eq. (4.15) there]

$$\begin{aligned}
\pi^{-1}A_{200}&=d_{00},\\
\pi^{-1}A_{220}&=d_{20}+3d_{00},\\
\pi^{-1}A_{221}&=d_{21}-2d_{00},\\
\pi^{-1}A_{240}&=d_{20}+d_{40}+\frac{3}{2}d_{00},\\
\pi^{-1}A_{241}&=d_{21}-\frac{11}{2}d_{00}+d_{41}-2d_{20},\\
\pi^{-1}A_{242}&=d_{42}-2d_{21}+3d_{00}.
\end{aligned} \tag{6.8}$$

Further details about our integration procedures, and our intermediate results, are provided in the Supplemental Material [51].

VII. EVALUATION OF THE $O(G^7)$ SCATTERING INTEGRALS

At the $O(G^7)$ level, i.e., for the integrals A_{mnk} with index $m=3$, the structure of the integrands $a_{mnk}(T,T')$ becomes more complex. The rational functions $R_{pq}^{mnk}(T,T')$ entering as coefficients in Eq. (4.16) involve higher-order polynomials in their numerators, but their most important feature, namely the location of the poles in the denominators, stays the same as at the $O(G^6)$ level. Again the poles are located at $T=T'$, $T=-1/T'$, $T=\pm 1$, $T=\pm i$, $T'=\pm 1$ and $T'=\pm i$. However, an important change concerns the powers p and q with which the functions $At(T,T')\equiv \arctan(T)-\arctan(T')$ and $Ath(T,T')\equiv \mathrm{arctanh}(T)-\mathrm{arctanh}(T')$ enter the numerator of $a_{mnk}(T,T')$. At the $O(G^7)$ level, we have the values $(p,q)=(0,0),(1,0),(2,0),(3,0),(0,1),(1,1)$. In particular, the highest value of $p+q$ is 3, and is reached via the presence of a term proportional to $Ath^3(T,T')$. We already noticed that both $Ath(T,T')$ and $At(T,T')$ are of weight 1. The integrand $R_{pq}^{3nk}(T,T')Ath^3(T,T')$ is therefore of weight 3. Its double integral over T and T' can therefore be *a priori* expected to be of weight 5.

We succeeded in finding the analytic expressions of the $O(G^7)$ integrals entering the 6PN nonlocal scattering angle by using several methods. As a preliminary method, we combined very-high-precision (200 digits) numerical computation of the integrals (using a double-exponential change of variables [52]) with the PSLQ algorithm [53] and a basis of transcendental constants indicated by the structure of the integrands. Let us recall that such an *experimental mathematics* strategy is often used in the realm of multiloop Feynman calculus, when a direct analytic integration seems prohibitive, see e.g., Refs. [48,54,55]. Previous uses of experimental mathematics and high-precision arithmetics within studies of binary systems include Refs. [56–58]. We note in passing that one of the integrals (in momentum space) contributing to the 4PN-static term of the two-body potential, used in [58] and originally obtained by analytic recognition [59], was later analytically confirmed by direct integration (in position space) [60].

The application of experimental mathematics to the A_{3nk} integrals has shown that, similarly to what happened for the A_{2nk} integrals, the final results were simpler than what was *a priori* expected. In particular, we found that the final results do not go beyond weight 4, and that the only weight-4 quantity entering (some of) the results is simply $\zeta(4)\propto\pi^4$.

Having obtained such simple semianalytic expressions for the A_{3nk} integrals, we embarked on confirming them by means of a purely analytical derivation. We found that an efficient method for doing so was to reformulate the time-domain integral defining the integrated action Eq. (3.3) in frequency space. Actually, when decomposing $W^{\rm nonloc,h}(E,j)$ in the two contributions entering Eq. (3.7), the most difficult one to evaluate is

$$W_1^{\rm tail,h}(E,j) \equiv -\alpha \left\langle {\rm Pf}_{2s/c} \int_{-\infty}^{\infty} \frac{dt'}{|t-t'|} \mathcal{F}^{\rm split}_{\rm 2PN}(t,t') \right\rangle_\infty . \quad (7.1)$$

In order to express $W_1^{\rm tail,h}(E,j)$ in the frequency domain, the first step is to Fourier transform[3] the multipole moments. For example,

$$I_{ab}(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \hat{I}_{ab}(\omega), \quad (7.2)$$

where

$$\hat{I}_{ab}(\omega) = \int_{-\infty}^{+\infty} dt e^{i\omega t} I_{ab}(t), \quad (7.3)$$

with the associated PN expansion

$$\hat{I}_{ab}(\omega) = \hat{I}^{\rm N}_{ab}(\omega) + \eta^2 \hat{I}^{\rm 1PN}_{ab}(\omega) + \eta^4 \hat{I}^{\rm 2PN}_{ab}(\omega) + O(\eta^6). \quad (7.4)$$

Inserting these Fourier representations into Eq. (7.1) then yields (see Sec. V of Ref. [25] for details)

$$W_1^{\rm tail,h}(E,j) = 2\frac{G^2 H_{\rm tot}}{\pi c^5} \int_0^\infty d\omega \mathcal{K}(\omega) \ln\left(\omega \frac{2s}{c} e^\gamma\right), \quad (7.5)$$

where

$$\mathcal{K}(\omega) = \frac{1}{5}\omega^6 |\hat{I}_{ab}(\omega)|^2 + \eta^2 \left[\frac{\omega^8}{189}|\hat{I}_{abc}(\omega)|^2 + \frac{16}{45}\omega^6 |\hat{J}_{ab}(\omega)|^2\right] + \eta^4 \left[\frac{\omega^{10}}{9072}|\hat{I}_{abcd}(\omega)|^2 + \frac{\omega^8}{84}|\hat{J}_{abc}(\omega)|^2\right], \quad (7.6)$$

and we have used the result

$$ {\rm Pf}_T \int_0^\infty d\tau \frac{\cos\omega\tau}{\tau} = -\ln(|\omega| T e^\gamma), \quad (7.7)$$

with $\gamma = 0.577215\ldots$. Note the close link between the expression (7.5) for $W_1^{\rm tail,h}(E,j)$ and the frequency-domain expression of the total energy flux emitted during the scattering process, namely

$$\Delta E_{\rm GW} = \frac{G}{\pi c^5} \int_0^\infty d\omega \mathcal{K}(\omega). \quad (7.8)$$

The difference between the two expressions is embodied in the logarithmic factor $\ln(\omega \frac{2s}{c} e^\gamma)$, which is characteristic of the tail in the frequency domain [61].

The relation between $\Delta E_{\rm GW}$ and $W_1^{\rm tail,h}(E,j)$ is clarified by stating them in the framework of the Mellin transform. Let us first note that it is convenient to replace the frequency ω by the variable u, using

$$\omega \equiv \frac{u}{e_r \bar{a}_r^{3/2}}, \quad (7.9)$$

so that Eq. (7.5) becomes[4]

$$W_1^{\rm tail,h}(E,j) = 2\ln\alpha_s G H_{\rm tot} \Delta E_{\rm GW} + 2\frac{G^2 H_{\rm tot}}{\pi c^5} \frac{1}{e_r \bar{a}_r^{3/2}} \int_0^\infty du \mathcal{K}(u) \ln u, \quad (7.10)$$

where

$$\Delta E_{\rm GW} = \frac{G}{\pi c^5} \frac{1}{e_r \bar{a}_r^{3/2}} \int_0^\infty du \mathcal{K}(u), \quad (7.11)$$

with

$$\mathcal{K}(u) = \mathcal{K}(\omega)|_{\omega = u/(e_r \bar{a}_r^{3/2})}, \quad (7.12)$$

and

$$\alpha_s = \frac{2s}{c e_r \bar{a}_r^{3/2}} e^\gamma. \quad (7.13)$$

We recall that the Mellin transform of a function $f(u)$ (with $u \in [0, +\infty[$) is defined as

$$g(s) \equiv \mathfrak{M}\{f(u); s\} = \int_0^\infty u^{s-1} f(u) du. \quad (7.14)$$

It is then easily seen that the first two terms of the Taylor expansion of $g(s)$ around $s = 1$ are respectively given by

$$g(1) = \int_0^\infty f(u) du, \quad (7.15)$$

and

$$\left.\frac{dg(s)}{ds}\right|_{s=1} = \int_0^\infty f(u) \ln u du. \quad (7.16)$$

[3]In the following, we use $GM = 1$, i.e., we work with GM-rescaled time and frequency variables.

[4]We take here e_r and $\bar{a}_r$, as fundamental variables. At any stage of the calculation these can be reexpressed, via Eq. (4.5), as functions of E and j.

This shows the possible usefulness of the Mellin transform in connecting $W_1^{\rm tail,h}$ to $\Delta E_{\rm GW}$.

We have indeed been able to use the Mellin transform to analytically compute all the scattering integrals at $O(G^7)$, i.e., the values of the integrals appearing in the double PN + PM (or $\eta^2 - e_r^{-1}$) expansion of

$$\int_0^\infty du \ln u [\mathcal{K}(u)]^{\rm PN+PM}, \tag{7.17}$$

where

$$\begin{aligned}
&[\mathcal{K}(u)]^{\rm PN+PM}\\
&= \mathcal{K}_{\rm N}^{\rm LO}(u) + \eta^2 \mathcal{K}_{\rm 1PN}^{\rm LO}(u) + \eta^4 \mathcal{K}_{\rm 2PN}^{\rm LO}(u)\\
&\quad + \frac{1}{e_r}\mathcal{K}_{\rm N}^{\rm NLO}(u) + \frac{\eta^2}{e_r}\mathcal{K}_{\rm 1PN}^{\rm NLO}(u) + \frac{\eta^4}{e_r}\mathcal{K}_{\rm 2PN}^{\rm NLO}(u)\\
&\quad + \frac{1}{e_r^2}\mathcal{K}_{\rm N}^{\rm NNLO}(u) + \frac{\eta^2}{e_r^2}\mathcal{K}_{\rm 1PN}^{\rm NNLO}(u) + \frac{\eta^4}{e_r^2}\mathcal{K}_{\rm 2PN}^{\rm NNLO}(u)\\
&\quad + \frac{1}{e_r^3}\mathcal{K}_{\rm N}^{\rm N^3LO}(u) + \frac{\eta^2}{e_r^3}\mathcal{K}_{\rm 1PN}^{\rm N^3LO}(u) + \frac{\eta^4}{e_r^3}\mathcal{K}_{\rm 2PN}^{\rm N^3LO}(u),
\end{aligned} \tag{7.18}$$

as well as their simpler analogs appearing in the double $\eta^2 - e_r^{-1}$ expansion of

$$\int_0^\infty du [\mathcal{K}(u)]^{\rm PN+PM}. \tag{7.19}$$

The starting point of this approach rests on the simple value of the Fourier transform of the multipole moments at the lowest PN order, i.e., at the Newtonian order $[O(\eta^0)]$, but at all orders in $\frac{1}{e_r}$:

$$[\mathcal{K}(u)]_{\rm N} = \mathcal{K}_{\rm N}^{\rm LO}(u) + \frac{1}{e_r}\mathcal{K}_{\rm N}^{\rm NLO}(u) + \frac{1}{e_r^2}\mathcal{K}_{\rm N}^{\rm NNLO}(u) + \cdots. \tag{7.20}$$

In the elliptic-motion case, it is well known that the (discrete) Fourier expansion of the Newtonian multipole moments involve ordinary Bessel functions, namely $J_{p+k}(pe_r)$, where p and k are integers. In the hyperbolic-motion case the (continuous) Fourier transform of the Newtonian-level multipole moments involve integrals of the form

$$\int_{-\infty}^\infty e^{q \sinh v - (p+k)v} dv = 2e^{-i\frac{\pi}{2}(p+k)} K_{p+k}(u), \tag{7.21}$$

involving the modified Bessel function $K_{p+k}(u)$ of real argument u, Eq. (7.9), but of order $p+k$, where $k = 0$, $\pm 1, \ldots$ is an integer, while p defined as

$$p \equiv \frac{q}{e_r}, \qquad q \equiv iu, \tag{7.22}$$

is purely imaginary, and u dependent. The Newtonian-level energy integrand $[\mathcal{K}(u)]_{\rm N}$ is quadratic in time derivatives of the Newtonian multipole moments. Remembering the fact that the variable u is proportional to the frequency, $[\mathcal{K}(u)]_{\rm N}$ therefore involves functions of the type

$$u^{k_1} K_{p+k_2}(u) K_{p+k_3}(u), \tag{7.23}$$

with some integers k_1, k_2, k_3.

There are several technical features which allow one to compute integrals involving bilinear quantities in Bessel K functions of the type (7.23). First, the Mellin transform $g_{\rm KK}(s;\mu,\nu)$ of the function $f_{\rm KK}(u;\mu,\nu) \equiv K_\mu(u)K_\nu(u)$ has a simple explicit expression, namely

$$\begin{aligned}
g_{\rm KK}(s;\mu,\nu) &= \frac{2^{s-3}}{\Gamma(s)}\Gamma\left(\frac{s+\mu+\nu}{2}\right)\Gamma\left(\frac{s-\mu+\nu}{2}\right)\\
&\quad \times \Gamma\left(\frac{s+\mu-\nu}{2}\right)\Gamma\left(\frac{s-\mu-\nu}{2}\right).
\end{aligned} \tag{7.24}$$

Differentiating the result (7.24) with respect to the Mellin parameter s then allows one to compute the $\ln u$-weighted integral of integrands of the form (7.23).

The situation becomes more involved when going beyond the Newtonian level. Indeed, the post-Newtonian-level Fourier-domain *integrands* $\mathcal{K}_{\rm 1PN}^{\rm LO}(u)$, $\mathcal{K}_{\rm 1PN}^{\rm NLO}(u)$, etc. can no longer be explicitly computed. For instance, the 1PN-level, $\frac{1}{e_r}$-NLO term $\mathcal{K}_{\rm 1PN}^{\rm NLO}(u)$ reads

$$\begin{aligned}
\mathcal{K}_{\rm 1PN}^{\rm NLO}(u) &= \frac{16}{21}u^3\left[\left(u^4 - 46u^2 - \frac{141}{5}\right)K_0(u)^2 + \frac{122}{5}u\left(u^2 - \frac{653}{122}\right)K_0(u)K_1(u) + \left(u^4 - \frac{333u^2}{10} - \frac{39}{5}\right)K_1(u)^2\right]\\
&\quad - \frac{48}{5\pi}u^4\int_{-\infty}^\infty dv \arctan\left(\tanh\frac{v}{2}\right)\Big[\sinh 2v(K_0(u) + 2uK_1(u))\cos(u\sinh v)\\
&\quad + \frac{1}{2}(\cosh 3v - 5\cosh v)(uK_0(u) + K_1(u))\sin(u\sinh v)\Big]\\
&\quad - \frac{64}{21}u^3\left[\left(u^4 - \frac{21u^2}{20} - \frac{3}{4}\right)K_0(u)^2 - \frac{6}{5}u\left(u^2 + \frac{95}{24}\right)K_0(u)K_1(u) + \left(u^4 - \frac{23u^2}{20} - \frac{21}{20}\right)K_1(u)^2\right]\nu.
\end{aligned} \tag{7.25}$$

TABLE II. Analytical results for the $O(G^7)$ scattering coefficients A_{3nk}.

Coefficient	Value
A_{300}	$\frac{79936}{225}-\frac{18688}{15}\ln(2)-\frac{88576}{75}\zeta(3)$
A_{320}	$\frac{2239456}{1575}-\frac{568448}{105}\ln(2)-\frac{64256}{525}\zeta(3)-\frac{621}{20}\pi^4$
A_{321}	$-\frac{384}{175}+\frac{96512}{15}\ln(2)+\frac{901632}{175}\zeta(3)$
A_{340}	$\frac{57597448}{51975}+\frac{1175968}{567}\ln(2)+\frac{135861232}{17325}\zeta(3)-\frac{16848}{25}\pi^2+\frac{31779}{448}\pi^4$
A_{341}	$-\frac{1677767408}{259875}+\frac{22912832}{1575}\ln(2)-\frac{12013696}{3465}\zeta(3)+\frac{14067}{112}\pi^4$
A_{342}	$-\frac{5455648}{2205}-\frac{237824}{15}\ln(2)-\frac{132771328}{11025}\zeta(3)$

Here, the terms on the second and third lines involve an integral over v which cannot be explicitly evaluated. [This v-integral comes from the original integral $\int dt e^{i\omega t} I_{ab\cdots}(t)=\int dv \frac{dt(v)}{dv} e^{i\omega t(v)} I_{ab\cdots}(t(v))$ defining the Fourier-transformed multipole moments, when inserting for the function $t(v)$ the PN + eccentricity expansion of the relativistic Kepler equation, see Eq. (4.2), which notably involves the (large-eccentricity-expanded) function $V(v)$, Eq. (4.3).] However, it is still possible to analytically evaluate the resulting double integral $\int du \ln u \int dv[\cdots]$ by integrating first over u (using Mellin-transform properties to replace the $\ln u$ factor by an s derivative), and then integrating over v. These computations could be done because we could obtain explicit expressions for the Mellin transforms $g_{\rm Kcos}(s;\nu,v)$ and $g_{\rm Ksin}(s;\nu,v)$ of the functions

$$f_{\rm Kcos}(u;\nu,v)\equiv K_\nu(u)\cos(u\sinh v), \qquad (7.26)$$

and

$$f_{\rm Ksin}(u;\nu,v)\equiv K_\nu(u)\sin(u\sinh v), \qquad (7.27)$$

which appear in Eq. (7.25), and also at higher PN orders. See Eq. (A12).

Last, but not least, we are interested in expanding the integrals in the large eccentricity limit, $e_r\to\infty$. In this limit, the u-dependent order $p=\frac{iu}{e_r}$ tends to zero, so that the $\frac{1}{e_r}$ expansion is equivalent to evaluating derivatives with respect to the *order*, ν, of Bessel K_ν functions.

Using all those technical features of the frequency-domain integrals [as well as the program HYPEXP2 [13], which allows one to evaluate the Taylor expansion of hypergeometric functions around half-integer values of their parameters, see Eq. (A26)], we were able to derive analytic expressions for all the scattering coefficients A_{3nk} (which confirmed the results previously obtained by experimental mathematics techniques). More technical details on our analytical derivations are given in Appendix A. The final results for the N^{3}LO scattering coefficients A_{3nk} appearing at the 6PN level are listed in Table II.

VIII. FINAL RESULTS FOR THE NONLOCAL CONTRIBUTIONS TO THE SCATTERING ANGLE AT $O(G^7)$

As briefly recalled in Sec. II, there are three types of contributions to the scattering angle, as displayed in Eq. (2.2): the f-route local contribution $\chi_n^{\rm loc,f}$, the h-route nonlocal contribution $\chi^{\rm nonloc,h}$, and the additional contribution $\chi^{\text{f-h}}$. The f-route local contribution, $\chi_n^{\rm loc,f}$ was computed (at the 6PN accuracy) up to G^7 included in Ref. [4] [see Eq. (8.2) there]. The two remaining contributions are related to nonlocal effects. Previous results [2–5] on $\chi_6^{\rm nonloc,h}$ and $\chi_6^{\text{f-h}}$ were complete only up to order G^5. Here, we shall give complete results up to order G^7, within the 6PN accuracy.

The h-route nonlocal contribution $\chi^{\rm nonloc,h}$, in Eq. (2.2), is directly linked [via Eq. (3.1)] to the integrated nonlocal action $W^{\rm nonloc,h}(E,J;\nu)$, Eqs. (3.2) and (3.3). The work done in the sections above has allowed us to derive the analytical expressions of the expansion coefficients A_{mnk} parametrizing $W^{\rm nonloc,h}(E,J;\nu)$, and therefore $\chi^{\rm nonloc,h}$ [as per Eq. (2.5)]. We gather the final results for the function $\chi^{\rm nonloc,h}(\gamma,j,\nu)$ in the following subsection.

The last contribution, $\chi^{\text{f-h}}$, in Eq. (2.2) to the scattering angle is indirectly related to nonlocal effects. As explained in Refs. [3–5], this additional contribution is defined as

$$\frac{1}{2}\chi^{\text{f-h}}(\gamma,j;\nu)=\frac{1}{2M^2\nu}\frac{\partial W^{\text{f-h}}(\gamma,j;\nu)}{\partial j}, \qquad (8.1)$$

where $W^{\text{f-h}}(\gamma,j;\nu)$ is the additional contribution to the integrated action related to the use of a suitably flexed partie-finie scale $f(t)\Delta t^h=2f(t)r_{12}^h(t)/c$ in the definition of the nonlocal Hamiltonian. This generates the following result for $W^{\text{f-h}}$:

$$W^{\text{f-h}}=+2\frac{GH_{\rm tot}}{c^5}\int dt\mathcal{F}^{\rm split}_{\rm 2PN}(t,t)\ln(f(t)). \qquad (8.2)$$

As discussed in Refs. [3–5], the flexibility factor is determined, modulo some gauge freedom, by the few contributions to the function $\chi^{\rm nonloc,h}(\gamma,j,\nu)$ that violate the simple ν-dependence rules [62] satisfied by the total scattering angle $\chi^{\rm tot}$. The resulting value of

$\chi^{\text{f-h}}(\gamma, j; \nu)$ will be discussed in the second subsection below.

Before listing our results for the various contributions to the scattering angle, let us recall our conventional definition of the expansion coefficients in the large-j limit (which include a factor $\frac{1}{2}$):

$$\frac{1}{2}\chi^{\text{tot}}(\gamma, j; \nu) = \sum_{n\geq 1}\frac{\chi_n(\gamma,\nu)}{j^n}, \tag{8.3}$$

with

$$\chi_n(\gamma,\nu) = \chi_n^{\text{loc,f}}(\gamma;\nu) + \chi_n^{\text{nonloc},f}(\gamma;\nu). \tag{8.4}$$

The various pieces of the nonlocal part

$$\chi_n^{\text{nonloc},f}(\gamma;\nu) = \chi_n^{\text{nonloc,h}}(\gamma;\nu) + \chi_n^{\text{f-h}}(\gamma;\nu), \tag{8.5}$$

with

$$\chi_n^{\text{nonloc,h}} = \chi_n^{\text{h},\alpha} + \chi_n^{\text{h},\alpha^2} + O(\alpha^3), \tag{8.6}$$

will be shown as a 6PN-accurate expansion (keyed by the powers of $p_\infty \equiv \sqrt{\gamma^2-1}$) of the type

$$\chi_n^{\text{h},\alpha} = \chi_n^{\text{h},\alpha,\text{4PN}} + \chi_n^{\text{h},\alpha,\text{5PN}} + \chi_n^{\text{h},\alpha,\text{6PN}}, \tag{8.7}$$

and

$$\chi_n^{\text{h},\alpha^2} = \chi_n^{\text{h},\alpha^2,\text{5.5PN}}. \tag{8.8}$$

Note that the third-order-tail contribution starts at the 7PN level, which is beyond the PN accuracy sought for in the present work.

A. The h-route first-order-tail contribution to the scattering angle

The $\frac{1}{j}$-expansion coefficients of the $4+5+6$PN contribution to the first-order-tail part of the scattering angle are given by

$$\begin{aligned}
\pi^{-1}\chi_4^{\text{h},\alpha,\text{4PN}} &= \left[-\frac{37}{5}\ln\left(\frac{p_\infty}{2}\right) - \frac{63}{4}\right]\nu p_\infty^4,\\
\pi^{-1}\chi_4^{\text{h},\alpha,\text{5PN}} &= \left[\left(-\frac{1357}{280} + \frac{111}{10}\nu\right)\ln\left(\frac{p_\infty}{2}\right) - \frac{2753}{1120} + \frac{1071}{40}\nu\right]\nu p_\infty^6,\\
\pi^{-1}\chi_4^{\text{h},\alpha,\text{6PN}} &= \left[\left(-\frac{27953}{3360} + \frac{2517}{560}\nu - \frac{111}{8}\nu^2\right)\ln\left(\frac{p_\infty}{2}\right) - \frac{155473}{8960} + \frac{109559}{40320}\nu - \frac{186317}{5040}\nu^2\right]\nu p_\infty^8,
\end{aligned} \tag{8.9}$$

$$\begin{aligned}
\chi_5^{\text{h},\alpha,\text{4PN}} &= \left[-\frac{6656}{45} - \frac{6272}{45}\ln\left(4\frac{p_\infty}{2}\right)\right]\nu p_\infty^3,\\
\chi_5^{\text{h},\alpha,\text{5PN}} &= \left[\left(-\frac{74432}{525} + \frac{13952}{45}\nu\right)\ln\left(4\frac{p_\infty}{2}\right) + \frac{114368}{1125} + \frac{221504}{525}\nu\right]\nu p_\infty^5,\\
\chi_5^{\text{h},\alpha,\text{6PN}} &= \left[\left(-\frac{881392}{11025} + \frac{288224}{1575}\nu - \frac{21632}{45}\nu^2\right)\ln\left(4\frac{p_\infty}{2}\right) + \frac{48497312}{231525} - \frac{5134816}{23625}\nu - \frac{25465952}{33075}\nu^2\right]\nu p_\infty^7,
\end{aligned} \tag{8.10}$$

$$\begin{aligned}
\pi^{-1}\chi_6^{\text{h},\alpha,\text{4PN}} &= \left[-122\ln\left(\frac{p_\infty}{2}\right) - \frac{99}{4} - \frac{2079}{8}\zeta(3)\right]\nu p_\infty^2,\\
\pi^{-1}\chi_6^{\text{h},\alpha,\text{5PN}} &= \left[\left(\frac{811}{2}\nu - \frac{13831}{56}\right)\ln\left(\frac{p_\infty}{2}\right) - \frac{41297}{112} - \frac{9216}{7}\ln(2) + \frac{49941}{64}\zeta(3) + \left(\frac{3303}{4}\zeta(3) + \frac{1937}{8}\right)\nu\right]\nu p_\infty^4,\\
\pi^{-1}\chi_6^{\text{h},\alpha,\text{6PN}} &= \left[\left(\frac{64579}{1008} - 785\nu^2 + \frac{75595}{168}\nu\right)\ln\left(\frac{p_\infty}{2}\right) - \frac{40711}{128}\zeta(3) + \frac{1033549}{4536} + \frac{10704}{7}\ln(2)\right.\\
&\quad\left. + \left(\frac{75520}{21}\ln(2) + \frac{8008171}{8064} - \frac{660675}{256}\zeta(3)\right)\nu + \left(-\frac{100935}{64}\zeta(3) - \frac{583751}{864}\right)\nu^2\right]\nu p_\infty^6,
\end{aligned} \tag{8.11}$$

and

$$
\begin{aligned}
\chi_7^{\mathrm{h},\alpha,4\mathrm{PN}} &= \left[-\frac{9344}{15}\ln\left(4\frac{p_\infty}{2}\right)+\frac{79936}{225}-\frac{88576}{75}\zeta(3)\right]\nu p_\infty,\\
\chi_7^{\mathrm{h},\alpha,5\mathrm{PN}} &= \left[\left(-\frac{284224}{105}+\frac{48256}{15}\nu\right)\ln\left(4\frac{p_\infty}{2}\right)-\frac{621}{20}\pi^4+\frac{2239456}{1575}-\frac{64256}{525}\zeta(3)+\left(\frac{901632}{175}\zeta(3)-\frac{384}{175}\right)\nu\right]\nu p_\infty^3,\\
\chi_7^{\mathrm{h},\alpha,6\mathrm{PN}} &= \left[\left(-\frac{118912}{15}\nu^2+\frac{11456416}{1575}\nu+\frac{587984}{567}\right)\ln\left(4\frac{p_\infty}{2}\right)+\frac{135861232}{17325}\zeta(3)+\frac{57597448}{51975}-\frac{16848}{25}\pi^2+\frac{31779}{448}\pi^4\right.\\
&\quad\left.+\left(-\frac{12013696}{3465}\zeta(3)+\frac{14067}{112}\pi^4-\frac{1677767408}{259875}\right)\nu+\left(-\frac{5455648}{2205}-\frac{132771328}{11025}\zeta(3)\right)\nu^2\right]\nu p_\infty^5. \qquad (8.12)
\end{aligned}
$$

B. The h-route second-order-tail contribution to the scattering angle

The second-order-tail contribution $W^{\mathrm{nonloc}}_{5.5\mathrm{PN}}$ to the nonlocal integrated action is given by

$$W^{\mathrm{nonloc}}_{5.5\mathrm{PN}} = \alpha^2\left\langle\frac{B}{2}\int_{-\infty}^{\infty}\frac{d\tau}{\tau}\mathcal{H}^{\mathrm{split}}(t,\tau)\right\rangle_\infty, \qquad (8.13)$$

where $B=-\frac{107}{105}$ and

$$\mathcal{H}^{\mathrm{split}}(t,\tau)=\frac{G}{5c^5}[I_{ij}^{(3)}(t)I_{ij}^{(4)}(t+\tau)-I_{ij}^{(3)}(t)I_{ij}^{(4)}(t-\tau)]. \qquad (8.14)$$

Working in the Fourier domain we find

$$W^{\mathrm{nonloc}}_{5.5\mathrm{PN}} = -\alpha^2 B\frac{G}{5c^5}\int_0^\infty d\omega\omega^7|\hat{I}_{ij}(\omega)|^2, \qquad (8.15)$$

where we have used the result

$$\int_{-\infty}^{\infty}d\tau\frac{\sin\omega\tau}{\tau}=\pi. \qquad (8.16)$$

At our present level of accuracy, it is enough to use the Newtonian approximation to the Fourier transform $\hat{I}_{ij}(\omega)$ of the quadrupole moment. Using the relations given in the previous section we have then

$$W^{\mathrm{nonloc}}_{5.5\mathrm{PN}} = \alpha^2\frac{107}{105}\frac{G}{c^5}\frac{1}{e_r^2\bar{a}_r^3}\int_0^\infty du u\mathcal{K}_{\mathrm{N}}(u). \qquad (8.17)$$

Using the results of Sec. V in [5], extending the large-eccentricity expansion to the NNLO order and using the frequency-domain integrals presented in Appendix A, one finds

$$
\begin{aligned}
\chi_5^{\mathrm{h},\alpha^2,5.5\mathrm{PN}} &= -\frac{47936}{675}\nu p_\infty^6,\\
\pi^{-1}\chi_6^{\mathrm{h},\alpha^2,5.5\mathrm{PN}} &= -\frac{10593}{560}\pi^2\nu p_\infty^5,\\
\chi_7^{\mathrm{h},\alpha^2,5.5\mathrm{PN}} &= -\left(\frac{499904}{1575}+\frac{4738816}{23625}\pi^2\right)\nu p_\infty^4. \qquad (8.18)
\end{aligned}
$$

C. The f-h additional contribution to the scattering angle

The flexibility factor $f(t)$ has been determined in terms of the 6PN-accurate, $O(G^6)$ h-route nonlocal scattering angle in Sec. VII of [5]. [Our new results at the $O(G^7)$ level do not change the determination of the flexibility factor.] The corresponding additional contribution

$$\Delta^{\mathrm{f\text{-}h}}H = 2\frac{GH_{\mathrm{tot}}}{c^5}\mathcal{F}^{\mathrm{split}}_{2\mathrm{PN}}(t,t)\ln(f(t)), \qquad (8.19)$$

to the f-route nonlocal Hamiltonian has been determined in [5] [Eq. (7.29) there] to be equal, modulo an irrelevant canonical transformation, to

$$\Delta^{\mathrm{f\text{-}h}}H'_{5+6\mathrm{PN}} = \Delta^{\mathrm{f\text{-}h}}H'^{\mathrm{min}}_{5+6\mathrm{PN}}+\Delta^{\mathrm{f\text{-}h}}H'^{CD}_{5+6\mathrm{PN}}. \qquad (8.20)$$

Here, $\Delta^{\mathrm{f\text{-}h}}H'^{\mathrm{min}}_{5+6\mathrm{PN}}$ denotes the *minimal* part of the canonically transformed $\Delta^{\mathrm{f\text{-}h}}H$ [built with the minimal solution, Eq. (7.28) there], while $\Delta^{\mathrm{f\text{-}h}}H'^{CD}_{5+6\mathrm{PN}}$ denotes the part that involves six arbitrary flexibility parameters; namely, C_2, C_3, D_2^0, D_3^0, and $D_4=D_4^0+\nu D_4^1$. Explicitly, the latter contribution reads

$$
\begin{aligned}
\frac{\Delta^{\mathrm{f\text{-}h}}H'^{CD}_{5+6\mathrm{PN}}}{M} &= C_2\frac{\nu^3p_r^2}{r^5}+C_3\frac{\nu^3}{r^6}+\left(D_2^0+\frac{14}{3}\nu C_2\right)\frac{\nu^3p_r^4}{r^5}\\
&\quad+\left[D_3^0+\nu\left(-\frac{3}{2}C_2+6C_3\right)\right]\frac{\nu^3p_r^2}{r^6}\\
&\quad+(D_4^0+\nu D_4^1)\frac{\nu^3}{r^7}. \qquad (8.21)
\end{aligned}
$$

On the other hand, the fully determined minimal Hamiltonian $\Delta^{\mathrm{f\text{-}h}}H'^{\mathrm{min}}_{5+6\mathrm{PN}}$ given in Eq. (7.30) of [5] involves the coefficient

$$D=\frac{1}{\pi}\left(\frac{5}{2}A_{221}+\frac{15}{8}A_{200}+A_{242}\right), \qquad (8.22)$$

which could not be analytically determined in [5]. Our new results, presented above, allow one to determine the exact analytical expression of the coefficient D. Though the individual scattering coefficients A_{2nk} entering D involve $\zeta(3)$, it is remarkably found that D turns out to be equal to the rational number

$$D = -\frac{12607}{108}, \tag{8.23}$$

which is compatible with the previous numerical estimate of [5], namely $D^{\rm num} = -116.73148147(1)$. The value of D then determines the minimal value of the flexibility coefficient $D_3^{\rm min}$ [see Eq. (7.28) in [5]], namely

$$D_3^{\rm min} = -\frac{68108}{945}\nu, \tag{8.24}$$

as well as the f-related, 6PN-level contribution to the periastron precession [see Eq. (8.30) in Ref. [5]]:[5]

$$K^{\text{f-h,circ,min}}(j) = +\frac{68108}{945}\frac{\nu^3}{j^{12}}. \tag{8.25}$$

Inserting the analytical value of D in Eq. (7.30) of [5] also determines the analytical value of $\Delta^{\text{f-h}}H'^{\rm min}_{5+6\rm PN}$, namely

$$\frac{\Delta^{\text{f-h}}H'^{\rm min}_{5+6\rm PN}}{M} = \nu^3\frac{168}{5}\frac{p_r^4}{r^4} + \nu^3\left(\frac{271066}{4725}+\frac{21736}{189}\nu\right)\frac{p_r^6}{r^4} - \nu^4\frac{39712}{189}\frac{p_r^4}{r^5} - \nu^4\frac{68108}{945}\frac{p_r^2}{r^6}. \tag{8.26}$$

Using the (canonically transformed) additional Hamiltonian (8.20), it is a straightforward matter to compute the large-eccentricity expansion of the corresponding integrated action

$$W^{\text{f-h}} = +2\frac{GH_{\rm tot}}{c^5}\int dt\mathcal{F}^{\rm split}_{2\rm PN}(t,t)\ln(f(t)) = \int dt\Delta^{\text{f-h}}H'_{5+6\rm PN}, \tag{8.27}$$

and the corresponding (halved) scattering angle contribution

$$\frac{1}{2}\chi^{\text{f-h}} = \frac{1}{2M^2\nu}\frac{\partial W^{\text{f-h}}(\gamma,j;\nu)}{\partial j}. \tag{8.28}$$

We find

$$\begin{aligned}
\pi^{-1}\chi_4^{\text{f-h}} &= -\frac{3}{32}C_1\nu^2p_\infty^6 + \left(\frac{27}{64}C_1\nu - \frac{3}{64}C_1 - \frac{15}{256}D_1\right)\nu^2p_\infty^8,\\
\chi_5^{\text{f-h}} &= \left(-\frac{8}{5}C_1 - \frac{8}{15}C_2\right)\nu^2p_\infty^5 + \left[\left(\frac{276}{35}C_1+\frac{32}{15}C_2\right)\nu - \frac{172}{35}C_1 - \frac{4}{15}C_2 - \frac{8}{7}D_1 - \frac{8}{35}D_2\right]\nu^2p_\infty^7,\\
\pi^{-1}\chi_6^{\text{f-h}} &= \left(-\frac{45}{32}C_1 - \frac{15}{16}C_2 - \frac{15}{16}C_3\right)\nu^2p_\infty^4\\
&\quad + \left[\left(\frac{495}{64}C_1 + \frac{275}{64}C_2 + \frac{105}{32}C_3\right)\nu - \frac{615}{64}C_1 - \frac{95}{32}C_2 - \frac{15}{32}C_3 - \frac{75}{64}D_1 - \frac{15}{32}D_2 - \frac{5}{32}D_3\right]\nu^2p_\infty^6,\\
\chi_7^{\text{f-h}} &= (-8C_1 - 8C_2 - 16C_3)\nu^2p_\infty^3\\
&\quad + \left[\left(\frac{252}{5}C_1 + \frac{216}{5}C_2 + \frac{344}{5}C_3\right)\nu - 100C_1 - \frac{292}{5}C_2 - \frac{264}{5}C_3 - 8D_1 - \frac{24}{5}D_2 - \frac{16}{5}D_3 - \frac{16}{5}D_4\right]\nu^2p_\infty^5,
\end{aligned} \tag{8.29}$$

with minimal values [for vanishing values of C_2, C_3, D_2^0, D_3^0, and $D_4 = D_4^0 + \nu D_4^1$, and the minimal values $C_1^{\rm min}, D_1^{\rm min}, D_2^{\rm min}, D_3^{\rm min}$ given in Eq. (7.28) in [5], with Eq. (8.24) above]

$$\begin{aligned}
\pi^{-1}\chi^{\text{f-h}}_{4\,\rm min} &= -\frac{63}{20}\nu^2p_\infty^6 + \left(-\frac{199037}{40320} + \frac{74959}{10080}\nu\right)\nu^2p_\infty^8,\\
\chi^{\text{f-h}}_{5\,\rm min} &= -\frac{1344}{25}\nu^2p_\infty^5 + \left(-\frac{7629872}{33075} + \frac{6004832}{33075}\nu\right)\nu^2p_\infty^7,\\
\pi^{-1}\chi^{\text{f-h}}_{6\,\rm min} &= -\frac{189}{4}\nu^2p_\infty^4 + \left(-\frac{786449}{2016} + \frac{50729}{216}\nu\right)\nu^2p_\infty^6,\\
\chi^{\text{f-h}}_{7\,\rm min} &= -\frac{1344}{5}\nu^2p_\infty^3 + \left(-\frac{18044528}{4725} + \frac{150944}{75}\nu\right)\nu^2p_\infty^5.
\end{aligned} \tag{8.30}$$

[5]In Eq. (8.27) of Ref. [5], first line: the numerical coefficient $-155/12$ of $\ln(2)$ should read $-155/112$. We thank Johannes Blümlein for noticing this typo.

IX. CONCLUSIONS

By using computing techniques developed for the evaluation of multiloop Feynman integrals, we have advanced the analytical knowledge of classical gravitational scattering at the seventh order in G, and at the sixth post-Newtonian accuracy, by fully determining the *nonlocal-in-time* contribution to the scattering angle. The present work has given a new instance of a fruitful synergy between classical GR and QFT techniques leading to an improved theoretical description of gravitationally interacting binary systems.

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APPENDIX A: DETAILS ON THE FREQUENCY DOMAIN COMPUTATION

The first step is to Fourier transform[6] the multipolar moments [see, e.g., Eq. (7.3)]. At the Newtonian level the computation is done by using the integral representation of the Hankel functions of the first kind of order p and argument q:

$$H_p^{(1)}(q) = \frac{1}{i\pi}\int_{-\infty}^{\infty} e^{q\sinh v - pv} dv. \qquad \text{(A1)}$$

As the argument $q = iu$ of the Hankel function is purely imaginary, the Hankel function becomes converted into a Bessel K function, according to the relation

$$H_p^{(1)}(iu) = \frac{2}{\pi} e^{-i\frac{\pi}{2}(p+1)} K_p(u). \qquad \text{(A2)}$$

Note that the order $p = iu/e_r$ of the Bessel functions is purely imaginary, and proportional to the (frequency-dependent) argument $u = \omega e_r \bar{a}_r^{3/2}$. However, the order p tends to zero when $e_r \to \infty$, which allows most integrals to be explicitly computed when performing a large-eccentricity expansion. A typical term at the Newtonian level $[O(\eta^0)]$ is of the kind $e^{q\sinh v-(p+k)v}$, the Fourier transform of which is

$$e^{q\sinh v-(p+k)v} \to 2e^{-i\frac{\pi}{2}(p+k)} K_{p+k}(u), \qquad \text{(A3)}$$

involving Bessel functions having the same argument u, but various orders differing by integers. However, standard identities valid for Bessel functions allow one to reduce the orders $p+k$ to either p or $p+1$. When taking the large-eccentricity expansion, one expands with respect to the order of the Bessel functions. This gives rise, at LO, to $K_0(u)$, and $K_1(u)$, and at NLO, NNLO, N^{3}NLO, to derivatives of $K_0(u)$, and $K_1(u)$ with respect to their orders.

Higher orders in the PN expansion $[O(\eta^2), O(\eta^4)]$ imply for the integration in v more complicated expressions like $v^n e^{q\sinh v-(p+k)v}$ and $e^{q\sinh v-(p+k)v} V(v)$. The Fourier transform of $v^n e^{q\sinh v-(p+k)v}$ leads to integrands involving

$$v^n e^{q\sinh v-(p+k)v} \to 2(-1)^n \frac{\partial^n}{\partial p^n}[e^{-i\frac{\pi}{2}(p+k)} K_{p+k}(u)], \qquad \text{(A4)}$$

while the Fourier transform of the terms $e^{q\sinh v-(p+k)v} V(v)$ requires to work with the large-e_r expansion of the V-term [see Eq. (4.3)], i.e.,

$$V(v) = 2\arctan\left(\tanh\frac{v}{2}\right) + \frac{1}{e_r}\tanh v + \frac{\sinh v}{e_r^2 \cosh^2 v} + O(e_r^{-3}). \qquad \text{(A5)}$$

One then generally has terms of the form $e^{q\sinh v-(p+k)v} f_j(v)$, involving nontrivial functions $f_j(v)$, which cannot be integrated analytically. However, in most cases one can overcome this difficulty by integrating over u, before integrating over v.

1. Integrating over the frequency spectrum and Mellin transform

The integrated nonlocal action $W_1^{\text{tail,h}}$ [Eq. (7.10)] and the GW energy ΔE_{GW} [Eq. (7.11)] are connected by the Mellin transform [Eq. (7.14)] of the function $\mathcal{K}(u)$, being defined in terms of the integrals

$$I_{W_1} = \int_0^\infty du \mathcal{K}(u) \ln u \qquad \text{(A6)}$$

and

$$I_{\Delta E} = \int_0^\infty du \mathcal{K}(u), \qquad \text{(A7)}$$

respectively. Denoting by $f(u) = \mathcal{K}(u)$ and by $g(s)$ its Mellin transform, we then have that $I_{\Delta E} = g(1)$ and $I_{W_1} = \frac{dg(s)}{ds}|_{s=1}$. Mellin transforms are well implemented in standard symbolic algebra manipulators.

At the Newtonian level, the function $\mathcal{K}(u)$ is expressed in terms of modified Bessel functions of the second kind. The typical term has the form

[6] In the following, we use $GM = 1$, i.e., we work with GM-rescaled time and frequency variables.

$$u^k K_\mu(u) K_\nu(u), \tag{A8}$$

so that it is enough to compute the Mellin transform $g_{\rm KK}(s;\mu,\nu)$ of the function

$$f_{\rm KK}(u;\mu,\nu) = K_\mu(u) K_\nu(u), \tag{A9}$$

[see Eq. (7.24)] and its first derivative with respect to s, further using the property $\mathfrak{M}\{x^k f(x); s\} = g(s+k)$.

At higher PN orders also appear terms like

$$u^k K_\nu(u) \cos(u \sinh v), \quad u^k K_\nu(u) \sin(u \sinh v), \tag{A10}$$

to be integrated both over u and v. Hence we also need the Mellin transforms $g_{\rm Kcos}(s,\nu,v)$ and $g_{\rm Ksin}(s,\nu,v)$ of the functions

$$\begin{aligned} f_{\rm Kcos}(u;\nu,v) &= K_\nu(u)\cos(u\sinh v), \\ f_{\rm Ksin}(u;\nu,v) &= K_\nu(u)\sin(u\sinh v), \end{aligned} \tag{A11}$$

and their first derivatives with respect to s. Their Mellin transforms are given by

$$\begin{aligned} g_{\rm Kcos}(s;\nu,v) &= \frac{2^{s-2}}{\cosh^{s-\nu} v}\Gamma\left(\frac{s+\nu}{2}\right)\Gamma\left(\frac{s-\nu}{2}\right) \\ &\quad \times {}_2F_1\left(\frac{1-s-\nu}{2}, \frac{s-\nu}{2}; \frac{1}{2}; \tanh^2 v\right), \\ g_{\rm Ksin}(s;\nu,v) &= \frac{2^{s-1}\sinh v}{\cosh^{1+s+\nu} v}\Gamma\left(\frac{s+\nu+1}{2}\right)\Gamma\left(\frac{s-\nu+1}{2}\right) \\ &\quad \times {}_2F_1\left(\frac{2-s+\nu}{2}, \frac{s+\nu+1}{2}; \frac{3}{2}; \tanh^2 v\right). \end{aligned} \tag{A12}$$

For each of them (i = KK, Kcos, Ksin) we need then

$$g_{is} = \frac{\partial}{\partial s} g_i, \quad g_{i\nu\nu} = \frac{\partial^2}{\partial \nu^2} g_i, \quad g_{is\nu\nu} = \frac{\partial^3}{\partial s \partial \nu^2} g_i \tag{A13}$$

(for example, $g_{{\rm KK}s} = g_{{\rm KK}s}(s;\mu,\nu)$, etc.) and higher derivatives with respect to the order ν for increasing PN accuracy as well as level of expansion in the eccentricity parameter. Explicit expressions can be obtained which generally involve HPLs, coming from the derivatives of the hypergeometric functions with respect to their parameters (s,ν) (see below).

2. Results

The function $\mathcal{K}(u)$ can be decomposed as in Eq. (7.18) (here in a conveniently rescaled form)

$$\mathcal{K}(u) = \mathcal{K}_{\rm N}(u) + \frac{\eta^2}{\bar{a}_r}\mathcal{K}_{\rm 1PN}(u) + \frac{\eta^4}{\bar{a}_r^2}\mathcal{K}_{\rm 2PN}(u), \tag{A14}$$

with

$$\begin{aligned} \mathcal{K}_{\rm nPN}(u) &= \frac{\nu^2}{e_r^2 \bar{a}_r^2}\left[\tilde{\mathcal{K}}^{\rm LO}_{\rm nPN}(u) + \frac{\pi}{e_r}\tilde{\mathcal{K}}^{\rm NLO}_{\rm nPN}(u)\right. \\ &\quad \left. + \frac{1}{e_r^2}\tilde{\mathcal{K}}^{\rm NNLO}_{\rm nPN}(u) + \frac{\pi}{e_r^3}\tilde{\mathcal{K}}^{\rm N^3LO}_{\rm nPN}(u)\right], \end{aligned} \tag{A15}$$

up to the 2PN order and to the N^{3}LO order in the large eccentricity.

At the Newtonian level we find

$$\begin{aligned} \tilde{\mathcal{K}}^{\rm LO}_{\rm N}(u) &= \frac{32}{5}u^2\left[\left(\frac{1}{3}+u^2\right)K_0^2(u) + 3uK_0(u)K_1(u) + (1+u^2)K_1^2(u)\right], \\ \tilde{\mathcal{K}}^{\rm NLO}_{\rm N}(u) &= u\tilde{\mathcal{K}}^{\rm LO}_{\rm N}(u), \\ \tilde{\mathcal{K}}^{\rm NNLO}_{\rm N}(u) &= \frac{\pi^2}{2}u^2\tilde{\mathcal{K}}^{\rm LO}_{\rm N}(u) - \frac{32}{5}u^2\left\{(1+3u^2)K_0^2(u) + 7uK_0(u)K_1(u) + (1+2u^2)K_1^2(u)\right. \\ &\quad \left. + u^2\left[\left(u^2+\frac{1}{3}\right)K_0(u) + \frac{3}{2}uK_1(u)\right]\frac{\partial^2 K_\nu(u)}{\partial\nu^2}\bigg|_{\nu=0} + u^2\left[\frac{3}{2}uK_0(u) + (u^2+1)K_1(u)\right]\frac{\partial^2 K_\nu(u)}{\partial\nu^2}\bigg|_{\nu=1}\right\}, \\ \tilde{\mathcal{K}}^{\rm N^3LO}_{\rm N}(u) &= u\tilde{\mathcal{K}}^{\rm NNLO}_{\rm N}(u) - \frac{\pi^2}{3}u^3\tilde{\mathcal{K}}^{\rm LO}_{\rm N}(u). \end{aligned} \tag{A16}$$

Using the same decomposition as above for $I_{\Delta E,{\rm N}}$ we find

$$\begin{aligned} I^{\rm LO}_{\Delta E,{\rm N}} &= \frac{32}{15}[3g_{\rm KK}(5;1,1) + 3g_{\rm KK}(3;1,1) + 9g_{\rm KK}(4;0,1) + 3g_{\rm KK}(5;0,0) + g_{\rm KK}(3;0,0)] \\ &= \frac{37}{15}\pi^2, \end{aligned}$$

$$
\begin{aligned}
I_{\Delta E,\mathrm{N}}^{\mathrm{NLO}} &= \frac{32}{15}[3g_{\mathrm{KK}}(6;1,1)+3g_{\mathrm{KK}}(4;1,1)+9g_{\mathrm{KK}}(5;0,1)+3g_{\mathrm{KK}}(6;0,0)+g_{\mathrm{KK}}(4;0,0)] \\
&= \frac{1568}{45}, \\
I_{\Delta E,\mathrm{N}}^{\mathrm{NNLO}} &= \frac{16}{15}[-9g_{\mathrm{KK}\nu\nu}(6;1,0)-6g_{\mathrm{KK}\nu\nu}(7;0,0)-2g_{\mathrm{KK}\nu\nu}(5;0,0)] \\
&\quad+\frac{16}{15}[-9g_{\mathrm{KK}\nu\nu}(6;0,1)-6g_{\mathrm{KK}\nu\nu}(7;1,1)-6g_{\mathrm{KK}\nu\nu}(5;1,1)] \\
&\quad+\frac{16}{15}[3\pi^2\left(g_{\mathrm{KK}}(7;0,0)+g_{\mathrm{KK}}(7;1,1)+g_{\mathrm{KK}}(5;1,1)+3g_{\mathrm{KK}}(6;0,1)+\frac{1}{3}g_{\mathrm{KK}}(5;0,0)\right) \\
&\quad-12g_{\mathrm{KK}}(5;1,1)-6g_{\mathrm{KK}}(3;1,1)-42g_{\mathrm{KK}}(4;0,1)-18g_{\mathrm{KK}}(5;0,0)-6g_{\mathrm{KK}}(3;0,0)] \\
&= \frac{281}{10}\pi^2, \\
I_{\Delta E,\mathrm{N}}^{\mathrm{N^3LO}} &= \frac{16}{45}[-27g_{\mathrm{KK}\nu\nu}(7;1,0)-18g_{\mathrm{KK}\nu\nu}(8;0,0)-6g_{\mathrm{KK}\nu\nu}(6;0,0)] \\
&\quad+\frac{16}{45}[-27g_{\mathrm{KK}\nu\nu}(7;0,1)-18g_{\mathrm{KK}\nu\nu}(8;1,1)-18g_{\mathrm{KK}\nu\nu}(6;1,1)] \\
&\quad+\frac{16}{45}[3\pi^2\left(g_{\mathrm{KK}}(8;1,1)+g_{\mathrm{KK}}(6;1,1)+3g_{\mathrm{KK}}(7;0,1)+\frac{1}{3}g_{\mathrm{KK}}(6;0,0)\right) \\
&\quad-18g_{\mathrm{KK}}(4;1,1)-36g_{\mathrm{KK}}(6;1,1)-126g_{\mathrm{KK}}(5;0,1)-54g_{\mathrm{KK}}(6;0,0)-18g_{\mathrm{KK}}(4;0,0)] \\
&= \frac{7808}{45}, \qquad \text{(A17)}
\end{aligned}
$$

where the values of the various Mellin transforms are listed in Table III. The corresponding result for $I_{W_1,\mathrm{N}}$ is obtained simply by replacing each of them by its derivative with respect to the Mellin parameter, leading to

$$
\begin{aligned}
I_{W_1,\mathrm{N}}^{\mathrm{LO}} &= \left(\frac{40}{3}-\frac{74}{5}\ln(2)-\frac{74}{15}\gamma\right)\pi^2, \\
I_{W_1,\mathrm{N}}^{\mathrm{NLO}} &= \frac{4448}{135}+\frac{3136}{45}\ln(2)-\frac{3136}{45}\gamma, \\
I_{W_1,\mathrm{N}}^{\mathrm{NNLO}} &= \left(\frac{2479}{30}-\frac{843}{5}\ln(2)-\frac{281}{5}\gamma+\frac{2079}{20}\zeta(3)\right)\pi^2, \\
I_{W_1,\mathrm{N}}^{\mathrm{N^3LO}} &= -\frac{23936}{675}+\frac{15616}{45}\ln(2)-\frac{15616}{45}\gamma+\frac{88576}{225}\zeta(3). \qquad \text{(A18)}
\end{aligned}
$$

Starting from the 1PN level, the Fourier transform of the multipolar moments can be explicitly done only partly, so that the resulting function $\mathcal{K}(u)$ is not fully determined in closed form. Consider, for instance, the NLO term

$$
\begin{aligned}
\tilde{\mathcal{K}}_{\mathrm{1PN}}^{\mathrm{NLO}}(u) &= \frac{16}{21}u^3\left[\left(u^4-46u^2-\frac{141}{5}\right)K_0(u)^2+\frac{122}{5}u\left(u^2-\frac{653}{122}\right)K_0(u)K_1(u)+\left(u^4-\frac{333u^2}{10}-\frac{39}{5}\right)K_1(u)^2\right] \\
&\quad-\frac{48}{5\pi}u^4\int_{-\infty}^{\infty}dv\arctan\left(\tanh\frac{v}{2}\right)\left[\sinh 2v(K_0(u)+2uK_1(u))\cos(u\sinh v)\right. \\
&\quad\left.+\frac{1}{2}(\cosh 3v-5\cosh v)(uK_0(u)+K_1(u))\sin(u\sinh v)\right] \\
&\quad-\frac{64}{21}u^3\left[\left(u^4-\frac{21u^2}{20}-\frac{3}{4}\right)K_0(u)^2-\frac{6}{5}u\left(u^2+\frac{95}{24}\right)K_0(u)K_1(u)+\left(u^4-\frac{23u^2}{20}-\frac{21}{20}\right)K_1(u)^2\right]\nu. \qquad \text{(A19)}
\end{aligned}
$$

It is convenient taking the Mellin transform first (i.e., integrating over u), and then integrating over v. We find

TABLE III. List of Mellin transforms (7.24) used in Eqs. 116 and (A18). The function $g_{\rm KK}(s;\mu,\nu)$ and its derivative with respect to the Mellin parameter s are both evaluated at $s=1+k$.

$1+k$	μ	ν	$g_{\rm KK}$	$g_{{\rm KK}s}$	$g_{{\rm KK}\nu\nu}$	$g_{{\rm KK}s\nu\nu}$
3	0	0	$\frac{\pi^2}{32}$	$-\frac{\pi^2(6\ln(2)-5+2\gamma)}{64}$	$\frac{\pi^2}{64}(\pi^2-8)$	$-\frac{7\pi^2\zeta(3)}{32}-\frac{\pi^2}{16}+\frac{\gamma\pi^2}{8}+\frac{5\pi^4}{128}-\frac{\gamma\pi^4}{64}+\frac{3}{8}\pi^2\ln(2)-\frac{3}{64}\pi^4\ln(2)$
3	0	1	$\frac{1}{2}$	$\frac{1}{4}(2\ln 2-2\gamma-1)$	$\frac{(3+\pi^2)}{12}$	$-\frac{\zeta(3)}{2}-\frac{3}{8}-\frac{\gamma}{4}-\frac{\pi^2}{24}-\frac{\gamma\pi^2}{12}+\frac{\ln(2)}{4}+\frac{1}{12}\pi^2\ln(2)$
3	1	1	$\frac{3\pi^2}{32}$	$-\frac{\pi^2(18\ln(2)-11+6\gamma)}{64}$	$\frac{\pi^2}{64}(3\pi^2-8)$	$-\frac{21\pi^2\zeta(3)}{32}+\frac{\pi^2}{16}+\frac{\gamma\pi^2}{8}+\frac{11\pi^4}{128}-\frac{3\gamma\pi^4}{64}+\frac{3}{8}\pi^2\ln(2)-\frac{9}{64}\pi^4\ln(2)$
4	0	0	$\frac{1}{3}$	$\frac{1}{3}\ln(2)+\frac{1}{18}-\frac{1}{3}\gamma$	$\frac{(\pi^2-6)}{18}$	$\frac{5}{18}+\frac{\gamma}{3}+\frac{\pi^2}{108}-\frac{\gamma\pi^2}{18}-\frac{\ln(2)}{3}+\frac{1}{18}\pi^2\ln(2)-\frac{1}{3}\zeta(3)$
4	0	1	$\frac{3\pi^2}{64}$	$-\frac{\pi^2(18\ln(2)-17+6\gamma)}{128}$	$\frac{1}{384}\pi^2(9\pi^2-68)$	$-\frac{21\pi^2\zeta(3)}{64}-\frac{77\pi^2}{576}+\frac{17\gamma\pi^2}{96}+\frac{17\pi^4}{256}-\frac{3\gamma\pi^4}{128}+\frac{17}{32}\pi^2\ln(2)-\frac{9}{128}\pi^4\ln(2)$
4	1	1	$\frac{2}{3}$	$\frac{2}{3}\ln(2)-\frac{1}{18}-\frac{2}{3}\gamma$	$\frac{(2\pi^2-3)}{18}$	$-\frac{2\zeta(3)}{3}+\frac{2}{9}+\frac{\gamma}{6}-\frac{\pi^2}{108}-\frac{\gamma\pi^2}{9}-\frac{\ln(2)}{6}+\frac{1}{9}\pi^2\ln(2)$
5	0	0	$\frac{27\pi^2}{512}$	$-\frac{27\pi^2(12\ln(2)-13+4\gamma)}{2048}$	$\frac{3\pi^2(-80+9\pi^2)}{1024}$	$-\frac{83}{256}\pi^2-\frac{189}{512}\pi^2\zeta(3)+\frac{351}{4096}\pi^4+\frac{15}{64}\gamma\pi^2-\frac{27}{1024}\pi^4\gamma+\frac{45}{64}\ln(2)\pi^2-\frac{81}{1024}\pi^4\ln(2)$
5	0	1	$\frac{2}{3}$	$\frac{2}{3}\ln(2)+\frac{5}{18}-\frac{2}{3}\gamma$	$\frac{(-21+4\pi^2)}{36}$	$-\frac{2\zeta(3)}{3}+\frac{55}{144}+\frac{7\gamma}{12}+\frac{5\pi^2}{108}-\frac{\gamma\pi^2}{9}-\frac{7\ln(2)}{12}+\frac{1}{9}\pi^2\ln(2)$
5	1	1	$\frac{45\pi^2}{512}$	$-\frac{3\pi^2(180\ln(2)-187+60\gamma)}{2048}$	$\frac{\pi^2(-368+45\pi^2)}{1024}$	$\frac{23}{64}\gamma\pi^2+\frac{561}{4096}\pi^4-\frac{315}{512}\pi^2\zeta(3)+\frac{69}{64}\ln(2)\pi^2-\frac{105}{256}\pi^2-\frac{45}{1024}\pi^4\gamma-\frac{135}{1024}\pi^4\ln(2)$
6	0	0	$\frac{16}{15}$	$\frac{16}{15}\ln(2)+\frac{172}{225}-\frac{16}{15}\gamma$	$-\frac{4}{3}+\frac{8}{45}\pi^2$	$\frac{11}{45}-\frac{16}{15}\zeta(3)+\frac{86}{675}\pi^2+\frac{4}{3}\gamma-\frac{8}{45}\gamma\pi^2-\frac{4}{3}\ln(2)+\frac{8}{45}\ln(2)\pi^2$
6	0	1	$\frac{135\pi^2}{1024}$	$-\frac{27\pi^2(60\ln(2)-69+20\gamma)}{4096}$	$\frac{3\pi^2(-1964+225\pi^2)}{10240}$	$\frac{1473}{2560}\gamma\pi^2+\frac{1863}{8192}\pi^4-\frac{945}{1024}\pi^2\zeta(3)-\frac{45853}{51200}\pi^2+\frac{4419}{2560}\ln(2)\pi^2-\frac{135}{2048}\pi^4\gamma-\frac{405}{2048}\pi^4\ln(2)$
6	1	0	$\frac{135\pi^2}{1024}$	$-\frac{27\pi^2(60\ln(2)-69+20\gamma)}{4096}$	$\frac{3\pi^2(-2036+225\pi^2)}{10240}$	$-\frac{49147}{51200}\pi^2+\frac{1527}{2560}\gamma\pi^2+\frac{1863}{8192}\pi^4-\frac{945}{1024}\pi^2\zeta(3)+\frac{4581}{2560}\ln(2)\pi^2-\frac{135}{2048}\pi^4\gamma-\frac{405}{2048}\pi^4\ln(2)$
6	1	1	$\frac{8}{5}$	$\frac{8}{5}\ln(2)+\frac{76}{75}-\frac{8}{5}\gamma$	$-\frac{5}{3}+\frac{4}{15}\pi^2$	$\frac{5}{3}\gamma+\frac{38}{225}\pi^2-\frac{8}{5}\zeta(3)-\frac{5}{3}\ln(2)+\frac{53}{90}-\frac{4}{15}\gamma\pi^2+\frac{4}{15}\ln(2)\pi^2$
7	0	0	$\frac{1125\pi^2}{4096}$	$-\frac{75\pi^2(180\ln(2)-221+60\gamma)}{16384}$	$\frac{5\pi^2(-2072+225\pi^2)}{8192}$	$-\frac{29023}{12288}\pi^2-\frac{7875}{4096}\pi^2\zeta(3)+\frac{16575}{32768}\pi^4+\frac{1295}{1024}\gamma\pi^2-\frac{1125}{8192}\pi^4\gamma+\frac{3885}{1024}\ln(2)\pi^2-\frac{3375}{8192}\pi^4\ln(2)$
7	0	1	$\frac{16}{5}$	$\frac{16}{5}\ln(2)+\frac{212}{75}-\frac{16}{5}\gamma$	$-\frac{172}{45}+\frac{8}{15}\pi^2$	$\frac{172}{45}\gamma+\frac{106}{225}\pi^2-\frac{16}{5}\zeta(3)-\frac{172}{45}\ln(2)+\frac{37}{225}-\frac{8}{15}\gamma\pi^2+\frac{8}{15}\ln(2)\pi^2$
7	1	0	$\frac{16}{5}$	$\frac{16}{5}\ln(2)+\frac{212}{75}-\frac{16}{5}\gamma$	$-\frac{188}{45}+\frac{8}{15}\pi^2$	$\frac{188}{45}\gamma+\frac{106}{225}\pi^2-\frac{16}{5}\zeta(3)-\frac{188}{45}\ln(2)-\frac{8}{15}\gamma\pi^2+\frac{8}{15}\ln(2)\pi^2-\frac{7}{225}$
7	1	1	$\frac{1575\pi^2}{4096}$	$-\frac{15\pi^2(1260\ln(2)-1523+420\gamma)}{16384}$	$\frac{\pi^2(-14072+1575\pi^2)}{8192}$	$\frac{1759}{1024}\gamma\pi^2+\frac{22845}{32768}\pi^4-\frac{11025}{4096}\pi^2\zeta(3)+\frac{5277}{1024}\ln(2)\pi^2-\frac{37283}{12288}\pi^2-\frac{1575}{8192}\pi^4\gamma-\frac{4725}{8192}\pi^4\ln(2)$
8	0	0	$\frac{288}{35}$	$\frac{288}{35}\ln(2)+\frac{10824}{1225}-\frac{288}{35}\gamma$	$-\frac{56}{5}+\frac{48}{35}\pi^2$	$-\frac{1294}{525}+\frac{1804}{1225}\pi^2+\frac{56}{5}\gamma-\frac{48}{35}\gamma\pi^2-\frac{56}{5}\ln(2)+\frac{48}{35}\ln(2)\pi^2-\frac{288}{35}\zeta(3)$
8	0	1	$\frac{7875\pi^2}{8192}$	$\frac{120525\pi^2}{32768}-\frac{7875\gamma\pi^2}{8192}-\frac{23625\pi^2\ln(2)}{8192}$	$\frac{5\pi^2(11025\pi^2-100628)}{114688}$	$-\frac{55125\pi^2\zeta(3)}{8192}-\frac{21098123\pi^2}{2408448}+\frac{125785\gamma\pi^2}{28672}+\frac{120525\pi^4}{65536}-\frac{7875\gamma\pi^4}{16384}+\frac{377355\pi^2\ln(2)}{28672}-\frac{23625\pi^4\ln(2)}{16384}$
8	1	0	$\frac{7875\pi^2}{8192}$	$\frac{120525\pi^2}{32768}-\frac{7875\gamma\pi^2}{8192}-\frac{23625\pi^2\ln(2)}{8192}$	$\frac{5\pi^2(11025\pi^2-102428)}{114688}$	$-\frac{55125\pi^2\zeta(3)}{8192}-\frac{21767273\pi^2}{2408448}+\frac{128035\gamma\pi^2}{28672}+\frac{120525\pi^4}{65536}-\frac{7875\gamma\pi^4}{16384}+\frac{384105\pi^2\ln(2)}{28672}-\frac{23625\pi^4\ln(2)}{16384}$
8	1	1	$\frac{384}{35}$	$\frac{384}{35}\ln(2)+\frac{13872}{1225}-\frac{384}{35}\gamma$	$-\frac{208}{15}+\frac{64}{35}\pi^2$	$\frac{208}{15}\gamma+\frac{2312}{1225}\pi^2-\frac{384}{35}\zeta(3)-\frac{208}{15}\ln(2)-\frac{2992}{1575}-\frac{64}{35}\gamma\pi^2+\frac{64}{35}\ln(2)\pi^2$

$$
\begin{aligned}
I^{\rm NLO}_{\Delta E,1{\rm PN}} = & -\frac{888}{35}g_{\rm KK}(6;1,1)-\frac{208}{35}g_{\rm KK}(4;1,1)+\frac{16}{21}g_{\rm KK}(8;1,1)+\frac{1952}{105}g_{\rm KK}(7;0,1)-\frac{10448}{105}g_{\rm KK}(5;0,1)\\
& +\frac{16}{21}g_{\rm KK}(8;0,0)-\frac{752}{35}g_{\rm KK}(4;0,0)-\frac{736}{21}g_{\rm KK}(6;0,0)\\
& -\frac{1}{\pi}\int dv\arctan\left(\tanh\left(\frac{v}{2}\right)\right)\\
& \times\left[\frac{48}{5}\sinh(2v)(2g_{\rm Kcos}(6;1,v)+g_{\rm Kcos}(5;0,v))+\frac{96}{5}\cosh(v)(\cosh(v)^2-2)(g_{\rm Ksin}(5;1,v)+g_{\rm Ksin}(6;0,v))\right]\\
& +\left[-\frac{64}{21}g_{\rm KK}(8;1,1)+\frac{368}{105}g_{\rm KK}(6;1,1)+\frac{16}{5}g_{\rm KK}(4;1,1)+\frac{128}{35}g_{\rm KK}(7;0,1)+\frac{304}{21}g_{\rm KK}(5;0,1)\right.\\
& \left.-\frac{64}{21}g_{\rm KK}(8;0,0)+\frac{16}{5}g_{\rm KK}(6;0,0)+\frac{16}{7}g_{\rm KK}(4;0,0)\right]\nu\\
= & -\frac{25616}{315}+\int dv\arctan\left(\tanh\left(\frac{v}{2}\right)\right)\frac{\sinh v}{\cosh^4 v}\left(-\frac{4032}{5}+\frac{2448}{\cosh^2 v}\right)-\frac{1136}{45}\nu\\
= & \frac{944}{1575}-\frac{1136}{45}\nu,
\end{aligned}
\tag{A20}
$$

where we have used

$$
\begin{aligned}
g_{\rm Kcos}(5;0,v) &= \frac{3\pi}{2\cosh^9 v}(8\cosh^4 v-40\cosh^2 v+35),\\
g_{\rm Kcos}(6;1,v) &= \frac{45\pi}{2\cosh^{11} v}(8\cosh^4 v-28\cosh^2 v+21),\\
g_{\rm Ksin}(5;1,v) &= -\frac{15\pi\sinh v}{2\cosh^9 v}(4\cosh^2 v-7),\\
g_{\rm Ksin}(6;0,v) &= \frac{15\pi\sinh v}{2\cosh^{11} v}(8\cosh^4 v-56\cosh^2 v+63).
\end{aligned}
\tag{A21}
$$

The corresponding result for $I^{\rm NLO}_{W_1,1{\rm PN}}$ is

$$
\begin{aligned}
I^{\rm NLO}_{W_1,1{\rm PN}} = & -\frac{3536}{135}-\frac{51232}{315}\ln(2)+\frac{51232}{315}\gamma\\
& +\int dv\arctan\left(\tanh\left(\frac{v}{2}\right)\right)\frac{\sinh v}{\cosh^2 v}\left[-\frac{576}{5}+\left(\frac{8064}{5}\ln(2)+\frac{8064}{5}\gamma+\frac{16128}{5}\ln(\cosh v)-\frac{34464}{5}\right)\frac{1}{\cosh^2 v}\right.\\
& \left.+\left(-9792\ln(\cosh v)+\frac{86592}{5}-4896\ln(2)-4896\gamma\right)\frac{1}{\cosh^4 v}\right]\\
& +\left(-\frac{77744}{945}-\frac{2272}{45}\ln(2)+\frac{2272}{45}\gamma\right)\nu\\
= & -\frac{56144}{3375}+\frac{1888}{1575}\ln(2)-\frac{1888}{1575}\gamma+\left(-\frac{77744}{945}-\frac{2272}{45}\ln(2)+\frac{2272}{45}\gamma\right)\nu,
\end{aligned}
\tag{A22}
$$

where we have used

$$
\begin{aligned}
g_{\mathrm{Kcos}s}(5;0,v) &= -\frac{12\pi}{\cosh^9 v}\left[\left(2\cosh^4 v - 10\cosh^2 v + \frac{35}{4}\right)\ln(\cosh v) + \left(\gamma - \frac{25}{6} + \ln(2)\right)\cosh^4 v \right.\\
&\quad \left. + \left(\frac{107}{6} - 5\ln(2) - 5\gamma\right)\cosh^2 v + \frac{35}{8}\ln(2) - \frac{44}{3} + \frac{35}{8}\gamma\right],\\
g_{\mathrm{Kcos}s}(6;1,v) &= -\frac{180\pi}{\cosh^{11} v}\left[\left(2\cosh^4 v - 7\cosh^2 v + \frac{21}{4}\right)\ln(\cosh v) + \frac{1}{15}\cosh^6 v + \left(\gamma + \ln(2) - \frac{127}{30}\right)\cosh^4 v \right.\\
&\quad \left. + \left(-\frac{7}{2}\ln(2) + \frac{1583}{120} - \frac{7}{2}\gamma\right)\cosh^2 v + \frac{21}{8}\ln(2) + \frac{21}{8}\gamma - \frac{563}{60}\right],\\
g_{\mathrm{Ksin}s}(5;1,v) &= \frac{\pi \sinh v}{2\cosh^9 v}[(120\cosh^2 v - 210)\ln(\cosh v) + 6\cosh^4 v\\
&\quad + (60\gamma - 229 + 60\ln(2))\cosh^2 v - 105\gamma + 352 - 105\ln(2)],\\
g_{\mathrm{Ksin}s}(6;0,v) &= -\frac{60\pi \sinh v}{\cosh^{11} v}\left[\left(\frac{63}{4} + 2\cosh^4 v - 14\cosh^2 v\right)\ln(\cosh v) + \left(\gamma - \frac{137}{30} + \ln(2)\right)\cosh^4 v \right.\\
&\quad \left. + \left(\frac{809}{30} - 7\gamma - 7\ln(2)\right)\cosh^2 v + \frac{63}{8}\ln(2) - \frac{563}{20} + \frac{63}{8}\gamma\right].
\end{aligned} \tag{A23}
$$

At the NNLO the derivatives of the hypergeometric functions entering the Mellin transforms (A12) also generate HPLs of weight 2. Consider, for instance, the Mellin transform $g_{\mathrm{Kcos}}(6;0,v)$ and its derivative $g_{\mathrm{Kcos}s}(6;0,v)$. We find

$$
g_{\mathrm{Kcos}}(6;0,v) = \left(-\frac{120}{\cosh^7 v} + \frac{840}{\cosh^9 v} - \frac{945}{\cosh^{11} v}\right) v \sinh v + \frac{274}{\cosh^6 v} - \frac{1155}{\cosh^8 v} + \frac{945}{\cosh^{10} v}, \tag{A24}
$$

and

$$
g_{\mathrm{Kcos}s}(6;0,v) = -g_{\mathrm{Kcos}}(6;0,v)\left(\ln(\cosh v) - \ln(2) + \gamma - \frac{3}{2}\right) + \frac{64}{\cosh^6 v}\frac{\partial}{\partial s}{}_2F_1\left(\frac{s}{2},\frac{1-s}{2};\frac{1}{2};\tanh(v)^2\right)\Big|_{s=6}, \tag{A25}
$$

respectively. The latter term can be computed, e.g., by using the tool HYPEXP2 [13], which allows for Taylor-expanding hypergeometric functions around their parameters. It reads

$$
\begin{aligned}
\frac{\partial}{\partial s}{}_2F_1\left(\frac{s}{2},\frac{1-s}{2};\frac{1}{2};\tanh(v)^2\right)\Big|_{s=6} &= \left[\left(-\frac{15}{8\cosh v} + \frac{105}{8\cosh^3 v} - \frac{945}{64\cosh^5 v}\right) v \sinh v \right.\\
&\quad \left. + \frac{945}{64\cosh^4 v} - \frac{1155}{64\cosh^2 v} + \frac{137}{32}\right]\ln(\cosh v)\\
&\quad + \left(\frac{945}{128\cosh^5 v} - \frac{105}{16\cosh^3 v} + \frac{15}{16\cosh v}\right)|\sinh v| H_{-,+}(|\tanh v|)\\
&\quad + \left(\frac{247}{8\cosh^3 v} - \frac{3921}{128\cosh^5 v} - \frac{23}{4\cosh v}\right) v \sinh v + \frac{141}{128\cosh^4 v} + \frac{39}{64} - \frac{219}{128\cosh^2 v},
\end{aligned} \tag{A26}
$$

where

$$
H_{-,+}(|\tanh v|) = 2\ln(2\cosh v)|v| + \mathrm{Li}_2\left(\frac{1}{2} - \frac{|\sinh v|}{2\cosh v}\right) - \mathrm{Li}_2\left(\frac{1}{2} + \frac{|\sinh v|}{2\cosh v}\right) \tag{A27}
$$

is an HPL with weights $\pm$, which can be in turn converted into HPLs with integer weights according to the rule

$$
H_{-,+}(x) = -H_{-1,-1}(x) - H_{-1,1}(x) + H_{1,-1}(x) + H_{1,1}(x). \tag{A28}
$$

Going to the 2PN level we get more involved expressions, but with the same structure (further including terms containing derivatives of the Bessel functions with respect to the order up to the fourth at $\mathrm{N^3LO}$ as well as HPLs of increasing weight).

APPENDIX B: SUMMARY OF FINAL RESULTS FOR THE INTEGRATED NONLOCAL ACTION

We recap below our final results for the integrated nonlocal action $W^{\rm tail,h}$ up to the N^{3}LO order in the large eccentricity expansion, showing also equivalent forms corresponding to different choices of orbital parameters used as independent variables, i.e., either $(\bar{a}_r, e_r)$ or $(\bar{E}, j)$, which are related by Eq. (4.5).

1. First-order-tail part

The 2PN-accurate values of the two contributions to the first-order tail $W^{\rm tail,h} = W_1^{\rm tail,h} + W_2^{\rm tail,h}$, i.e.,

$$W_{1,2}^{\rm tail,h} = W_{1,2}^{\rm tail,hLO} + W_{1,2}^{\rm tail,hNLO} + W_{1,2}^{\rm tail,hNNLO} + W_{1,2}^{\rm tail,hN^3LO} + O(e_r^{-7}), \tag{B1}$$

TABLE IV. Expressions for the various coefficients $W_1^{{\rm tail,h}n{\rm LO}}$ of the large-e_r expansion (B1) of the first-order-tail $W_1^{\rm tail,h}$.

Coefficient	Expression
$W_1^{\rm tail,hLO}$	$\frac{2}{15}\frac{\pi M\nu^2}{e_r^3\bar{a}_r^{7/2}}H_{\rm tot}\{100+37\ln(\frac{s}{4e_r\bar{a}_r^{3/2}})+[\frac{685}{4}-\frac{1017}{14}\nu+(\frac{3429}{56}-\frac{37}{2}\nu)\ln(\frac{s}{4e_r\bar{a}_r^{3/2}})]\frac{\eta^2}{\bar{a}_r}$ $+[\frac{3656939}{8064}-\frac{18181}{72}\nu+\frac{235453}{4032}\nu^2+(\frac{114101}{672}-\frac{7055}{112}\nu+\frac{111}{8}\nu^2)\ln(\frac{s}{4e_r\bar{a}_r^{3/2}})]\frac{\eta^4}{\bar{a}_r^2}\}$
$W_1^{\rm tail,hNLO}$	$\frac{2}{15}\frac{M\nu^2}{e_r^4\bar{a}_r^{7/2}}H_{\rm tot}\{\frac{2224}{9}+\frac{1568}{3}\ln(\frac{4s}{e_r\bar{a}_r^{3/2}})+[-\frac{28072}{225}-\frac{38872}{63}\nu+(\frac{944}{105}-\frac{1136}{3}\nu)\ln(\frac{4s}{e_r\bar{a}_r^{3/2}})]\frac{\eta^2}{\bar{a}_r}$ $+[-\frac{67489874}{77175}-\frac{3115726}{3675}\nu+\frac{165086}{315}\nu^2+(\frac{419036}{735}-\frac{3244}{7}\nu+\frac{764}{3}\nu^2)\ln(\frac{4s}{e_r\bar{a}_r^{3/2}})]\frac{\eta^4}{\bar{a}_r^2}\}$
$W_1^{\rm tail,hNNLO}$	$\frac{2}{15}\frac{\pi M\nu^2}{e_r^5\bar{a}_r^{7/2}}H_{\rm tot}\{\frac{2479}{4}+\frac{6237}{8}\zeta(3)+\frac{843}{2}\ln(\frac{s}{4e_r\bar{a}_r^{3/2}})$ $+[\frac{112309}{224}+\frac{27648}{7}\ln(2)-\frac{299511}{64}\zeta(3)+(-\frac{7332}{7}-918\zeta(3))\nu$ $+(-\frac{66999}{112}-\frac{1827}{4}\nu)\ln(\frac{s}{4e_r\bar{a}_r^{3/2}})]\frac{\eta^2}{\bar{a}_r}$ $+[-\frac{26903663}{16128}-\frac{59760}{7}\ln(2)+\frac{571467}{128}\zeta(3)+(\frac{2338541}{2688}-\frac{20224}{7}\ln(2)+\frac{918657}{256}\zeta(3))\nu$ $+(\frac{321719}{384}+\frac{35613}{64}\zeta(3))\nu^2+(-\frac{442237}{1344}+\frac{28735}{32}\nu+\frac{4497}{16}\nu^2)\ln(\frac{s}{4e_r\bar{a}_r^{3/2}})]\frac{\eta^4}{\bar{a}_r^2}\},$
$W_1^{\rm tail,hN^3LO}$	$\frac{2}{15}\frac{M\nu^2}{e_r^6\bar{a}_r^{7/2}}H_{\rm tot}\{-\frac{11968}{45}+\frac{44288}{15}\zeta(3)+\frac{7808}{3}\ln(\frac{4s}{\bar{a}_r^{3/2}e_r})$ $+[-\frac{753568}{1575}-\frac{505984}{35}\zeta(3)+\frac{621}{8}\pi^4+(-\frac{87136}{45}-\frac{577408}{105}\zeta(3))\nu$ $+(-\frac{763712}{105}-\frac{11584}{3}\nu)\ln(\frac{4s}{\bar{a}_r^{3/2}e_r})]\frac{\eta^2}{\bar{a}_r}\}$ $+[\frac{261560660008}{22920975}+\frac{5617672}{3465}\zeta(3)+\frac{8424}{5}\pi^2-\frac{367551}{896}\pi^4$ $+(\frac{954967984}{72765}+\frac{11922016}{495}\zeta(3)-\frac{26865}{224}\pi^4)\nu+(\frac{20142088}{6615}+\frac{7589024}{2205}\zeta(3))\nu^2$ $+(-\frac{119081008}{19845}+\frac{7488736}{315}\nu+2384\nu^2)\ln(\frac{4s}{\bar{a}_r^{3/2}e_r})]\frac{\eta^4}{\bar{a}_r^2}\}$

TABLE V. Expressions for the various coefficients $W_2^{\rm tail,h\,nLO}$ of the large-e_r expansion (B1) of the first-order-tail $W_2^{\rm tail,h}$.

Coefficient	Expression
$W_2^{\rm tail,h\,LO}$	$\frac{2}{15}\frac{\pi M\nu^2}{e_r^3\bar{a}_r^{7/2}}H_{\rm tot}\{-\frac{85}{4}-37\ln(\frac{s}{2e_r\bar{a}_r})+[-\frac{9679}{224}+\frac{981}{56}\nu+(-\frac{3429}{56}+\frac{37}{2}\nu)\ln(\frac{s}{2e_r\bar{a}_r})]\frac{\eta^2}{\bar{a}_r}$ $+[-\frac{1830565}{16128}+\frac{54899}{1152}\nu-\frac{29969}{4032}\nu^2+(-\frac{114101}{672}+\frac{7055}{112}\nu-\frac{111}{8}\nu^2)\ln(\frac{s}{2e_r\bar{a}_r})]\frac{\eta^4}{\bar{a}_r^2}\}$
$W_2^{\rm tail,h\,NLO}$	$\frac{2}{15}\frac{M\nu^2}{e_r^4\bar{a}_r^{7/2}}H_{\rm tot}\{\frac{2768}{9}-\frac{1568}{3}\ln(\frac{2s}{e_r\bar{a}_r})+[-\frac{64904}{225}-\frac{5992}{45}\nu+(-\frac{944}{105}+\frac{1136}{3}\nu)\ln(\frac{2s}{e_r\bar{a}_r})]\frac{\eta^2}{\bar{a}_r}$ $+[-\frac{2925494}{77175}-\frac{542014}{2205}\nu+\frac{145498}{735}\nu^2+(-\frac{419036}{735}+\frac{3244}{7}\nu-\frac{764}{3}\nu^2)\ln(\frac{2s}{e_r\bar{a}_r})]\frac{\eta^4}{\bar{a}_r^2}\}$
$W_2^{\rm tail,h\,NNLO}$	$\frac{2}{15}\frac{\pi M\nu^2}{e_r^5\bar{a}_r^{7/2}}H_{\rm tot}\{-\frac{3419}{8}-\frac{843}{2}\ln(\frac{s}{2e_r\bar{a}_r})$ $+[\frac{103645}{448}+\frac{56559}{112}\nu+(\frac{66999}{112}+\frac{1827}{4}\nu)\ln(\frac{s}{2e_r\bar{a}_r})]\frac{\eta^2}{\bar{a}_r}$ $+[\frac{2467109}{13824}-\frac{3706175}{5376}\nu-\frac{1577635}{8064}\nu^2+(\frac{442237}{1344}-\frac{28735}{32}\nu-\frac{4497}{16}\nu^2)\ln(\frac{s}{2e_r\bar{a}_r})]\frac{\eta^4}{\bar{a}_r^2}\}$
$W_2^{\rm tail,h\,N^3LO}$	$\frac{2}{15}\frac{M\nu^2}{e_r^6\bar{a}_r^{7/2}}H_{\rm tot}\{\frac{4384}{9}-\frac{7808}{3}\ln(\frac{2s}{e_r\bar{a}_r})$ $+[-\frac{7182736}{1575}-\frac{210928}{315}\nu+(\frac{763712}{105}+\frac{11584}{3}\nu)\ln(\frac{2s}{e_r\bar{a}_r})]\frac{\eta^2}{\bar{a}_r}$ $+[-\frac{2135067428}{2083725}+\frac{392856316}{33075}\nu+\frac{1895524}{1323}\nu^2$ $+(\frac{119081008}{19845}-\frac{7488736}{315}\nu-2384\nu^2)\ln(\frac{2s}{e_r\bar{a}_r})]\frac{\eta^4}{\bar{a}_r^2}\}$

TABLE VI. Coefficients $\mathcal{W}_n$ entering the large-j expansion (B2) of the first-order-tail $W^{\rm tail,h}=W_1^{\rm tail,h}+W_2^{\rm tail,h}$.

Coefficient	Expression
$\mathcal{W}_3$	$\frac{\pi(2\bar E)^2}{15}M^2\nu^2\{\frac{315}{2}+37\ln(\frac{\bar E}{2})+[\frac{12609}{112}-\frac{719}{4}\nu+(\frac{2393}{56}-37\nu)\ln(\frac{\bar E}{2})](2\bar E)\eta^2$ $+[\frac{189347}{896}-\frac{295831}{2016}\nu+\frac{10603}{63}\nu^2+(\frac{745}{12}-\frac{7179}{224}\nu+\frac{481}{16}\nu^2)\ln(\frac{\bar E}{2})](2\bar E)^2\eta^4\}$
$\mathcal{W}_4$	$\frac{(2\bar E)^{3/2}}{15}M^2\nu^2\{\frac{3328}{3}+\frac{1568}{3}\ln(8\bar E)+[-\frac{16184}{75}-\frac{274856}{105}\nu+(\frac{25468}{35}-\frac{2900}{3}\nu)\ln(8\bar E)](2\bar E)\eta^2$ $+[-\frac{28646896}{15435}-\frac{931192}{1575}\nu+\frac{7860556}{2205}\nu^2+(\frac{1894327}{2940}-\frac{67069}{70}\nu+\frac{4353}{4}\nu^2)\ln(8\bar E)](2\bar E)^2\eta^4\}$
$\mathcal{W}_5$	$\frac{\pi(2\bar E)}{15}M^2\nu^2\{\frac{297}{2}+\frac{6237}{4}\zeta(3)+366\ln(\frac{\bar E}{2})$ $+[\frac{65547}{28}+\frac{55296}{7}\ln(2)-\frac{137349}{32}\zeta(3)+(-\frac{10593}{8}-\frac{73035}{16}\zeta(3))\nu+(\frac{46617}{56}-1125\nu)\ln(\frac{\bar E}{2})](2\bar E)\eta^2$ $+[-\frac{388793}{6048}-\frac{36576}{7}\ln(2)-\frac{13845}{32}\zeta(3)+(-\frac{7540333}{1344}-\frac{123392}{7}\ln(2)+\frac{1390239}{128}\zeta(3))\nu$ $+(\frac{218503}{72}+\frac{223533}{32}\zeta(3))\nu^2+(\frac{14975}{84}-\frac{24671}{16}\nu+\frac{6987}{4}\nu^2)\ln(\frac{\bar E}{2})](2\bar E)^2\eta^4\},$
$\mathcal{W}_6$	$\frac{(2\bar E)^{1/2}}{15}M^2\nu^2\{-\frac{79936}{45}+\frac{88576}{15}\zeta(3)+\frac{4672}{3}\ln(8\bar E)$ $+[-\frac{145784}{21}+\frac{28352}{21}\zeta(3)+\frac{621}{4}\pi^4+(\frac{56152}{315}-\frac{2627392}{105}\zeta(3))\nu+(\frac{146200}{21}-7848\nu)\ln(8\bar E)](2\bar E)\eta^2$ $+[-\frac{19308599}{2970}-\frac{19317988}{495}\zeta(3)+\frac{16848}{5}\pi^2-\frac{132813}{448}\pi^4+(\frac{1528632323}{51975}+\frac{28349672}{3465}\zeta(3)-\frac{127629}{224}\pi^4)\nu$ $+(\frac{9153757}{882}+\frac{111368548}{2205}\zeta(3))\nu^2+(-\frac{75949}{1134}-\frac{5855873}{315}\nu+\frac{33581}{2}\nu^2)\ln(8\bar E)](2\bar E)^2\eta^4\}$

are listed in Tables IV and V. It is easily seen that the intermediate scale s cancels between the two contributions.

Reexpressing $\bar a_r$ and e_r in terms of $\bar E$ and j we get

$$W^{\rm tail,h}=\frac{\mathcal{W}_3}{j^3}+\frac{\mathcal{W}_4}{j^4}+\frac{\mathcal{W}_5}{j^5}+\frac{\mathcal{W}_6}{j^6}+O(j^{-7}) \tag{B2}$$

with coefficients $\mathcal{W}_k$, $k=3$, 4, 5, 6 listed in Table VI.

The f-induced additional contribution reads

$$W^{\rm f\text{-}h}=\frac{M\nu^3}{\bar a_r^{9/2}}H_{\rm tot}\eta^2\left[\frac{\pi}{e_r^3}\mathcal{W}^{\rm f\text{-}h\,LO}+\frac{1}{e_r^4}\mathcal{W}^{\rm f\text{-}h\,NLO}+\frac{\pi}{e_r^5}\mathcal{W}^{\rm f\text{-}h\,NNLO}+\frac{1}{e_r^6}\mathcal{W}^{\rm f\text{-}h\,N^3LO}\right], \tag{B3}$$

with the various $\mathcal{W}^{\rm f\text{-}h\,nLO}$ listed in Table VII.

Using the *minimal* solution of the $5+6$PN constraints

$$C_1^{\rm min}=\frac{168}{5},\qquad C_2^{\rm min}=0,\qquad C_3^{\rm min}=0,$$
$$D_1^{\rm min}=\frac{271066}{4725}+\frac{21736}{189}\nu,\qquad D_2^{\rm min}=-\frac{39712}{189}\nu,$$
$$D_3^{\rm min}=-\frac{68108}{945}\nu,\qquad D_4^{\rm min}=0, \tag{B4}$$

the previous expression becomes

TABLE VII. Coefficients $\mathcal{W}^{\rm f\text{-}h\,nLO}$ entering the large-e_r expansion (B3) of f-h contribution $W^{\rm f\text{-}h}$ to the first-order tail.

Coefficient	Expression
$\mathcal{W}^{\rm f\text{-}h\,LO}$	$\frac{1}{16}C_1+(-\frac{7}{32}\nu C_1+\frac{7}{32}C_1+\frac{5}{128}D_1)\frac{\eta^2}{\bar a_r}$
$\mathcal{W}^{\rm f\text{-}h\,NLO}$	$\frac{4}{5}C_1+\frac{4}{15}C_2+[(-\frac{2}{3}C_2-\frac{96}{35}C_1)\nu+\frac{4}{7}D_1+\frac{2}{5}C_2+\frac{4}{35}D_2+\frac{114}{35}C_1]\frac{\eta^2}{\bar a_r}$
$\mathcal{W}^{\rm f\text{-}h\,NNLO}$	$\frac{21}{32}C_1+\frac{3}{8}C_3+\frac{3}{8}C_2$ $+[(-\frac{141}{64}C_1-\frac{31}{32}C_2-\frac{9}{16}C_3)\nu+\frac{13}{16}C_2+\frac{1}{16}D_3+\frac{195}{64}C_1+\frac{3}{16}D_2-\frac{3}{16}C_3+\frac{135}{256}D_1]\frac{\eta^2}{\bar a_r}$
$\mathcal{W}^{\rm f\text{-}h\,N^3LO}$	$\frac{64}{15}C_1+\frac{16}{3}C_3+\frac{16}{5}C_2$ $+[(-\frac{128}{15}C_2-\frac{48}{5}C_3-\frac{1472}{105}C_1)\nu+\frac{16}{15}D_3+\frac{64}{35}D_2+\frac{8}{5}C_3+\frac{16}{15}D_4+\frac{136}{15}C_2+\frac{2336}{105}C_1+\frac{80}{21}D_1]\frac{\eta^2}{\bar a_r}$

TABLE VIII. Coefficients $W_n^{\text{f-h}}$ entering the large-e_r expansion (B6) of f-h contribution $W^{\text{f-h}}$ to the first-order tail.

Coefficient	Expression
$W_3^{\text{f-h}}$	$\pi(2\bar{E})^3M^2\nu^3\eta^2[\frac{1}{16}C_1+(\frac{5}{64}C_1+\frac{5}{128}D_1-\frac{15}{64}\nu C_1)(2\bar{E})\eta^2]$
$W_4^{\text{f-h}}$	$(2\bar{E})^{5/2}M^2\nu^3\eta^2\{\frac{4}{5}C_1+\frac{4}{15}C_2+[(-\frac{241}{70}C_1-\frac{9}{10}C_2)\nu+\frac{4}{7}D_1+\frac{207}{70}C_1+\frac{4}{35}D_2+\frac{3}{10}C_2](2\bar{E})\eta^2\}$
$W_5^{\text{f-h}}$	$\pi(2\bar{E})^2M^2\nu^3\eta^2\{\frac{3}{8}C_2+\frac{9}{16}C_1+\frac{3}{8}C_3$
	$+[(-\frac{9}{8}C_3-\frac{45}{16}C_1-\frac{49}{32}C_2)\nu+\frac{1}{16}D_3+\frac{3}{8}C_3+\frac{33}{8}C_1+\frac{11}{8}C_2+\frac{15}{32}D_1+\frac{3}{16}D_2](2\bar{E})\eta^2\}$
$W_6^{\text{f-h}}$	$(2\bar{E})^{3/2}M^2\nu^3\eta^2\{\frac{8}{3}C_1+\frac{8}{3}C_2+\frac{16}{3}C_3$
	$+[(-\frac{67}{5}C_2-\frac{314}{15}C_3-\frac{79}{5}C_1)\nu+\frac{103}{3}C_1+\frac{16}{15}D_3+\frac{8}{5}D_2+\frac{307}{15}C_2+\frac{98}{5}C_3+\frac{16}{15}D_4+\frac{8}{3}D_1](2\bar{E})\eta^2\}$

$$W^{\text{f-h}}_{\text{min}}=\frac{M\nu^3}{\bar{a}_r^{9/2}}H_{\text{tot}}\eta^2\left\{\frac{\pi}{e_r^3}\left[\frac{21}{10}+\left(-\frac{43207}{15120}\nu+\frac{580061}{60480}\right)\frac{\eta^2}{\bar{a}_r}\right]+\frac{1}{e_r^4}\left[\frac{672}{25}+\left(-\frac{1668832}{33075}\nu+\frac{4703992}{33075}\right)\frac{\eta^2}{\bar{a}_r}\right]\right.$$
$$\left.+\frac{\pi}{e_r^5}\left[\frac{441}{20}+\left(-\frac{1732117}{30240}\nu+\frac{594173}{4480}\right)\frac{\eta^2}{\bar{a}_r}\right]+\frac{1}{e_r^6}\left[\frac{3584}{25}+\left(-\frac{3267904}{6615}\nu+\frac{95857952}{99225}\right)\frac{\eta^2}{\bar{a}_r}\right]\right\}. \tag{B5}$$

Reexpressing $\bar{a}_r$ and e_r in terms of $\bar{E}$ and j we find

$$W^{\text{f-h}}=\frac{W_3^{\text{f-h}}}{j^3}+\frac{W_4^{\text{f-h}}}{j^4}+\frac{W_5^{\text{f-h}}}{j^5}+\frac{W_6^{\text{f-h}}}{j^6}+O(j^{-7}), \tag{B6}$$

with the coefficients $W_n^{\text{f-h}}$ listed in Table VIII below.

Using the minimal value solutions of the C_i and D_i we find

$$W^{\text{f-h}}_{\text{min}}=\frac{(2\bar{E})^3}{j^3}M^2\nu^3\eta^2\left\{\pi\left[\frac{21}{10}+\left(\frac{294293}{60480}-\frac{10229}{3024}\nu\right)(2\bar{E})\eta^2\right]\right.$$
$$+\frac{(2\bar{E})^{-1/2}}{j}\left[\frac{672}{25}+\left(\frac{4370596}{33075}-\frac{2446756}{33075}\nu\right)(2\bar{E})\eta^2\right]$$
$$+\frac{\pi(2\bar{E})^{-1}}{j^2}\left[\frac{189}{10}+\left(\frac{834077}{5040}-\frac{22813}{270}\nu\right)(2\bar{E})\eta^2\right]$$
$$\left.+\frac{(2\bar{E})^{-3/2}}{j^3}\left[\frac{448}{5}+\left(\frac{18520808}{14175}-\frac{143384}{225}\nu\right)(2\bar{E})\eta^2\right]\right\}. \tag{B7}$$

2. Second-order-tail part

Finally, the second-order-tail contribution turns out to be

$$W^{\text{tail,h,5.5PN}}=\frac{M^2\nu^2}{e_r^4\bar{a}_r^5}\left[\frac{23968}{675}+\frac{10593}{1400}\frac{\pi^3}{e_r}+\left(\frac{835456}{4725}+\frac{4738816}{70875}\pi^2\right)\frac{1}{e_r^2}\right], \tag{B8}$$

or equivalently

$$W^{\text{tail,h,5.5PN}}=\frac{M^2\nu^2(2\bar{E})^2}{j^4}\left[\frac{23968}{675}(2\bar{E})+\frac{10593}{1400}\frac{\pi^3(2\bar{E})^{1/2}}{j}+\left(\frac{499904}{4725}+\frac{4738816}{70875}\pi^2\right)\frac{1}{j^2}\right]. \tag{B9}$$

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