


## Asymptotic flatness and Hawking quasilocal mass

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We point out an association between anomalies in the Hawking quasilocal mass (or, in spherical symmetry, in its better known version, the Misner-Sharp-Hernandez mass) and unphysical properties of the spacetime geometry. While anomalous behaviors show up in certain quantum-corrected black holes, they are not unique to this context and signal serious physical pathologies of isolated gravitating systems in general.

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### I. INTRODUCTION

According to the equivalence principle, which constitutes the foundation of general relativity (GR) and of metric theories of gravity [1,2], the gravitational field can be eliminated locally and it is impossible to assign a local energy density to the gravitational field. For isolated systems, one can consider the notion of mass at spatial infinity, which is embodied by the Arnowitt-Deser-Misner (ADM) construct. This concept, however, is not defined for nonisolated systems (for example, massive objects embedded in cosmological spacetimes), and it is only defined asymptotically. It is, however, possible to define the mass energy of a gravitating system in a quasilocal way. In the presence of spherical symmetry, the Misner-Sharp-Hernandez mass [3,4] has been used for a long time, especially in the context of the gravitational collapse of fluids. The Misner-Sharp-Hernandez mass finds a generalization to nonspherically symmetric spacetimes in the Hawking quasilocal mass [5,6], and several other definitions of quasilocal energy have been proposed (see Ref. [7] for a review).

The Hawking quasilocal mass is not normally associated with asymptotic flatness, however one can associate certain “anomalies” in the behavior of the Hawking mass when the gravitational field exhibits pathologies. The purpose of this work is to illustrate this association and to discuss how the Hawking/Misner-Sharp-Hernandez mass can signal unphysical properties of spacetime.

The first occurrence of this association is in the context of regular black holes. In the quest to avoid spacetime singularities, proposals have been made to quantize the full GR theory or, from more phenomenological points of view, at

least its black holes to remove the timelike singularities hiding inside them. Naturally, much attention has focused on removing the singularity of the prototypical Schwarzschild black hole and the quantum-corrected black holes proposed in the literature are usually static and spherically symmetric geometries. Often, these quantum-corrected black holes do not describe isolated systems *in vacuo* and, sometimes, they are not even asymptotically flat. The Bardeen regular black hole [8] can be construed as a solution of the Einstein equations coupled to nonlinear electrodynamics, thus it is not a vacuum solution [9]. Many other examples of regular black holes have been provided over the years, including the more recent Planck star proposal ([10], see [11] for a review) and the subject is a mature one with a relatively large literature devoted to it. Quantum correcting the Schwarzschild black hole according to loop quantum gravity produces a geometry [12–14] that fails to be truly asymptotically flat [15,16]. This fact causes the black hole geometry to exhibit unexpected unphysical properties, due to the fact that the small quantum gravity corrections actually dominate in regions in which gravity is weak, as well as in strong gravity regions near the singularity that they are designed to eliminate [15,16]. This fact is responsible for unphysical properties, which include a vanishing quasilocal mass as seen from spatial infinity, instead of the positive Schwarzschild mass that one expects to recover far away from the black hole [16]. In addition, no initially outgoing timelike geodesic can reach  $r = +\infty$ , where  $r$  is the areal radius [16].

Motivated by the example of quantum corrected and regular black holes, we consider the more general question of whether possible variations in the definition of asymptotic flatness (i.e., in the falloff rate of the fields) can be physically meaningful. We use the ADM mass at infinity and the Hawking quasilocal mass as tools to discuss physical properties of the gravitating systems described. The result is that the falloff rates of the physical fields required in the definition of asymptotic flatness are strictly

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necessary and relaxing them causes physical pathologies, which will be discussed.

In Sec. II we recall the definition of ADM mass and discuss the physical implications of relaxing the falloff rates of the fields in it. Since the ADM mass is only defined at infinity, in Sec. III we seek further physical insight by using the Hawking quasilocal mass [5,6], which is defined at any finite distance from a self-gravitating body but reduces to the ADM mass at spatial infinity. It is also defined in nonasymptotically flat geometries, which allows us to explore easily geometries that relax the requirements of asymptotic flatness.

We first consider spherical symmetry, in which case the Hawking mass reduces to the better known Misner-Sharp-Hernandez mass used in fluid mechanics and in gravitational collapse [3,4]. Then, in Sec. IV, we relax the assumption of spherical symmetry. Predictably, it is much more difficult to prove precise statements in this general situation, but we provide an argument in general (i.e., nonspherically symmetric) geometries pointing again to the fact that the conditions in the definition of asymptotic flatness cannot be relaxed without introducing physical pathologies. These pathologies are reflected in anomalies in the Hawking mass, such as its vanishing or divergence at spatial infinity, or the fact that it receives a contribution from matter but not from the gravitational field.

Throughout this work, we follow the notation of Ref. [1]. Units are such that the speed of light and Newton's constant are unity.

## II. ASYMPTOTIC FLATNESS AND ADM MASS

Let us consider the 3 + 1 foliation of a general spacetime  $(\mathcal{M}, g)$ , with  $g$  denoting the metric tensor, in terms of three-dimensional spacelike hypersurfaces  $\Sigma_t = \{x^\mu | t(x^\mu) = \text{const}\}$ , with  $t$  denoting a time function. The time evolution of the system is, therefore, generated by the vector field  $\partial/\partial t$  that can be split into a component tangent to  $\Sigma_t$  and a normal to the hypersurface, i.e.,

$$\left(\frac{\partial}{\partial t}\right)^a = Nn^a + N^a, \quad (2.1)$$

with  $N$  the lapse function,  $N^a$  the shift vector, and  $n^a$  the normal to  $\Sigma_t$  (in the coordinate representation  $n_\alpha \sim \partial_\alpha t$ ).

The pullback of  $g$  onto  $\Sigma_t$  defines the induced metric  $\gamma_{ab} = \varphi^* g_{ab}$ , with  $\varphi$  denoting the embedding of  $(\Sigma_t, \gamma)$  into  $(\mathcal{M}, g)$ .  $\gamma_{ab}$ , adapted to the coordinates on  $(\mathcal{M}, g)$ , reads

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu, \quad (2.2)$$

in fact  $\gamma_\nu^\mu$  acts as a tangential projector onto  $\Sigma_t$ , i.e., if  $V^a \in T\mathcal{M}$ , then  $\gamma_\nu^\mu V^\nu$  belongs to  $T\Sigma_t$ . In a similar way, the Levi-Civita connection  $\nabla$  defined on  $(\mathcal{M}, g)$  induces the Levi-Civita connection  $D$  on  $(\Sigma_t, \gamma)$ . Furthermore, if  $\epsilon = \sqrt{-g} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$  denotes the volume form

on  $(\mathcal{M}, g)$ , then  $\bar{\epsilon} = \sqrt{\gamma} dy^1 \wedge dy^2 \wedge dy^3$ . If we adapt our chart on  $\Sigma_t$  so that  $y^i = x^i$  for  $i = 1, 2, 3$ , then  $\bar{\epsilon}_{\alpha\beta\gamma} = n^\mu \epsilon_{\mu\alpha\beta\gamma}$ .

Finally, the extrinsic curvature of  $(\Sigma_t, \gamma)$  in  $(\mathcal{M}, g)$  is defined as

$$K_{ab} := \frac{1}{2} \mathcal{L}_n \gamma_{ab}. \quad (2.3)$$

In a coordinate chart of  $(\mathcal{M}, g)$ , this reads

$$K_{\mu\nu} = \gamma_\mu^\alpha \nabla_\alpha n_\nu = \gamma_\mu^\alpha \gamma_\nu^\beta \nabla_\alpha n_\beta. \quad (2.4)$$

Similarly to the case of spacelike three surfaces  $\Sigma$ , one can embed closed two surfaces  $\mathcal{S}$  into  $\Sigma$ . The normal bundle  $T^\perp \mathcal{S}$  of  $\mathcal{S}$  can be spanned by a timelike vector field  $n^a$  and a spacelike vector field  $s^a$ . Usually, one also conventionally chooses these two vectors to be orthogonal, i.e.,  $n^a s_a = 0$ . Thus, if  $\Sigma$  is a spacelike three surface embedded in the spacetime  $(\mathcal{M}, g)$ , one can identify  $n^a$  with the (timelike) normal to  $\Sigma$ , whereas  $s^a$  will be the normal to  $\mathcal{S}$  tangent to  $\Sigma$ , i.e.,  $s^a \in T\Sigma$  and  $n^a \in T^\perp \Sigma$ . (Alternatively,  $T^\perp \mathcal{S}$  can be split at each  $p \in \mathcal{S}$  in terms of two null normal vectors tangent to ingoing and outgoing null geodesics.) Hence,  $(\mathcal{S}, q)$  is an embedded closed two surface in  $(\Sigma_t, \gamma)$ , with  $q$  being the pullback of  $\gamma$  to  $\mathcal{S}$  that, in a coordinate chart of  $(\mathcal{M}, g)$  adapted to the 3 + 1 splitting, reads

$$q_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu - s_\mu s_\nu = \gamma_{\mu\nu} - s_\mu s_\nu. \quad (2.5)$$

The induced Levi-Civita connection on  $(\mathcal{S}, q)$  is denoted by  ${}^2D$ , while the surface two form is  ${}^2\epsilon = \sqrt{q} dz^1 \wedge dz^2$ . If one considers a coordinate chart of  $(\mathcal{M}, g)$  adapted to the 3 + 1 splitting, it yields  ${}^2\epsilon_{\mu\nu} = n^\alpha s^\beta \epsilon_{\alpha\beta\mu\nu}$ . Then, we denote the deformation tensor  $\Theta_{ab}^{(v)}$  associated with the vector field  $v^a$  normal to  $\mathcal{S}$  as

$$\Theta_{\mu\nu}^{(v)} = q^\alpha_\mu q^\beta_\nu \nabla_\alpha v_\beta, \quad (2.6)$$

in the usual coordinate chart of  $(\mathcal{M}, g)$  adapted to the 3 + 1 splitting. In particular, we denote by

$$k_{ab} \equiv \Theta_{ab}^{(s)}, \quad (2.7)$$

the extrinsic curvature of  $(\mathcal{S}, q)$  inside the three slice  $(\Sigma_t, \gamma)$  corresponding to the spacelike normal  $s^a$ .

Let us now move on to the notion of asymptotic flatness and the 3 + 1 decomposition using a coordinate-based approach (see [17] for further details). Let  $\Sigma$  be a three-dimensional spacelike slice of  $(\mathcal{M}, g)$  with induced metric  $\gamma_{ab}$ .  $\Sigma$  is an asymptotically flat slice if there exists a Riemannian background metric  $f_{ij}$  such that

- (i)  $f_{ij}$  is flat, except on a compact domain  $\mathcal{D} \subset \Sigma$ .

- (ii)  $\exists$  a Cartesian-like chart  $\{x^i: \mathcal{M} \rightarrow \mathbb{R}^3\}$  such that, outside  $\mathcal{D}$ , one has  $f_{ij} = \text{diag}(1, 1, 1)$  and  $r \equiv \sqrt{x^2 + y^2 + z^2}$  can take arbitrary large value;
- (iii) as  $r \rightarrow \infty$ , one has

$$\gamma_{ij} = f_{ij} + \mathcal{O}(1/r), \quad (2.8)$$

$$\partial_k \gamma_{ij} = \mathcal{O}(1/r^2), \quad (2.9)$$

$$K_{ij} = \mathcal{O}(1/r^2), \quad (2.10)$$

$$\partial_k K_{ij} = \mathcal{O}(1/r^3). \quad (2.11)$$

Given an asymptotically flat spacetime foliated by asymptotically flat (or Euclidean) slices  $\Sigma_t$ , one defines spatial infinity as  $r \rightarrow \infty$  and denotes it by  $i^0$ .

Let  $\mathcal{V} \subset \mathcal{M}$  be a four-dimensional spacetime region with boundary  $\partial\mathcal{V}$  such that

$$\partial\mathcal{V} = \Sigma_{t_1} \cup (-\Sigma_{t_2}) \cup \mathcal{T}, \quad (2.12)$$

with  $t_1 < t_2$ ,  $\Sigma_{t_1}, \Sigma_{t_2}$  two spacelike three slices (as above) with metric and extrinsic curvature  $(\gamma_{ab}, K_{ab})$ ,  $\mathcal{T}$  an outer timelike tube, and let the boundary condition be  $\delta g_{ab}|_{\partial\mathcal{V}} = 0$ . Note that  $\mathcal{S}_t \equiv \Sigma_{t_2} \cap \mathcal{T}$  forms a closed space-like two surface with induced metric and extrinsic curvature  $(q_{ab}, k_{ab})$ .

The Einstein-Hilbert action, including also the Gibbons-Hawking-York boundary term, reads

$$S = \frac{1}{16\pi} \int_{\mathcal{V}} \epsilon R + \frac{1}{8\pi} \int_{\partial\mathcal{V}} \bar{\epsilon} (K - K_0), \quad (2.13)$$

with  $K_0$  denoting the extrinsic curvature of the boundary embedded in flat spacetime. This action then reduces to

$$S = \frac{1}{16\pi} \int_{t_1}^{t_2} dt \left[ \int_{\Sigma_t} N({}^3R + K_{ij}K^{ij} - K^2) \sqrt{\gamma} d^3x + 2 \oint_{\mathcal{S}_t} (k - k_0) N \sqrt{q} d^2x \right], \quad (2.14)$$

with  $k$  and  $k_0$  being the trace of the extrinsic curvature of  $\mathcal{S}_t$  embedded in  $(\Sigma_t, \gamma)$  and  $(\Sigma_t, f)$ , respectively.

Moving to the Hamiltonian formalism, one finds the total Hamiltonian

$$H = -\frac{1}{16\pi} \left\{ \int_{\Sigma_t} (N\mathcal{H} + 2N^i \mathcal{H}_i) \sqrt{\gamma} d^3x + 2 \oint_{\mathcal{S}_t} [N(k - k_0) - N^i (K_{ij} - K\gamma_{ij}) s^j] \sqrt{q} d^2x \right\}, \quad (2.15)$$

with  $\mathcal{H} = {}^3R - K_{ij}K^{ij} + K^2$  and  $\mathcal{H}_i = D_j K_i^j - D_i K$ .

*In vacuo*, it is  $\mathcal{H} = \mathcal{H}_i = 0$  (Hamiltonian and momentum constraints) on solutions of the Einstein equation. Hence, on shell, one has

$$H_{\text{on shell}} = -\frac{1}{8\pi} \oint_{\mathcal{S}_t} [N(k - k_0) - N^i (K_{ij} - K\gamma_{ij}) s^j] \sqrt{q} d^2x. \quad (2.16)$$

Choosing  $\partial/\partial t$  so that it is associated with some asymptotically inertial observer, i.e.,  $N = 1$  and  $N^i = 0$  when  $r \rightarrow \infty$ , yields the ADM mass

$$M = -\frac{1}{8\pi} \lim_{\mathcal{S}_t(r \rightarrow \infty)} \oint_{\mathcal{S}_t} (k - k_0) \sqrt{q} d^2x, \quad (2.17)$$

and then using the asymptotically flat slicing one finds

$$M = \frac{1}{16\pi} \lim_{\mathcal{S}_t(r \rightarrow \infty)} \oint_{\mathcal{S}_t} (\partial_j \gamma_i^j - \partial_i \gamma_j^j) s^i \sqrt{q} d^2x \quad (2.18)$$

(see Ref. [18] for an explicit derivation). The asymptotic flatness conditions guarantee the convergence of this integral.

To appreciate the effect of metric components decaying slower than  $1/r$ , it is useful to contemplate the analogous situation in Newtonian gravity. *In vacuo*, the Newtonian potential  $\phi$  solves the Laplace equation  $\nabla^2 \phi = 0$  and can be expressed as the sum of a monopole term, a dipole term, etc., which makes the first integral in Eq. (2.15) converge. The fact that  $\phi$  decays slower than  $1/r$  signals the presence of matter (or, possibly, effective matter<sup>1</sup>) in space, in which case the Laplace equation turns into the Poisson equation  $\nabla^2 \phi = 4\pi\rho$ . A similar property holds in GR: *in vacuo* and for a stationary self-gravitating and isolated source, the general metric is necessarily given by a multipole expansion with the first term scaling as  $1/r$  and no terms scaling as  $r^{-(1-\epsilon)}$  (with  $\epsilon > 0$ ) are possible [19]. The curvature tensor coincides with the Weyl tensor  $C^a{}_{bcd}$ , which exhibits the peeling property along null geodesics [20,21]. The failure to satisfy this property for a stationary spacetime signals the presence of matter (or effective matter) and a nonvanishing Ricci tensor  $R_{ab}$  (see Sec. IV).

<sup>1</sup>This is the case, for example, if a cosmological constant is introduced into the Laplace equation.

In the presence of matter fields,  $\mathcal{H} \propto \rho$  and  $\mathcal{H}^i \propto J^i$  (where  $\rho$  and  $J^i$  are the energy density and energy current density, respectively) and the first integral in the right-hand side of Eq. (2.15) converges only if the matter fields decay sufficiently fast. This is the case, for example, for exact solutions of the Einstein equations describing relativistic stars with energy density that is not a function with compact support but decays very fast as  $r \rightarrow \infty$  (see [22] for a review). If this integral diverges, there cannot be asymptotic flatness and the ADM mass is not defined. What is more, any pathologies in the energy density or effective density (for example, a negative sign, as in certain quantum-corrected black holes) will leave an imprint in the ADM mass (when the latter is well defined).

### III. QUASILOCAL MASS—SPHERICAL SYMMETRY

Let us turn now to a different concept of mass, the Hawking quasilocal mass, which has the potential to provide extra information with respect to the ADM mass. In fact, the quasilocal mass is defined using topological two spheres of finite size, while the ADM mass is necessarily defined only at spatial infinity. For simplicity, we restrict to spherically symmetric and static geometries  $g_{ab}$ . The line element can be written as

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\Omega_{(2)}^2 \quad (3.1)$$

without loss of generality, where  $r$  is the areal radius defined by the two spheres of symmetry and  $d\Omega_{(2)}^2 \equiv d\vartheta^2 + \sin^2\vartheta d\varphi^2$  is the line element on the unit two sphere.

In spherical symmetry, the Hawking quasilocal mass [5,6] reduces [23] to the better known Misner-Sharp-Hernandez mass  $M_{\text{MSH}}$  defined by [3,4]

$$M_{\text{MSH}} = \frac{r}{2} (1 - \nabla^c r \nabla_c r), \quad (3.2)$$

which, in the gauge (3.1), assumes the form

$$M_{\text{MSH}} = \frac{r}{2} \left( 1 - \frac{1}{B} \right). \quad (3.3)$$

The loop quantum gravity black hole of [12–14] fails to be asymptotically flat [15,16] and this feature is reflected in a vanishing quasilocal mass at large (areal) radii [16]. Other quantum-corrected black holes have the correct asymptotic flatness. For example, the Kehagias-Sfetsos geometry is a solution of Hořava-Lifschitz gravity [24] in the presence of plasma, with line element [25]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{(2)}^2, \quad (3.4)$$

where

$$f(r) = 1 + \omega_{KS} r^2 \left[ 1 - \left( 1 + \frac{4m}{\omega_{KS} r^3} \right)^{1/2} \right]. \quad (3.5)$$

By expanding for  $m/r \ll 1$ , one obtains  $f(r) \simeq 1 - 2m/r + \mathcal{O}(1/r^2)$ , which is the correct asymptotics for asymptotic flatness.

Let us discuss the relation between Misner-Sharp-Hernandez mass and asymptotic flatness more in general. In asymptotically flat spacetimes, the metric component  $g_{rr}$  has the asymptotics

$$g_{rr} = 1 + \mathcal{O}\left(\frac{1}{r}\right), \quad (3.6)$$

which implies that also  $g^{rr} = 1 + \mathcal{O}(1/r)$ ; then the quasilocal mass (3.3) is finite since the prefactor  $r$  cancels the only remaining term in the round brackets, which is of order  $1/r$ . This situation is physical and occurs, for example, in the Schwarzschild geometry

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{1 - 2m/r} + r^2 d\Omega_{(2)}^2, \quad (3.7)$$

for which  $M_{\text{MSH}}$  does not depend on the position  $r$  and coincides with the Schwarzschild mass  $m$  everywhere outside the horizon  $r = 2m$ , and with the ADM and the Newtonian mass as  $r \rightarrow +\infty$ .

If the metric is not asymptotically flat, say

$$g_{rr} = 1 + \mathcal{O}\left(\frac{1}{r^{1+\epsilon}}\right), \quad (3.8)$$

with  $\epsilon > 0$ , then  $M_{\text{MSH}}(r) \rightarrow 0$  as  $r \rightarrow \infty$ . This is the situation, e.g., for the quantum-corrected Schwarzschild black hole of [12–14], for which  $g_{rr} = [1 - (2m/r)^{1+\epsilon}]^{-1}$ , where  $\epsilon$  is a small positive number (dependent on the black hole mass) that, for a solar mass black hole, assumes the value  $\sim 10^{-26}$  [14]. In this case the mass  $M_{\text{MSH}}$  (which is always defined in spherical symmetry) vanishes as  $r \rightarrow +\infty$ . In this limit, the Newtonian potential  $\phi_N$  is given by

$$1 + 2\phi_N = 1 - \left(\frac{2m}{r}\right)^{1+\epsilon} \equiv 1 - \frac{2M(r)}{r}, \quad (3.9)$$

and one obtains the position-dependent Newtonian mass

$$M(r) = \left(\frac{2m}{r}\right)^\epsilon m, \quad (3.10)$$

which does not coincide with the mass obtained from the monopole term of the expansion of the metric in multipoles, as it should.

If instead  $g_{rr} = 1 + \mathcal{O}(1/r^{1-\epsilon})$  (again, with  $\epsilon > 0$ ), then the quasilocal mass  $M_{\text{MSH}}(r)$  is again position-dependent

and diverges as  $r \rightarrow +\infty$ , another unphysical situation for an isolated object.

What is more, if the asymptotics required by the definition of asymptotic flatness is not satisfied, the Newtonian limit is jeopardized. In an asymptotically flat system, at large spatial distances from the source of gravity one ought to recover the post-Newtonian approximation [2] in which the line element reduces to

$$ds^2 = -(1 + \phi_N)dt^2 + (1 - \phi_N)(dr^2 + r^2 d\Omega_{(2)}^2). \quad (3.11)$$

The dominant term in the Newtonian potential  $\phi_N$  must be a monopole, and this term must be present. Contrary to electrostatics, in which electric charge can have positive or negative sign and one could have a dipole with zero total charge, mass cannot be negative and the first term in a multipole expansion of  $\phi_N$  must necessarily be the monopole term scaling as  $1/r$ . The failure to obtain such a term means that the geometry does not admit a Newtonian limit. While this possibility is fine for, e.g., gravitational waves that do not have a counterpart in Newtonian gravity, it is unacceptable for an isolated black hole.

Another example is given by a Reissner-Nordstrom naked singularity with electric charge and vanishing mass parameter,

$$ds^2 = -\left(1 + \frac{Q^2}{r^2}\right)dt^2 + \frac{dr^2}{1 + Q^2/r^2} + r^2 d\Omega_{(2)}^2, \quad (3.12)$$

which has Misner-Sharp-Hernandez quasilocal mass

$$M_{\text{MSH}}(r) = -\frac{Q^2}{2r}. \quad (3.13)$$

A silly object like an electric charge without mass violates the positivity of the quasilocal energy everywhere and should not exist. Although, superficially, the metric reduces to the Minkowski one away from the central object, it does so with the wrong asymptotics  $g_{rr} = 1 + \mathcal{O}(1/r^2)$ , which creates a negative Misner-Sharp-Hernandez mass everywhere. Although  $M_{\text{MSH}}$  is defined independent of the energy conditions, a deviation from the correct asymptotics signals the presence of a distribution of mass-energy incompatible with an isolated object and true asymptotic flatness, or some physical pathology. If the energy density and stresses of the latter do not fall off sufficiently rapidly, then the notion of asymptotic flatness as referring to isolated energy distributions fails. What is more, if this energy distribution corresponds to negative energies, it leaves an imprint on the quasilocal mass and may make it negative. Of course, this is not the only way to violate the positivity of the MSH mass: for example, the Schwarzschild solution with negative mass (another naked singularity) does that, but it has the correct asymptotics required by asymptotic flatness.

The spherically symmetric Bardeen regular black hole [8] is asymptotically flat and the MSH mass is well behaved, and so are the Hayward regular black hole [26] and its modification describing a Planck star [27], the Peltola-Kunstatter black hole arising in polymer quantization of the Schwarzschild geometry [28], and the Gambini-Olmedo-Pullin regular black hole [29]. Therefore, quantum corrections do not necessarily spoil asymptotic flatness or introduce physical pathologies or mass anomalies.

We can add some insight by recasting the spherical line element in a particular gauge exhibiting explicitly the Misner-Sharp-Hernandez mass. Any spherically symmetric metric can be rewritten in the Abreu-Visser gauge

$$ds^2 = -e^{-2\Phi} \left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - 2M/r} + r^2 d\Omega_{(2)}^2, \quad (3.14)$$

where  $\Phi = \Phi(t, r)$ ,  $M = M(t, r)$  and, *a posteriori*,  $M$  is shown to be the Misner-Sharp-Hernandez mass [30]. It follows immediately from this line element that, as  $r \rightarrow +\infty$ , the asymptotic flatness conditions (2.8)–(2.11) require that  $2M/r = \mathcal{O}(1/r)$  and  $M$  tends to a finite limit  $M_\infty$ , or  $M = \mathcal{O}(1)$ .

Let us consider now the stress energy tensor  $T_{ab}$  associated with this geometry, which is given by

$$G_{00} = 8\pi T_{00} = \frac{2M'}{r^2}, \quad (3.15)$$

$$G_{01} = \frac{2\dot{M}e^\Phi}{r^2(1 - 2M/r)}, \quad (3.16)$$

$$G_{11} = -\frac{2M'}{r^2} - \frac{2\Phi'}{r} \left(1 - \frac{2M}{r}\right), \quad (3.17)$$

$$G_{22} = G_{33} = \frac{M''}{r} - \frac{e^{-\Phi}}{r} \frac{\partial}{\partial t} \left[ \frac{\dot{M}e^\Phi}{(1 - 2M/r)^2} \right] - \frac{e^\Phi}{r\sqrt{1 - 2M/r}} \frac{\partial}{\partial r} \left[ r \left(1 - \frac{2M}{r}\right)^{3/2} e^{-\Phi} \Phi' \right], \quad (3.18)$$

where a prime and an overdot denote differentiation with respect to radius and time, respectively. Although these expressions are too cumbersome to draw general conclusions, we can restrict to static ( $\dot{M} = 0$ ) geometries for which  $\Phi \equiv 0$ . Almost all the quantum-corrected black holes proposed in the literature (but not Planck stars [27]) have this form. Then, the energy density of matter is simply

<sup>2</sup>This conclusion agrees with the recent Ref. [31].

$$\rho = \frac{M'}{4\pi r^2} \quad (3.19)$$

and we conclude immediately that vacuum corresponds to constant  $M$  (as in the case of the Schwarzschild black hole) and, in the presence of matter,  $\rho > 0$  if and only if the Misner-Sharp-Hernandez mass increases with radius,  $M' > 0$ . Furthermore, the fact that  $M$  decreases with  $r$ , i.e.,  $M' < 0$ , signals the presence of a negative energy density, which decreases the value due to a central object that would be constant in the absence of this energy distribution in its exterior (this is exactly the case of the quantum-corrected Schwarzschild black hole of [12–14]). Therefore, pathologies in the behavior of the Misner-Sharp-Hernandez mass signal physically pathological behavior of the geometry.

#### IV. QUASILOCAL MASS—GENERAL SPACETIMES

Let us remove now the assumption that the spacetime is spherically symmetric or stationary. The Misner-Sharp-Hernandez mass is then generalized by the Hawking quasilocal mass [5,6], defined as follows.

Let  $S$  be a spacelike, compact, and orientable two surface; denote with  $\mathcal{R}$  the induced Ricci scalar on  $S$ , and let  $\theta_{(\pm)}$  and  $\sigma_{ab}^{(\pm)}$  be the expansions and shear tensors of a pair of null geodesic congruences (outgoing and ingoing from the surface  $S$ ). Let  $h_{ab}$  be the two metric induced on  $S$  by  $g_{ab}$ , let  $\mu$  be the volume two form on the surface  $S$ , while  $\mathcal{A}$  is the area of  $S$ ; then [5]

$$M_H \equiv \frac{1}{8\pi} \sqrt{\frac{\mathcal{A}}{16\pi}} \int_S \mu \left( \mathcal{R} + \theta_{(+)}\theta_{(-)} - \frac{1}{2} \sigma_{ab}^{(+)} \sigma_{(-)}^{ab} \right). \quad (4.1)$$

As a consequence of the Riemann tensor splitting into Ricci and Weyl parts [1]

$$R_{abcd} = C_{abcd} + g_{a[c}R_{d]b} - g_{b[c}R_{d]a} - \frac{R}{3}g_{a[c}g_{d]b} \quad (4.2)$$

(where  $R_{ab}$  and  $C_{abcd}$  are the Ricci and Weyl tensors, respectively, and  $R \equiv R^c_c$  is the Ricci scalar), the Hawking mass splits into two contributions, one coming from matter and one from the vacuum gravitational field, respectively. We recall this decomposition, performed in Ref. [32]. We use the contracted Gauss equation [6]

$$\mathcal{R}^{(h)} + \theta_{(+)}\theta_{(-)} - \frac{1}{2} \sigma_{ab}^{(+)} \sigma_{(-)}^{ab} = h^{ac}h^{bd}R_{abcd} \quad (4.3)$$

to compute the integral defining the Hawking mass. Using then the Einstein equations

$$R_{ab} = 8\pi G \left( T_{ab} - \frac{1}{2}g_{ab}T \right) \quad (4.4)$$

and  $R = -8\pi GT$  (where  $T \equiv T^c_c$ ), one obtains

$$h^{ac}h^{bd}R_{abcd} = h^{ac}h^{bd}C_{abcd} + 8\pi Gh^{ac}h^{bd} \left[ g_{a[c}T_{d]b} - g_{b[c}T_{d]a} - \frac{T}{2}(g_{a[c}g_{d]b} - g_{b[c}g_{d]a}) \right]. \quad (4.5)$$

Then,

$$h^{ac}h^{bd}(g_{a[c}g_{d]b} - g_{b[c}g_{d]a}) = 2, \quad (4.6)$$

$$h^{ac}h^{bd}(g_{a[c}T_{d]b} - g_{b[c}T_{d]a}) = h^{ab}T_{ab} \quad (4.7)$$

give the Hawking mass as<sup>3</sup> [32]

$$M_H = \sqrt{\frac{\mathcal{A}}{16\pi}} \int_S \mu \left( h^{ab}T_{ab} - \frac{2T}{3} \right) + \frac{1}{8\pi G} \sqrt{\frac{\mathcal{A}}{16\pi}} \int_S \mu h^{ac}h^{bd}C_{abcd}, \quad (4.8)$$

where the first integral on the right hand side is the matter contribution and the second integral is the Weyl free field contribution, and the only one present *in vacuo*. Since we have used the Einstein equations, the rest of this discussion applies only to geometries that solve these equations.

If the matter content of spacetime consists of a single perfect fluid with stress-energy tensor

$$T_{ab} = (P + \rho)u_a u_b + P g_{ab}, \quad (4.9)$$

energy density  $\rho$ , pressure  $P$ , and four velocity  $u^c$ , then one can choose the two surface  $S$  comoving with the fluid (i.e., the unit normal  $n^a$  to  $S$  pointing outside of  $\Sigma_t$  is parallel to the timelike fluid four velocity  $u^a$ ),  $h_{ac}u^c$  vanishes, and

$$h^{ab}T_{ab} - \frac{2T}{3} = \frac{2\rho}{3}. \quad (4.10)$$

In the case of an imperfect fluid, the stress-energy tensor is instead

$$T_{ab} = \rho u_a u_b + P \gamma_{ab} + q_a u_b + q_b u_a + \Pi_{ab}, \quad (4.11)$$

where  $\gamma_{ab}$  is the three metric on the three space orthogonal to  $u^a$ , as in

$$g_{ab} = -u_a u_b + \gamma_{ab}, \quad (4.12)$$

$q^a$  is a purely spatial heat current vector ( $q^c u_c = 0$ ), and  $\Pi_{ab}$  is the symmetric, trace-free, shear tensor. The trace is  $T = -\rho + 3P$  and now [32]

<sup>3</sup>This splitting and the corresponding equation (4.8) occur also in scalar-tensor gravity [33].

$$\begin{aligned} h^{ab}T_{ab} - \frac{2T}{3} &= \frac{2}{3}\rho + h^{ab}\Pi_{ab} \\ &= \frac{2}{3}\rho + \Pi^2_2 + \Pi^3_3 = \frac{2}{3}\rho - \Pi^1_1 \end{aligned} \quad (4.13)$$

[where  $(x^2, x^3)$  are coordinates on  $\mathcal{S}$ ].

Let us consider vacuum, in which case  $M_H$  given by Eq. (4.8) coincides with the Weyl contribution. In asymptotically flat spacetimes according to the definition of Sec. II, the Weyl tensor enjoys the well-known peeling property [20,21]. Let  $\gamma$  denote null geodesics going from a finite point to null infinity,  $\lambda$  be an affine parameter along such a geodesic, and  $k^a$  its four tangent. Then, the Weyl tensor splits according to

$$\begin{aligned} C^a{}_{bcd} &= \frac{(C^a{}_{bcd})^{(I)}}{\lambda} + \frac{(C^a{}_{bcd})^{(II)}}{\lambda^2} + \frac{(C^a{}_{bcd})^{(III)}}{\lambda^3} \\ &\quad + \frac{(C^a{}_{bcd})^{(IV)}}{\lambda^4} + \mathcal{O}\left(\frac{1}{\lambda^5}\right), \end{aligned} \quad (4.14)$$

where, in the algebraic classification of Ref. [1],  $(C^a{}_{bcd})^{(I)}$  is of type IV,  $(C^a{}_{bcd})^{(II)}$  of type III,  $(C^a{}_{bcd})^{(III)}$  of type II or II-II, and  $k^a$  is the repeated principal null vector.  $(C^a{}_{bcd})^{(IV)}$  is of type I and  $k^a$  is one of the principal null directions of  $C^a{}_{bcd}$ .

This asymptotics in terms of an affine null geodesic parameter may not seem illuminating in general, but there is a situation in which it is, and which includes most of the regular black holes proposed in the literature. Let the spacetime be stationary and spherically symmetric, with the extra requirement that  $g_{tt}g_{rr} = -1$ ; that is, the line element assumes the form (3.4). As shown in Ref. [34], this extra requirement is equivalent to the areal radius  $r$  being an affine parameter along radial null geodesics. Now consider the surface  $\mathcal{S}$  to be a two sphere orbit of the spherical symmetry, and  $\gamma$  to be radial outgoing null geodesics emanating from  $\mathcal{S}$ . Then, the peeling property (4.14) of the Weyl tensor can be rewritten using  $r$  instead of  $\lambda$ . This equation then shows that no terms decreasing slower than  $1/r$  are possible in the integrand of  $M_H$  in *vacuo*. Such terms may be created when a form of matter (or effective matter) with  $T_{ab} \neq 0$ , responsible for the first integral in the right hand side of Eq. (2.15), produces a nonvanishing Ricci tensor  $R_{ab}$ . Similarly, no fractional powers of  $1/r$  are possible in the Weyl tensor in *vacuo*.

In general (i.e., nonspherically symmetric) geometries, the affine parameter  $\lambda$  along null geodesics does not coincide with the radial coordinate (assuming that polar coordinates are used). However, in asymptotically flat spacetimes, the dominant term as  $r \rightarrow \infty$  is the monopole one [19] and the property  $g_{tt}g_{rr} = -1$  is satisfied with better and better accuracy further and further away from the source. Since the metric components  $g_{\theta\theta}, g_{\varphi\varphi}$  in polar coordinates scale as  $r^2$  and  $r^2 \sin^2 \vartheta$ , respectively, we have

$$C_{2323} \sim r^2 C^2{}_{323} \sim r(C^2{}_{323})^{(I)} \quad (4.15)$$

and

$$h^{ac}h^{bd}C_{abcd} \simeq \frac{2C_{2323}}{r^4 \sin^2 \vartheta} \sim \frac{(C^2{}_{323})^{(I)}}{r^3}. \quad (4.16)$$

Then, in *vacuo*,

$$\begin{aligned} M_H &= \frac{1}{8\pi} \sqrt{\frac{A}{16\pi}} \int_{\mathcal{S}} \mu h^{ac}h^{bd}C_{abcd} \simeq \frac{r}{16\pi} \frac{(C^2{}_{323})^{(I)}}{r} \\ &\sim \frac{(C^2{}_{323})^{(I)}}{16\pi}. \end{aligned} \quad (4.17)$$

If the system is not asymptotically flat, there will be the contribution to  $M_H$  from the matter stress-energy tensor  $T_{ab}$  and the peeling property of the Weyl tensor will not be satisfied. Then, the dominant term will not be of order  $\mathcal{O}(1/r)$  and the Weyl contribution to  $M_H$  will diverge or vanish. The latter situation corresponds to zero contribution to  $M_H$  from the gravitational field, with  $M_H$  reducing solely to the matter contribution. Both cases are unphysical.

## V. CONCLUSIONS

Physical anomalies in the general-relativistic gravitational field can be signaled by anomalies of the Hawking quasilocal mass  $M_H$  [5,6] or, in spherical symmetry, of its better known version, the Misner-Sharp-Hernandez mass [3,4]. These anomalies include situations in which the quasilocal mass becomes negative, zero, or diverges. While this association is brought about by certain quantum-corrected black holes, the association between anomalies in  $M_H$  and physical pathologies is more general, as shown by the examples discussed in this work. In particular, a monopole term scaling as  $1/r$  is a necessity for isolated gravitating systems and for their Newtonian counterparts (GR solutions which do not have Newtonian counterparts, or nonasymptotically flat analytical solutions that are not realized in nature, such as infinitely long cylindrical solutions, or *pp*-waves, escape this requirement).

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