# Implications of Kleinian relativity

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Inspired in metamaterials, we present a covariant mechanics for particles in Kleinian spacetime and show some of its effects, such as time contraction and length dilatation. We present the new expressions for relativistic momentum and energy for a pointlike particle. To illustrate the new mechanics, we describe the particle motion under a uniform Newtonian gravitational field. We also revisit the free spin-half particle problem in Kleinian spacetime, discuss some quantum implications, like the constraint on the dispersion relation for Weyl fermions, and adapt a metamaterial analog system to Klein spacetime.

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## I. INTRODUCTION

Metamaterials are artificial structures that have peculiar physical properties used in many applications like electromagnetic wave cloaking, the construction of hyper lenses, the reproduction of light concentration phenomena, and the reverse Doppler effect [1–5]. Furthermore, they provide insights for new analog models in physics. For instance, special relativity, cosmology, black hole physics, and celestial mechanics can be mimicked using metamaterial systems [6–9]. However, one special feature is that metamaterials also make possible the effective realization of some properties of the so-called Kleinian spacetime (KST) [10].

Unlike the Minkowski spacetime, which is associated with a hyperbolic geometry, the Klein spacetime is associated with an ultrahyperbolic geometry [11]. In four dimensions, this spacetime is characterized by the $(+,+,-,-)$ metric signature and appears in the mathematical theory of ultrahyperbolic differential equations [12]. Such geometry has also appeared in discussions of metric signature change [13,14], which is an issue present in classical and quantum cosmological models [15–17]. Moreover, KST is particularly useful in the construction of $\mathcal{N} = 2$ super-string models [18,19].

Examples of ultrahyperbolic geometry are found in theories with more than one time dimensions. This kind of theoretical proposal can lead to closed timelike curves and consequently to issues involving causality. Nevertheless, consideration of this double coordinate time hypothesis is recurrent in the literature. For instance, in Kaluza-Klein models, compactified timelike extra dimensions have been considered to avoid the cosmological constant problem [20] and in the brane world context, the timelike extra dimension could give the *graceful exit* from the de Sitter initial expansion of the Universe [21]. A good general discussion of the number of timelike dimensions and spacelike dimensions in a given manifold is presented in [22] and other interesting theoretical questions, like the realization of two-time physics, have been investigated in [23,24].

Motivated by the "special relativity" theories constructed in non-Minkowskian, like the de Sitter [25,26] and the anti–de Sitter [27,28] spacetimes, and by the possibility that metamaterials might simulate some properties of the behavior of particles and fields in KST, we ask ourselves what could be the relativistic physical implications of the Kleinnian geometry. To mainly investigate the $x$-direction particle motion in KST, where the Lorentz boost fails, and to give an answer to the above question, we construct a covariant mechanical model and show some of its classical consequences. Then we apply it in quantum relativistic mechanics to describe the Weyl fermions and other features related with one-half spin particles. We also briefly discuss a metamaterial analog model for these fermions.

This paper is organized as follows. In Sec. II, we start presenting how the Kleinian metric appears effectively in metamaterials. In Sec. III, by applying the relativity principles, we find the transformations between the $xt$ coordinates of two inertial reference frames, that we call Klein transformations. In Sec. IV, we analyze kinematic relativistic effects, such as dilatation of length and contraction of time. In Sec. V, we discuss some dynamical aspects. In Sec. VI, we show the behavior of a particle under an uniform Newtonian gravitational field in KST.

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In Secs. VII and VIII, we analyze the free and the analog fermions, respectively. In the last section, we close the paper with our final remarks and conclusions.

## II. METAMATERIALS AND KLEIN SPACETIMES

Metamaterials were idealized by Veselago, in 1968 [29], which proposed a hypothetical material admitting simultaneously negative permittivity $\epsilon_r$ and permeability $\mu_r$, keeping a real refractive index $n = \pm\sqrt{\epsilon_r \mu_r}$. In the last years this idea has been realized, since structures have been designed with negative effective permittivity and/or negative effective permeability, such as a metamaterial composed of copper split ring resonators or any other left-hand metamaterial. Beyond this achievement, other systems, the so-called hyperbolic metamaterials, with $\epsilon_r < 0$ and $\mu_r > 0$ or $\epsilon_r > 0$ and $\mu_r < 0$ have also been realized using appropriate technology.

Considering anisotropic metamaterials, where the permittivity and permeability are tensors, we find interesting physical possibilities. For example, multilayer metal-dielectric or metal wire array structures have been theoretically used to simulate effective Kleinian spacetimes [10]. In such systems the permittivity tensor $\epsilon_{ij}$ is given by

$$\epsilon_{ij} = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_2 \end{pmatrix}, \tag{1}$$

where $\epsilon_1 > 0$ and $\epsilon_2 < 0$. This tensor works as an effective metric for the electromagnetic wave. As a consequence, an incident electromagnetic wave has its "extraordinary" component propagating as a real scalar Klein-Gordon field over a Kleinian spacetime. Another way to simulate KST is to use electronic metamaterials as done in [30].

Other examples of anisotropic metamaterial, in our perspective, are the ones based on nematic liquid crystals [31]. In a particular case [32], light rays have been studied in terms of the effective metric in cylindrical coordinates

$$ds^2_{\text{eff}} = -c^2dt^2 + \epsilon_0 dr^2 + \epsilon_e r^2 d\phi^2 + \epsilon_0 dz^2, \tag{2}$$

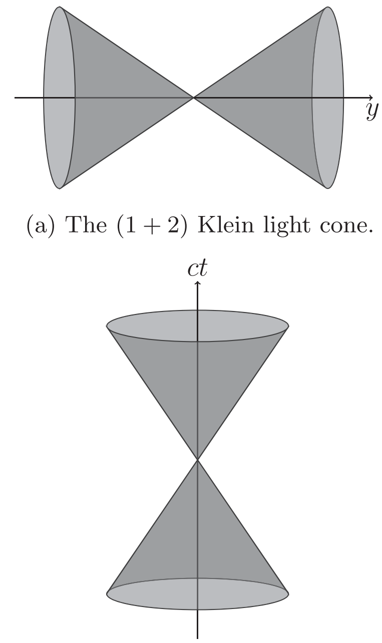
in a configuration where the permittivities $\epsilon_0 > 0$ and $\epsilon_e < 0$, corresponding to a Klein-like spacetime signature.

## III. THE SPACETIME AND THE KLEINIAN TRANSFORMATIONS

In a more careful way, a Klein metric is a flat pseudo-Riemannian metric, that for simplicity can be stated according to the following line element

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \tag{3}$$

where $\eta_{\mu\nu} = \text{diag}(+c^2, +1, -1, -1)$ and $c$ denotes the speed of light in vacuum. From the point of view of




(a) The $(1+2)$ Klein light cone.

(b) The $(1+2)$ Minkowski light cone.

FIG. 1. Representations of the light cone in the Klein and Minkowski spacetimes.

Einstein's general relativity, this metric is a simple mathematical non-Lorentzian solution. For the sake of illustration of the difference between the Lorentzian and Kleinian spacetimes, we show the $(1+2)$ light cones, respectively, in Figs. 1(a) and 1(b).

From these figures we infer that the noncausal regions of one cone are the exact causal regions of the other. In this way, a timelike vector in the Kleinian light cone corresponds to a spacelike vector outside of the Minkowski light cone, and a spacelike vector in Klein cone corresponds to a timelike vector in the other cone.

Inspired by special relativity theory, and in order to present a covariant model, we postulate that the metric is invariant under a class of linear transformations, which connects different inertial observers in Kleinian spacetime. Following this assumption, $ds = ds'$, we deduce a suitable class of coordinate transformations $(t, x, y, z) \to (t', x', y', z')$, which we call Kleinian transformations. In particular, taking $y' = y$, $z' = z$, the "boost" in $x$ direction is given by

$$x' = \frac{x \mp vt}{\sqrt{1 + v^2/c^2}}, \tag{4}$$

$$t' = \frac{t \pm \frac{vx}{c^2}}{\sqrt{1 + v^2/c^2}}. \tag{5}$$

The Klein factor, $\gamma \doteq \frac{1}{\sqrt{1+v^2/c^2}}$, has a bell-shape profile, as illustrated in Fig. 2.

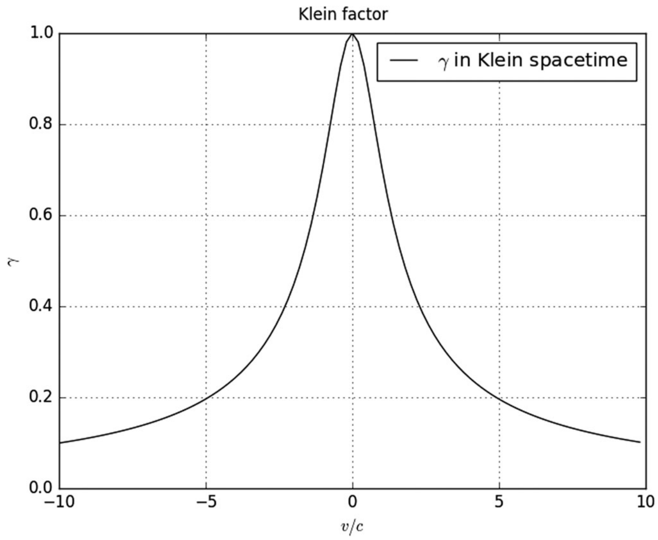



FIG. 2. The $\gamma$ factor against speed $v/c$.

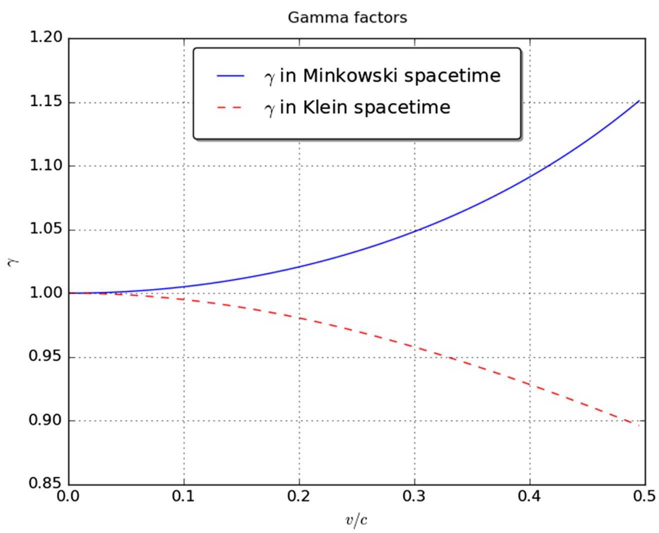



FIG. 3. Behavior of Lorentz and Klein factors in the two spacetimes

In Fig. 3 we compare the Klein factor with the Lorentz factor profile, and note that for small speed values the two factors are almost the same. However, for larger values of $v$ it is clear that the Klein factor decreases and the Lorentz factor increases, and the difference between them becomes more evident.

As in special relativity we define a parametric transformation

$$ct = \sin(\phi), \qquad x - c_0 = \cos(\phi), \tag{6}$$

where $\phi = \arctan(\frac{v}{c})$. This leads to $\gamma_k = \cos(\phi)$, and the Lorentz-like transformations (4) and (5) can be seen as an ordinary rotation by $\phi$, which is very different from the hyperbolic rotation which characterizes the Lorentz transformation and is certainly not a Galilean transformation. Furthermore, to illustrate how rich KST is, we point out that the metric (3) allows Lorentz transformations as well if we boost in the $y$ or $z$ direction, for instance. For a general transformation see [33].

## IV. KINEMATICS AND CONSEQUENCES OF KLEIN TRANSFORMATIONS

Now, we describe some physical consequences of Klein transformations, starting by kinematics. The first effect that we discuss in this spacetime, is the length dilatation. Let $S$ and $S'$ be two inertial frames moving with respect to each other along the $x$ direction with relative speed $v$. Suppose that $L_0$ is the proper length of a horizontal rod measured by an observer in $S$. An observer in the $S'$ frame will measure a larger length, $L$, for the same rod. Both quantities, $L$ and $L_0$, are related by the equation

$$L = L_0\sqrt{1 + \frac{v^2}{c^2}}. \tag{7}$$

This relation shows that the rod dilates, its length being increased in the direction of the motion of $S$.

To illustrate that the time interval is also a relative quantity in this scenario, suppose that $T_0$ is the proper time measured by an observer in the $S$ frame, and that it indicates the time interval between two events. Suppose also that $T$ is the time interval between the same two events measured by an observer in the $S'$ frame. Then, the intervals are different and are related by the expression

$$T = \frac{T_0}{\sqrt{1 + \frac{v^2}{c^2}}}, \tag{8}$$

which indicates a time contraction for the $S'$ observer. In this way, both results are the inverse of the corresponding effects of the contraction of length and the time dilatation that occur in special relativity.

### A. Muon decay

In order to better characterize this time interval contraction effect, we theoretically analyze the muon decay. As it is well known, muons ($\mu^+$ and $\mu^-$) are unstable particles that spontaneously decay as

$$\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu,$$
$$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu.$$

They have been used to demonstrate the relativistic time dilation effect (see the Frisch-Smith experiment [34]).

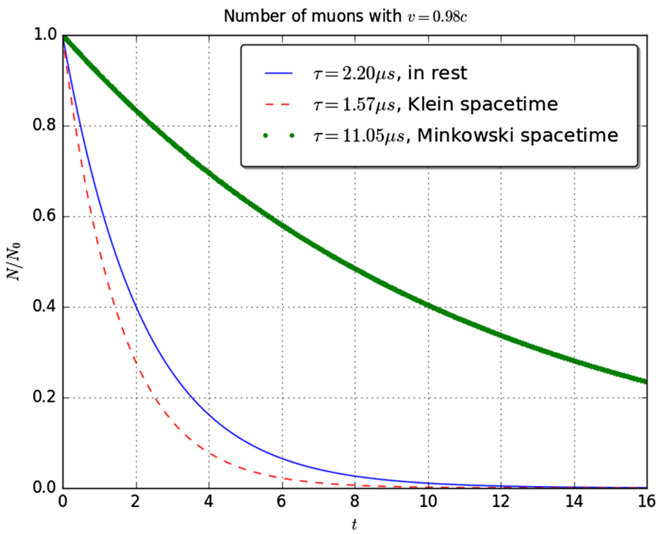



FIG. 4. The decay of the number of muons with time for different mean lifetimes

The number $N$ of muons at a time $t$ is given by

$$N = N_0 \exp\left(-\frac{t}{\tau}\right), \tag{9}$$

where $N_0$ is the number of muons at $t = 0$ and $\tau = 2.2\ \mu$s is the value of the mean lifetime of the muon in rest (proper frame or comoving frame). Supposing the muons are moving at speed $0.98c$, and taking the frame $S$ to be attached to a laboratory on Earth and the frame $S'$ to be the rest frame of the muon, it follows from (8) that $\tau = 1.57\ \mu$s. See Fig. 4.

Therefore, in the case of Klein spacetime, the value of the mean lifetime of a moving muon measured by a frame in rest will be less than that measured by a comoving frame (proper time). This is an interesting behavior that can be used as a probe to detect Kleinian spacetime regions.

### B. Transformation of velocities

As it could be expected, the velocity transformation is also different from that of special relativity. For one-dimensional motion, the relativistic velocity addition formula is

$$u'_x = \frac{u_x - v}{1 + vu_x/c^2}, \tag{10}$$

where $u_x$ is the speed of a particle in an inertial frame $S$, and $u'_x$ is the speed of the same particle in another inertial frame, $S'$, and $v$ is the relative speed between these two frames. With this transformation it is possible to find an observer where the light is at rest or an observer where the light speed is faster than $c$, these facts are not possible in special relativity. Although the transformation (10) gives these strange results, the Galilean velocity composition is recovered for $v \ll c$.

Observing the transverse velocity transformation,

$$u'_y = \frac{u_y\sqrt{1 + \frac{v^2}{c^2}}}{(1 + \frac{vu_x}{c^2})}, \tag{11}$$

we see that for $v \ll c$, then $u'_y = u_y$ and the Galilean velocity transformation is again recovered for low velocities.

## V. DYNAMICAL CONSEQUENCES

Here we show that the Klennian metric leads us to a new particle dynamics. Using the metric (3), we see that $\vec{v} \cdot \vec{v} = -v_x^2 + v_y^2 + v_z^2$ and therefore the free particle Lagrangian becomes

$$L = -m_0 c^2 \sqrt{1 + \frac{v_x^2}{c^2} - \frac{v_y^2}{c^2} - \frac{v_z^2}{c^2}}. \tag{12}$$

Now, we can find the particle momentum $\vec{p}$. From the definition $p_i = \frac{\partial L}{\partial v_i}$, we get

$$p_x = \frac{-m_0 v_x}{\sqrt{1 + \frac{v_x^2}{c^2} - \frac{v_y^2}{c^2} - \frac{v_z^2}{c^2}}}, \tag{13}$$

$$p_y = \frac{m_0 v_y}{\sqrt{1 + \frac{v_x^2}{c^2} - \frac{v_y^2}{c^2} - \frac{v_z^2}{c^2}}}, \tag{14}$$

$$p_z = \frac{m_0 v_z}{\sqrt{1 + \frac{v_x^2}{c^2} - \frac{v_y^2}{c^2} - \frac{v_z^2}{c^2}}}. \tag{15}$$

If we assume that the relation $p = mv$ is valid in Klein spacetime, it follows that the rest mass is a diagonal 3D tensor $m = \mathrm{diag}(-m_0, m_0, m_0)$, where $m_0 > 0$. The fact that the inertial response is tensorial reflects the anisotropic nature of Klein spacetime.

If the particle is moving along the $x$ direction, then $p_x = p$, $v_x = v$, $v_y = v_z = 0$, and we are left with

$$p = -\frac{m_0 v}{\sqrt{1 + \frac{v^2}{c^2}}}. \tag{16}$$

From this equation we note that $p$ goes to the inverse Newtonian momentum as $v \to 0$. Conversely, it goes to $\mp m_0 c$ as $v \to \pm\infty$. In Fig. 5 it is possible to see the unusual behavior of the momentum, for high values of $v$.

Since the Lagrangian (12) does not depend explicitly on the coordinates, the particle momentum is conserved. From the Hamiltonian definition $H = p\dot{q} - L$, we find the relativistic energy for the particle moving in the $x$ direction,

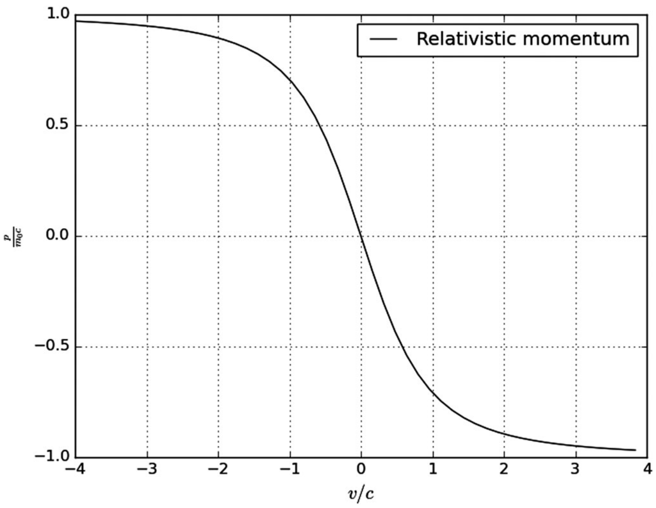



FIG. 5. In the figure we see the rescaled momentum $\frac{p}{m_0 c}$ against normalized speed $v/c$.

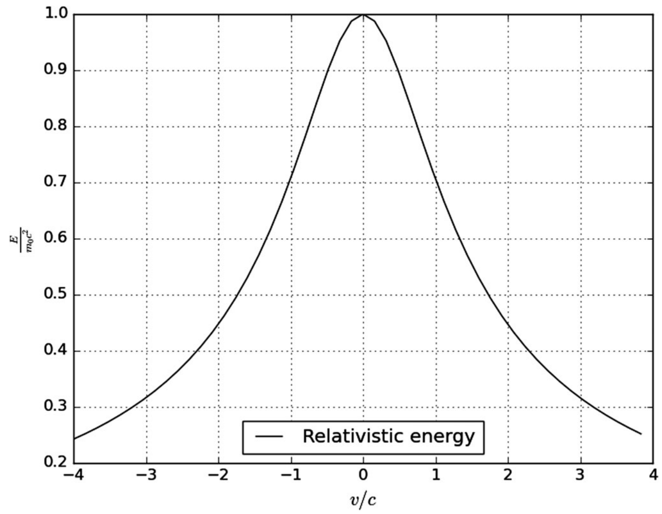



FIG. 6. The figure shows the point particle rescaled energy $E/m_0c^2$ against the normalized speed $v/c$.

$$E = \frac{m_0 c^2}{\sqrt{1+\frac{v^2}{c^2}}}. \tag{17}$$

We see from this equation that, as long as the particle speeds up, the total energy approaches zero. Also, differently from the special relativity case, where a finite rest mass particle would have an infinite momentum if $v \to \pm c$, in the present case the momentum is always finite, even for a particle with superluminal speed.

If we expand (17), for small values of $v/c$, we find the relation

$$E \approx m_0 c^2 - \frac{m_0 v^2}{2}, \tag{18}$$

where we can identify the kinetic energy as $T = m_0 v^2/2$ and the rest energy by $m_0 c^2$. This is very different from the special relativity case, where both energies are added but reinforces the anisotropic character of the inertial mass. Note that, for motion along the $y$ or $z$ directions, Eq. (18) comes with the usual $+$ sign in front of the kinetic energy term.

For a general motion, it is easily deduced that the energy becomes

$$E = \frac{m_0 c^2}{\sqrt{1+v_x^2 - v_y^2 - v_z^2}}, \tag{19}$$

as described by [33].

Taking the four momentum as $p^\mu = (E/c, p, 0, 0)$ the conservation equation, $p^\mu p_\mu = m_0^2 c^2$, has the explicit form

$$E^2 + p^2c^2 = m_0^2 c^4, \tag{20}$$

if the motion happens along the $x$ direction. It is very important to note that this relation corresponds to an exotic conservation equation, that was previously discussed in [35], who has also considered negative masses. The energy behavior (17) can be seen in the Fig. 6.

## VI. UNIDIMENSIONAL MOTION IN AN UNIFORM NEWTONIAN FIELD

As an application of the new mechanical ideas developed above, we present a single particle motion due to a uniform Newtonian gravitational field. Using a more geometrical approach, let us consider a particle moving along some world line $x^\mu(\lambda)$ under a uniform Newtonian gravitational potential $V(r)$. The associated Lagrangian is

$$L = -m_0 c \sqrt{\eta_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}} - V(r). \tag{21}$$

It is clear that the above Lagrangian is not covariant, but for our proposes this fact is not important. With $x^\mu = (t, x)$, the Lagrangian provides the following Euler-Lagrange equations of motion for the particle

$$\frac{d}{d\lambda}\left(\frac{\dot{t}}{\sqrt{\eta_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}}\right) = 0, \tag{22}$$

$$m_0 c^2 \frac{d}{d\lambda}\left(\frac{\dot{x}}{\sqrt{\eta_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}}\right) = -\frac{\partial V(r)}{\partial r}. \tag{23}$$

Since we want to investigate the motion of a particle under a constant gravitational field, we choose $V = m_0 g x$, and $\lambda = t$, then Eq. (23) gets the following form

$$\frac{d}{dt}\left(\frac{v}{\sqrt{1+\frac{v^2}{c^2}}}\right) = g. \tag{24}$$

In fact, this equation describes the motion of a particle under constant acceleration seen by an observer in a rest frame $S$, which is implied in the solution

$$x = k_1 \pm \sqrt{\frac{c^4}{g^2} - c^2(t-k_2)^2}, \tag{25}$$

where $k_1$ and $k_2$ are integration constants. This equation can be easily cast in the form

$$(x-k_1)^2 + c^2(t-k_2)^2 = \frac{c^4}{g^2}, \tag{26}$$

which describes an ellipse in the $t$-$x$ plane. For simplicity, we admit $k_2 = 0$, then the velocity of the particle can be expressed as

$$v = \mp \frac{c^2 t}{\sqrt{\frac{c^4}{g^2} - c^2 t^2}}. \tag{27}$$

Observing this equation, we see a singular behavior for $t = \pm\frac{c^2}{g}$. This fact leads us to restrict the problem to the time interval $(-\frac{c^2}{g}, +\frac{c^2}{g})$. We see that for higher values of $g$, this time interval becomes smaller.

Figure 7 illustrates the particle motion under constant acceleration as expected from special relativity and from our case (elliptic motion). We can see, for $t \to \pm\frac{c^2}{g}$, that the particle seems to disappear. However, if we plot the full curve we can interpret the "forbidden" regions as the particle traveling back in time (antiparticle) as it is the case of tachyons moving in Minkowski spacetime. Accordingly, at any given instant of time, particle and antiparticle appear at opposing sides of the ellipse shown in Fig. 8.

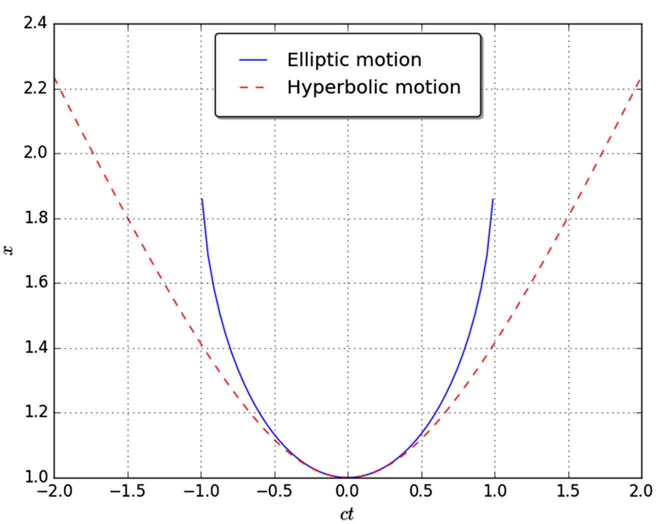



FIG. 7. Considering $c = g = 1$, $k_1 = 1$, and $k_2 = 0$, we show a comparison between the uniformly accelerated motion in the $x$ direction of Klein spacetime and in the corresponding direction in Minkowski spacetime.

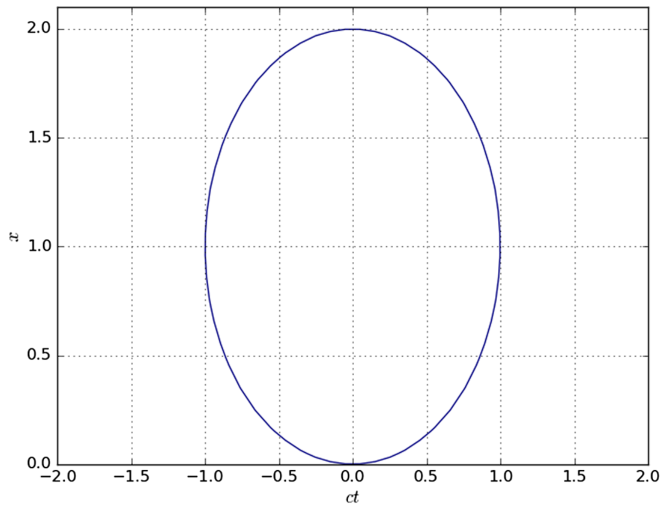



FIG. 8. Considering $c = g = 1$, $k_1 = 1$, and $k_2 = 0$, we plot the complete elliptic "trajectory" according to (26).

This example is an illustration of how KST can give rise to physical issues as causality, unbound velocities, closed timelike curves, and so on. Nevertheless, from the geometrical point of view, this spacetime carries interesting features.

## VII. SOME QUANTUM CONSEQUENCES

It is well known that the principles of Einstein's special relativity gave rise to new quantum mechanical models. The Klein-Gordon equation, for example, is a simple generalization of Schrödinger quantum mechanics in the framework of special relativity. Dirac, however, proposed an equation that has fundamental importance in the understanding of the relativistic spinning particle dynamics [36]. In this section, we dedicate a few words about the Dirac equation in Klein spacetime. First, we revisit some ideas already discussed in this direction and then we develop complementary new ones.

Following Alty's results [13], we adopt a prescription for Kleinian gamma matrices, $\gamma_K^\mu$, as

$$\gamma_K^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{28}$$

$$\gamma_K^1 = \begin{pmatrix} 0 & i\sigma_1 \\ -i\sigma_1 & 0 \end{pmatrix}, \tag{29}$$

$$\gamma_K^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \tag{30}$$

where $I$ is the identity matrix. With this choice the relation between the Kleinian gamma matrices and the spacetime metric remains the usual,

$$\{\gamma_K^\mu, \gamma_K^\nu\} = 2\eta^{\mu\nu} I.$$

We note that, for $\mu = 0, 2, 3$, the gamma matrices $\gamma_K^\alpha$ are in the Pauli-Dirac representation. However, $\gamma_K^1$ is not. This last one is related to the usual gamma prescription in Lorentz spacetime, $\gamma_L^1$, by the transformation $\gamma_K^1 = i\gamma_L^1$.

### A. Free particles

Now that we have pointed out these settings, we analyze the free spin-$\frac{1}{2}$ particle motion. In this spacetime the Dirac equation gets the form

$$(i\hbar\gamma_K^\mu \partial_\mu - m_0 c)\psi = 0, \tag{31}$$

where $x^0 = ct$. For this case, we assume that the particle has the plane wave profile

$$\psi = N e^{i(\vec{p}\cdot\vec{r} - Et)} \begin{bmatrix} \phi(p) \\ \chi(p) \end{bmatrix}, \tag{32}$$

where $N$ is the normalization factor, $\phi(p)$ and $\chi(p)$ are the spinor components. In terms of the Pauli matrices, $\vec{\sigma}$, the Dirac equation becomes

$$\begin{bmatrix} E - m_0 c & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -E - m_0 c \end{bmatrix} \begin{bmatrix} \phi(p) \\ \chi(p) \end{bmatrix} = 0. \tag{33}$$

Here we are concerned with the one-dimensional motion along the $x$ direction, such that $p_y = p_z = 0$, that leads us to the following equation

$$\begin{bmatrix} E/c - m_0 c^2 & -ip_x\sigma_1 \\ ip_x\sigma^x & -E/c - m_0 c^2 \end{bmatrix} \begin{bmatrix} \phi(p) \\ \chi(p) \end{bmatrix} = 0. \tag{34}$$

The nontrivial solutions of the above system occur if the matrix determinant is zero. Applying this condition we again arrive at the dispersion relation (20)

$$E^2 = m_0^2 c^4 - p^2 c^2, \tag{35}$$

where $p = p_x$.

As usual, we have two solutions for negative energy states expressed as

$$\psi_{(-)}^{\mathrm{up}} = N_{(-)} e^{i(\vec{p}\cdot\vec{x} - E^{(-)}t)} \begin{bmatrix} 0 \\ \frac{-ip_x c}{E^{(-)} - m_0 c^2} \\ 1 \\ 0 \end{bmatrix}, \tag{36}$$

$$\psi_{(-)}^{\mathrm{down}} = N_{(-)} e^{i(\vec{p}\cdot\vec{x} - E^{(-)}t)} \begin{bmatrix} \frac{-ip_x c}{E^{(-)} - m_0 c^2} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \tag{37}$$

where $\psi_{(-)}^{\mathrm{up}}$ represents the spin up particle, and $\psi_{(-)}^{\mathrm{down}}$ represents the spin down particle and we have defined

$$N_{(-)} \doteq \sqrt{\frac{m_0 c - E^{(-)}/c}{2Vm_0 c}}.$$

Similarly, we also have two positive energy state solutions

$$\psi_{(+)}^{\mathrm{up}} = N_{(+)} e^{i(\vec{p}\cdot\vec{x} - E^{(+)}t)} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \frac{ip_x c}{E^{(+)} + m_0 c^2} \end{bmatrix}, \tag{38}$$

$$\psi_{(+)}^{\mathrm{down}} = N_{(+)} e^{i(\vec{p}\cdot\vec{x} - E^{(+)}t)} \begin{bmatrix} 0 \\ 1 \\ \frac{ip_x c}{E^{(+)} + m_0 c^2} \\ 0 \end{bmatrix}, \tag{39}$$

and we have defined

$$N_{(+)} \doteq \sqrt{\frac{m_0 c + E^{(+)}/c}{2Vm_0 c}}.$$

### B. Forbidden states

Looking at the free particle dispersion relation (35), we infer that there is just one band of allowed energy levels, which lays between $-m_0c^2$ and $m_0c^2$, as illustrated in Fig. 9. In Fig. 10 is another representation of the energy spectrum showing again the single energy band shown in Fig. 9. This is the complement of the usual Dirac spectrum of the free particle. Because of this gapless energy structure there is no minimum energy requirement to promote a fermion from a negative energy state to a positive one, but there are lower and upper bounds for the energy. As a consequence, there is a limit in the amount of energy that a particle can either absorb or emit, quite differently from the usual relativistic case. We also note that due to these energy

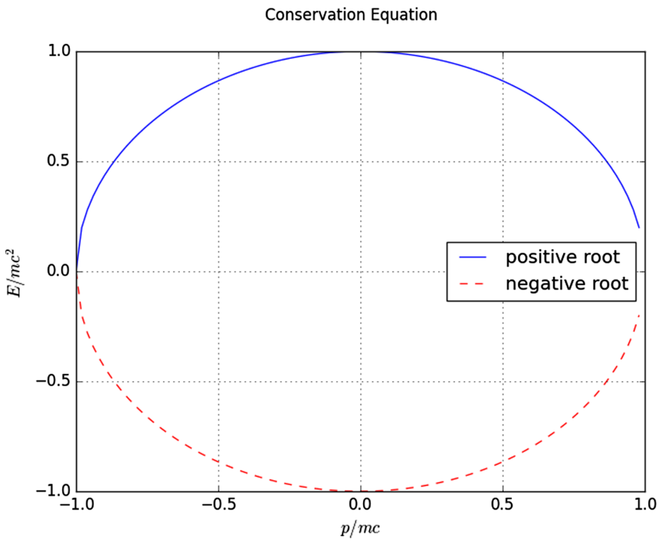



FIG. 9. Dispersion relation for a free fermion moving in the $x$ direction, as given by (35), showing a single continuous energy band restricted to the interval $-m_0c^2 < E < m_0c^2$.

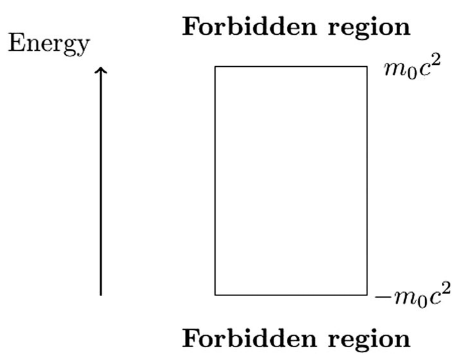



FIG. 10. Single energy band of the free unidimensional Dirac particle moving along $x$ direction in Klein spacetime.

limits, the catastrophic electronic transition of relativistic quantum mechanics [37] is prevented.

By considering the motion in the $x$-$y$ plane, the dispersion relation becomes

$$\frac{E}{m_0c^2} = \pm\sqrt{1 - \left(\frac{p_x}{m_0c}\right)^2 + \left(\frac{p_y}{m_0c}\right)^2}, \qquad (40)$$

a hyperboloid, as depicted in Fig. 11. This type of behavior is characteristic of hyperbolic metamaterials [38]. From this hyperboloid, we see that there is no $p_x$-$p_y$ rotational symmetry. Also note that by taking any $p_y$, the energy band increases its bounds. If we consider the photons, for example, the dispersion relation changes and gets the form

$$\frac{E}{c} = \pm\sqrt{p_y^2 - p_x^2}, \qquad (41)$$

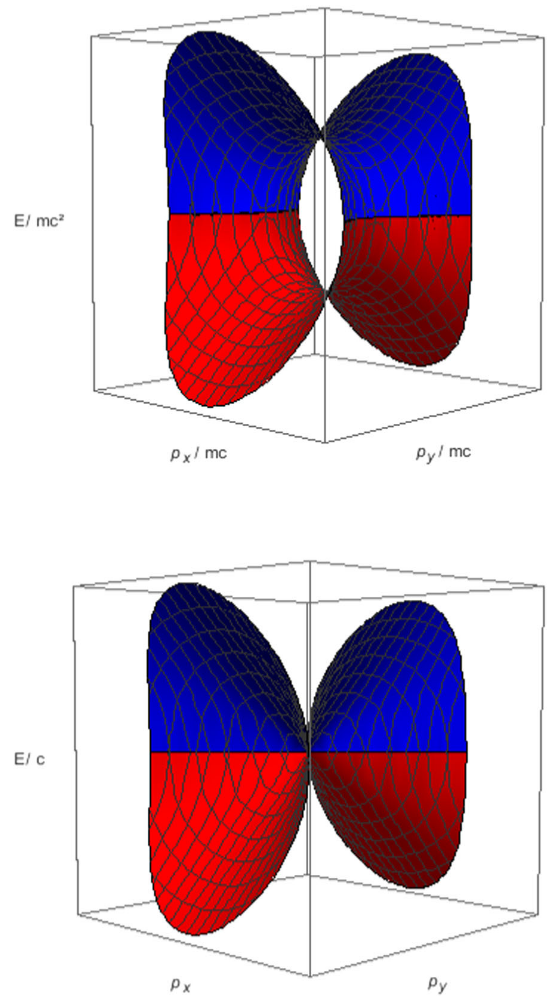



FIG. 11. Two-dimensional energy surface representing the dispersion relation (40). The blue portion of the surface represents the positive root and the red one the negative root of the dispersion relation.

valid or $p_y \geq p_x$. The corresponding diagrams are represented in Fig. 11, and for a comparison we represent the usual dispersion relations in Minkowski spacetime, Eqs. (42) and (43), in Fig. 12.

The second figure, the elliptical cone, corresponds to Eq. (41).

$$\frac{E}{m_0c^2} = \pm\sqrt{1 + \left(\frac{p_x}{m_0c}\right)^2 + \left(\frac{p_y}{m_0c}\right)^2}, \qquad (42)$$

$$\frac{E}{c} = \pm\sqrt{p_y^2 + p_x^2}, \qquad (43)$$

### C. Weyl fermions and helicity

Now we look at a particular case where we make $m_0 = 0$ in the Dirac equation, the so-called Weyl spinor case, which was related to the neutrino theoretical description [39].

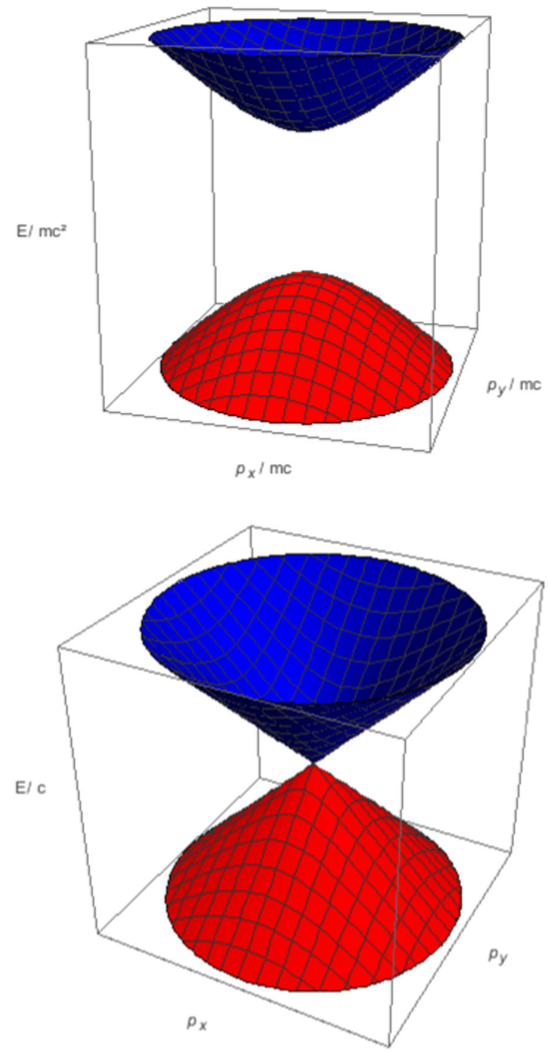



FIG. 12. Two-dimensional energy surface representing the dispersion relations (42) and (43). The blue portion of the surface represents the positive root and the red one the negative root of the dispersion relation.

The motion of Weyl fermions in Kleinian spacetime is severely restricted by Eq. (40) which limits the $x$ component of the momentum. It is also clear that there cannot be Weyl fermions moving only in the $x$ direction.

Still thinking only in $x$-direction movements, all other one-half spinning particles moving in this direction have a frame-dependent helicity, $h$. Furthermore, a right-handed massive particle, $h > 0$, has its spin in the same direction of its momentum $p$ but in the opposite direction of its motion. Alternatively, a massive left-handed particle, $h < 0$, has its spin in the same direction of its motion but in the opposite direction of its momentum. These facts are consequences of the unusual tensorial mass character.

## VIII. ANALOG KLEINIAN FERMIONS

The dispersion relation in Kleinian relativity can be realized using the analog Dirac equation designed for metamaterials, developed in [40,41]. To apply their ideas here, we start this section with the following Maxwell equations,

$$\vec{\nabla} \times \vec{E} = \mu_r \mu_0 \frac{\partial \vec{H}}{\partial t}, \tag{44}$$

$$\vec{\nabla} \times \vec{H} = -\epsilon_r \epsilon_0 \frac{\partial \vec{E}}{\partial t}, \tag{45}$$

which describe electromagnetic waves in one-dimensional metamaterials. Assuming that the wave propagates in the $x$ direction, the equations can be simplified and written in the form

$$\partial_x E_z = i\omega \mu_r \mu_0 H_y, \tag{46}$$

$$\partial_x H_y = -i\omega \epsilon_r \epsilon_0 E_z, \tag{47}$$

where $\epsilon_r$ and $\mu_r$ are the constitutive parameters. We define the analog fermion in Kleinian relativity $\psi$ in terms of the electromagnetic wave component through the relation

$$\psi = \begin{pmatrix} \sqrt{\epsilon_0} E_z \\ \sqrt{\mu_0} H_y \end{pmatrix}. \tag{48}$$

The wave propagation equation may be stated as

$$[-i\sigma_x \partial_x + \sigma_z m(\omega)]\psi = E(\omega)\psi. \tag{49}$$

This is the one-dimensional Dirac equation in natural units ($c = \hbar = 1$). The effective energy, $E(\omega)$, and effective mass $m(\omega)$, and the wave number $k$ are functions of the constitutive parameters in the following form

$$m(\omega) = \frac{\omega\sqrt{\mu_0 \epsilon_0}}{2}[\epsilon_r(\omega) - \mu_r(\omega)], \tag{50}$$

$$E(\omega) = -\frac{\omega\sqrt{\mu_0 \epsilon_0}}{2}[\epsilon_r(\omega) + \mu_r(\omega)], \tag{51}$$

$$k^2 = \omega^2 \epsilon_0 \epsilon_r(\omega) \mu_0 \mu_r(\omega). \tag{52}$$

From these equations it is straightforward to show that the dispersion relation is

$$E(\omega)^2 = m(\omega)^2 + k^2. \tag{53}$$

### A. Mimicking the Kleinian fermions

Here we assume, for our proposes, that the relative permeability and the relative permittivity have opposite signs, this is $\epsilon_r(\omega) < 0$, $\mu_r(\omega) > 0$ [or the other case $\epsilon_r(\omega) > 0$, $\mu_r(\omega) < 0$]. As a result from (52), we have the following dispersion relation

$$E(\omega)^2 = -k^2 + m(\omega)^2 \tag{54}$$

that characterize the Kleinnian conservation equation, where the effective parameters are

$$m(\omega) = \frac{\omega\sqrt{\mu_0\epsilon_0}}{2}[\epsilon_r(\omega) + |\mu_r(\omega)|], \tag{55}$$

$$E(\omega) = -\frac{\omega\sqrt{\mu_0\epsilon_0}}{2}[\epsilon_r(\omega) - |\mu_r(\omega)|]. \tag{56}$$

From Eq. (55), no negative or null mass analog particles are allowed, then no analog Weyl fermions must move in this direction. In addition, for this case, negative energy states are reproduced when $\epsilon_r(\omega) > |\mu_r(\omega)|$, while the positive energy states are simulated when $\epsilon_r(\omega) < |\mu_r(\omega)|$.

In the opposite way, if we assume that $\epsilon_r(\omega) < 0$, $\mu_r(\omega) > 0$, the relation (54) still holds, then the analog mass and energy expressions are written as

$$m(\omega) = -\frac{\omega\sqrt{\mu_0\epsilon_0}}{2}[|\epsilon_r(\omega)| + \mu_r(\omega)], \tag{57}$$

$$E(\omega) = -\frac{\omega\sqrt{\mu_0\epsilon_0}}{2}[\mu_r(\omega) - |\epsilon_r(\omega)|]. \tag{58}$$

Both cases are interesting since they provide different signs for $m_0$, which could describe the motion of a fermion along the $x$ direction in Klein spacetime. However, there are some limitations in this analog model as stated by [41]. This adaptation indicates a way to simulate the Klein-Minkowski metric signature transition using fermions as test particles, complementing the discussion of [30].

## IX. FINAL REMARKS

In this work, we developed a relativistic mechanical model for single point particles in Klein spacetime. A summary of some of our results and a comparison between known mechanical models is presented in Table I.

From this table we see that the Kleinian free particles do not have an upper or lower speed limit, but their momentum is bound. This is in accordance with the fact that these particles have a limited range of energy values, and this is completely different from the tachyon and special relativity description of particles. However, free tachyons and free Klein particles moving in the $x$ direction share some common features as the possibility to reach superluminal speeds and as the fact that they could have zero total energy and still carry a momentum $m_0c$. Furthermore, it is also important to stress the following:

TABLE I. Single particle mechanical properties in different models for one-dimensional movement in the $x$ direction.

| Model | $v_x$ | $p_x$ | $E$ |
|---|---|---|---|
| Minkowskian | $\|v_x\| < c$ | $\|p_x\| < \infty$ | Unbound |
| Kleinian | $0 < \|v_x\| < \infty$ | $\|p_x\| < m_0c$ | Bound |
| Tachyonic | $c < \|v_x\| < \infty$ | $m_0c < \|p_x\| < \infty$ | Unbound |
| Newtonian | $0 < \|v_x\| < \infty$ | $0 < \|p_x\| < \infty$ | Unbound |

1. The existence of the so-called exotic matter, as stated by Wang [35], seems to be a consequence of Klein spacetime, since it allows a negative mass component.
2. The first postulate of Einstein's special relativity does not agree with this new scenario, only the second postulate is in agreement with it. This is a clear consequence of the non-Minkowskian geometry.
3. If, somehow, the time contraction by muon decay were measured then it should be an evidence of the existence of a patch of KST somewhere.
4. There is an upper bound for the $x$-direction momentum of a Weyl fermion in Kleinian spacetime, even if the particle also moves in other directions, as suggested by Eq. (11).
5. Metamaterials appear as an arena where some proprieties of the KST may be mimicked, but one cannot expect that all the metric-derived effects can be reproduced in laboratories with this technology. Some are intrinsic to the special geometry of KST. In particular, Kleinnian relativity can only happen in a true, noneffective, Klein geometry.

Perhaps the main contribution of this article is to complement Alty's discussion [13] on Kleinian spacetime effects, adding some relativistic analyses to it. Furthermore, we expect that our developments will inspire new analog models for Klein spacetime, and new investigations, such as in the quantum field theory realm, in special, in order to clarify the unusual Dirac sea picture. In addition, we hope that our results constitute an important step in the development of this curious and pathological scenario, which we call "Kleinian Relativity."

## ACKNOWLEDGMENTS

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