

# Gravitational decoupling for axially symmetric systems and rotating black holes

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We introduce a systematic and direct procedure to generate hairy rotating black holes by deforming a spherically symmetric seed solution. We develop our analysis in the context of the gravitational decoupling approach, without resorting to the Newman-Janis algorithm. As examples of possible applications, we investigate how the Kerr black hole solution is modified by a surrounding fluid with conserved energy-momentum tensor. We find nontrivial extensions of the Kerr and Kerr-Newman black holes with primary hair. We prove that a rotating and charged black hole can have the same horizon as Kerr's, Schwarzschild's, or Reissner-Nordström's, thus showing possible observational effects of matter around black holes.

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## I. INTRODUCTION

Black holes (BHs) have been considered more than mere exotic solutions of the Einstein equations for quite some time now [1,2]. Nonetheless, it is only very recently that their direct existence was detected, mainly due to the spectacular results of both the LIGO [3] and Event Horizon Telescope [4] Collaborations. It is also fair to mention that some ultracompact stellar models could act as “black-hole mimickers” [5,6], although the existence of such objects would not necessarily exclude the existence of BHs [7], as it could be naively concluded.

Starting with Kerr's celebrated work [8], the interest in BHs has increased notably, and a large number of solutions have been found in various contexts (for some recent notable works, see e.g., [9–15]). Despite this diversity, in four-dimensional space-time, we can group all cases into two large groups: (i) static spherically symmetric solutions and (ii) stationary rotating solutions. (Note that, if we include non-Abelian matter fields, we can also find axisymmetric static BHs [16].) The study of these rotating and nonrotating BH metrics, and the shadow they produce,

has been extensively explored in recent years [17–47], the Newman-Janis algorithm [48] and its version without complexification [49] being tremendously useful tools to generate rotating systems.

In all cases, it is well known that the presence of matter around BHs could produce a significant distortion of the shadow in a highly model-dependent fashion (see e.g., [36,50–52] and references therein). The resolution of the first BH image is not enough to support or discard any of these models; hence, it is important to study this distortion with a minimum set of assumptions.<sup>1</sup> This is precisely the topic of the present work. Namely, we will consider a Kerr BH surrounded by an axially symmetric “tensor-vacuum” (analogous to the electro-vacuum and scalar-vacuum cases) represented by a conserved energy-momentum tensor  $S_{\mu\nu}$  which could account for one or more fundamental fields (scalar, vector, or tensor fields representing any phenomenologically viable form of matter energy, such as dark matter or dark energy). The only restriction we require is that  $S_{\mu\nu}$  satisfies either the strong energy condition (SEC) or the dominant energy condition (DEC) in the region outside the event horizon. Since the gravitational decoupling (GD) approach [53,54] is precisely designed for describing

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<sup>1</sup>Indeed, a theoretical model with the minimum amount of assumptions is always desirable in any context.

deformations of known solutions of general relativity induced by additional sources, we will study this problem by first extending the GD to axially symmetric systems.

The GD is originally based on the so-called minimal geometric deformation (MGD) [55,56] (for some earlier works on the MGD, see e.g., [57–65] and Refs. [66–84] for some recent applications). The GD has been shown to be particularly useful for at least three tasks [85–118]: (i) to generate new and more complex solutions from known (seed) solutions of the Einstein field equations, (ii) to systematically reduce (decouple) a complex energy-momentum tensor  $T_{\mu\nu}$  into simpler components, and (iii) to find solutions in gravitational theories beyond general relativity. Despite the above, one of the apparent limitations of the GD is that the decoupling of gravitational sources has only been achieved in the spherically symmetric case so far. One of the goals of this paper is to show that indeed the GD can be implemented beyond spherical symmetry. In particular, we will show how the GD can be obtained for axially symmetric systems, which is of particular importance for the study of rotating stellar systems and BHs.

The paper is organized as follows: in Sec. II, we first review the fundamentals of the GD approach for a spherically symmetric system containing two sources, and then we show in detail how to extend the GD approach for the axially symmetric case; in Sec. III, we apply our results to generate the axially symmetric version of two spherically symmetric hairy BH solutions, without implementing the Newman-Janis algorithm. The first solution contains a source satisfying the SEC and is an extension of the Kerr metric, while the DEC holds for the source in the second solution, which represents an extension of the Kerr-Newman BH; finally, we summarize our conclusions in Sec. IV.

## II. GRAVITATIONAL DECOUPLING

We start this section by briefly reviewing the key aspects of the GD for spherically symmetric systems (described in detail in Ref. [54]). A particularly simple case of GD is given by the MGD [55,56], which will guide us to introduce a GD for the axially symmetric case.

We start by considering the Einstein field equation<sup>2</sup>

$$\tilde{G}_{\mu\nu} \equiv \tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{R}\tilde{g}_{\mu\nu} = k^2\tilde{T}_{\mu\nu}, \quad (1)$$

with a total energy-momentum tensor containing two contributions,

$$\tilde{T}_{\mu\nu} = T_{\mu\nu} + S_{\mu\nu}, \quad (2)$$

where  $T_{\mu\nu}$  is usually associated with a known solution of general relativity, whereas  $S_{\mu\nu}$  may contain new fields or a new gravitational sector. Since the Einstein tensor  $\tilde{G}_{\mu\nu}$

satisfies the Bianchi identity, the total source  $\tilde{T}_{\mu\nu}$  must be covariantly conserved.

### A. Spherically symmetric case

For spherically symmetric and static systems, the metric  $\tilde{g}_{\mu\nu}$  can be written as

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2d\Omega^2, \quad (3)$$

where  $\nu = \nu(r)$  and  $\lambda = \lambda(r)$  are functions of the areal radius  $r$  only and  $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ . The Einstein tensor in Eq. (1) then reads

$$\tilde{G}_0^0 = \frac{1}{r^2} - e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right), \quad (4)$$

$$\tilde{G}_1^1 = \frac{1}{r^2} - e^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu'}{r}\right), \quad (5)$$

$$\tilde{G}_2^2 = -\frac{e^{-\lambda}}{4}\left(2\nu'' + \nu'^2 - \lambda'\nu' + 2\frac{\nu' - \lambda'}{r}\right), \quad (6)$$

where  $f' \equiv \partial_r f$  and  $\tilde{T}_3^3 = \tilde{T}_2^2$  due to the spherical symmetry. By simple inspection, we can identify an effective density

$$\tilde{\epsilon} = T_0^0 + S_0^0, \quad (7)$$

an effective radial pressure

$$\tilde{p}_r = -T_1^1 - S_1^1, \quad (8)$$

and an effective tangential pressure

$$\tilde{p}_t = -T_2^2 - S_2^2. \quad (9)$$

Since the anisotropy  $\Pi \equiv \tilde{p}_t - \tilde{p}_r$  usually does not vanish, the system of Eqs. (4)–(6) may be viewed as an anisotropic fluid.

We next consider a solution to Eq. (1) generated by the seed source  $T_{\mu\nu}$  alone (i.e., for  $S_{\mu\nu} = 0$ ), which we write as

$$ds^2 = e^{\xi(r)}dt^2 - e^{\mu(r)}dr^2 - r^2d\Omega^2, \quad (10)$$

where

$$e^{-\mu(r)} \equiv 1 - \frac{k^2}{r} \int_0^r x^2 T_0^0(x) dx = 1 - \frac{2m(r)}{r} \quad (11)$$

is the standard general relativity expression containing the Misner-Sharp mass function  $m = m(r)$ . Adding the source  $S_{\mu\nu}$  can then be accounted for by the deformation of the seed metric (10) given by

<sup>2</sup>We use units with  $c = 1$  and  $k^2 = 8\pi G_N$ , where  $G_N$  is Newton's constant.

$$\xi \rightarrow \nu = \xi + \alpha g, \quad (12)$$

$$e^{-\mu} \rightarrow e^{-\lambda} = e^{-\mu} + \alpha f, \quad (13)$$

where the parameter  $\alpha$  is introduced to keep track of these deformations.

By means of Eqs. (12) and (13), the Einstein equations (1) split into the Einstein field equations for the seed metric (10), that is

$$G_{\mu}^{\nu}(\xi, \mu) = k^2 T_{\mu}^{\nu}, \quad (14)$$

where

$$G_0^0 = \frac{1}{r^2} - e^{-\mu} \left( \frac{1}{r^2} - \frac{\mu'}{r} \right), \quad (15)$$

$$G_1^1 = \frac{1}{r^2} - e^{-\mu} \left( \frac{1}{r^2} + \frac{\xi'}{r} \right), \quad (16)$$

$$G_2^2 = -\frac{e^{-\mu}}{4} \left( 2\xi'' + \xi'^2 - \mu'\xi' + 2\frac{\xi' - \mu'}{r} \right), \quad (17)$$

and a second set containing the source  $S_{\mu\nu}$ , which reads

$$\alpha G_{\mu}^{\nu}(\xi, \mu; f, g) = k^2 S_{\mu}^{\nu}, \quad (18)$$

where

$$\mathcal{G}_0^0 = -\frac{f}{r^2} - \frac{f'}{r}, \quad (19)$$

$$\mathcal{G}_1^1 = -f \left( \frac{1}{r^2} + \frac{\nu'}{r} \right) - Z_1, \quad (20)$$

$$\mathcal{G}_2^2 = -\frac{f}{4} \left( 2\nu'' + \nu'^2 + 2\frac{\nu'}{r} \right) - \frac{f'}{4} \left( \nu' + \frac{2}{r} \right) - Z_2, \quad (21)$$

and

$$Z_1 = \frac{e^{-\mu} g'}{r}, \quad (22)$$

$$4Z_2 = e^{-\mu} \left( 2g'' + g'^2 + \frac{2g'}{r} + 2\xi'g' - \mu'g' \right). \quad (23)$$

One clearly sees that the tensor  $S_{\mu\nu}$  must vanish when the metric deformations vanish ( $\alpha = 0$ ). On assuming  $g = 0$ , we have  $Z_1 = Z_2 = 0$  and Eq. (18) reduces to the simpler “quasi-Einstein” system of the MGD of Refs. [55,56], in which the deformation  $f$  is only determined by the source  $S_{\mu\nu}$  and the seed metric (10).

What makes the GD work is the fact that, under the transformations (12) and (13), the Einstein tensor changes as

$$G_{\gamma}^{\sigma}(\xi, \mu) \rightarrow G_{\gamma}^{\sigma}(\nu, \lambda) = G_{\gamma}^{\sigma}(\xi, \mu) + \alpha G_{\gamma}^{\sigma}(\nu, \lambda). \quad (24)$$

That is to say, Eqs. (12) and (13) yield a *linear* decomposition of the Einstein tensor in the parameter  $\alpha$ , like the two sources add linearly in the rhs of Eq. (1). We therefore expect that a similar GD can be introduced for any given space-time, independently of its symmetries, if we can implement a linear decomposition for the Einstein tensor of the form in Eq. (24). A natural application is then to consider axially symmetric systems.

## B. Axially symmetric case

Let us start with the simplest extension of the Kerr metric, given by [119]

$$ds^2 = \left[ 1 - \frac{2r\tilde{m}(r)}{\tilde{\rho}^2} \right] dt^2 + \frac{4\tilde{a}r\tilde{m}(r)\sin^2\theta}{\tilde{\rho}^2} dt d\phi - \frac{\tilde{\rho}^2}{\tilde{\Delta}} dr^2 - \tilde{\rho}^2 d\theta^2 - \frac{\tilde{\Sigma}\sin^2\theta}{\tilde{\rho}^2} d\phi^2, \quad (25)$$

with

$$\tilde{\rho}^2 = r^2 + \tilde{a}^2 \cos^2\theta, \quad (26)$$

$$\tilde{\Delta} = r^2 - 2r\tilde{m}(r) + \tilde{a}^2, \quad (27)$$

$$\tilde{\Sigma} = (r^2 + \tilde{a}^2)^2 - \tilde{a}^2 \tilde{\Delta} \sin^2\theta, \quad (28)$$

and

$$\tilde{a} = \tilde{J}/\tilde{M}, \quad (29)$$

where  $\tilde{J}$  is the angular momentum and  $\tilde{M}$  the total mass of the system. Note that the line element (25) reduces to the Kerr solution when the metric function  $\tilde{m} = \tilde{M}$ . Moreover, when  $\tilde{a} = 0$ , we obtain the Schwarzschild-like metric in Eq. (3) with

$$e^{\nu} = e^{-\lambda} = 1 - \frac{2\tilde{m}(r)}{r}. \quad (30)$$

Hence, the correspondence between the metrics (3) and (25) is clear, with the latter being a rotational version of a Kerr-Schild spherically symmetric space-time (see e.g., Refs. [120–123]). Although Eq. (25) is not the most general axially symmetric line element, it can be used to describe rotating compact objects, like BHs and gravastars, among many others.

The components of the Einstein tensor for the metric (25) read

$$\tilde{G}_0^0 = 2 \frac{r^4 + (\rho^2 - r^2)^2 + \tilde{a}^2(2r^2 - \rho^2)}{\rho^6} \tilde{m}' - \frac{r\tilde{a}^2 \sin^2 \theta}{\rho^4} \tilde{m}'', \quad \tilde{\epsilon} = -\tilde{p}_r = \frac{2r^2}{\rho^4} \tilde{m}', \quad (40)$$

$$\tilde{G}_1^1 = 2 \frac{r^2}{\rho^4} \tilde{m}', \quad (42)$$

$$\tilde{G}_2^2 = 2 \frac{\rho^2 - r^2}{\rho^4} \tilde{m}' + \frac{r}{\rho^2} \tilde{m}'', \quad (43)$$

$$\tilde{G}_3^3 = 2 \frac{2r^2(\rho^2 - r^2) + \tilde{a}^2(\rho^2 - 2r^2)}{\rho^6} \tilde{m}' + \frac{r(\tilde{a}^2 + r^2)}{\rho^4} \tilde{m}'', \quad (44)$$

$$\tilde{G}_0^3 = 2 \frac{\tilde{a}(2r^2 - \rho^2)}{\rho^6} \tilde{m}' - \frac{\tilde{a}r}{\rho^4} \tilde{m}''. \quad (45)$$

The key observation now is that this Einstein tensor is linear in derivatives<sup>3</sup> of the mass function  $\tilde{m}(r)$ , whereas the rotational parameter  $\tilde{a}$  appears in a convoluted form. Any linear decomposition of the mass function,

$$\tilde{m} = m(r) + \alpha m_s(r), \quad (46)$$

will therefore generate a linear decomposition of the Einstein tensor of the form in Eq. (24) with  $\mathcal{G}_\gamma^\sigma = G_\gamma^\sigma$ , provided the rotational parameter  $\tilde{a}$  is left unaffected, that is

$$\tilde{G}_\gamma^\sigma(\tilde{m}, \tilde{a}) = G_\gamma^\sigma(m, \tilde{a}) + \alpha G_\gamma^\sigma(m_s, \tilde{a}). \quad (47)$$

Like for the spherically symmetric case in Sec. II A, we will assume that the mass functions  $m$  and  $m_s$  are generated by the energy-momentum tensor  $T_{\mu\nu}$  and  $S_{\mu\nu}$  in Eq. (2), respectively. It is convenient to introduce the tetrads [124]

$$\tilde{e}_t^\mu = \frac{(r^2 + \tilde{a}^2, 0, 0, \tilde{a})}{\sqrt{\rho^2 \Delta}}, \quad \tilde{e}_r^\mu = \frac{\sqrt{\Delta}(0, 1, 0, 0)}{\sqrt{\rho^2}},$$

$$\tilde{e}_\theta^\mu = \frac{(0, 0, 1, 0)}{\sqrt{\rho^2}}, \quad \tilde{e}_\phi^\mu = -\frac{(\tilde{a} \sin^2 \theta, 0, 0, 1)}{\sqrt{\rho^2} \sin \theta}, \quad (48)$$

so that the total source  $\tilde{T}_{\mu\nu}$  generating the metric (25) can be written as

$$\tilde{T}^{\mu\nu} = \tilde{\epsilon} \tilde{e}_t^\mu \tilde{e}_t^\nu + \tilde{p}_r \tilde{e}_r^\mu \tilde{e}_r^\nu + \tilde{p}_\theta \tilde{e}_\theta^\mu \tilde{e}_\theta^\nu + \tilde{p}_\phi \tilde{e}_\phi^\mu \tilde{e}_\phi^\nu, \quad (49)$$

where the energy density  $\tilde{\epsilon}$  and the pressures  $\tilde{p}_r$ ,  $\tilde{p}_\theta$ , and  $\tilde{p}_\phi$  are given by

<sup>3</sup>Of course, when  $\tilde{m} = M$  is constant, the Einstein tensor  $\tilde{G}_{\mu\nu} = 0$ , since Eq. (25) is the vacuum Kerr metric.

$$\tilde{p}_\theta = \tilde{p}_\phi = -\frac{r}{\rho^2} \tilde{m}'' + \frac{2(r^2 - \rho^2)}{\rho^4} \tilde{m}', \quad (41)$$

which are also, consistently, linear in (derivatives of) the mass function.

We next consider a solution to the Eq. (1) for the seed source  $T_{\mu\nu}$  alone, which we write as

$$ds^2 = \left[ 1 - \frac{2rm(r)}{\rho^2} \right] dt^2 + \frac{4arm(r) \sin^2 \theta}{\rho^2} dt d\phi - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - \frac{\Sigma \sin^2 \theta}{\rho^2} d\phi^2, \quad (42)$$

where the expressions for  $\rho$ ,  $\Sigma$ , and  $\Delta$  are the same as those in Eqs. (26)–(28) but contain  $m$  and  $a$  instead of  $\tilde{m}$  and  $\tilde{a}$ . The addition of the second source  $S_{\mu\nu}$  can then be accounted for by the GD of the seed metric (42) given by

$$m(r) \rightarrow \tilde{m} = m(r) + \alpha m_s(r), \quad (43)$$

with the parameter  $\alpha$  introduced to keep track of the deformation as usual. In order to achieve the decoupling (37) of Eqs. (31)–(35), we must also demand

$$\tilde{a} = a = a_s, \quad (44)$$

that is to say, the length scales  $a$  and  $a_s$  associated, respectively, with the sources  $T_{\mu\nu}$  and  $S_{\mu\nu}$  must be and remain equal. Finally, notice that the mass deformation (43) corresponds to the particular metric deformation  $f$  in Eq. (13) given by

$$f(r) = -\frac{2m_s(r)}{r}. \quad (45)$$

Unlike the general GD for the spherically symmetric case, Eqs. (43) and (44) split the Einstein equations (1) in two equal sets: (i) one is given by Einstein field equations with the energy-momentum tensor  $T_{\mu\nu}$ , that is

$$G_\mu{}^\nu(m, a) = k^2 T_\mu{}^\nu, \quad (46)$$

whose solution is the seed metric (42), and (ii) the second set contains the source  $S_{\mu\nu}$  and reads

$$\alpha G_\mu{}^\nu(m_s, a) = k^2 S_\mu{}^\nu, \quad (47)$$

whose solution has the same form as the one in Eq. (42) but with  $m(r) \rightarrow \alpha m_s(r)$ . In this case, we conclude that the two sources  $T_{\mu\nu}$  and  $S_{\mu\nu}$  can be decoupled by means of the metric deformations (43) and the length-scale invariant condition (44). Notice that the energy and pressures in Eqs. (40) and (41) can be written as [49]

$$\tilde{\epsilon} = \epsilon + \alpha\epsilon_S, \quad (48)$$

$$\tilde{p}_i = p_i + \alpha p_{Si} \quad (i = r, \theta, \phi), \quad (49)$$

where  $\epsilon_S$  and  $p_{Si}$  are the energy and pressures of the source  $S_{\mu\nu}$ . Finally, we see that for  $a = 0$  the sets (46) and (47) reduce to those in (14) and (18), respectively. We want to emphasize once again that this procedure is exact and does not require a perturbative expansion in the parameter  $\alpha$ .

### 1. Strategy

We can now detail our scheme to generate new axially symmetric metrics from known solutions of the Einstein field equations which are as follows:

- (1) Consider the field equations  $\tilde{G}_{\mu\nu} = k^2\tilde{T}_{\mu\nu}$  which determine the metric  $\tilde{g}_{\mu\nu}(\tilde{m}, \tilde{a})$  in Eq. (25), where  $\tilde{T}_{\mu\nu} = T_{\mu\nu}^1 + \dots + T_{\mu\nu}^n$ , and  $T_{\mu\nu}^i$  is the energy-momentum tensor of the  $i$ th gravitational source.
- (2) Solve  $G_{\mu\nu}^i = k^2T_{\mu\nu}^i$  for each  $T_{\mu\nu}^i$  to find their respective  $g_{\mu\nu}^i(m_i, a_i)$  in Eq. (42) [of the same form as (25)], namely,

$$G_{\mu\nu}^1 = k^2T_{\mu\nu}^1 \Rightarrow g_{\mu\nu}^1(m_1, a_1)$$

$$\vdots$$

$$G_{\mu\nu}^n = k^2T_{\mu\nu}^n \Rightarrow g_{\mu\nu}^n(m_n, a_n).$$

- (3) The solution  $\tilde{g}_{\mu\nu}(\tilde{m}, \tilde{a})$  in Eq. (25) of the original problem  $\tilde{G}_{\mu\nu} = k^2\tilde{T}_{\mu\nu}$  is obtained by setting

$$\tilde{m} = m_1 + \dots + m_n$$

$$\tilde{a} = a_1 = \dots = a_n$$

in the line element (25).

The previous scheme can be simplified even further by noting that step 2 actually amounts to computing just the mass functions  $m_i = m_i(r)$ . We can thus do that for  $a_i = 0$  for each  $G_{\mu\nu}^i = k^2T_{\mu\nu}^i$ , that is solve the spherically symmetric cases

$$G_{\mu\nu}^i = k^2T_{\mu\nu}^i \Rightarrow g_{\mu\nu}^i(m_i, a_i = 0) \quad (50)$$

and generate the axially symmetric version by plugging the mass function  $\tilde{m} = m_1 + \dots + m_n$  into Eq. (25) with the asymptotic angular momentum parameter  $\tilde{a}$  of choice.

### 2. Decoupling Einstein-Maxwell

With the aim of testing the consistency of our approach, let us consider a well-known case, namely, the Einstein-Maxwell system. In particular, we will consider the axially symmetric electro-vacuum, for which the result must be the well-known Kerr-Newman solution.

Following our strategy, we start by identifying the sources  $T_{\mu\nu}^1 = T_{\mu\nu} = 0$  and  $T_{\mu\nu}^2 = S_{\mu\nu}$  of relevance for the case at hand, that is

$$\tilde{T}_{\mu\nu} = \overset{0}{\mathcal{X}}_{\mu\nu} + S_{\mu\nu}, \quad (51)$$

where

$$S_{\mu\nu} = \frac{1}{4\pi} \left( F_{\mu\alpha} F_{\nu}^{\alpha} + \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right) \quad (52)$$

is the Maxwell tensor.

Next, we solve the Einstein equations for each source separately, in the particularly simple case  $a_1 = a_2 = 0$ . For the vacuum  $T_{\mu\nu} = 0$ , we find the Schwarzschild solution with mass

$$m_1 = M. \quad (53)$$

For the source  $S_{\mu\nu}$ , we find the Reissner-Nordström solution, whose mass function is given by

$$m_s(r) = A - \frac{Q^2}{2r}, \quad (54)$$

where  $A$  and  $Q$  are integration constants, with  $Q$  eventually identified as the electric charge.

The total mass function is given by

$$m(r) = M + A - \frac{Q^2}{2r} \equiv \mathcal{M} - \frac{Q^2}{2r}, \quad (55)$$

which, plugged into the metric (25), yields the well-known Kerr-Newman solution with

$$\tilde{\Delta} = r^2 - 2r\mathcal{M} + a^2 + Q^2. \quad (56)$$

We see that the method is straightforward, and that we do not need to use the Newman-Janis algorithm to map the spherically symmetric solution into the axially symmetric one.

## III. ROTATING BLACK HOLE SOLUTIONS

In a recent paper [125], we developed a new method to generate spherically symmetric hairy BHs by imposing a minimal set of requirements consisting of (i) the existence of a well-defined event horizon and (ii) the SEC or DEC for the hair outside the horizon. In particular, we considered a Schwarzschild BH surrounded by a spherically symmetric tensor-vacuum represented by a conserved energy-momentum tensor  $S_{\mu\nu}$ , which is dealt with as explained in Sec. II A. We will here use those solutions as seeds to generate axially symmetric systems according to our strategy in Sec. II B 1.

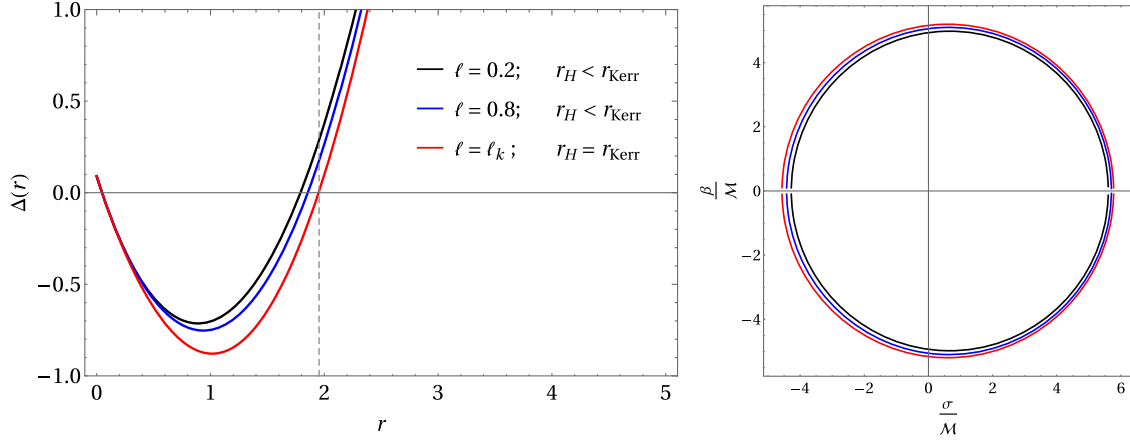


FIG. 1. Extended Kerr solution: function  $\tilde{\Delta}$  (left panel) and silhouette (right panel) of the shadow cast for different values of  $\ell$  with  $\alpha = 0.6$ ,  $a = 0.3$ , and  $\mathcal{M} = 1$ . The Kerr horizon  $r_{\text{Kerr}}$  here corresponds to the saturated case of Eq. (58).

### A. Extended Kerr solution

When we demand that  $S_{\mu\nu}$  satisfies the SEC in the region outside the event horizon, we found the extended Schwarzschild BH metric

$$\begin{aligned} e^\nu = e^{-\lambda} &= 1 - \frac{1}{r}(2\mathcal{M} + \ell_0) + \alpha e^{-r/M} \\ &= 1 - \frac{2\mathcal{M}}{r} + \alpha e^{-r/(\mathcal{M}-\ell_0/2)}, \end{aligned} \quad (57)$$

where  $\ell_0 = \alpha\ell$  measures the increase of entropy from the minimum Schwarzschild value  $S = 4\pi\mathcal{M}^2$  caused by the hair and must satisfy

$$\ell_0 \leq 2\mathcal{M} \equiv \ell_K \quad (58)$$

in order to ensure asymptotic flatness.

In order to extend this metric to the axially symmetric case, we just need to identify the mass function from Eq. (57), that is

$$\tilde{m} = \mathcal{M} - \alpha \frac{r}{2} e^{-r/(\mathcal{M}-\ell_0/2)}, \quad (59)$$

which we then plug into the metric (25). This yields

$$\tilde{\Delta} = r^2 + a^2 - 2r\mathcal{M} + \alpha r^2 e^{-r/(\mathcal{M}-\ell_0/2)}. \quad (60)$$

The equation determining the horizon  $r = r_H$  of the metric (25) is given by  $0 = \tilde{g}^{rr} \sim \tilde{\Delta}$ , which yields

$$r_H^2 + a^2 - 2\mathcal{M}r_H + \alpha r_H^2 e^{-r_H/(\mathcal{M}-\ell_0/2)} = 0. \quad (61)$$

We see that the Kerr horizon

$$r_{\text{Kerr}} = \mathcal{M} + \sqrt{\mathcal{M}^2 - a^2} \quad (62)$$

is recovered for  $\alpha = 0$  and also when the inequality (58) is saturated.

The function (60) is plotted in Fig. 1 for a few values of  $\ell_0$  at fixed  $\alpha$  and  $a$ . We see that the horizon shifts to larger radii when  $\ell_0$  increases, reaching a maximum value corresponding to the Kerr horizon for  $\ell_0 = \ell_K$ . The silhouette of the BH is also shown in Fig. 1 where  $\sigma$  and  $\beta$  are the usual celestial coordinates (see the Appendix).

We conclude that the metric (25) with the mass function (59) represents a family of rotating hairy BHs described by the parameters  $\{\mathcal{M}, a, \ell_0\}$ , where  $\ell_0 = \alpha\ell$  represents a charge associated with primary hair.

Finally, the spherically symmetric metric with components (57) satisfies the SEC, that is

$$\begin{aligned} \epsilon_S + p_{Sr} + 2p_{S\theta} &\geq 0 \\ \rho_S + p_{Sr} &\geq 0 \\ \rho_S + p_{S\theta} &\geq 0. \end{aligned} \quad (63)$$

It can be checked straightforwardly that this property is inherited by the rotating solution.

### B. Extended Kerr-Newman solution

The second case we will consider is the rotating version of the spherically symmetric solution [125]

$$\begin{aligned} e^\nu = e^{-\lambda} &= 1 - \frac{2\mathcal{M} + \alpha\ell}{r} + \frac{Q^2}{r^2} - \frac{\alpha\mathcal{M}e^{-r/M}}{r} \\ &= 1 - \frac{2\mathcal{M}}{r} + \frac{Q^2}{r^2} - \frac{\alpha}{r} \left( \mathcal{M} - \frac{\alpha\ell}{2} \right) e^{-r/(\mathcal{M}-\alpha\ell/2)}, \end{aligned} \quad (64)$$

which extends a Reissner-Nordström-like metric<sup>4</sup> to include a conserved source  $S_{\mu\nu}$  satisfying the DEC. From (64), we read out the mass function

<sup>4</sup>We remark that  $Q$  is not necessarily the electric charge, but could be a tidal charge of extra-dimensional origin or any other charge for the tensor  $S_{\mu\nu}$ .

$$\tilde{m} = \mathcal{M} - \frac{Q^2}{2r} + \frac{\alpha}{2} \left( \mathcal{M} - \frac{\ell_0}{2} \right) e^{-r/(\mathcal{M}-\ell_0/2)}, \quad (65)$$

which immediately yields the rotating version (25) with

$$\tilde{\Delta} = r^2 + a^2 + Q^2 - 2r\mathcal{M} - \alpha r \left( \mathcal{M} - \frac{\ell_0}{2} \right) e^{-r/(\mathcal{M}-\ell_0/2)}, \quad (66)$$

where again the inequality (58) must hold to ensure asymptotic flatness.

The horizon is again determined by  $\tilde{\Delta} = 0$  or

$$\begin{aligned} r_{\text{H}}^2 + a^2 + Q^2 - 2\mathcal{M}r_{\text{H}} \\ = \alpha r_{\text{H}} \left( \mathcal{M} - \frac{\ell_0}{2} \right) e^{-r_{\text{H}}/(\mathcal{M}-\ell_0/2)}. \end{aligned} \quad (67)$$

The Kerr-Newman horizon

$$r_{\text{KN}} = \mathcal{M} + \sqrt{\mathcal{M}^2 - a^2 - Q^2} \quad (68)$$

is found for  $\alpha = 0$  (hence  $\ell_0 = 0$ ) and also when the inequality in (58) is saturated.

The metric function (66) is plotted in Fig. 2 for given values of  $\alpha$ ,  $a$ , and  $Q$ . For the range of  $\ell$  shown there, the horizon shrinks to smaller radii when  $\ell$  increases, reaching a minimum value corresponding to the Kerr-Newman horizon when Eq. (58) is saturated. We conclude that the metric (25) with the mass function (65) represents rotating hairy BHs depending on the parameters  $\{\mathcal{M}, a, Q, \alpha, \ell_0\}$ , where  $\ell_0 = \alpha\ell$  represents a charge associated with primary hair.

Like with the SEC in Sec. III A, one can check straightforwardly that the rotating metrics inherit the DEC satisfied by the spherically symmetric metric functions in Eq. (64),

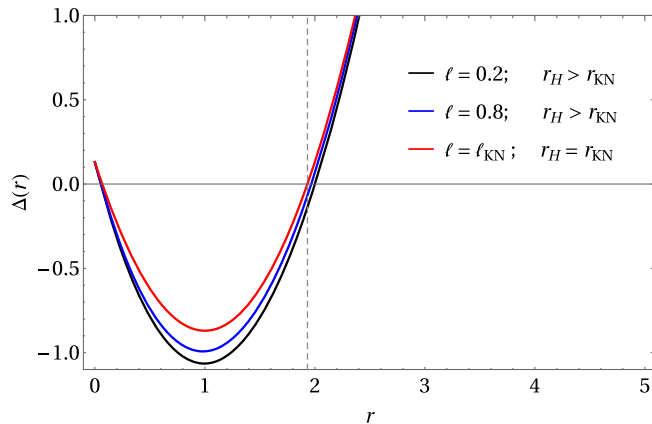


FIG. 2. Extended Kerr-Newman solution: function  $\tilde{\Delta}$  for different values of  $\ell$  with  $\alpha = 0.6$ ,  $a = 0.3$ ,  $Q = 0.2$ , and  $\mathcal{M} = 1$ . The Kerr-Newman horizon  $r_{\text{KN}}$  here corresponds to the saturated case of Eq. (58).

$$\rho_S \geq 0, \quad (69)$$

$$\rho_S \geq |\tilde{p}_{Si}| \quad (i = r, \theta, \phi). \quad (70)$$

It is impossible to find analytical solutions to Eq. (67), except for some particular cases. Three of them are shown below.

### 1. Case 1

If the charge and angular momentum parameters satisfy the condition

$$a^2 + Q^2 = \alpha r_{\text{H}} \left( \mathcal{M} - \frac{\ell_0}{2} \right) e^{-r_{\text{H}}/(\mathcal{M}-\ell_0/2)}, \quad (71)$$

we have

$$\begin{aligned} \tilde{\Delta} = r^2 - 2r\mathcal{M} + \alpha \left( \mathcal{M} - \frac{\ell_0}{2} \right) \\ \times [r_{\text{H}} e^{-r_{\text{H}}/(\mathcal{M}-\ell_0/2)} - r e^{-r/(\mathcal{M}-\ell_0/2)}], \end{aligned} \quad (72)$$

and the event horizon is located at the Schwarzschild radius  $r_{\text{H}} = 2\mathcal{M}$ . This indicates that the source  $S_{\mu\nu}$  filling the “electro-vacuum” produces a screening effect on the charges  $a$  and  $Q$  in such a way that an external observer will see a rotating space-time with the horizon apparently generated by a nonrotating and neutral distribution.

### 2. Case 2

Next, if the charge satisfies

$$Q^2 = \alpha r_{\text{H}} \left( \mathcal{M} - \frac{\ell_0}{2} \right) e^{-r_{\text{H}}/(\mathcal{M}-\ell_0/2)}, \quad (73)$$

the metric function

$$\begin{aligned} \tilde{\Delta} = r^2 + a^2 - 2r\mathcal{M} + \alpha \left( \mathcal{M} - \frac{\ell_0}{2} \right) \\ \times [r_{\text{H}} e^{-r_{\text{H}}/(\mathcal{M}-\ell_0/2)} - r e^{-r/(\mathcal{M}-\ell_0/2)}], \end{aligned} \quad (74)$$

and the event horizon is located at  $r_{\text{H}} = r_{\text{Kerr}}$  given in Eq. (62), provided  $a^2 \leq \mathcal{M}^2$ . This indicates a screening effect of the charge  $Q$  only, so that an external observer will detect a horizon corresponding to a neutral distribution.

### 3. Case 3

Last, if the angular momentum satisfies

$$a^2 = \alpha r_{\text{H}} \left( \mathcal{M} - \frac{\ell_0}{2} \right) e^{-r_{\text{H}}/(\mathcal{M}-\ell_0/2)}, \quad (75)$$

which leads to

$$\Delta = r^2 + Q^2 - 2r\mathcal{M} + \alpha \left( \mathcal{M} - \frac{\ell_0}{2} \right) \times [r_{\text{H}} e^{-r_{\text{H}}/(\mathcal{M}-\ell_0/2)} - r e^{-r/(\mathcal{M}-\ell_0/2)}], \quad (76)$$

the event horizon is given by Eq. (77) with  $a = 0$ , namely, the Reissner-Nordström horizon

$$r_{\text{RN}} = \mathcal{M} + \sqrt{\mathcal{M}^2 - Q^2}, \quad (77)$$

provided of course the charge  $Q^2 \leq \mathcal{M}^2$ . In this case, the screening effect occurs on the rotational charge  $a$ . An external observer will see a rotating BH with effective horizon corresponding to a nonrotating charged distribution.

The three screening cases above can be described collectively by the metric function

$$\tilde{\Delta} = r^2 + Z_i^2 - 2r\mathcal{M} + \alpha L (r_{\text{H}} e^{-r_{\text{H}}/L} - r e^{-r/L}), \quad (78)$$

where  $L = \mathcal{M} - \ell_0/2$  and

$$Z_i^2 = \{Z_S^2, Z_K^2, Z_{\text{RN}}^2\} = \{0, a^2, Q^2\} \quad (79)$$

for the three effective horizons, namely, Schwarzschild, Kerr, and Reissner-Nordström, respectively. Notice that the hair charge  $\ell_0 \equiv \alpha \ell$  in the expressions (71), (73), and (75) is related with the charges  $\mathcal{M}$ ,  $a$ , and  $Q$  by the Lambert  $\mathcal{W}$  function as

$$\ell_{0i} = 2\mathcal{M} - \frac{2r_{\text{H}i}}{\mathcal{W}\left(\frac{\alpha r_{\text{H}i}^2}{a^2 + Q^2 - Z_i^2}\right)}, \quad (80)$$

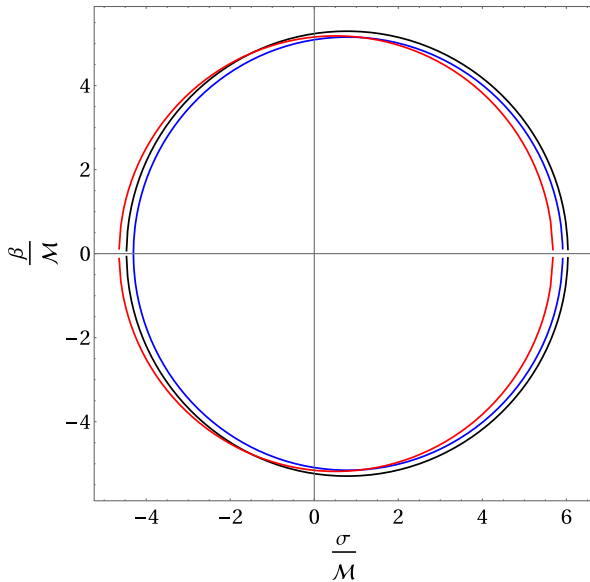


FIG. 3. Special extended Kerr-Newman solutions: BH shadow for (a) Schwarzschild horizon (black), (b) Kerr horizon (blue), and (c) Reissner-Nordström horizon (red), for  $\alpha = 0.9$ ,  $\ell = 0.7$ ,  $a = 0.4$ ,  $Q = 0.2$ , and  $\mathcal{M} = 1$ .

where the index  $i$  runs on the three cases in Eq. (79). As an example, Fig. 3 shows the shadow cast in the three cases for a given choice of parameters.

#### IV. CONCLUSIONS

Using the GD approach and the simplest extension of the Kerr metric (25), which could be generated by the Newman-Janis algorithm without complexification [49], we have proven that the decoupling of gravitational sources in general relativity is possible in the axially symmetric case, as long as the metric takes the form (25) and the asymptotic angular momentum parameter  $a$  in Eq. (29) satisfies the critical condition (44). As a direct consequence, we provided a simple and systematic strategy to generate axially symmetric BHs departing from a spherically symmetric seed solution, without implementing any variant of the Newman-Janis algorithm.

On a formal level, our results stem from observing that the Einstein tensor (31)–(35) for the metric (25) is linear in derivatives of the mass function. This property could be at the heart of other known methods to generate axially symmetric solutions of the Einstein equations from spherically symmetric solutions. Moreover, the GD could help in investigating mass functions in nonspherically symmetric systems [126–131].<sup>5</sup>

Following the aforementioned approach, we showed how the Kerr BH, given by the metric (25) with mass function equal to the total mass  $\mathcal{M}$ , is modified when a fluid with conserved energy-momentum tensor  $S_{\mu\nu}$  fills the axially symmetric vacuum. We thus find nontrivial extensions of the Kerr BH, given by the mass function (59), and Kerr-Newman BH, with mass function (65). Both solutions can support a primary hair  $\ell_0 \leq 2\mathcal{M}$ , whose impact on the silhouette of the shadow is displayed in Figs. 1 and 3.

Finally, in the case of the extended Kerr-Newman BH, whose horizon is found by solving Eq. (67), we identify special cases describing a screening effect induced by the source  $S_{\mu\nu}$  on the charges  $a$  and  $Q$ , in such a way that an external observer would see a rotating BH with effective horizon corresponding to (i) nonrotating and neutral distribution, (ii) neutral distribution, and (iii) nonrotating but charged distribution. This clearly indicates that the matter around BHs may have a significant observational impact and separating different models of BHs could remain a very hard task. In this respect, we notice that these results do not contradict the conclusions of Ref. [132] about no hairs for BHs in astrophysical environments, since the (effective) fluid modifying the Kerr geometry overlaps the BH horizon in our case.

Although it is not the main goal of the present work, we would like to conclude by mentioning that both rotating solutions [characterized by the metric functions in Eqs. (60)

<sup>5</sup>We plan to investigate these issues further in separate works.



and (66), respectively] could be investigated further, in particular, for possible observational constraints on the primary hairs  $\ell_0$  and  $Q$ . However, this is beyond the purpose of the present work.

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### APPENDIX: NULL GEODESICS AROUND ROTATING BHS

We briefly review how to study null geodesics in a rotating space-time like the one in Eq. (25) and find the celestial coordinates describing the BH shadow. Waves propagate along characteristic curves described by the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial \lambda} = \frac{1}{2} g^{\mu\nu} \partial_\mu S \partial_\nu S, \quad (\text{A1})$$

where  $\lambda$  is a parameter along the curve and  $S$  the Jacobi action. Given the symmetries of the space-time (25), Eq. (A1) is separable and one has [133]

$$S = -Et + \Phi\phi + S_r(r) + S_\theta(\theta), \quad (\text{A2})$$

with  $E$  and  $\Phi$  being the conserved energy and angular momentum, respectively. Replacing (A2) in Eq. (A1), we obtain

$$\begin{aligned} S_r &= \int \frac{\sqrt{R(r)}}{\Delta} dr \\ S_\theta &= \int \sqrt{\Theta(\theta)} d\theta, \end{aligned} \quad (\text{A3})$$

where

$$\begin{aligned} R &= [(r^2 + a^2)E - a\Phi]^2 - \Delta[Q + (\Phi - aE)^2] \\ \Theta &= Q - (\Phi^2 \csc^2 \theta - a^2 E^2) \cos^2 \theta, \end{aligned} \quad (\text{A4})$$

with  $Q$  the Carter constant.

The (unstable) circular photon orbits are determined by  $R = R' = 0$ , namely,

$$\begin{aligned} (a^2 - a\xi + r^2)^2 - (a^2 + r^2 F)[(a - \xi)^2 + \eta] &= 0 \\ 4(a^2 - a\xi + r^2) - [(a - \xi)^2 + \eta](rF' + 2F) &= 0, \end{aligned} \quad (\text{A5})$$

where  $\xi = \Phi/E$  and  $\eta = Q/E^2$  are the impact parameters. Accordingly,

$$\begin{aligned} \xi &= a + \frac{r^2}{a} - \frac{4(a^2 + r^2 F)}{a(rF' + 2F)}, \\ \eta &= \frac{r^2[r^2 + 2a(a - \xi) - (a - \xi)^2 F]}{a^2 + r^2 F}, \\ F &= 1 - \frac{2\tilde{m}}{r}, \end{aligned} \quad (\text{A6})$$

where  $r$  is the radius of the unstable photon orbit.

The apparent shape of the shadow is finally described by the celestial coordinates [134]

$$\sigma \equiv \lim_{r_0 \rightarrow \infty} \left( -r_0^2 \sin \theta_0 \frac{d\phi}{dr} \Big|_{(r_0, \theta_0)} \right) = -\frac{\xi}{\sin \theta_0} \quad (\text{A7})$$

and

$$\begin{aligned} \beta &\equiv \lim_{r_0 \rightarrow \infty} \left( r_0^2 \frac{d\theta}{dr} \Big|_{(r_0, \theta_0)} \right) \\ &= \sqrt{\eta - \xi^2 \cot^2 \theta_0 + a^2 \cos^2 \theta_0}, \end{aligned} \quad (\text{A8})$$

where  $(r_0, \theta_0)$  are the coordinates of the observer.

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