

Spontaneous scalarization of a conducting sphere in Maxwell-scalar models

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We study the spontaneous scalarization of a standard conducting charged sphere embedded in Maxwell-scalar models in flat spacetime, wherein the scalar field ϕ is nonminimally coupled to the Maxwell electrodynamics. This setup serves as a toy model for the spontaneous scalarization of charged (vacuum) black holes in Einstein-Maxwell-scalar (generalized scalar-tensor) models. In the Maxwell-scalar case, unlike the black hole cases, closed-form solutions exist for the scalarized configurations. We compute these configurations for three illustrations of nonminimal couplings: one that *exactly* linearizes the scalar field equation, and the remaining two that produce nonlinear continuations of the first one. We show that the former model leads to a runaway behavior in regions of the parameter space and neither the Coulomb nor the scalarized solutions are stable in the model; but the latter models can heal this behavior producing stable scalarized solutions that are dynamically preferred over the Coulomb one. This parallels reports on black hole scalarization in the extended-scalar-Gauss-Bonnet models. Moreover, we analyze the impact of the choice of the boundary conditions on the scalarization phenomenon. Dirichlet and Neumann boundary conditions accommodate both (linearly) stable and unstable parameter space regions, for the scalar-free conducting sphere; but radiative boundary conditions always yield an unstable scalar-free solution and preference for scalarization. Finally, we perform numerical evolution of the full Maxwell-scalar system, following dynamically the scalarization process. They confirm the linear stability analysis and reveal that the scalarization phenomenon can occur in qualitatively distinct ways.

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I. INTRODUCTION

Compact objects such as neutron stars and black holes can undergo a spontaneous scalarization phenomenon in asymptotically flat spacetimes, in scalar-tensor theories of gravitation, see e.g., [1–7] for scalarization of relativistic stars and e.g., [8–40] for black hole scalarization. This is a strong gravity phase transition, wherein the nonminimal coupling between the scalar field and a curvature invariant (or matter density) plays a key role that triggers a tachyonic growth of the scalar field in high curvature (or high density) backgrounds.

Spontaneous scalarization, however, is not exclusive of either scalar-tensor models or even of strong gravity¹; it

occurs in a broader class of models. For instance, if the scalar field nonminimally couples to the Maxwell (rather than a curvature) invariant $F^{\mu\nu}F_{\mu\nu}$, with $F_{\mu\nu}$ being the electromagnetic field strength, charged black holes can scalarize in Einstein-Maxwell-scalar models [46–57], wherein both the scalar and electromagnetic fields *minimally* couple to Einstein’s gravity (see also [58,59] for scalar field couplings in nonlinear electrodynamics). Additionally, even in the absence of gravity, a nonminimal coupling of the scalar field to Maxwell’s electromagnetism, $f(\phi)F^{\mu\nu}F_{\mu\nu}$, where $f(\phi)$ is a regular function of ϕ , may allow scalarization of a charged object, such as a conducting sphere [46] (see also [60]). This occurs in Maxwell-scalar (MS models on Minkowski spacetime, which therefore provide one of the simplest arenas to study the spontaneous scalarization phenomenon. Studying this phenomenon in detail will be the subject of this paper.

MS models are rather simple for the static and spherically symmetric configurations, which possess an integrable structure [61]. Then, despite being nontrivially coupled to the electromagnetic dynamics via a nonminimal

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¹Also, a similar phenomenon exists for vector and tensor (rather than scalar) fields, see e.g., [41–45].

coupling function $f(\phi)$, the scalar field equation can be fully decoupled from Maxwell's equations via considering the first integral that arises from the latter, the electric charge. Consequently, specifying $f(\phi)$, one can fully grasp the linear/nonlinear structure of the scalar field equation. This is unlike what happens in scalar-tensor or Einstein-Maxwell-scalar models, in which the impact of the geometry on the scalar field equation and the precise structure of the corresponding nonlinearities are not so explicit in general, since the geometry is sourced by the scalar field.

The first investigation of spontaneous scalarization in an MS model was briefly mentioned in [46], wherein a closed-form solution describing a scalarized conducting sphere was reported. This was found for the specific nonlinear form of $f(\phi)$ given by Eq. (26) below. As it turns out, the particular simplicity induced by this coupling can be traced to the fact that it *exactly* linearizes the scalar field equation. Using appropriate variables, the latter is, in fact, simply the standard harmonic equation. In [46], it was observed that the corresponding scalarized conductor was energetically preferred over the Coulomb solution with the same charge in a subset of the parameter space, but no dynamical or stability analysis was performed in Ref. [46]. Such an analysis will be performed in this paper, leading to the conclusion that this model is afflicted by a runaway mode: there are regions of the parameter space wherein neither the scalar-free (Coulomb) nor the scalarized conducting sphere are stable.

A physically more reasonable behavior, with a stable (classical) vacuum, can be obtained by augmenting the previous model with sufficient nonlinearities. This will be illustrated by considering two examples of nonlinear couplings $f(\phi)$ given by Eqs. (37) and (45) below, which can be regarded as nonlinear continuations of the first choice that led to the linear model. In fact, the scalar field equation now becomes that of a nonlinear harmonic oscillator. The nonlinearities that these couplings induce in the latter equation will be shown to be able to quench the runaway mode, by introducing an effective potential with new extrema, yielding stable scalarized configurations.

This dichotomy between the linear and nonlinear models and (non)existence of a stable scalarized vacuum, which was just explained above, finds a close parallelism with the situation found in black hole spontaneous scalarization in extended-scalar-tensor-Gauss-Bonnet models. The simplest nonminimal coupling allowing for black hole scalarization [8], which produces a linear scalar field equation,² yields static scalarized black holes that are not entropically preferred over the Schwarzschild solution with the same mass and, moreover, which were later found to be radially unstable [12]. But couplings which produces a

²This linearity is only apparent, since the scalar field backreacts on the geometry which then impacts on the scalar field equation.

manifestly nonlinear scalar field equation could give rise to perturbatively stable scalarized solutions [9,12,13,17,30,31]. We suggest that the aforementioned MS models, described herein, are a simple toy model to study this behavior and the dynamical formation process of scalarized solutions.

In the MS models, unlike the black hole case, however, there is a larger freedom in choosing the boundary conditions for the scalar field on the surface of the conducting sphere. We will show that the choice of the boundary conditions can impact significantly on the instability of the conducting sphere. These findings will result from a linear perturbation theory analysis, which is also performed for the scalarized solutions, but will also be confirmed by fully nonlinear numerical evolution. Moreover, the latter will help us understand *how* the scalarized solution is reached. We will see that the tachyonic scalarization instability is always stronger in the immediate vicinity of the surface of the conducting sphere, and then propagates towards large radial distances. Moreover, in some cases the scalarization instability proceeds in a rather monotonic and straightforward manner to form the scalarized solutions, whereas in other cases the scalar field oscillates considerably before approaching the end point.

This paper is organized as follows. MS models are briefly described in Sec. II, where the Coulomb solution and its stability, when embedded in these models, is considered for different boundary conditions. Section III describes the *linear* model and its scalarized solution, establishing they are always unstable. Section IV describes the two *nonlinear* models and their scalarized solution, establishing the regimes in which they are stable. In Sec. V we describe the numerical simulations to study the time evolution of the scalarization phenomenon. Some conclusions are presented in Sec. VI. Two Appendices provide some further technical details.

II. MS MODELS

A. Action and field equations

Let us consider the following family of MS models, wherein a real scalar field ϕ is nonminimally coupled to the electromagnetic field³:

$$S = - \int d^4x [2\partial^\mu\phi\partial_\mu\phi + f(\phi)F^{\mu\nu}F_{\mu\nu}], \quad (1)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength, A_μ is the $U(1)$ gauge field, and $f(\phi)$ is a free function of the scalar field ϕ , specifying the nonminimal coupling. For $f = 1$ the scalar field is free and it is not sourced by the electromagnetic field; we shall be

³The standard normalizations of the scalar and Maxwell kinetic terms correspond to multiplying this action by $1/4$.

interested in nontrivial functions $f(\phi)$ that allow a scalarization phenomenon, triggered by charged configurations. Throughout this work, dynamics will take place on flat Minkowski spacetime, whose metric is taken in the standard spherical coordinates

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\bar{\varphi}^2), \quad (2)$$

where t , r , and $(\theta, \bar{\varphi})$ are the time, radial, and angular coordinates, respectively. By varying the action (1) with respect to ϕ and A_μ , we obtain the MS field equations:

$$\square\phi - \frac{1}{4}f_\phi(\phi)F^{\mu\nu}F_{\mu\nu} = 0, \quad (3)$$

$$D_\mu(f(\phi)F^{\mu\nu}) = 0, \quad (4)$$

where $f_\phi := df/d\phi$. We shall be interested in spherical dynamics and spherical static configurations. As such, for static solutions we assume a spherically symmetric ansatz for both the gauge potential and the scalar field:

$$A_\mu = (A_t, A_r, A_\theta, A_{\bar{\varphi}}) := (A_0(r), 0, 0, 0), \quad (5)$$

$$\phi = \phi_0(r). \quad (6)$$

In this spherical sector, the MS model is integrable [61]. The nontrivial components of the vector and scalar field equations of motion, Eqs. (3) and (4), are given by

$$\phi_0''(r) + \frac{2}{r}\phi_0'(r) + \frac{1}{2}f_\phi(\phi_0)A_0'(r)^2 = 0, \quad (7)$$

$$A_0''(r) + \frac{2}{r}A_0'(r) + \frac{f_\phi(\phi_0)}{f(\phi_0)}\phi_0'(r)A_0'(r) = 0, \quad (8)$$

where ‘‘prime’’ denotes radial derivative. Integrating the second equation yields a first integral:

$$A_0'(r) = \frac{Q}{r^2 f(\phi_0)}, \quad (9)$$

where the integration constant Q is the total electric charge. The scalar field equation (7) then reduces to

$$r^2 \frac{d}{dr} \left(r^2 \frac{d\phi_0}{dr} \right) = \frac{d}{d\phi_0} \left(\frac{Q^2}{2f(\phi_0)} \right). \quad (10)$$

Defining a new radial coordinate $x := 1/r$ and an effective potential for the motion

$$\mathcal{V}_{\text{eff}}(\phi_0) := -\frac{Q^2}{2f(\phi_0)}, \quad (11)$$

Eq. (10) becomes

$$\frac{d^2\phi_0}{dx^2} = -\frac{d\mathcal{V}_{\text{eff}}}{d\phi_0}. \quad (12)$$

This describes a mechanical model of a particle with degree of freedom ϕ_0 moving under the influence of a one-dimensional potential (11), with the inverse radius x playing the role of ‘‘time.’’ This potential is, up to a constant, the inverse of the coupling function. The model can be solved for any choice of the nonminimal coupling function $f(\phi)$. We remark that a second integral can be obtained from (10) [61].

B. The Coulomb solution embedded in MS models and its stability

The Coulomb solution in standard electromagnetism, plus a trivial scalar field,

$$A_0'(r) = \frac{Q}{r^2}, \quad \phi_0(r) = 0, \quad (13)$$

solve the MS theory (1) as long as the coupling function obeys

$$f(0) = 1, \quad f_\phi(0) = 0. \quad (14)$$

If $f(\phi)$ admits a Taylor series around $\phi = 0$, then the general form of the coupling satisfying (14) is given by

$$f(\phi) = 1 + a\phi^2 + \sum_{n=3}^{\infty} f_n\phi^n, \quad (15)$$

where $a > 0$ and f_n ($n = 3, 4, \dots$) are constants.

Embedding the Coulomb solution in a MS theory raises the question *whether it is the preferred vacuum in spherical symmetry with charge Q . Or, could there be another dynamically preferred vacuum? And, in that case, do scalar perturbations promote the evolution of the Coulomb solution towards the preferred vacuum?*

In order to address these questions, the physical setup herein is to consider a conducting charged sphere with a radius $r = r_s$. This circumvents the well-known issue in classical electromagnetism that the energy of the Coulomb solution diverges as $r \rightarrow 0$. Having such a sphere, the Coulomb solution holds on and outside the conductor. In our study, we shall always focus on the behavior on and outside the conductor only.

In Appendix A, we discuss the linear radial perturbations in general backgrounds in MS models. For the coupling given by Eq. (15), scalar and vector field perturbations decouple in Eq. (A1); the scalar field radial perturbation $\phi_1(r)$ obeys

$$-\ddot{\phi}_1 + \phi_1'' + \frac{2}{r}\phi_1' + \frac{aQ^2}{r^4}\phi_1 = 0, \quad (16)$$

where “dot” stands for a time derivative. Fourier decomposing the solution, $\phi_1(t, r) = \int d\omega e^{-i\omega t} \phi_\omega(r)$, each mode obeys

$$\phi_\omega''(r) + \frac{2}{r} \phi_\omega'(r) + \frac{aQ^2}{r^4} \phi_\omega(r) = -\omega^2 \phi_\omega(r). \quad (17)$$

An instability is present if there are modes with purely imaginary eigenvalues, $\omega^2 < 0$ (see Appendix B for the proof). Thus, instabilities are absent if the mode $\omega^2 = 0$ is the lowest eigenmode; in this case, the solution $\phi_{\omega=0}$ has no nodes between the conductor at $r = r_s$ and spatial infinity $r \rightarrow \infty$. For convenience, we introduce the dimensionless quantities,

$$\bar{r} := \frac{r}{r_s}, \quad \bar{Q} := \frac{Q\sqrt{a}}{r_s}, \quad (18)$$

so that the conductor surface is located at $\bar{r} = 1$. Imposing the physical boundary condition that the perturbation should vanish at spatial infinity, $\phi_\omega \rightarrow 0$, we obtain the following solution for the $\omega = 0$ mode:

$$\phi_{\omega=0} = C_0 \sin\left(\frac{\bar{Q}}{\bar{r}}\right), \quad (19)$$

where C_0 is an integration constant. At the surface of the conductor different boundary conditions can be chosen. The Dirichlet (D) or Neumann (N) boundary conditions,

$$\phi_0(r_s) = 0(D), \quad \phi_0'(r_s) = 0(N), \quad (20)$$

lead to

$$\sin(\bar{Q}) = 0 \quad (D), \quad \cos(\bar{Q}) = 0 \quad (N); \quad (21)$$

consequently, these boundary conditions yield, respectively, the following constraints on the parameters:

$$\bar{Q}|_D = \pi(1+n), \quad \bar{Q}|_N = \frac{\pi}{2}(1+2n), \quad n \in \mathbb{N}_0. \quad (22)$$

If \bar{Q} is greater than the value given by (22) for $n = 0$, the lowest mode of ϕ_1 has nodes, implying there are eigenmodes with $\omega^2 < 0$. Then, the Coulomb solution is unstable. Instability is therefore present if the effective coupling \bar{Q} is larger than a threshold value, which depends on the boundary condition chosen, i.e., when

$$\bar{Q} > \bar{Q}_D := \pi, \quad \bar{Q} > \bar{Q}_N := \frac{\pi}{2}. \quad (23)$$

A third type of boundary condition can be defined, which will make contact with the numerical evolution in Sec. V

$$(r\phi_{\omega=0})'|_{r=r_s} = 0. \quad (24)$$

This will be called a *radiative* (R) boundary condition, which corresponds to the ingoing boundary condition Eq. (59) for $\omega = 0$. From (19) it leads to the condition

$$\tan(\bar{Q}) = \bar{Q}. \quad (25)$$

In this case, the lowest mode gives $\bar{Q} = 0$. Thus for any value of $a, Q > 0$, the Coulomb solution is *always* unstable for these radiative boundary conditions.

III. THE LINEAR MODEL

Let us now consider several specific choices of the coupling function $f(\phi)$. The first choice leads to an exact solution already discussed in [46]. Albeit not discussed explicitly therein, this choice linearizes the ϕ equation. Thus, we call it the *linear* model.

A. The scalarized solution

The linear model has the coupling function:

$$f(\phi) = \frac{1}{1 - a\phi^2}, \quad \stackrel{(11)}{\Rightarrow} \quad \mathcal{V}_{\text{eff}}(\phi_0) = \frac{Q^2}{2}(a\phi_0^2 - 1). \quad (26)$$

The effective potential \mathcal{V}_{eff} informs us that this coupling leads to a one-dimensional mechanical model with a harmonic potential. It has a minimum at $\phi_0 = 0$; this is not, however, the only solution to the equations of motion, as we shall promptly see. For this model, the electric field (9) reduces to

$$A_0' = \frac{Q}{r^2}(1 - a\phi_0^2). \quad (27)$$

One possible viewpoint is to face the scalarized solutions as introducing a sort of medium around the charged sphere. This interpretation has been used, say, in [61]. Then, the impact on the gauge field can be regarded as a screening of the charge. The scalar field equation (12) then becomes the exactly linear (harmonic) equation

$$\frac{d^2\phi_0}{dx^2} = -aQ^2\phi_0. \quad (28)$$

The solution satisfying the boundary condition $\phi(r \rightarrow \infty) = 0$ is given by, in terms of the r variable,

$$\phi_0 = \zeta \sin\left(\frac{\sqrt{a}Q}{r}\right), \quad (29)$$

where ζ is an undetermined amplitude. Observe that this *exact* solution for the scalar field in the MS model defined by the coupling (26), precisely coincides with the solution for the $\omega = 0$ *linear perturbation* around the Coulomb solution in Eq. (19). In the limit of $r \rightarrow \infty$ Eq. (29) leads to

$$\phi_0 = \frac{Q_s}{r} + \mathcal{O}\left(\frac{1}{r^3}\right), \quad (30)$$

where the scalar “charge” Q_s ⁴ has been identified as $Q_s := \sqrt{a}\zeta Q$. Since the equation of motion (28) is linear, Q_s also grows linearly with the amplitude ζ .

Imposing the boundary conditions (D) and (N) on the solution given by Eq. (29) at the surface of the conductor, leads to, precisely, the conditions in Eq. (22). For each type of boundary conditions, there are, therefore, solutions with $n \in \mathbb{N}_0$ nodes between the conductor and infinity. The lowest state, the $n = 0$ node solution, branches off from the pure Coulomb solution at the threshold of instability against scalar perturbations; that is, when $\bar{Q} = \bar{Q}_N$ or $\bar{Q} = \bar{Q}_D$, for the boundary conditions (N) or (D), respectively, given by Eq. (23). The branching is controlled by the amplitude ζ ; the electrostatic potential of the scalarized solution becomes [46]

$$A_0 = - \left[\left(1 - \frac{a\zeta^2}{2}\right) \frac{Q}{r} + \frac{\sqrt{a}\zeta^2}{4} \sin\left(\frac{2\sqrt{a}Q}{r}\right) \right]. \quad (31)$$

As we shall see in the next subsection, however, the scalarized solution Eqs. (29) and (31) for the model (26) turns out to be unstable. This should be no surprise, considering the effective potential (26). Indeed, from (12), the (inverse) radial coordinate plays the role of “time” in the one-dimensional mechanical model. Thus the sinusoidal oscillations in the radial coordinate of solution (29) correspond to the time oscillations of a harmonic oscillator. Since such an oscillator has only a stable equilibrium point at the minimum of the potential, we anticipate that only the solution with $\phi_0 = 0$ everywhere could be stable. One can also relate this instability to the linear nature of (28), which allows the amplitude of the scalar field to grow without bound. This is reminiscent of the instability observed for the scalarized black holes in extended-scalar-tensor-Gauss-Bonnet gravity, for a quadratic coupling function, that also leads to a scalar field equation which is linear in the scalar field [8,9,12,13,17,30,31].

B. Instability of the scalarized charged conductor in the linear model

To assess the linear stability of the scalarized solution given by Eqs. (29) and (31), we consider small radial perturbations about it:

$$\begin{aligned} A_t &= A_0(r) + \epsilon a_t(t, r), & A_r &= \epsilon a_r(t, r), \\ \phi &= \phi_0(r) + \epsilon \phi_1(t, r), \end{aligned} \quad (32)$$

⁴Using common terminology, the coefficient of the leading asymptotic term of the scalar field, Q_s , is called “charge”; but, in fact, it is not associated to any conservation law.

where $\epsilon \ll 1$ is the benchmark parameter for the perturbations. The details of the perturbation analysis are shown in Appendix A. By integrating out the gauge field perturbations a_t and a_r , one can obtain the master equation for the scalar field perturbation. Fourier decomposing the master variable Φ , $\Phi = \int d\omega \Phi_\omega e^{-i\omega t}$, each mode obeys a time independent Schrödinger-type equation

$$-\Phi''_\omega + V_{\text{eff}}(r)\Phi_\omega = \omega^2\Phi_\omega, \quad (33)$$

where we have defined the *effective potential for the perturbations*

$$\begin{aligned} V_{\text{eff}}(r) &:= -\frac{Q^2}{2r^4 f(\phi_0(r))^3} \\ &\times [f(\phi_0(r))f_{\phi\phi}(\phi_0(r)) - 2f_\phi(\phi_0(r))^2], \end{aligned} \quad (34)$$

with $f_{\phi\phi}(\phi) := d^2f(\phi)/d\phi^2$. For the model (26), this effective potential simplifies to

$$V_{\text{eff}}(r) = -\frac{Q^2 a}{r^4}, \quad (35)$$

which is negative definite. This implies that unstable modes ($\omega^2 < 0$, see Appendix B) may exist. In fact, they will exist if the perturbation with $\omega = 0$ has nodes. The analysis then reduces to that of Sec. II B, and the conclusion is that the scalarized charged conductors are unstable in the linear model whenever the Coulomb solution, with the same charge and the same boundary conditions, is unstable. Thus, the scalarized solution cannot be the end point of the instability of the Coulomb solution.

It is interesting to notice, in fact, that allowing for both a radial *and* a time dependence in the ansatz (5), the model (26) admits the *exact* time dependent solution

$$\begin{aligned} \phi_0 &= \zeta t \sin\left(\frac{\sqrt{a}Q}{r}\right), \\ A_0 &= - \left[\left(1 - \frac{a\zeta^2 t^2}{2}\right) \frac{Q}{r} + \frac{\sqrt{a}\zeta^2 t^2}{4} \sin\left(\frac{2\sqrt{a}Q}{r}\right) \right], \end{aligned} \quad (36)$$

which simply amounts to giving the amplitude ζ a linear time dependence $\zeta \rightarrow \zeta t$. The existence of such an unbound (secular), growing solution, is again, related to the linear nature of the model.

To obtain a physically self-consistent model, in the next Sec. IV we shall consider scalarized charged conductor solutions for more general coupling functions, wherein the nonlinear corrections in the scalar field equation are able to quench the tachyonic instability.

IV. NONLINEAR MODELS

We shall now consider two examples of coupling functions leading to models where the scalar equation of

motion is nonlinear. As we shall see, in such cases the scalarized solutions may be free of instabilities.

A. The inverse quartic polynomial model

The first model is a nonlinear continuation of the linear model (26). Specifically, we consider model (26) augmented by a quartic term:

$$f(\phi) = \frac{1}{1 - a\phi^2 + k^2 a^2 \phi^4 / 4} \quad (11)$$

$$\mathcal{V}_{\text{eff}}(\phi_0) = \frac{Q^2}{2} \left(-\frac{k^2 a^2}{4} \phi_0^4 + a\phi_0^2 - 1 \right). \quad (37)$$

The effective potential has extrema at

$$\phi_0 = 0 \quad (\text{local minimum}), \quad \phi_0 = \pm \sqrt{\frac{2}{ak^2}} \quad (\text{maxima}). \quad (38)$$

With the choice (37), the electric field (9) becomes

$$A'_0 = \frac{Q}{r^2} \left(1 - a\phi_0^2 + \frac{a^2 k^2}{4} \phi_0^4 \right). \quad (39)$$

Substituting this expression for the electric field into (7), the scalar equation becomes that of a nonlinear harmonic oscillator. Its solution, satisfying the boundary condition $\phi_0(r \rightarrow \infty) = 0$ reads

$$\bar{\phi}(\bar{r}) = \sqrt{1 - \sqrt{1 - \bar{C}_0} \text{sn}} \times \left[\frac{\bar{Q}}{\sqrt{2\bar{r}}} \sqrt{1 + \sqrt{1 - \bar{C}_0}}, \frac{1 - \sqrt{1 - \bar{C}_0}}{1 + \sqrt{1 - \bar{C}_0}} \right], \quad (40)$$

where “sn” denotes a Jacobi elliptic function,

$$\bar{\phi} := \sqrt{\frac{ak^2}{2}} \phi_0 \quad (k \neq 0), \quad (41)$$

\bar{r} and \bar{Q} were introduced in Eq. (18), and \bar{C}_0 is an integration constant, which will be fixed by the boundary conditions at the surface of the conductor $r = r_s$. This form of the solution applies for $k \neq 0$; for $k = 0$ the coupling (37) reduces to the coupling (26) and hence we recover the linear model.

At first glance, the function (40) looks imaginary for $\bar{C}_0 > 1$. However, we numerically confirmed that $\bar{\phi}$ remains real even if Eq. (40) is analytically continued into the parameter region of $\bar{C}_0 > 1$ and satisfies the boundary condition $\bar{\phi}(\bar{r} \rightarrow \infty) = 0$, similarly to what holds for $0 \leq \bar{C}_0 \leq 1$, although we did not find an alternative analytic form of (40) in which the arguments are

explicitly real. In this sense the solution (40) is valid for $\bar{C}_0 > 0$. We exclude the case of $\bar{C}_0 < 0$, for which Eq. (40) becomes imaginary. As shown in Fig. 1, the boundary conditions (N) and (D) (20) admit the solutions $0 \leq \bar{C}_0 \leq 1$ for any \bar{Q} . On the other hand, the solutions adapted to the radiative boundary condition used in the numerical simulations (see Sec. V) include the case $\bar{C}_0 > 1$. We can then resort to numerical methods to obtain the corresponding profiles of the scalarized solution of Eq. (7). We also remark that for $\bar{C}_0 = 0$ one recovers the Coulomb solution with $\bar{\phi} = 0$; for $\bar{C}_0 = 1$ one obtains

$$\bar{\phi} \rightarrow \tanh\left(\frac{\bar{Q}}{\sqrt{2\bar{r}}}\right). \quad (42)$$

The asymptotic ($\bar{r} \rightarrow \infty$) behavior of (40) gives

$$\bar{\phi} = \frac{\bar{Q}_s}{\bar{r}} + \mathcal{O}\left(\frac{1}{\bar{r}^3}\right); \quad (43)$$

then, considering (41), the scalar “charge” \bar{Q}_s is given by

$$\bar{Q}_s = \frac{\sqrt{\bar{C}_0} \bar{Q}}{\sqrt{2}}. \quad (44)$$

Solution (40) contains two independent integration constants, \bar{C}_0 and \bar{Q} . These are, however, related by the boundary condition on the conductor, $\bar{r} = 1$. In Fig. 1 this relation is exhibited: \bar{C}_0 is shown as a function of \bar{Q} . The solid and dashed curves correspond to the boundary conditions (D) and (N) at the surface of the conductor, respectively, where each curve is truncated at $\bar{C}_0 = 0.9995$.

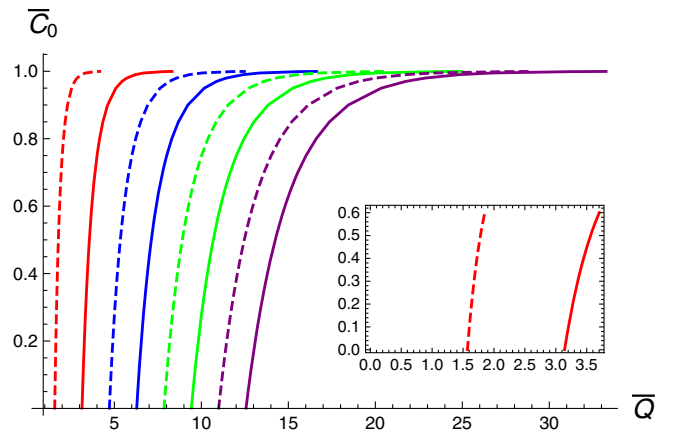


FIG. 1. \bar{C}_0 as a function of \bar{Q} for the scalar field solution (40), in the model with an inverse quartic coupling (37), and for the boundary condition (D) (solid curves) or the boundary condition (N) (dashed curves). Red, blue, green, and purple curves correspond to solutions with 0-, 1-, 2- and 3-nodes, respectively. Each curve is truncated at $\bar{C}_0 = 0.9995$. Inset: enlargement of the 0-node solutions around the origin.

Red, blue, green, and purple curves correspond to the solutions with 0-, 1-, 2- and 3-nodes, respectively. The inset provides a magnification of the 0-node solutions around the bifurcation point from the Coulomb solution. The critical couplings (23) given by $\bar{Q}_N = \pi/2 \approx 1.5708$ and $\bar{Q}_D = \pi \approx 3.1416$, are in good agreement with the bifurcation points of the 0-node solutions, given in the inset of the figure.

Figure 1 shows that for the 0-nodes solutions, \bar{C}_0 approaches unity very quickly as a function of \bar{Q} . The limit $\bar{C}_0 \rightarrow 0$ reproduces Eq. (21) giving rise to the solutions (22) which correspond to vertical straight lines starting from the bifurcation points in Fig. 1.

B. The inverse cosine coupling

As a second nonlinear completion of the coupling (26) (which led to the linear model), we consider the inverse cosine coupling (45),

$$f(\phi) = \frac{1}{\cos(\sqrt{2a}\phi)} \stackrel{(11)}{\Rightarrow} \mathcal{V}_{\text{eff}}(\phi_0) = -\frac{Q^2}{2} \cos(\sqrt{2a}\phi_0). \quad (45)$$

The effective potential has now infinite countable set of extrema, at

$$\sin(\sqrt{2a}\phi_0) = 0. \quad (46)$$

The electric field (9) is now

$$A'_0 = \frac{Q}{r^2} \cos(\sqrt{2a}\phi_0). \quad (47)$$

The scalar field equation (12) becomes a one-dimensional sine-Gordon equation. Its solution, satisfying the boundary condition $\phi_0(r \rightarrow \infty) = 0$, can be expressed in terms of the Jacobi amplitude, denoted “am(k, m)” as:

$$\bar{\phi}(r) = \text{am}\left(\sqrt{\frac{C_1 \bar{Q}}{2\bar{r}}}, \frac{2}{C_1}\right), \quad (48)$$

where now

$$\bar{\phi} := \sqrt{\frac{a}{2}} \phi_0, \quad (49)$$

and C_1 is an integration constant. The function (48) takes a real value and satisfies the boundary condition $\bar{\phi}(\bar{r} \rightarrow \infty) = 0$ for any $C_1 > 0$. On the other hand, we exclude the case of $C_1 < 0$, for which Eq. (48) becomes imaginary. In order to impose the (D) or (N) boundary conditions (20), we restrict to $0 \leq C_1 \leq 2$ (see Fig. 2). In the lower limit, $C_1 \rightarrow 0$, one verifies that $\phi \rightarrow 0$, which represents the

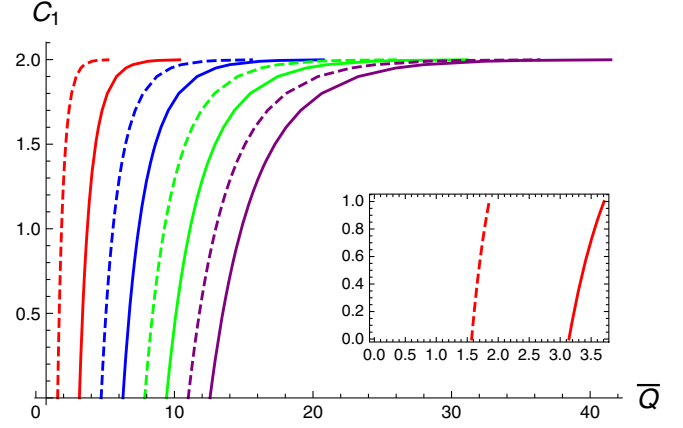


FIG. 2. C_1 as a function of \bar{Q} for the scalar field solution (48), in the model with an inverse quartic coupling (45), and for the boundary condition (D) (solid curves) or the boundary condition (N) (dashed curves). Red, blue, green, and purple curves correspond to solutions with 0-, 1-, 2-, and 3-nodes, respectively. Each curve is truncated at $C_1 = 1.999$. Inset: enlargement of the 0-node solutions around the origin.

Coulomb solution. In the upper limit, $C_1 = 2$, the solution becomes

$$\bar{\phi}(r) = 2 \arctan^{-1}(e^{\bar{Q}/\bar{r}}) - \frac{\pi}{2}, \quad (50)$$

which coincides with the 1-soliton solution of the sine-Gordon equation with no time dependence [62].

In the large distance limit, (48) yields

$$\bar{\phi} = \frac{\bar{Q}_s}{\bar{r}} + \mathcal{O}\left(\frac{1}{\bar{r}^2}\right). \quad (51)$$

Thus, the scalar charge is $\bar{Q}_s := \sqrt{C_1 \bar{Q}}/\sqrt{2}$.

As in the previous subsection, C_1 is related to \bar{Q} by imposing a boundary condition at the surface of the conductor $\bar{r} = 1$. This relation is exhibited in Fig. 2, where C_1 is shown as a function of \bar{Q} for the boundary condition (D) (solid curves) and the boundary condition (N) (dashed curves). The red, blue, green, and purple curves correspond to the solutions of 0-, 1-, 2-, and 3-nodes, respectively. Each curve is truncated at $C_1 = 1.999$. The inset corresponds to the magnification of the 0-node solutions around the origin. Thus, the behavior of this model is similar to the previous one, defined by (37). The threshold coupling for which the Coulomb solution becomes unstable is given by Eq. (23).

C. Stability of the scalarized charged conductor in the nonlinear models

In order for the scalarized solution to be the end state emerging from the evolution of the (unstable) Coulomb solution, it must be the preferred state. Let us first assess energetic preference, and then the dynamical one.

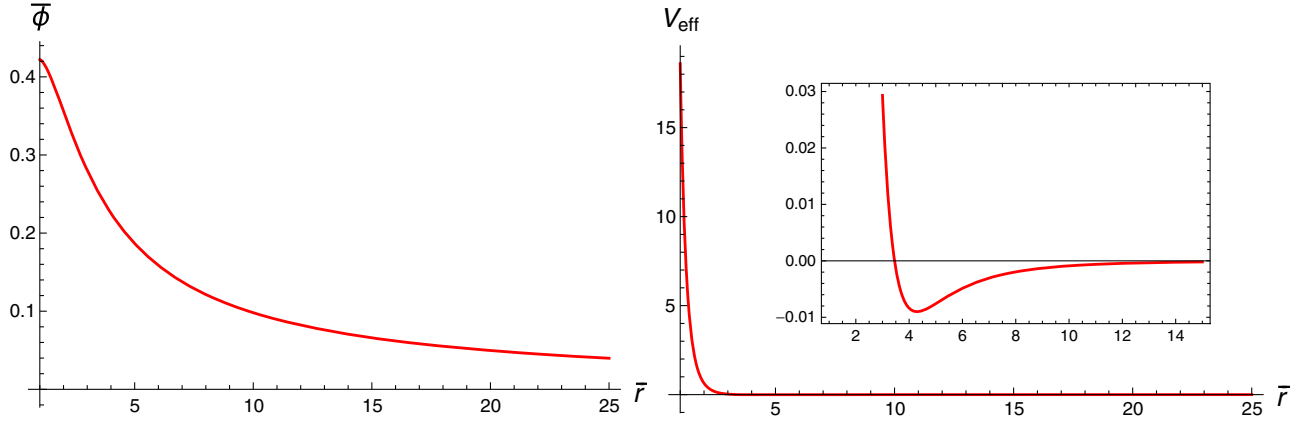


FIG. 3. $\bar{\phi}_0$ (left panel) and V_{eff} (right panel) as functions of \bar{r} for the model (37) and the boundary condition (N). We set $\bar{Q} = 3.22738$ and $\bar{C}_0 = 0.995$. The inset shows an enlargement of V_{eff} to clearly exhibit the negative region.

The energy of the scalarized solution can be computed as

$$\begin{aligned}
 E &= 4\pi \int_{r_s}^{\infty} dr r^2 \left[\frac{1}{2} \phi'^2 + \frac{1}{2} f(\phi) (A'_0)^2 \right] \\
 &= 2\pi \int_{r_s}^{\infty} dr \frac{Q^2}{r^2} \left[C_i - 1 + \frac{2}{f(\phi(r))} \right], \\
 &= E_{\text{Coulomb}} (C_i - 1) + 4\pi Q^2 \int_{r_s}^{\infty} dr \frac{1}{r^2 f(\phi(r))}, \quad (52)
 \end{aligned}$$

where $i = 0, 1$, to account simultaneously for each of the two models in the previous subsections, and $C_0 := \bar{C}_0/k^2$.

To illustrate which one is the *energetically* preferred state, let us take concrete values for C_i . If we choose $\bar{C}_0 = 1$, for the inverse quartic polynomial model, we obtain

$$\begin{aligned}
 E &= E_{\text{Coulomb}} - 2\pi \bar{Q}^2 \frac{r_s}{a} \\
 &\times \left\{ 1 - \frac{2\sqrt{2}}{\bar{Q}} \tanh\left(\frac{\bar{Q}}{\sqrt{2}}\right) \left(1 - \frac{1}{3} \tanh^2\left(\frac{\bar{Q}}{\sqrt{2}}\right) \right) \right\}. \quad (53)
 \end{aligned}$$

For the solutions satisfying $\bar{Q}/\sqrt{2} > 1.2964$, the second term is negative, so that $E < E_{\text{Coulomb}}$. If we choose $C_1 = 2$, for the inverse cosine model, then we obtain

$$E = E_{\text{Coulomb}} - 4\pi \bar{Q} \frac{r_s}{a} \left(\frac{4}{1 + e^{2\bar{Q}}} + \bar{Q} - 2 \right). \quad (54)$$

If $\bar{Q} > 1.9150$, the second term is, again, negative. Thus, this illustrates that, at least in some part of the parameter space, the scalarized solutions have less energy than the Coulomb solution.

Next, we consider the linear stability of the scalarized solutions to assess if they can be, dynamically, the end state. For the models (37) and (45) with the nonlinear corrections in the scalar field equations of motion,

introducing the dimensionless quantities (18) the effective potential for the radial perturbations (34) is given by, respectively,

$$V_{\text{eff}}(\bar{r}) = -\frac{\bar{Q}^2}{\bar{r}^4} (1 - 3\bar{\phi}_0(\bar{r})^2), \quad (55)$$

$$V_{\text{eff}}(\bar{r}) = -\frac{\bar{Q}^2}{\bar{r}^4} \cos(2\bar{\phi}_0(\bar{r})). \quad (56)$$

This informs us that when the amplitude of the background scalar field $\bar{\phi}_0(r)$ becomes sufficiently large, the effective potential becomes non-negative in part of the spacetime region, unlike that for the linear model (35), which is everywhere negative. We can therefore understand the (potential) healing effect of the nonlinear terms, which may indeed quench the instability.

Next, we investigate if for concrete solutions the above effective potential indeed becomes positive. In Figs. 3 and 4 we show the plots for the effective potential for radial perturbations for the models (37) and (45); $\bar{\phi}_0$ and V_{eff} are shown as functions of \bar{r} for illustrations of scalarized solutions satisfying the boundary condition (N).

Figures 3 and 4 confirm that, for both the models, the effective potential becomes positive in the vicinity of the conductor and hence there is no tachyonic mode trapped in the vicinity of $r = r_s$. This confirms the healing potential of the nonlinear models, concerning the instability of the scalarized solution. However, the effective potential is still (mildly) negative at larger radius (see the insets in Figs. 3 and 4).⁵ Thus, we must further analyze the perturbation equation in order to establish the stability of the scalarized solutions.

By the same procedure that was followed for the linear model, introducing the dimensionless quantities (18), we

⁵This is unlike what happens for black hole scalarization, see e.g., [49].

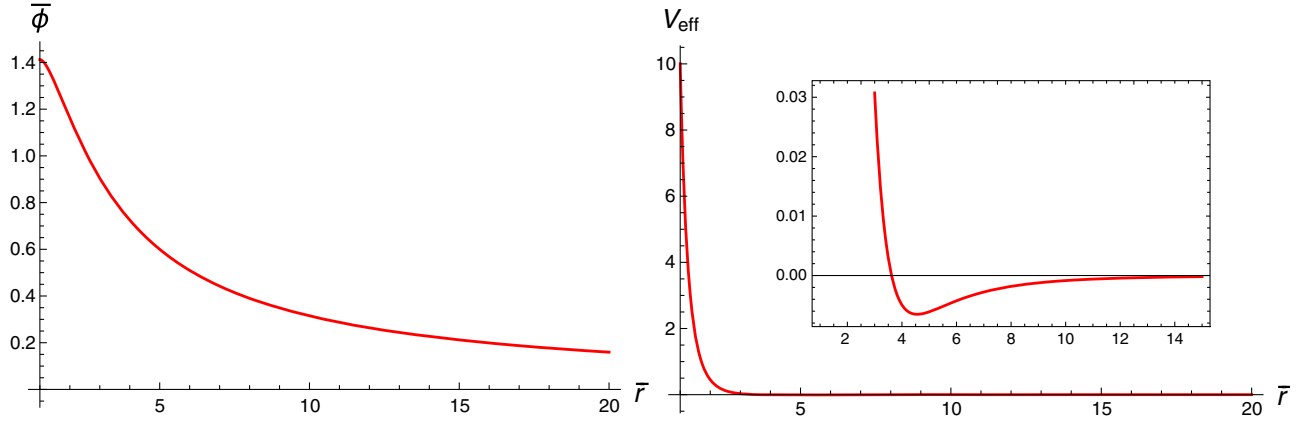


FIG. 4. Same as Fig. 3 but now for the model (45) and the boundary condition (N). We set $\bar{Q} = 3.24484$ and $C_1 = 1.95$.

have obtained the scalar field perturbation equation around the scalarized solution in a Schrödinger-like form:

$$-\Phi''_{\omega}(\bar{r}) + V_{\text{eff}}\Phi_{\omega} = \bar{\omega}^2\Phi_{\omega}, \quad (57)$$

where now the prime denotes the derivative with respect to \bar{r} and V_{eff} is given by Eqs. (55) or (56), and we have introduced $\bar{\omega} := \omega r_s$.

To solve (57) we need to impose appropriate boundary conditions for the perturbations. A first possibility are standard quasinormal mode-like boundary conditions: we impose outgoing (ingoing) boundary conditions at infinity (on the conductor's surface), for the perturbations:

$$(\Phi'_{\omega} - i\bar{\omega}\Phi_{\omega})|_{\bar{r}=\infty} = 0, \quad \text{outgoing condition,} \quad (58)$$

$$(\Phi'_{\omega} + i\bar{\omega}\Phi_{\omega})|_{\bar{r}=1} = 0, \quad \text{ingoing condition.} \quad (59)$$

This generalizes the radiative boundary conditions introduced before for $\omega = 0$, cf. Eq. (24), since $\Phi := r\phi_1$, cf. Appendix A, and justifies the terminology “radiative” boundary condition. As two other possibilities, we impose a Dirichlet (Neumann) boundary condition at the conductor's surface instead of an ingoing boundary condition, which is equivalent to the boundary conditions (D), (N) used for obtaining the static solution and the numerical simulations in Sec. V and hence useful for interpreting the result of the latter,

$$(\Phi'_{\omega} - i\bar{\omega}\Phi_{\omega})|_{\bar{r}=\infty} = 0, \quad \text{outgoing condition,} \quad (60)$$

$$\Phi_{\omega}|_{\bar{r}=1} = 0, \quad \text{Dirichlet condition.} \quad (61)$$

$$(\Phi'_{\omega} - i\bar{\omega}\Phi_{\omega})|_{\bar{r}=\infty} = 0, \quad \text{outgoing condition,} \quad (62)$$

$$\left(\frac{\Phi_{\omega}}{r}\right)' \Big|_{\bar{r}=1} = 0, \quad \text{Neumann condition.} \quad (63)$$

In order to obtain the parameter space regions of (in) stability, we follow the method already used in Sec. II B.

Focusing on the static solutions (for the perturbations, $\bar{\omega} = 0$) and confirming that the solution does not cross the horizontal axis, we obtain the stability regions in Fig. 5. The instability regions are highlighted as the shaded blue regions in the different panels. In this analysis we have left the constants \bar{C}_0 and C_1 as arbitrary (in terms of \bar{Q}), in order to accommodate an arbitrary boundary condition for the scalarized solution. Then, to identify the (background) scalarized solutions with the different boundary conditions, in the different plots we added solid, dashed, and dot-dashed lines, corresponding to the scalarized solutions with boundary conditions (D), (N), and (R),⁶ respectively. The colors of these lines define the number of nodes of the solution, being the same as in Figs. 1 and 2 for (D) and (N) boundary conditions; in particular red and blue colors correspond to 0 and 1 node solutions, respectively.

Figure 5 (left panels) exhibits the behavior for the inverse quartic polynomial model and ingoing/radiative boundary conditions (top), (D) boundary conditions (middle), and (N) boundary conditions (bottom) for the perturbations. Each panel also shows the static scalarized solutions with corresponding boundary conditions (with 0, 1, or 2 nodes) respectively. The gray dashed line shows the result obtained from Eq. (52) about energetic preference: scalarized solutions to the right (left) have less (more) energy than the Coulomb solution with the same charge. The following conclusions can be drawn from these three panels:

- (i) Top panel: the (in)stability boundary line crosses $\bar{C}_0 = 1$ at some value of \bar{Q} , below which all scalarized solutions with $\bar{C}_0 \leq 1$ are unstable. This may be interpreted from the fact that the effective potential approaches the Coulomb solution's one for small charge or coupling and, as we have seen, the Coulomb solution is always unstable against radiative boundary conditions. On the other hand, the

⁶The last ones require the field value that extremizes the effective potential to hold at the conductor; they correspond to the radiative boundary condition used in the numerical evolution of the next section.

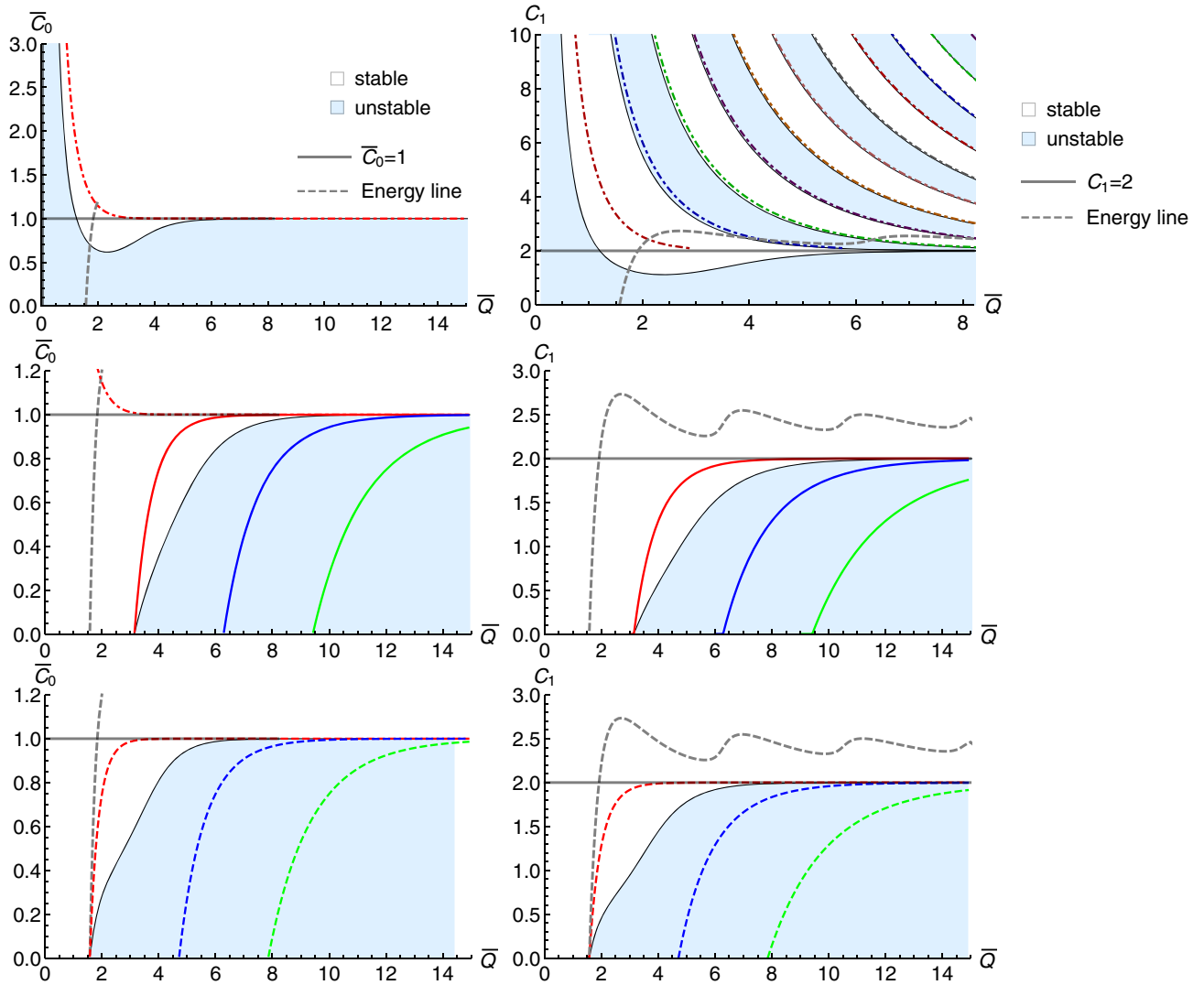


FIG. 5. (Un)stable region of the scalarized solution in the two considered nonlinear models: Eq. (40) (left panels) and Eq. (48) (right panels). The boundary condition for the perturbation, at the conductor, is taken as the ingoing condition (top panels), Dirichlet condition (middle panels), and Neumann condition (bottom panels). The gray dashed lines (“energy line”) correspond to the critical value of \bar{Q} above which the scalarized solution has less energy than the Coulomb solution. The curved lines correspond to the static solutions satisfying the boundary conditions (D) (solid curves), or (N) (dashed curves), or (R) [also enforcing a potential minimum Eqs. (38) and (46) at the surface $\bar{r} = 1$] (dot-dashed curves). The correspondence between the line color and the node of the solution is the same as Figs. 1 and 2 for the boundary condition (D) and the boundary condition (N). For the potential minimum condition, we can write the solution of Eq. (46) as $\bar{\phi} = n\pi/2$, where n is an integer, and red, blue, green, purple... lines correspond to the solution with $n = 1, 2, 3, \dots$

boundary line approaches $\bar{C}_0 = 1$ at large charge or coupling.

- (ii) Middle and bottom panels: there is no unstable region above $\bar{C}_0 = 1$. Since the limit $\bar{C}_0 \rightarrow 0$ yields the Coulomb solution, the boundary line of the stability reproduces the previous analysis of the stability of the Coulomb solution: along the \bar{Q} axis, instability holds for $\bar{Q} > \bar{Q}_D = \pi$ (middle) and $\bar{Q} > \bar{Q}_N = \pi/2$ (bottom) [see Eq. (23)].
- (iii) All panels: only the 0-nodes scalarized solutions fit into the stable areas. This fits the generic expectation

that 0-node solutions are fundamental states and nonzero node solutions are excited states. Moreover, there is no implication (either way) between energy preference and dynamical stability.

Figure 5 (right panels) exhibits the behavior for the inverse cosine model and ingoing/radiative boundary conditions (top), (D) boundary conditions (middle), and (N) boundary conditions (bottom) for the perturbations. Again, some observations can be made:

- (i) Top panel: stable regions now alternate with unstable regions, due to the oscillating behavior

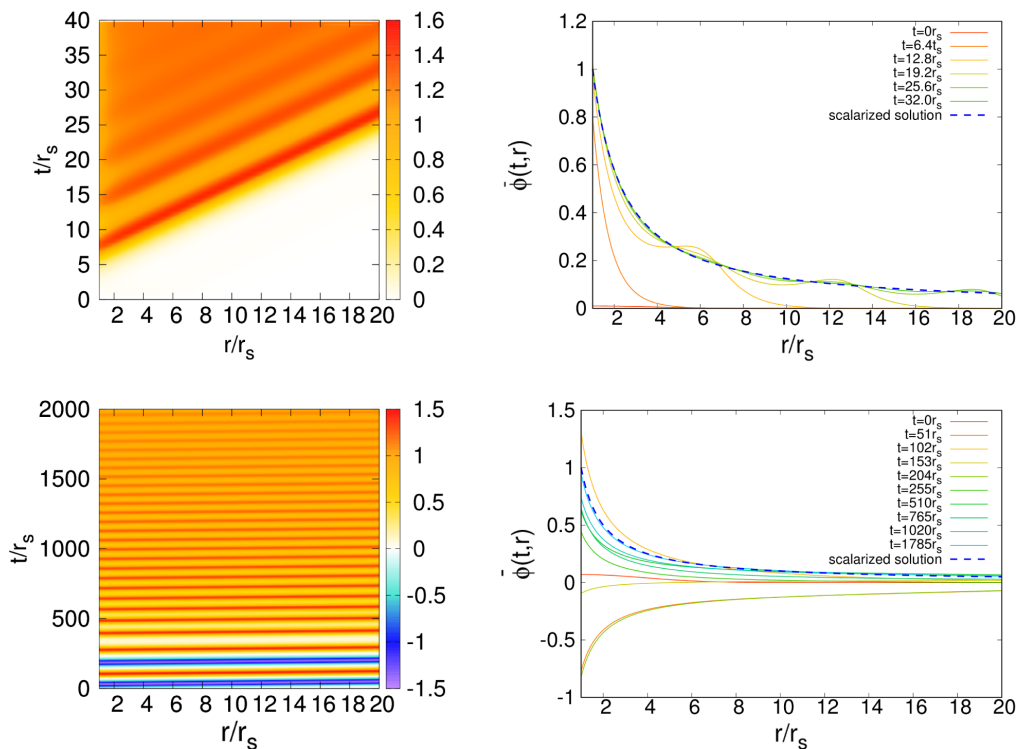


FIG. 6. Two illustrations of time evolution in the inverse quartic polynomial model (37) with a radiative boundary condition (68)–(69). The initial data are a momentarily static Gaussian initial data (75) with $w = 4r_s$ and (top panels) $A\sqrt{\frac{ak^2}{2}} = 0.01$, $\bar{Q} = \sqrt{2}$ or (bottom panels) $A\sqrt{\frac{ak^2}{2}} = 0.07$, $\bar{Q} = 0.1\sqrt{2}$. Left panels: contour plot of the time evolution for $r\bar{\phi}/r_s$. Right panels: radial profiles from snapshots of the time evolution at different times and of the final scalarized solution (blue dashed line) which has $\bar{C}_0 \simeq 1.559$ (top right panel) or $\bar{C}_0 \simeq 100$ (bottom right panel).

of the effective potential, Eq. (45). Now, the radiative boundary condition [imposing also that exterminizes the effective potential (45) and thus obey (46) [equivalently, $\sin(2\bar{\phi}_0) = 0$] at the conductor, a requirement that will become clear in the next section] leads to $2\bar{\phi}_0 = n\pi$, where n is an integer. We can see that odd (even) n solutions lie in the stable (unstable) region. The energy preference line (the gray dashed line) behaves in a similar manner to the previous model for small \bar{Q} . However, due to the existence of unstable regions, the line roughly follows the boundary lines of the stability.

- (ii) Middle and bottom panels: there is no unstable region above $C_1 = 2$ and only 0-nodes solutions are stable. Again, the limit $C_1 \rightarrow 0$ yields the Coulomb solution; as such along the \bar{Q} axis instability occurs for $\bar{Q} > \bar{Q}_D = \pi(\bar{Q}_N = \pi/2)$ [see Eq. (23)].

To conclude: scalarized solutions with 0-nodes are possible final states of the scalarization instability of the Coulomb solution, in some ranges of the parameter space and for the different boundary conditions. The next section

will confirm this expectation via fully dynamical numerical evolutions.

V. TIME EVOLUTIONS

In the previous sections we have constructed scalarized conducting spheres in MS models, under specific choices of the nonminimal coupling function $f(\phi)$, namely Eqs. (26), (37), and (45). Then, we have investigated the linear stability of these solutions. For the nonlinear models (37) and (45), this analysis suggests that there are stable scalarized solutions in appropriate parameter regions, which could be the end state of the evolution of the unstable Coulomb conducting sphere, when embedded in the corresponding MS model. In order to establish if such scalarized solutions are indeed the final state of the dynamics, in this section we numerically solve the time evolution of the MS system. This will also allow us to investigate *how* the final state is reached.

Performing time evolution implies dropping the stationarity assumption, which has been used in the previous sections to compute the scalarized solutions. Nonetheless, even dropping this assumption, the only relevant component of the electromagnetic field remains the radial

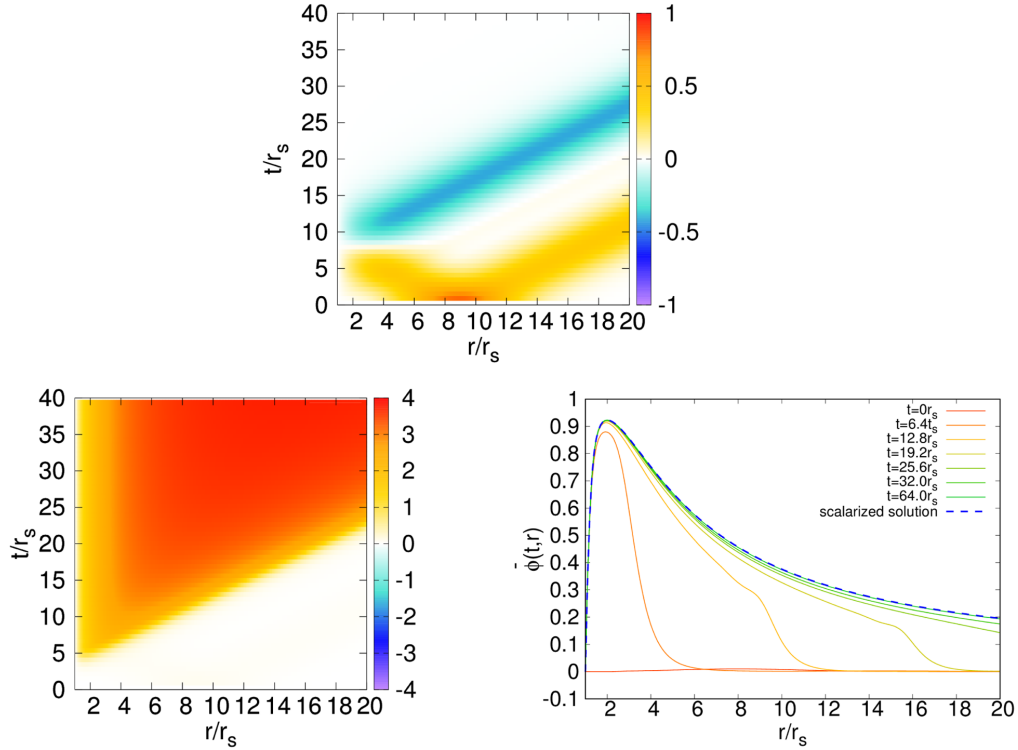


FIG. 7. Two illustrations of time evolution in the inverse quartic polynomial model (37) with the boundary condition (D) (71). The initial data are a momentarily static Gaussian initial data (75) with $w = 4r_s$, $r_0 = 8r_s$, and (top panel) $A\sqrt{\frac{ak^2}{2}} = 0.1$, $\bar{Q} = \sqrt{2}$ or (bottom panels) $A\sqrt{\frac{ak^2}{2}} = 0.01$, $\bar{Q} = 4\sqrt{2}$. Top and bottom left panels: contour plot of the time evolution for $r\bar{\phi}/r_s$. Bottom right panel: radial profiles from snapshots of the time evolution at different times and of the final scalarized solution (blue dashed line) which has $\bar{C}_0 \simeq 0.978$. In the simulation of the top panel, the final state is the Coulomb solution.

one, $E^r = F^{tr}$, due to spherical and gauge symmetries. Therefore, the expression for the electromagnetic field is the same as in the static case and we obtain the scalar field equation from Eqs. (3) and (4) as follows:

$$-\ddot{\phi}(t, r) + \phi''(t, r) + \frac{2}{r}\phi'(t, r) = -\frac{Q^2 f_\phi(\phi)}{2r^4 f(\phi)^2}. \quad (64)$$

In the following, we shall solve the time evolution of Eq. (64) numerically around the conductor. To do so, let us define new variables φ , π as

$$\varphi := r\phi, \quad \pi := r\dot{\phi}. \quad (65)$$

Using these variables, Eq. (64) is brought to a first order system of two equations:

$$\dot{\varphi} = \pi, \quad (66)$$

$$\dot{\pi} = \varphi'' + \frac{Q^2 f_\phi(\varphi/r)}{2r^3 f(\varphi/r)^2}. \quad (67)$$

The conducting sphere's radius is $r_s (> 0)$; thus, the physical domain within the radial direction is $[r_s, \infty[$. The boundary condition imposed on the scalar field at

the conductor must be chosen, as in the previous sections. In the subsequent analysis, we consider three different kind of boundary conditions during the numerical evolution. The first boundary condition is

$$\dot{\pi} = \pi' + \frac{f_\phi Q^2}{f^2 2r^3}, \quad (68)$$

$$\dot{\varphi} = \pi, \quad (69)$$

on the surface of the conductor. At the end state ($\dot{\pi} = \dot{\varphi} = 0$), (68)–(69) implies that, at the conductor,

$$\frac{f_\phi}{f^2} = 0. \quad (70)$$

This boundary condition for the perturbation around the Coulomb solution is compatible with Eq. (24)⁷ and hence it

⁷To check this point, let us consider perturbations around the Coulomb solution under the boundary condition given by Eq. (69). The perturbations can be expressed as a sum of an ingoing mode $\varphi_+(t+r)$ and an outgoing mode $\varphi_-(t-r)$. From the boundary condition, we can show the outgoing mode vanishes on the conductor; thus the boundary condition Eq. (69) corresponds to Eq. (59).

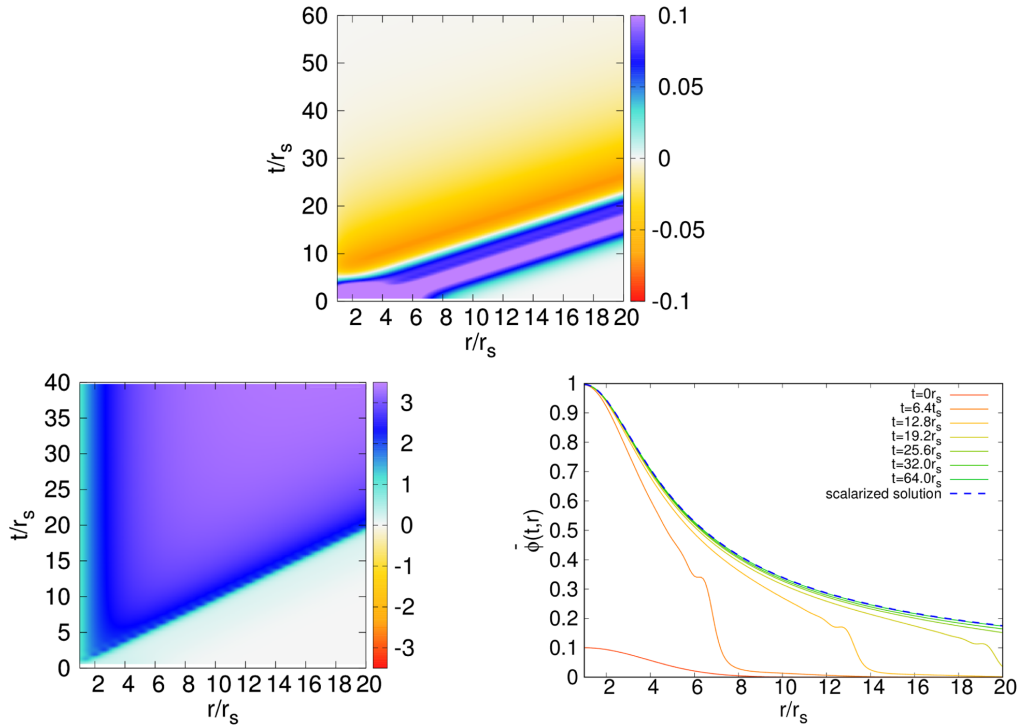


FIG. 8. Two illustrations of time evolution in the inverse quartic polynomial model (37) with the boundary condition (N) (73). The initial data are a momentarily static Gaussian initial data (75) with $r_0 = r_s$ and (top panel) $A\sqrt{\frac{ak^2}{2}} = 0.1$, $w = 4r_s$, $\bar{Q} = 1$ or (bottom panels) $A\sqrt{\frac{ak^2}{2}} = 0.1$, $w = 4r_s$, $\bar{Q} = 5$. Top and bottom left panels: contour plot of the time evolution for $r\bar{\phi}/r_s$. Bottom right panel: radial profiles from snapshots of the time evolution at different times and of the final scalarized solution (blue dashed line) which has $\bar{C}_0 \simeq 0.99995$. In the simulation of the top panel, the final state is the Coulomb solution.

is radiative/outgoing. Moreover, (70) informs us that at the endpoint the scalar field is at an extremum of the effective potential (11) at the surface of the conductor. This makes direct contact with the stability analysis discussed in the top panels of Fig. 5. The second boundary condition is a Dirichlet (D) boundary condition:

$$\dot{\pi} = 0, \quad \dot{\phi} = \pi, \quad (71)$$

on the surface of the conductor. In this case, the scalar field value is always zero at the conductor. Then, scalarized solutions with the boundary condition (D) are realized as the end states. This makes direct contact with the stability analysis discussed in the bottom panels of Fig. 5. The third boundary condition is a Neumann (N) boundary condition:

$$\partial_r \left(\frac{\phi}{r} \right) = 0, \quad (72)$$

$$\partial_r \left(\frac{\pi}{r} \right) = 0, \quad (73)$$

on the surface of the conductor. In this case, the radial derivative of the scalar field ϕ is always zero at the

conductor, and scalarized solutions with the boundary condition (N) are realized as the end states. In the asymptotic region, we impose an outgoing boundary condition.

To perform the evolution a C++ time evolution code was written. Time integration is calculated using a fourth order Runge-Kutta method, and spatial derivatives are evaluated using a fourth order central difference scheme. Ghost zones (grid areas beyond inner boundary and outer boundary) for each boundary are evaluated using a second order Lagrange extrapolation. The numerical code has, overall, a second order convergence.

The general behavior of the evolution does not strongly depend on the chosen initial data. Here, we use momentarily static Gaussian initial data:

$$\phi(r, t = 0) = A r e^{-\left(\frac{r-r_0}{w}\right)^2}, \quad (74)$$

$$\pi(r, t = 0) = 0, \quad (75)$$

where A , r_0 , and w are the amplitude, position, and width of the Gaussian pulse, respectively. In the graph of the simulation, we use $\bar{\phi} = \sqrt{\frac{ak^2}{2}}\phi$ for the inverse quartic

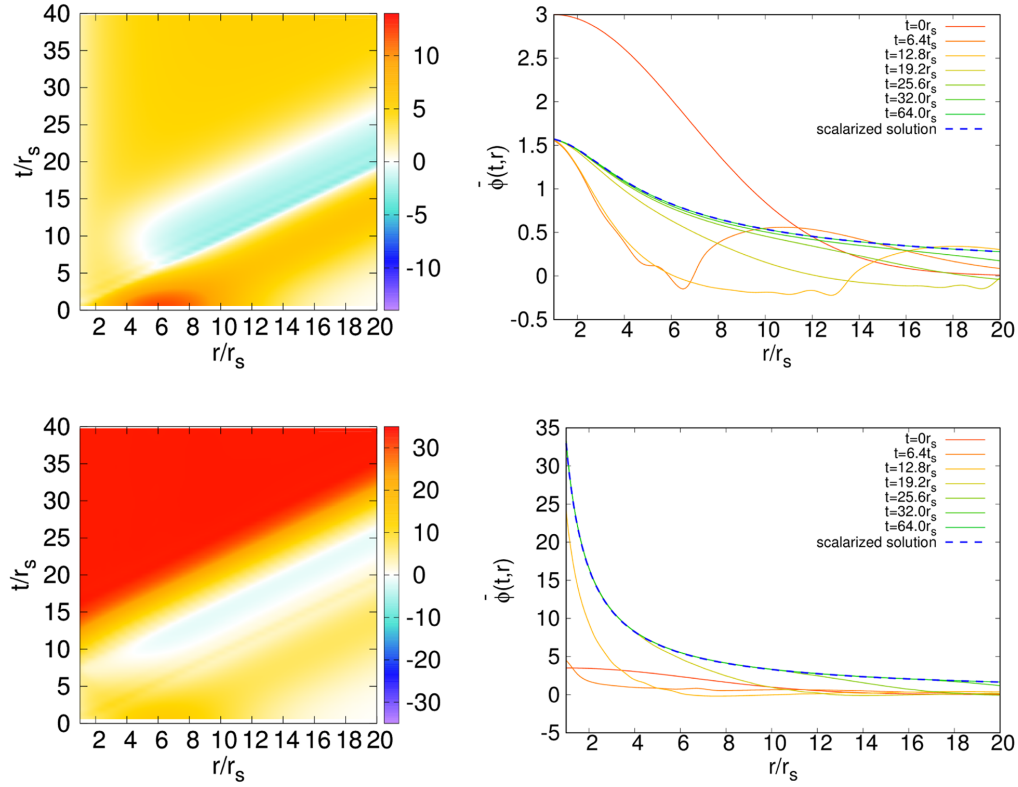


FIG. 9. Two illustrations of time evolution in the inverse cosine model (45) with a radiative boundary condition (68)–(69). The initial data are a momentarily static Gaussian initial data (75) with $r_0 = r_s$, $w = 8.0r_s$, $\bar{Q} = 4\sqrt{2}$ and (top panels) $A\sqrt{a/2} = 3.0$, or (bottom panels) $A\sqrt{a/2} = 3.5$. Left panels: contour plot of the time evolution for $r\bar{\phi}/r_s$. Right panels: radial profiles from snapshots of the time evolution at different times and of the final scalarized solution (blue dashed line) which has $C_1 \simeq 2.0004$ (top right panel) or $C_1 \simeq 69.0$ (bottom right panel).

polynomial model, and $\bar{\phi} = \sqrt{a/2}\phi$ for the inverse cosine model.

A. Numerical results

We shall now report the results obtained from the numerical simulations. We shall use the scaled charge \bar{Q} , cf. Eq. (18), in order to compare the results with the scalarized solutions obtained in the previous sections.

1. Inverse quartic polynomial model

Let us first report the evolutions for the inverse quartic polynomial model (37) and for the radiative boundary conditions (68)–(69). We have observed two qualitatively distinct types of evolution, both leading to a scalarized solution. For large \bar{Q} and a small initial perturbation, scalarization proceeds in a direct manner. That, is the tachyonic instability sets in, the scalar field grows, it briefly oscillates at each radial position but quickly settles down to the scalarized solution. This is illustrated by the top panels in Fig. 6, for which the initial parameters are $(A, w) = (0.01\phi_{\min}, 4r_s)$ and the charge is \bar{Q} is $0.1\sqrt{2}$. The left panel shows that the scalarization starts in the vicinity of the

sphere and propagates outwards. This is expected, as the strength of the tachyonic instability of the Coulomb solution decays with r . The right panel shows the radial profile, obtained from the evolution at different time slices, which can be observed to approach the one of a scalarized solution with $\bar{C}_0 = 1.559$, such that the relaxation time is longer for larger r , as expected. Moreover, the final value of $\bar{\phi}$ at the conductor is $\bar{\phi} = 1$ corresponding to an extremum of the effective potential, as expected from the radiative boundary condition [cf. (38) and (41)]. Thus, the final scalarized solution interpolates (from the conductor to infinity) between two extrema of the effective potential: $\bar{\phi} = 1$ and $\bar{\phi} = 0$.

The bottom panels in Fig. 6 show a qualitatively different behavior. After the tachyonic instability settles in, the scalar field initially evolves in the “wrong” direction; in this case it acquires negative values. Then, it oscillates around $\phi = 0$, losing energy and decreasing the oscillation amplitude. At some point, the scalar field stops crossing $\phi = 0$, and starts to oscillate around its final nontrivial value (in this case a positive value), approaching the scalarized solution. This occurs for a smaller value of the charge, $\bar{Q} = 0.1\sqrt{2}$, and the scalarization process takes

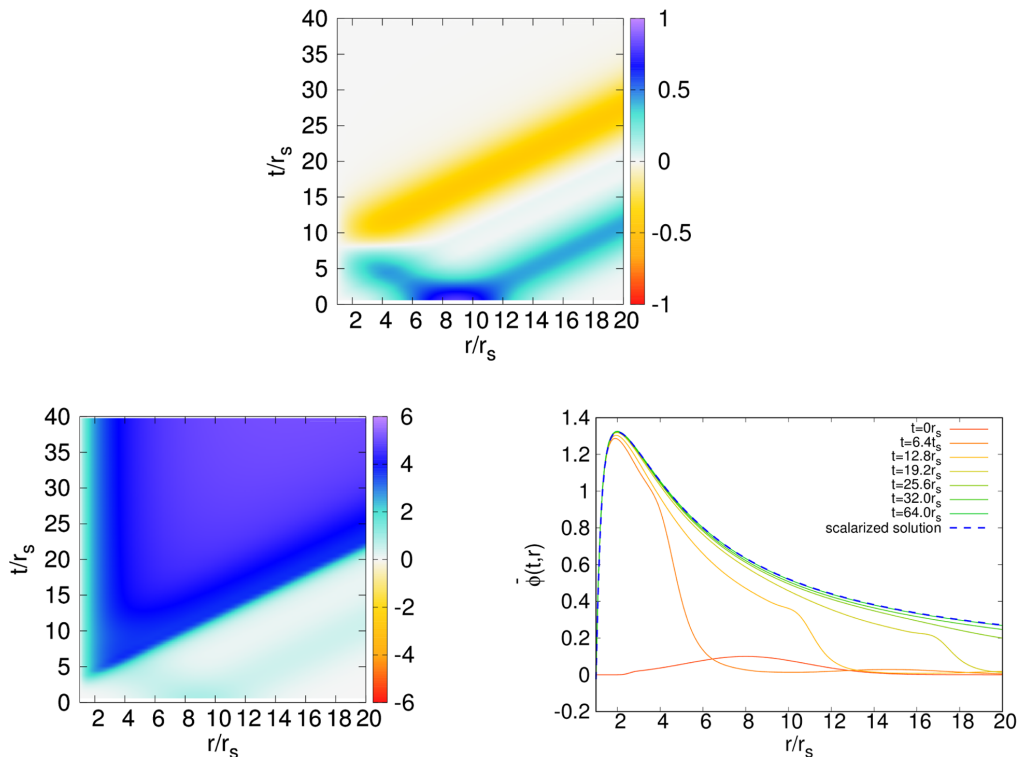


FIG. 10. Two illustrations of time evolution in the inverse cosine model (45) with the boundary condition (D) (71). The initial data are a momentarily static Gaussian initial data (75) with $r_0 = 8r_s$ and (top panel) $A\sqrt{\frac{a}{2}} = 0.1$, $w = 4r_s$, $\bar{Q} = \sqrt{2}$ or (bottom panels) $A\sqrt{\frac{a}{2}} = 0.1$, $w = 4r_s$, $\bar{Q} = 4\sqrt{2}$. Top and left bottom panels: contour plot of the time evolution for $r\bar{\phi}/r_s$. Right bottom panel: radial profiles from snapshots of the time evolution at different times and of the final scalarized solution (blue dashed line) which has $C_1 \simeq 1.88$. In the simulation of the top panel, the final state is the Coulomb solution.

considerably longer. Hence, in the left bottom panel of Fig. 6, the slope is barely visible (but still present), indicating that, as before, scalarization proceeds faster in the immediate vicinity of the conductor, then spreading towards larger values of r . Again, the final value at the conductor is $\bar{\phi} = 1$.

Next, we consider evolution for the inverse quartic polynomial model (37) and for the boundary condition (D) (71). We shall also illustrate two qualitatively different types of behavior. For sufficiently small charge, scalarization does not occur. This is illustrated by the simulation shown in the top panel of Fig. 7. One observes that the initial Gaussian perturbation produces some brief oscillations, but it essentially propagate outwards leaving behind an unscalarized Coulomb conductor. The lack of scalarization for small charges agrees with the analysis in Sec. II. For values of \bar{Q} that can fit the stability window in Fig. 5, we expect such a stable scalarized solution to form. This is illustrated by the bottom panel of Fig. 7. As in Fig. 6 we can see scalarization to proceed first in the vicinity of the conductor and then propagating outwards, this time in a direct manner, without oscillations. The end point can be identified as a scalarized solution with $\bar{C}_0 \simeq 0.978$ (right

panel). Observe that in this case the scalarized solution has $\bar{\phi} = 0$ at the conductor, and thus an extremum outside the conducting sphere, unlike the previous case, which is to be expected for the boundary conditions (D).

Finally, we consider evolutions for the inverse quartic polynomial model (37) and for the boundary condition (N) (73). The situation is analogous to that of the boundary condition (D) and we shall illustrate the same two qualitatively different behaviors. For a sufficiently small charge, scalarization does not occur. This is illustrated by the simulation shown in the top panel of Fig. 8. As in the previous case, the initial Gaussian perturbation produces some brief oscillations, but then dissipates away. This is in agreement with the analysis in Sec. II. But for values of \bar{Q} that can fit the stability window in Fig. 5, a scalarized solution forms. This is illustrated by the bottom panel of Fig. 8. As in the (D) case, scalarization proceeds in a rather monotonic fashion, without oscillations. Unlike the (D) case, however, the scalarized solution has no extremum outside the conducting sphere and has $\bar{\phi} = 1$ on the conductor, again, interpolating between the two extrema of the potential at the conductor ($\bar{\phi} = 1$) and infinity ($\bar{\phi} = 0$).

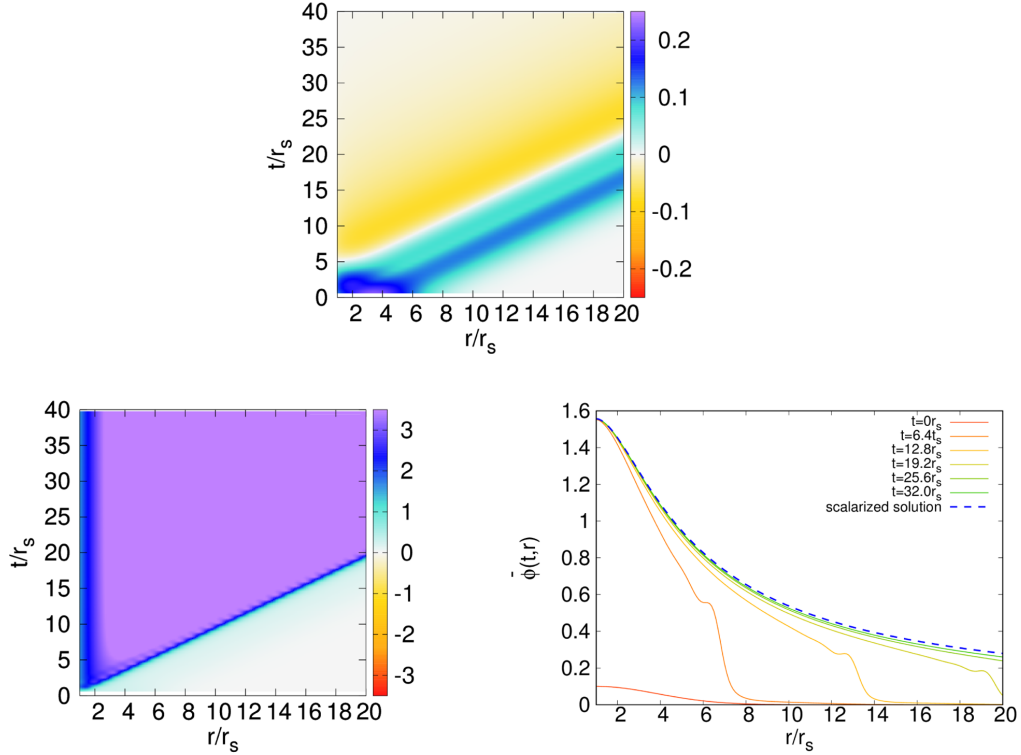


FIG. 11. Two illustrations of time evolution in the inverse cosine model (45) with the boundary condition (N) (73). The initial data are a momentarily static Gaussian initial data (75) with $r_0 = r_s$, $A\sqrt{q/2} = 0.1$, $w = 4r_s$ and (top panel) $\bar{Q} = \sqrt{2}$ or (bottom panels) $\bar{Q} = 4\sqrt{2}$. Top and left bottom panels: contour plot of the time evolution for $r\bar{\phi}/r_s$. Right bottom panel: radial profiles from snapshots of the time evolution at different times and of the final scalarized solution (blue dashed line) which has $C_1 \simeq 1.9996$. In the simulation of the top panel, the final state is the Coulomb solution.

2. Inverse cosine model

The evolution for the inverse cosine model (45) shows qualitatively similar features to the ones of the inverse quartic polynomial model (37) discussed in the previous subsection. These are illustrated for the radiative boundary conditions (68)–(69) in Fig. 9, for the boundary condition (D) in Fig. 10 and for the boundary condition (N) in Fig. 11. However, as is shown in Fig. 5, there are many stable scalarized solutions for the inverse cosine model with radiative boundary condition. This fact suggests that the final state may depend on the initial data. Figure 9 shows the two simulations with same \bar{Q} and different initial data, and the final scalarized solution is different. The simulations with the boundary conditions (D) and (N) in Figs. 10–11, each considers one case where scalarization occurs (bottom panel) and one where it does not (top panel).

VI. CONCLUSIONS

In this paper we have investigated spontaneous scalarization of a charged, conducting sphere in MS models. These models provide a simple, yet sufficiently rich, arena to study the spontaneous scalarization phenomenon, which has recently received considerable attention in the context

of strong gravity. Indeed, as we have explained, MS models can parallel features observed in the gravitational context, such as the coupling dependence on the existence (or not) of stable scalarized solutions. We believe that, in this respect, MS models are insightful on the behavior of the strong gravity models. On the other hand, MS models for scalarization of a conductor have features absent in strong gravity, such as the freedom to choose boundary conditions on the conductor, which in the gravitational case is fixed by requiring regularity at the event horizon. We have explored how the choice of boundary conditions impacts on the physical properties of the system, showing that such impact can be of considerable relevance. For instance, in the case of radiative boundary condition, the Coulomb solution is always unstable against scalarization, as shown in Sec. II. This is not the case for Dirichlet or Neumann boundary conditions, for which there are regions of stability and regions of instability.

The simplicity of the MS model allows also to perform numerical evolution of the scalarization phenomenon that is more challenging in the strong gravity case, although they have been performed for charged black holes in, e.g., [46,49]. The simulation herein helps us to understand how scalarization proceeds, when it occurs. We can distinguish two main behaviors: (i) direct scalarization, the scalar field

directly grows to form the scalarized solution, from the Coulomb solution; (ii) oscillating scalarization, the scalar field oscillates between the two scalarized solutions before settling into one of the solutions. If the charge is sufficiently small, on the other hand, the Coulomb solution is stable against scalarization (except for radiative boundary conditions).

For black hole scalarization in scalar-gravity systems, the boundary conditions in the limit of the event horizon are determined by regularity, and the derivative of the scalar field in the horizon limit is automatically fixed once the horizon field value is specified, and thus the flexibility seen in this paper has no parallelism. However, scalar-gravity systems allow the scalarization of sufficiently compact objects, albeit non-black holes, such as neutron stars. In this case, there will be a similar freedom in the choice of boundary conditions; for instance, in the case of neutron stars the derivative of the scalar field at the center of the star also depends on the energy density and pressure at the center of the star obeying the given equation of state. Therefore, one of the lessons from this work is that neutron star scalarization may depend on the choice of the boundary condition on the surface of such a compact object.

Adding the mass term $2m^2\phi^2$ or self-interaction potential $V(\phi)$ in the action (1) may significantly modify our results. As in the case of spontaneous scalarization of neutron stars [63–65] and black holes [22], adding the mass term to the action (1) may change the asymptotic behavior of the scalar field, while it may be possible to stabilize scalarized solutions even in the linear model by including appropriate self-interactions. The investigation of these generalized models is, however, beyond the scope of this paper.

Finally, as a direction for further research, the simplicity of the MS model may allow us to consider spontaneous scalarization beyond spherical symmetry, analyzing, in particular, the role of dissipation played by the electromagnetic radiation.

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APPENDIX A: THE RADIAL STABILITY ANALYSIS IN MS MODELS

To assess the linear stability of the solutions in MS models given by Eqs. (29) and (31), we consider small radial perturbations given by Eq. (32). We then expand the field equations up to first order in ϵ , and obtain the perturbed scalar field equation given by

$$-\ddot{\phi}_1 + \phi_1'' + \frac{2}{r}\phi_1' + \frac{Q^2 f_{\phi\phi}(\phi_0)}{2r^4 f(\phi_0)^2} \phi_1 - \frac{Qf_\phi(\phi_0)}{r^2 f(\phi_0)} (\dot{a}_r - a_r') = 0. \quad (\text{A1})$$

Integrating the t component of the perturbed vector field equation provides the relation

$$\dot{a}_r = a_r' + \frac{Qf_\phi(\phi_0)}{r^2 f(\phi_0)^2} \phi_1, \quad (\text{A2})$$

where we set the integration constant to be zero; using the last relation in the perturbed scalar field equation (A1) yields

$$-\ddot{\phi}_1 + \phi_1'' + \frac{2}{r}\phi_1' + \frac{Q^2}{2r^4 f(\phi_0)^3} \times [f(\phi_0)f_{\phi\phi}(\phi_0) - 2f_\phi(\phi_0)^2] \phi_1 = 0. \quad (\text{A3})$$

Introducing $\Phi := r\phi_1$, one can eliminate the first derivative terms

$$\Phi'' + \frac{Q^2}{2r^4 f(\phi_0)^3} [f(\phi_0)f_{\phi\phi}(\phi_0) - 2f_\phi(\phi_0)^2] \Phi = \ddot{\Phi}. \quad (\text{A4})$$

Fourier decomposing the field Φ , $\Phi = \int d\omega \Phi_\omega e^{-i\omega t}$, each mode obeys a time independent Schrödinger-type equation

$$-\Phi_\omega'' + V_{\text{eff}}(r)\Phi_\omega = \omega^2 \Phi_\omega, \quad (\text{A5})$$

where the effective potential $V_{\text{eff}}(r)$ is given by Eq. (34).

APPENDIX B: ON THE UNSTABLE MODES

Since the linear stability analysis depends on the chosen boundary conditions, in this Appendix we revisit the standard argument of the analysis for our setup. We start from Eq. (57) with a time independent Schrödinger-type form

$$-\Phi''_{\omega} + V_{\text{eff}}(\bar{r})\Phi_{\omega} = \bar{\omega}^2\Phi_{\omega}, \quad (\text{B1})$$

where “prime” here denotes the derivative with respect to \bar{r} . Multiplying Φ_{ω}^* and integrating from the surface of the conductor to infinity, we have

$$\begin{aligned} \bar{\omega}^2 \int_1^{\infty} d\bar{r} |\Phi_{\omega}|^2 &= -\Phi_{\omega}^* \Phi'_{\omega} |_{r \rightarrow \infty} + \Phi_{\omega}^* \Phi'_{\omega} |_{\bar{r} \rightarrow 1} \\ &+ \int_1^{\infty} d\bar{r} [|\Phi'_{\omega}|^2 + V_{\text{eff}}(\bar{r})|\Phi_{\omega}|^2], \\ &= -i\bar{\omega} |\Phi_{\omega}(\infty)|^2 + \Phi_{\omega}^* \Phi'_{\omega} |_{\bar{r} \rightarrow 1} \\ &+ \int_1^{\infty} d\bar{r} [|\Phi'_{\omega}|^2 + V_{\text{eff}}(\bar{r})|\Phi_{\omega}|^2], \quad (\text{B2}) \end{aligned}$$

where we have imposed the outgoing boundary condition at spatial infinity:

$$\Phi'_{\omega}(\infty) \rightarrow i\bar{\omega}\Phi_{\omega}(\infty). \quad (\text{B3})$$

On the other hand, as in Eqs. (59) and (61), we consider three different boundary conditions on the conductor:

$$\Phi_{\omega} |_{\bar{r} \rightarrow 1} = 0 \quad (D), \quad (\text{B4})$$

$$\left. \frac{d}{d\bar{r}} \left(\frac{\Phi_{\omega}(\bar{r})}{\bar{r}} \right) \right|_{\bar{r} \rightarrow 1} = 0 \quad (N), \quad (\text{B5})$$

$$\Phi'_{\omega}(1) \rightarrow -i\bar{\omega}\Phi_{\omega}(1) \quad (R). \quad (\text{B6})$$

For the boundary condition (R), Eq. (B2) reduces to

$$\begin{aligned} \bar{\omega}^2 \int_1^{\infty} d\bar{r} |\Phi_{\omega}|^2 + i\bar{\omega} (|\Phi_{\omega}(\infty)|^2 + |\Phi_{\omega}(1)|^2) \\ = \int_1^{\infty} d\bar{r} [|\Phi'_{\omega}|^2 + V_{\text{eff}}(\bar{r})|\Phi_{\omega}|^2]. \quad (\text{B7}) \end{aligned}$$

Since the right-hand side is real, the imaginary part of the left-hand side has to vanish, i.e.,

$$\bar{\omega}_R \left(2\bar{\omega}_I \int_1^{\infty} d\bar{r} |\Phi_{\omega}|^2 + (|\Phi_{\omega}(\infty)|^2 + |\Phi_{\omega}(1)|^2) \right) = 0, \quad (\text{B8})$$

and we can show that $\bar{\omega}_R$ vanishes for unstable modes with $\bar{\omega}_I > 0$. For the boundary condition (D), a similar analysis leads to

$$\bar{\omega}_R \left(2\bar{\omega}_I \int_1^{\infty} d\bar{r} |\Phi_{\omega}|^2 + |\Phi_{\omega}(\infty)|^2 \right) = 0, \quad (\text{B9})$$

and again $\bar{\omega}_R = 0$ for $\bar{\omega}_I > 0$. Finally, for the boundary condition (N), Eq. (B2) reduces to

$$\begin{aligned} \bar{\omega}^2 \int_1^{\infty} d\bar{r} |\Phi_{\omega}|^2 + i\bar{\omega} |\Phi_{\omega}(\infty)|^2 \\ = |\Phi_{\omega}(1)|^2 + \int_1^{\infty} d\bar{r} [|\Phi'_{\omega}|^2 + V_{\text{eff}}(\bar{r})|\Phi_{\omega}|^2]. \quad (\text{B10}) \end{aligned}$$

Since the right-hand side is real, a similar analysis leads to the same condition as (B9), and hence shows that $\bar{\omega}_R = 0$ vanishes for $\bar{\omega}_I > 0$. Thus, for all the boundary conditions considered, we have shown that the eigenvalues of the unstable modes are always purely imaginary: $\bar{\omega}_R = 0$.

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