

## Aspects of the dyonic Kerr-Sen-AdS<sub>4</sub> black hole and its ultraspinning version

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We explore some (especially, thermodynamical) properties of the dyonic Kerr-Sen-AdS<sub>4</sub> black hole and its ultraspinning counterpart, and check whether or not both black holes satisfy the first law and Bekenstein-Smarr mass formulas. To this end, new Christodoulou-Ruffini-like squared-mass formulas for the usual dyonic Kerr-Sen-AdS<sub>4</sub> solution and its ultraspinning cousin are deduced. Similar to the ultraspinning Kerr-Sen-AdS<sub>4</sub> black hole case, we demonstrate that the ultraspinning dyonic Kerr-Sen-AdS<sub>4</sub> black hole does not always violate the reverse isoperimetric inequality (RII) since the value of the isoperimetric ratio can either be larger/smaller than, or equal to unity, depending upon the range of the solution parameters, as is the case only with an electric charge. This property is apparently distinct from that of the superentropic dyonic Kerr-Newman-AdS<sub>4</sub> black hole, which always strictly violates the RII, although both of them have some similar properties in other aspects, such as the horizon geometry and conformal boundary.

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### I. INTRODUCTION

Recently, a new class of ultraspinning anti-de Sitter (AdS) black holes [1–3], in which one of their rotation angular velocities is boosted to the speed of light, has attracted a lot of attention. This class of black hole has a finite horizon area but with a noncompact horizon topology because there are two punctures at the north and south poles of its spherical horizon. Interestingly, the ultraspinning black holes violate the reverse isoperimetric inequality (RII) [4,5], and this means that the Schwarzschild-AdS black hole has the maximum upper entropy. Because the ultraspinning black hole can exceed the maximum entropy bound, it is therefore often called a “superentropic” black hole. Moreover, it is pointed out [1] that one can obtain the corresponding superentropic black hole solution by taking a simple ultraspinning limit from its usual rotating AdS black hole. Such a solution generating trick is very simple: first, rewrite the metric of the rotating AdS black hole in the rotating frame at infinity, then boost one of the rotation angular velocities to the speed of light, and finally compactify the corresponding azimuthal direction.

Since then, a dozen of new superentropic black hole solutions [6–10] have been constructed from the known rotating AdS black holes. Very recently, it has been suggested that the superentropic black hole can also be obtained by running a conical deficit through the usual rotating AdS black hole [11]. On the other hand, various aspects of the superentropic black holes, including thermodynamic properties [1,6,8–10,12–15], horizon geometry [3,6,8], Kerr/conformal field theory (CFT) correspondence [7–9], geodesic motion [16], etc., have also been extensively studied.

Quite recently, we have studied some interesting properties of the Kerr-Sen-AdS<sub>4</sub> black holes and their ultraspinning cousin in the four-dimensional gauged Einstein-Maxwell-dilaton-axion (EMDA) theory [17]. However, the black hole solution studied there only carries an electric charge and is just a special and relatively simple case of the four-dimensional gauged EMDA theory. It is then natural for us to extend that work to the more general dyonic case, which serves as our motivation of the present work. First, we shall present the dyonic generalization of the Kerr-Sen black hole solution and then include a nonzero negative cosmological constant into it to obtain a dyonic Kerr-Sen-AdS<sub>4</sub> black hole. After that, we will turn to investigate its ultraspinning counterpart. In the meanwhile, we will mainly study their thermodynamical properties and verify that all the thermodynamical quantities obtained for them perfectly obey both the extended law and the Bekenstein-Smarr mass formulas.

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The organization of this paper is outlined as follows. In Sec. II, we first give a brief introduction of the four-dimensional ungauged and gauged EMDA theories and summarize the current already-known exact rotating charged black hole solutions in these supergravity theories. In Sec. III, we present the dyonic Kerr-Sen black hole solution and its AdS<sub>4</sub> extension, and then turn to explore its thermodynamics. In Sec. IV, with the ultraspinning dyonic Kerr-Sen-AdS<sub>4</sub> black hole solution in hand, its thermodynamical properties, horizon topology and conformal boundary, RII, etc., are subsequently discussed. To this end, we derive new Christodoulou-Ruffini-like squared-mass formulas for the dyonic Kerr-Sen-AdS<sub>4</sub> black hole and its ultraspinning cousin. By differentiating them with respect to their individual thermodynamical variable, we get the expected thermodynamical quantities which obey both the first law and the Bekenstein-Smarr mass formulas without employing the chirality condition ( $J = MI$ ). After that, we impose the chirality condition and derive the reduced form of the mass formulas. Finally, we show that this ultraspinning dyonic Kerr-Sen-AdS<sub>4</sub> black hole does not always obey the RII, since the value of the isoperimetric ratio can either be larger/smaller than, or equal to unity, depending upon where the solution parameters lie in the parameters space. This property is very similar to that of the ultraspinning Kerr-Sen-AdS<sub>4</sub> black hole [17]; however, it signals a remarkable difference from the superentropic dyonic Kerr-Newman-AdS<sub>4</sub> black hole. Finally, the paper is ended up with our summaries in Sec. V.

## II. EMDA SUPERGRAVITY THEORIES AND THEIR ALREADY-KNOWN ROTATING CHARGED SOLUTIONS

### A. Brief introduction to ungauged and gauged EMDA supergravity theories

In 1992, Sen [18] presented a stationary and axially symmetric solution that describes a four-dimensional black hole beyond the Einstein-Maxwell theory. It is the *first* exact rotating charged solution in the low-energy effective field theory for the heterotic string theory. The bosonic sector of the four-dimensional low-energy heterotic string theory contains the metric field  $g_{\mu\nu}$ , the U(1) Abelian gauge field  $A_\mu$ , the dilaton scalar field  $\phi$ , and the three-order totally antisymmetric tensor field  $H_{\mu\nu\rho}$ . Its Lagrangian reads

$$\check{\mathcal{L}} = \sqrt{-g} \left[ R - \frac{1}{2} (\partial\phi)^2 - e^{-\phi} F^2 - \frac{1}{12} e^{-2\phi} H^2 \right], \quad (1)$$

where  $R$  is the Ricci scalar,  $F_{\mu\nu}$  is the Faraday-Maxwell electromagnetic tensor defined by  $F = dA$ ,  $F^2 = F_{\mu\nu} F^{\mu\nu}$ ,  $(\partial\phi)^2 = (\partial_\mu\phi)(\partial^\mu\phi)$ , and  $H^2 = H_{\mu\nu\rho} H^{\mu\nu\rho}$ .

In order to construct exact black hole solutions with a nonzero cosmological constant, the three-form field must

be dualized to an axion pseudoscalar field  $\chi$  via the relation  $H \equiv d\mathcal{B} - A \wedge F/4 = -e^{2\phi} \star d\chi$ , where  $\mathcal{B}$  is an antisymmetric two-form potential and the star operator represents the Hodge duality. Then the resulted theory is also known as the EMDA supergravity theory, and accordingly, the above Lagrangian can be rewritten in a different but completely equivalent form,

$$\hat{\mathcal{L}} = \sqrt{-g} \left[ R - \frac{1}{2} (\partial\phi)^2 - \frac{1}{2} e^{2\phi} (\partial\chi)^2 - e^{-\phi} F^2 \right] + \frac{\chi}{2} \epsilon^{\mu\nu\rho\lambda} F_{\mu\nu} F_{\rho\lambda}, \quad (2)$$

where  $\epsilon^{\mu\nu\rho\lambda}$  is the four-dimensional Levi-Civita antisymmetric tensor density. From the equation of motion derived for the Abelian gauge potential  $A$ , one can define its dual potential  $B$  by  $e^{-\phi} \star F + \chi F = -dB$ . It is very convenient to use them to compute the electrostatic and magnetostatic potentials.

In the gauged version corresponding to the above EMDA theory, the corresponding Lagrangian has the following form:

$$\begin{aligned} \mathcal{L} &= \hat{\mathcal{L}} + \mathcal{V}(\phi, \chi) \\ &= \hat{\mathcal{L}} + \sqrt{-g} [4 + e^{-\phi} + e^\phi (1 + \chi^2)] / l^2, \end{aligned} \quad (3)$$

with  $l$  being the cosmological scale or the reciprocal of the gauge coupling constant. Since the above Lagrangian is supplemented by a potential term related to the dilaton and axion scalar fields, it is no longer possible to reexpress this Lagrangian into a dualized version in terms of the three-form field that appeared in the ungauged version again.

The origin of the kinetic terms of matter fields and the scalar potential in the Lagrangian (3) of four-dimensional EMDA gauged supergravity can be attributed to the  $S^7$  reduction of 11-dimensional supergravity, which gives rise to SO(8) gauged  $\mathcal{N} = 8$  supergravity in four dimensions. A successive consistent truncation leads to  $\mathcal{N} = 2$ , D = 4 U(1) Fayet-Iliopoulos gauged supergravity that admits a prepotential formalism [19], of which the well-known STU model [20,21] endows with the prepotential  $\mathcal{F} = -X^1 X^2 X^3 / X^0$ . Further completion of the U(1)<sup>2</sup> truncation of this STU model (by setting the charge parameters to be pairwise equal), one arrives at the SU(1,1)/U(1) model whose prepotential is given by  $\mathcal{F} = -iX^0 X^1$ , sometimes also known as “ $-iX^0 X^1$  supergravity” [22]. Finally, the D = 4 EMDA gauged supergravity is obtained by setting one of the gauge fields (or charge parameter) to zero. See Ref. [22] for a relatively recent review, and particularly Fig. 1 therein that provides a schematic diagram to sketch the consistent reductions.

## B. Present status of stationary and axially symmetric solutions to the theories

It is well known that the most general stationary and axially symmetric class of type D solution of the four-dimensional Einstein-Maxwell equations with an aligned electromagnetic field is given by the seven-parameter family of the Plebański-Demiański solution [23], which contains the mass, NUT charge (dual mass), electric and magnetic charges, rotation and acceleration parameters, and a nonzero cosmological constant. Until now, no analogous extension of this general seven-parameter Plebański-Demiański solution has been found yet beyond the Einstein-Maxwell framework, including the gauged EMDA theory that we are interested in here.

Soon after Sen [18] gave the *first* rotating charged black hole solution in the above EMDA theory, a lot of attention has been paid to including more parameters into it. In the ungauged EMDA theory, the existing methods to generalize the nonextremal Kerr-Sen solution can be roughly classified into the following three categories:

- (1) The brute force solving approach. As a representative of this method, the work [24] adopted a suitable ansatz for the line element but a very restrictive one for the U(1) gauge potential, namely, the non-null electromagnetic field is not the most general aligned one. The solution presented in Ref. [24] can be thought of as a very special dyonic generalization of the Kerr-Sen solution.
- (2) Three different solution generating methods that correspond to different coset (potential) spaces depending upon dimensional reduction from distinct superstring and supergravity theories as follows:
  - (a) The Hassan-Sen method [25]: The Kerr-Sen solution was initially generated [18] from the famous Kerr solution via this approach by using a  $9 \times 9$  coset matrix. Later, the same method was subsequently applied in Refs. [26,27] to generate accelerating, rotating, charged black holes in the low-energy heterotic string theory, respectively, from some type D (Plebański-Demiański) metrics and its special case, namely, the accelerating Kerr solution. However, it should be pointed out here that it is a very difficult or rather challenging task to further include a nonzero cosmological constant into the so-obtained accelerating, rotating, charged solution.
  - (b) The  $\text{Sp}(4, \mathbb{R})/\text{U}(2)$  potential space method [28,29]. Starting from the Kerr-NUT solution, a new rotating dyonic black hole solution was generated in Ref. [30] by using the symmetry of this potential space. Its solution generating process is completely equivalent to the

Hassan-Sen method when employed to the Kerr-NUT (or only Kerr) solution first and then followed by implementing a necessary gauge transformation and the generalized electromagnetic duality transformation [30] to the obtained gauge potential. After being reexpressed in terms of observable physical quantities, namely, the mass, NUT charge, electric and magnetic charge, as well as rotation parameters, the resulted solution can be thought of as a dyonic NUT extension of the Kerr-Sen solution, which shall be named as the dyonic Kerr-Sen-NUT solution hereafter. However, the gauge potential had not been explicitly given in Ref. [30], and its spatial component still needs to be worked out via the dual of the magnetic scalar potential.

- (c) Subgroup of larger coset spaces:  $\text{O}(4, 4)/\text{O}(1, 1)^4$  [31] and  $\text{SO}(4, 4)/\text{SL}(2, \mathbb{R})^4$  [22,32]. The coset matrix representation of the EMDA theory can be viewed as a subset of these more complicated coset matrix representations of the four-dimensional STU supergravity theory, and the dyonic Kerr-Sen-NUT solution [30] is just a special case of the pairwise equal charge parameters that was introduced in Refs. [22,31]. The charging parameters used in [31] are two electric and two magnetic ones, and the seed metric adopted in the main context of that paper is the Kerr solution, but in the Appendix, the authors had already considered the Kerr-NUT solution and the most general Plebański-Demiański metric as the seed solution too. In particular, they had presented the explicit expressions for the obtained solutions in the case of the pairwise equal charge parameters and mentioned that they failed to make a generalization so as to include a nonzero cosmological constant when a nonvanishing acceleration parameter is turned on. On the other hand, the charging parameters chosen in [22] are enlarged to four electric and four magnetic ones, but their seed metric is only the Kerr-NUT solution. Incidentally, it should be pointed out that the accelerating Kerr-Sen and accelerating Kerr-NUT-Sen solution generated in Ref. [27] are just special cases of those were previously obtained in Ref. [31] to possess the pairwise equal charge parameters, although they are rederived by using a simpler Hassan-Sen method.
- (3) The Belinsky-Zakharov inverse scattering technique [33] was extended in [34] to act on the  $\text{Sp}(4, \mathbb{R})/\text{U}(2)$  coset space to get the dyonic Kerr-Sen-NUT solution that had already been obtained in Ref. [30].

In the case of the four-dimensional gauged STU supergravity theory, no solution generating technique can be used to get the rotating charged AdS solution, of which some subclasses had been already constructed [31,35] only in the special case of the pairwise equal charge parameters,<sup>1</sup> and in the consistent truncation cases with single-charge [37,38] and two-charge [39], respectively. However, they are not written in the simple or concise expressions, but expressed in the complicated forms in terms of charging parameters [31,35]. The most general rotating charged AdS<sub>4</sub> solution with generic unequal values of four pairs of electromagnetic charging parameters that generalizes the generic Chow-Compère solution [22] still remain elusive till now, let alone further introducing a nonzero acceleration parameter. The latter generic solution to four-dimensional gauged STU supergravity theory should contain 13 parameters that represent the mass, NUT charge, rotation and acceleration, four electric and four magnetic charge parameters, and a nonzero cosmological constant.

### III. DYONIC KERR-SEN BLACK HOLE AND ITS AdS<sub>4</sub> EXTENSION

#### A. A new simple form of dyonic Kerr-Sen solution

Although the dyonic NUT generalization of the Kerr-Sen black hole solution was already given in Ref. [30] sixteen years ago, we feel that its expressions for the solution are not suitable to our aim here. In particular, the angular component for the U(1) gauge potential was not explicitly given there. The solution contains a minimal set of five physical quantities, which correspond to the mass, NUT charge, electric and magnetic charges, as well as rotation parameter, respectively, while the dilaton scalar and axion pseudoscalar charges are not independent parameters but related to these five parameters mentioned above. In this paper, we will consider a slightly simpler case without the NUT charge, namely, the dyonic extension of the Kerr-Sen solution. In other words, we shall extend our previous work [17] to a more general case by adding only a nonzero magnetic charge to the Kerr-Sen black hole.

Written in terms of the Boyer-Lindquist coordinates, the line element, the Abelian gauge potential and its dual, as well as the dilaton scalar and axion pseudoscalar fields of the dyonic Kerr-Sen black hole are given in the following exquisite forms:

$$\begin{aligned} d\hat{s}^2 &= -\frac{\hat{\Delta}(r)}{\hat{\Sigma}}\hat{X}^2 + \frac{\hat{\Sigma}}{\hat{\Delta}(r)}dr^2 + \hat{\Sigma}d\theta^2 + \frac{\sin^2\theta}{\hat{\Sigma}}\hat{Y}^2, \\ \hat{A} &= \frac{q(r-p^2/m)}{\hat{\Sigma}}\hat{X} - \frac{p\cos\theta}{\hat{\Sigma}}\hat{Y}, \\ \hat{B} &= \frac{p(r-p^2/m)}{\hat{\Sigma}}\hat{X} + \frac{q\cos\theta}{\hat{\Sigma}}\hat{Y}, \\ e^{\hat{\phi}} &= \frac{r^2 + (a\cos\theta + k)^2}{\hat{\Sigma}}, \quad \hat{\chi} = 2\frac{kr - d(a\cos\theta + k)}{r^2 + (a\cos\theta + k)^2}, \end{aligned} \quad (4)$$

where

$$\begin{aligned} \hat{X} &= dt - a\sin^2\theta d\hat{\phi}, \quad \hat{Y} = a dt - (r^2 - 2dr - k^2 + a^2)d\hat{\phi}, \\ \hat{\Delta}(r) &= r^2 - 2dr - 2m(r-d) - k^2 + a^2 + p^2 + q^2, \\ \hat{\Sigma} &= r^2 - 2dr - k^2 + a^2\cos^2\theta, \end{aligned}$$

in which  $d = (p^2 - q^2)/(2m)$  and  $k = pq/m$  represent the dilaton scalar and axion pseudoscalar charges, and the mass, electric and magnetic charges as well as angular momentum of the black hole are  $M = m$ ,  $Q = q$ ,  $P = p$ , and  $J = ma$ , respectively. When the magnetic charge vanishes ( $p = 0$ ), the axion charge vanishes too ( $k = 0$ ), and then the solution reduces to the Kerr-Sen case previously considered in our previous work [17] with  $b = -d = q^2/(2m)$ . On the other hand, in the special case when  $p = q$ , the dilaton charge will completely vanish ( $d = 0$ ). Note that there is a useful quadratic relation:  $d^2 + k^2 = (p^2 + q^2)^2/(4m^2)$ . In addition, if wishes, one can work with another radial coordinate by shifting  $r \rightarrow r + p^2/m$ , or  $r \rightarrow r + d$  to make the expressions more symmetric about  $(p, q, d, k)$  (especially in the case when the NUT parameter is turned on).

It should be emphasized that in the above, we have chosen a concrete gauge choice so that the temporal components of both Abelian gauge potentials completely vanish at infinity; in the meanwhile, their angular components simultaneously become  $p\cos\theta$  and  $-q\cos\theta$  there, respectively. Our arguments for this gauge choice go as follows.

Generally speaking, the Abelian gauge potential  $\hat{A}$  at the infinity has the asymptotic form  $\hat{A}_\infty = \Phi_\infty dt + p(\cos\theta \pm C)d\hat{\phi}$ . In the standard monopole gauge theory, the constant  $C$  may usually take three different values:  $C = 0$  on the equator, and  $C = \pm 1$  at the north and south poles ( $\theta = 0, \pi$ ) to eliminate the Dirac string singularities at two poles, respectively. In the static case, the constant  $C$  can take any one of these three values, and this does not lead to any serious problem. But in the rotating case, if the constant  $C$  takes the above gauge choice at infinity, namely,  $C = 0, \pm 1$  on the equator and at the north and south poles, respectively, then it would give rise to an odd result for the expressions of the electrostatic potential on the horizon: they are not identical to each other, for instance, on the

<sup>1</sup>Rotating charged AdS solutions with one parameter less or more than those of Ref. [35] were successively constructed in [2,36]. It is declared in [36] that the extra parameter  $\beta$  represents a scalar hair, but it is unclear to us whether or not it is a redundant one.

equator, at two poles, and elsewhere. This obviously contradicts the common sense that the electrostatic potential should be a unique constant everywhere on the event horizon, and it means that the electric charge is not uniformly distributed on the horizon. To ensure that the electrostatic potential in the rotating case is equal everywhere on the horizon, one can only set  $C = 0$  so that  $\hat{A}_\phi = p \cos \theta$  at infinity. On the other hand, any choice of  $\Phi_\infty$  does not change the difference of the electrostatic potential on the event horizon and that measured by an observer at infinity:  $\Phi = \Phi_+ - \Phi_\infty$ . Frequently, two conventional options for the temporal component of the gauge potential  $\hat{A}$  are either  $\Phi_\infty = 0$  or  $\Phi_+ = 0$  (equivalently,  $\Phi_\infty = -\Phi_+$ ). Similar discussions completely apply to the dual Abelian gauge potential  $\hat{B}$  too. Here we would like to work with the temporal gauge choice  $\Phi_\infty = \Psi_\infty = 0$ . However, it should be noted that different choices of temporal gauge of the U(1) potentials correspond to different thermodynamical ensembles and therefore different thermodynamical grand potentials. One can have a total of four possibilities by fixing any one combination of the charges, electrostatic and magnetostatic potentials:  $(Q, P)$ ,  $(Q, \Psi)$ ,  $(P, \Phi)$ , and  $(\Phi, \Psi)$ . Taking into account of these, our discussions below about thermodynamics in the dyonic case should correspond to the thermodynamical ensemble with the static potentials  $(\Phi, \Psi)$  fixed.

### B. Dyonic Kerr-Sen-AdS<sub>4</sub> solution

We now add a nonzero negative cosmological constant into the above dyonic Kerr-Sen black hole solution and obtain an exact AdS<sub>4</sub> black hole solution to the gauged EMDA theory. Expressed in terms of the Boyer-Lindquist coordinates with the frame rotating at infinity, the dyonic Kerr-Sen-AdS<sub>4</sub> black hole solution can be written as follows:

$$\begin{aligned}
 d\bar{s}^2 &= -\frac{\bar{\Delta}_r}{\bar{\Sigma}} \bar{X}^2 + \frac{\bar{\Sigma}}{\bar{\Delta}_r} dr^2 + \frac{\bar{\Sigma}}{\bar{\Delta}_\theta} d\theta^2 + \frac{\bar{\Delta}_\theta \sin^2 \theta}{\bar{\Sigma}} \bar{Y}^2, \\
 \bar{A} &= \frac{q(r - p^2/m)}{\bar{\Sigma}} \bar{X} - \frac{p \cos \theta}{\bar{\Sigma}} \bar{Y}, \\
 \bar{B} &= \frac{p(r - p^2/m)}{\bar{\Sigma}} \bar{X} + \frac{q \cos \theta}{\bar{\Sigma}} \bar{Y}, \\
 e^{\bar{\phi}} &= \frac{r^2 + (a \cos \theta + k)^2}{\bar{\Sigma}}, \quad \bar{\chi} = 2 \frac{kr - d(a \cos \theta + k)}{r^2 + (a \cos \theta + k)^2}, \quad (5)
 \end{aligned}$$

where a bar is designed to distinct the present expressions from their corresponding ones in the ungauged case. Note that  $\bar{\Sigma} = \hat{\Sigma} = r^2 - 2dr - k^2 + a^2 \cos^2 \theta$  as before, but now we have

$$\begin{aligned}
 \bar{X} &= dt - \frac{a \sin^2 \theta}{\Xi} d\bar{\varphi}, \quad \bar{Y} = a dt - \frac{r^2 - 2dr - k^2 + a^2}{\Xi} d\bar{\varphi}, \\
 \bar{\Delta}_r &= \left(1 + \frac{r^2 - 2dr - k^2}{l^2}\right) (r^2 - 2dr - k^2 + a^2) \\
 &\quad - 2m(r - d) + p^2 + q^2, \\
 \bar{\Delta}_\theta &= 1 - \frac{a^2}{l^2} \cos^2 \theta, \quad \Xi = 1 - \frac{a^2}{l^2}.
 \end{aligned}$$

Clearly, the above solution (5) simply reduces to the dyonic Kerr-Sen black hole solution (4) when the cosmological constant vanishes.

Since the gauged EMDA theory is a successive consistent truncation of the four-dimensional gauged STU supergravity, therefore, similar to the purely electric-charged case as mentioned in our previous paper [17], the above AdS<sub>4</sub> black hole solution can be thought of as a special case of those obtained in Refs. [31,35], where more general solutions with the pairwise equal charge parameters have been constructed in the gauged  $-iX^0 X^1$  supergravity model. However, the solution presented here is slightly simpler than those given there especially by the radial structure function and is more convenient for further investigations.

### C. Thermodynamics

Now we would like to explore thermodynamics of the dyonic Kerr-Sen-AdS<sub>4</sub> black hole (5). One can compute all associated thermodynamic quantities via the standard method and express them as follows:

$$\begin{aligned}
 \bar{M} &= \frac{m}{\Xi}, \quad \bar{J} = \frac{ma}{\Xi^2}, \quad \bar{Q} = \frac{q}{\Xi}, \quad \bar{P} = \frac{p}{\Xi}, \\
 \bar{S} &= \frac{\pi}{\Xi} (r_+^2 - 2dr_+ - k^2 + a^2), \quad \bar{\Omega} = \frac{a\Xi}{r_+^2 - 2dr_+ - k^2 + a^2}, \\
 \bar{\Phi} &= \frac{q(r_+ - p^2/m)}{r_+^2 - 2dr_+ - k^2 + a^2}, \quad \bar{\Psi} = \frac{p(r_+ - p^2/m)}{r_+^2 - 2dr_+ - k^2 + a^2}, \\
 \bar{T} &= \frac{\bar{\Delta}'_{r_+}}{4\pi(r_+^2 - 2dr_+ - k^2 + a^2)} \\
 &= \frac{(r_+ - d)(2r_+^2 - 4dr_+ - 2k^2 + a^2 + l^2) - ml^2}{2\pi(r_+^2 - 2dr_+ - k^2 + a^2)l^2}, \quad (6)
 \end{aligned}$$

where the location of the event horizon  $r_+$  is the largest root of equation  $\bar{\Delta}_{r_+} = 0$ .

In the above, we have only listed the final expressions for these thermodynamic quantities while omitting the computing process. In particular, we have adopted the conformal completion method provided in Ref. [40] (especially its subsection IV.A is very useful and relevant to our aim) to compute the conserved charges (mass and angular momentum) given in Eqs. (6) and (13) in the rotating and rest frames at infinity, respectively. On the other hand, due to good falloff behaviors of the vector and scalar fields, the

electric and magnetic charges can be computed by the Gauss-type law integral

$$\begin{aligned} Q &= \frac{1}{4\pi} \int \star d\bar{A} = \frac{1}{4\pi} \int d\bar{B}, \\ P &= \frac{1}{4\pi} \int \star d\bar{B} = \frac{-1}{4\pi} \int d\bar{A}. \end{aligned} \quad (7)$$

In addition, by virtue of the specific gauge choice argued in the last section, the electrostatic and magnetostatic potentials are simply given by

$$\bar{\Phi} = (A_{\mu}\bar{\chi}^{\mu})|_{r=r_+}, \quad \bar{\Psi} = (B_{\mu}\bar{\chi}^{\mu})|_{r=r_+}, \quad (8)$$

where  $\bar{\chi} = \partial_t + \bar{\Omega}\partial_{\bar{\varphi}}$  is the Killing vector normal to the event horizon. Similarly, one can compute their corresponding thermodynamic expressions in the rest frame at infinity also.

It is not difficult to check that the thermodynamic quantities (6) obey the Bekenstein-Smarr mass formula

$$\bar{M} = 2\bar{T}\bar{S} + 2\bar{\Omega}\bar{J} + \bar{\Phi}\bar{Q} + \bar{\Psi}\bar{P} - 2\bar{V}\bar{P}, \quad (9)$$

where  $\bar{V}$  is the thermodynamic volume,

$$\begin{aligned} \bar{V} &= \frac{4}{3}(r_+ - d)\bar{S} \\ &= \frac{4\pi}{3\Xi}(r_+ - d)(r_+^2 - 2dr_+ - k^2 + a^2), \end{aligned} \quad (10)$$

which is conjugate to the pressure  $\bar{P} = 3/(8\pi l^2)$  and can be alternatively evaluated via the method put forward in Ref. [4] (see III.B therein),

$$\bar{V} = \frac{l^2}{3} \int_d^{r_+} dr \int_0^{2\pi} d\bar{\varphi} \int_0^{\pi} d\theta \sqrt{-\bar{g}} \mathcal{V}(\bar{\phi}, \bar{\chi}), \quad (11)$$

where the lower limit of  $r$ -integration must be taken to be  $r = d$ . However, the first law becomes a differential identity only,

$$d\bar{M} = \bar{T}d\bar{S} + \bar{\Omega}d\bar{J} + \bar{\Phi}d\bar{Q} + \bar{\Psi}d\bar{P} + \bar{V}d\bar{P} + \bar{J}d\Xi/(2a). \quad (12)$$

The reason for this is that we have worked with a frame rotating at infinity.

One can transform the above dyonic Kerr-Sen-AdS<sub>4</sub> solution into the frame rest at infinity via a simple coordinate transformation:  $\bar{\varphi} \rightarrow \tilde{\varphi} - a l^{-2} t$ . After a cumbersome computation (using the method mentioned above) for those thermodynamic quantities in this rest frame, it is easy to observe that only the mass, the angular velocity, and the thermodynamic volume are different from those given in Eq. (6) and related by the following expressions:

$$\tilde{M} = \bar{M} + \frac{a}{l^2} \bar{J} = \frac{m}{\Xi^2}, \quad \tilde{\Omega} = \bar{\Omega} + \frac{a}{l^2}, \quad \tilde{V} = \bar{V} + \frac{4\pi}{3} a \bar{J}. \quad (13)$$

Now it is easy to verify that thermodynamic quantities can indeed fulfil both the standard forms of the first law and the Bekenstein-Smarr mass formula simultaneously,

$$\begin{aligned} d\tilde{M} &= \tilde{T}d\tilde{S} + \tilde{\Omega}d\tilde{J} + \tilde{\Phi}d\tilde{Q} + \tilde{\Psi}d\tilde{P} + \tilde{V}d\tilde{P}, \\ \tilde{M} &= 2\tilde{T}\tilde{S} + 2\tilde{\Omega}\tilde{J} + \tilde{\Phi}\tilde{Q} + \tilde{\Psi}\tilde{P} - 2\tilde{V}\tilde{P}. \end{aligned} \quad (14)$$

In addition, one can show that the above differential and integral mass formulas can be derived from the following Christodoulou-Ruffini-like squared-mass formulas:

$$\tilde{M}^2 = \left(1 + \frac{8\tilde{P}\tilde{S}}{3}\right) \left[ \left(1 + \frac{8\tilde{P}\tilde{S}}{3}\right) \frac{\tilde{S}}{4\pi} + \frac{\pi\tilde{J}^2}{S} + \frac{\tilde{P}^2 + \tilde{Q}^2}{2} \right]. \quad (15)$$

When the magnetic charge  $P$  vanishes, all the above thermodynamic formulas can consistently reduce to those obtained in Ref. [17] for the purely electric-charged Kerr-Sen-AdS<sub>4</sub> case.

#### IV. ULTRASPINNING DYONIC KERR-SEN-ADS<sub>4</sub> BLACK HOLE

##### A. The ultraspinning dyonic solution

Following [17], we redefine  $\varphi = \bar{\varphi}/\Xi$  and then just need to set  $a = l$  in the above dyonic Kerr-Sen-AdS<sub>4</sub> black hole solution (5) to construct its corresponding ultraspinning version as follows:

$$\begin{aligned} ds^2 &= -\frac{\Delta(r)}{\Sigma} X^2 + \frac{\Sigma}{\Delta(r)} dr^2 + \frac{\Sigma}{\sin^2\theta} d\theta^2 + \frac{\sin^4\theta}{\Sigma} Y^2, \\ A &= \frac{q(r - p^2/m)}{\Sigma} X - \frac{p \cos\theta}{\Sigma} Y, \\ B &= \frac{p(r - p^2/m)}{\Sigma} X + \frac{q \cos\theta}{\Sigma} Y, \\ e^{\phi} &= \frac{r^2 + (l \cos\theta + k)^2}{\Sigma}, \quad \chi = 2 \frac{kr - d(l \cos\theta + k)}{r^2 + (l \cos\theta + k)^2}, \end{aligned} \quad (16)$$

where

$$\begin{aligned} X &= dt - l \sin^2\theta d\varphi, \quad Y = l dt - (r^2 - 2dr - k^2 + l^2) d\varphi, \\ \Delta(r) &= (r^2 - 2dr - k^2 + l^2)^2 l^{-2} - 2m(r - d) + p^2 + q^2 \\ &= [(r + q^2/m)(r - p^2/m) + l^2]^2 l^{-2} - 2m(r - p^2/m), \\ \Sigma &= r^2 - 2dr - k^2 + l^2 \cos^2\theta \\ &= (r + q^2/m)(r - p^2/m) + l^2 \cos^2\theta. \end{aligned}$$

Note that in the above ultraspinning dyonic solution, the period of  $\varphi$  is now assumed to take a dimensionless parameter  $\mu$  rather than  $2\pi$ .

With an exact solution of the ultraspinning dyonic Kerr-Sen-AdS<sub>4</sub> black hole in hand, the remaining main task of this work is to study its various interesting basic properties, such as its thermodynamical properties, the horizon topology and conformal boundary, the RII, etc.

### B. Various mass formulas

First, let us focus on thermodynamics of the ultraspinning dyonic Kerr-Sen-AdS<sub>4</sub> black hole. As before, one can obtain the following expressions of its fundamental thermodynamic quantities through the standard method:

$$\begin{aligned} M &= \frac{\mu}{2\pi} m, & J &= \frac{\mu}{2\pi} ml = Ml, & Q &= \frac{\mu}{2\pi} q, & P &= \frac{\mu}{2\pi} p, \\ S &= \frac{\mu}{2} (r_+^2 - 2dr_+ - k^2 + l^2), & \Omega &= \frac{l}{r_+^2 - 2dr_+ - k^2 + l^2}, \\ \Phi &= \frac{q(r_+ - p^2/m)}{r_+^2 - 2dr_+ - k^2 + l^2}, & \Psi &= \frac{p(r_+ - p^2/m)}{r_+^2 - 2dr_+ - k^2 + l^2}, \\ T &= \frac{\Delta'(r_+)}{4\pi(r_+^2 - 2dr_+ - k^2 + l^2)} \\ &= \frac{r_+ - d}{\pi l^2} - \frac{m}{2\pi(r_+^2 - 2dr_+ - k^2 + l^2)}, \end{aligned} \quad (17)$$

in which the location of the event horizon  $r_+$  is now determined by the largest root of equation  $\Delta(r_+) = 0$ .

As mentioned in the last section, one good way to compute the mass and angular momentum in the ultraspinning case is to adopt the conformal completion technique that is explicitly elucidated in Ref. [40] since the line element of the conformal boundary is not a diagonal metric. It is worthy to point out that all of the thermodynamic quantities given in Eq. (17) are obtained by applying the same method as mentioned above. Because a detailed discussion about the computation of all the thermodynamic quantities for the ultraspinning AdS solution had already addressed in subsection II. A of our previous work [14], so we omit the computing process here.

Note that there is a chirality condition ( $J = Ml$ ) that constrains the angular momentum and the mass, and the angular velocity  $\Omega$  is that of the event horizon because the ultraspinning dyonic black hole is rotating at the speed of light at infinity.

Now it can be shown that the above thermodynamical quantities completely fulfil both the first law and the Bekenstein-Smarr mass formulas,

$$dM = TdS + \Omega dJ + \Phi dQ + \Psi dP + Vd\mathcal{P} + Kd\mu, \quad (18)$$

$$M = 2TS + 2\Omega J + \Phi Q + \Psi P - 2V\mathcal{P}, \quad (19)$$

in which the thermodynamic volume and a new chemical potential,

$$V = \frac{4}{3}(r_+ - d)S = \frac{2}{3}\mu(r_+ - d)(r_+^2 - 2dr_+ - k^2 + l^2), \quad (20)$$

$$K = m \frac{l^2 - (r_+ + q^2/m)(r_+ - p^2/m)}{4\pi(r_+^2 - 2dr_+ - k^2 + l^2)}, \quad (21)$$

are conjugate to the pressure  $\mathcal{P} = 3/(8\pi l^2)$  and the dimensionless parameter  $\mu$ , respectively. Note that  $V$  can be also computed as

$$V = \frac{l^2}{3} \int_d^{r_+} dr \int_0^\mu d\varphi \int_0^\pi d\theta \sqrt{-g} \mathcal{V}(\phi, \chi), \quad (22)$$

where the upper limit of  $\varphi$ -integration is changed to  $\varphi = \mu$ , while the lower limit of  $r$ -integration still is  $r = d$ . Presumably, the chemical potential  $K$  might be obtained alternatively in the context of an angular gravitational tension [41] associated to the rotational symmetry of the black hole or in term of the thermodynamic length [42] related to the conical defect.

In the subsection II. E of our previous work [14] (and then III. C of Ref. [17]), we have made, for the first time, an attempt to relate the thermodynamic quantities of the ultraspinning solution to those of its usual rotating AdS solution. This is based upon such a naive belief that the ultraspinning solution is obtained by taking the  $a \rightarrow l$  limit, so were their thermodynamic quantities also. Similar to what was done in Refs. [14,17], now we simply generalize to the dyonic case, and proceed to assume the following simple relations:

$$\begin{aligned} M &= \frac{\mu\Xi\bar{M}}{2\pi}, & J &= \frac{\mu\Xi^2\bar{J}}{2\pi}, & Q &= \frac{\mu\Xi\bar{Q}}{2\pi}, & P &= \frac{\mu\Xi\bar{P}}{2\pi}, \\ \Omega &= \frac{\bar{\Omega}}{\Xi}, & S &= \frac{\mu\Xi\bar{S}}{2\pi}, & V &= \frac{\mu\Xi\bar{V}}{2\pi}, \\ T &= \bar{T}, & \Phi &= \bar{\Phi}, & \mathcal{P} &= \bar{\mathcal{P}}, \end{aligned} \quad (23)$$

and take the ultraspinning limit  $a \rightarrow l$ . Then we can find that the above thermodynamic quantities presented in Eq. (17) for the ultraspinning dyonic Kerr-Sen-AdS<sub>4</sub> black hole can also be obtained straightforwardly from those of its corresponding usual black hole in a frame rotating at infinity.

In Ref. [17], a new Christodoulou-Ruffini-like squared-mass formulas was derived for the ultraspinning Kerr-Sen-AdS<sub>4</sub> black hole. Now we expect to generalize it to the ultraspinning dyonic case. Rewriting the event horizon equation ( $\Delta_{r_+} = 0$ ) as

$$\frac{S^2}{\pi l^2} + \pi(P^2 + Q^2) = \mu M(r_+ - d), \quad (24)$$

then exploiting  $3/l^2 = 8\pi\mathcal{P}$ , we can obtain  $r_+ = d + [8\mathcal{P}S^2 + 3\pi(P^2 + Q^2)]/(3\mu M)$ . Now, we substitute it into the entropy  $S = \mu(r_+^2 - 2dr_+ - k^2 + l^2)/2$  and

use  $d = \pi(P^2 - Q^2)/(\mu M)$  and  $k = 2\pi PQ/(\mu M)$  as well as the chirality condition ( $J = MI$ ) to get a useful identity,

$$M^2 = \frac{8PS}{3\mu} \left[ \frac{4P}{3} S^2 + \pi(P^2 + Q^2) \right] + \frac{\mu J^2}{2S}, \quad (25)$$

which is our expected Christodoulou-Ruffini-like squared-mass formulas for the ultraspinning dyonic Kerr-Sen-AdS<sub>4</sub> black hole. We point out that this squared-mass formula (25) consistently reduces to the one obtained in the ultraspinning Kerr-Sen-AdS<sub>4</sub> black hole case [17] when the magnetic charge  $P$  is turned off.

Supposing temporarily that there exists no chirality condition ( $J = MI$ ) at all, then it is clear from Eq. (25) that the thermodynamical quantities  $S$ ,  $J$ ,  $Q$ ,  $P$ ,  $\mathcal{P}$ , and  $\mu$  can be treated as independent thermodynamical variables and consist of an entire set of extensive variables for the fundamental functional relation  $M = M(S, J, Q, P, \mathcal{P}, \mu)$ . In this way, as is done in Refs. [14,17,43–46], the differentiation of the above squared-mass formula (25) with respect to one formal variable of the whole set ( $S, J, Q, P, \mathcal{P}, \mu$ ) and simultaneously fixing the remaining ones, respectively, lead to their corresponding conjugate quantities as expected. Subsequently, one can obtain the differential first law (18) and the integral Bekenstein-Smarr relation (19) with the conjugate thermodynamic potentials correctly reproduced by the ordinary Maxwell relations.

Let us now demonstrate the above conclusion in more detail. Differentiating the squared-mass formula (25) with respect to the entropy  $S$  yields the conjugate Hawking temperature,

$$\begin{aligned} T &= \left. \frac{\partial M}{\partial S} \right|_{(J,Q,P,\mathcal{P},\mu)} = \frac{8P}{3\mu M} \left[ \frac{8P}{3} S^2 + \pi(P^2 + Q^2) \right] - \frac{M}{2S} \\ &= \frac{r_+ - d}{\pi l^2} - \frac{m}{2\pi(r_+^2 - 2dr_+ - k^2 + l^2)}, \end{aligned} \quad (26)$$

and the corrected angular velocity, the electrostatic and magnetostatic potentials, which are conjugate to  $J$ ,  $Q$ , and  $P$ , respectively, can be computed as

$$\Omega = \left. \frac{\partial M}{\partial J} \right|_{(S,Q,P,\mathcal{P},\mu)} = \frac{\mu J}{2MS} = \frac{l}{r_+^2 - 2dr_+ - k^2 + l^2}, \quad (27)$$

$$\Phi = \left. \frac{\partial M}{\partial Q} \right|_{(S,J,P,\mathcal{P},\mu)} = \frac{8\pi PQ}{3\mu M} S = \frac{q(r_+ - p^2/m)}{r_+^2 - 2dr_+ - k^2 + l^2}, \quad (28)$$

$$\Psi = \left. \frac{\partial M}{\partial P} \right|_{(S,J,Q,\mathcal{P},\mu)} = \frac{8\pi PP}{3\mu M} S = \frac{p(r_+ - p^2/m)}{r_+^2 - 2dr_+ - k^2 + l^2}. \quad (29)$$

Similarly, by the differentiation of the squared-mass formula (25) with respect to the pressure  $\mathcal{P}$  and the dimensionless parameter  $\mu$ , one can obtain the thermodynamical volume and a new chemical potential,

$$\begin{aligned} V &= \left. \frac{\partial M}{\partial \mathcal{P}} \right|_{(S,J,Q,P,\mu)} = \frac{4S}{3\mu M} \left[ \frac{8P}{3} S^2 + \pi(P^2 + Q^2) \right] \\ &= \frac{4}{3}(r_+ - d)S, \end{aligned} \quad (30)$$

$$\begin{aligned} K &= \left. \frac{\partial M}{\partial \mu} \right|_{(S,J,Q,P,\mathcal{P})} = \frac{M}{2\mu} - \frac{8PS}{3\mu^2 M} \left[ \frac{4P}{3} S^2 + \pi(P^2 + Q^2) \right] \\ &= m \frac{l^2 - (r_+ + q^2/m)(r_+ - p^2/m)}{4\pi(r_+^2 - 2dr_+ - k^2 + l^2)}. \end{aligned} \quad (31)$$

All the above conjugate quantities correctly reproduce those expressions previously presented in Eqs. (17), (20), and (21). Using all these thermodynamical conjugate pairs, it is trivial to check that both the differential first law (18) and the integral mass formula (19) are completely satisfied at the same time.

### C. Chirality condition and reduced mass formulas

Now we would like to discuss in details about the impact of the chirality condition ( $J = MI$ ) on the thermodynamical relations of the ultraspinning dyonic Kerr-Sen-AdS<sub>4</sub> black hole. By virtue of the existence of the chirality condition ( $J = MI$ ), three thermodynamical quantities ( $M$ ,  $J$ ,  $\mathcal{P}$ ) are, in fact, not truly independent of each other, and there is a constraint relation among them,

$$J^2 = 3M^2/(8\pi\mathcal{P}), \quad (32)$$

which implies that the above ultraspinning dyonic black hole is actually a degenerate thermodynamical system and there are not enough parameters to hold completely fixed when performing the differential operations in the last subsection. Once taking into account this chirality condition physically, the differential first law (18) and the integral Bekenstein-Smarr formula (19) should be constrained by Eq. (32), and actually depict a degenerate thermodynamical system.

Choosing  $J$  as a redundant variable (although it is a real observable quantity)<sup>2</sup> and eliminating it from the differential and integral mass formulas with the help of  $l^2 = 3/(8\pi\mathcal{P})$ , the first law (18) and the Bekenstein-Smarr relation (19) now boil down to the following nonstandard forms:

$$\begin{aligned} (1 - \Omega l) dM &= T dS + V' d\mathcal{P} + \Phi dQ + \Psi dP + K d\mu, \\ (1 - \Omega l) M &= 2(TS - V'\mathcal{P}) + \Phi Q + \Psi P, \end{aligned} \quad (33)$$

where

<sup>2</sup>Alternately, one can try to eliminate  $\mathcal{P}$  rather than  $J$  via Eq. (32) also.



$$V' = V - \frac{J\Omega}{2\mathcal{P}} = V - \frac{4\pi}{3}\Omega M l^3.$$

It is clear that the thermodynamic quantities in the above formulas cannot comprise the ordinary canonical conjugate pairs due to the existence of a factor  $(1 - \Omega l)$  in front of  $dM$  and  $M$ .

Meanwhile, the squared-mass formula (25) reduces to

$$M^2 \left(1 - \frac{\mu}{16\pi\mathcal{P}S}\right) = \frac{8\mathcal{P}S}{3\mu} \left[\frac{4\mathcal{P}}{3}S^2 + \pi(P^2 + Q^2)\right]. \quad (34)$$

In this way, one actually regards the enthalpy  $M$  as the fundamental functional relation  $M = M(S, Q, P, \mathcal{P}, \mu)$ . Resembling the strategy employed in the last subsection, one can deduce the above nonstandard differential and integral mass formulas from Eq. (34) by using the standard Maxwell rule.

#### D. Horizon geometry and conformal boundary

From now on, we shall concentrate on other basic properties, such as the horizon geometry and conformal boundary, and RII of the ultraspinning dyonic Kerr-Sen-AdS<sub>4</sub> black hole, as well as bounds on the mass and horizon radius in the extremal case. It is suggestive to recast the metric (16) into another helpful form,

$$ds^2 = -\frac{\Delta(r)\Sigma dt^2}{[2m(r-d) - p^2 - q^2]l^2} + \frac{\Sigma dr^2}{\Delta(r)} + \frac{\Sigma d\theta^2}{\sin^2\theta} + \frac{2m(r-d) - p^2 - q^2}{\Sigma} \left\{ l \sin^2\theta d\varphi - dt \right. \\ \left. + \frac{(r^2 - 2dr - k^2 + l^2)\Sigma}{[2m(r-d) - p^2 - q^2]l^2} dt \right\}^2. \quad (35)$$

To ensure that the spacetime outside the event horizon has the correct Lorentzian signature, the following inequalities must be simultaneously satisfied:

$$\Sigma \geq 0, \quad \Delta(r) \geq 0, \\ 2m(r-d) - p^2 - q^2 = 2m(r - p^2/m) \geq 0. \quad (36)$$

Given that the mass parameter of the ultraspinning dyonic black hole is absolutely positive ( $m > 0$ ), it is immediately required that

$$r \geq p^2/m. \quad (37)$$

And this also meets the requirement:  $\Sigma = (r + q^2/m) \times (r - p^2/m) + l^2 \cos^2\theta \geq 0$ . Then it can be checked that  $g_{\varphi\varphi} = 2ml^2(r - p^2/m) \sin^4\theta/\Sigma \geq 0$  is strictly guaranteed outside the event horizon, and thus the geometry is free of any closed timelike curve (CTC). Finally, the condition  $\Delta(r) \geq 0$  results in the following inequalities:

$$[(r + q^2/m)(r - p^2/m) + l^2]^2 \geq 2m(r - p^2/m)l^2 \geq 0. \quad (38)$$

Only when the above two requirements (37) and (38) are met, the spacetime outside the event horizon is Lorentzian and free of CTC.

To explore the geometry of the event horizon, let us study the constant  $(t, r)$  surface on which the induced metric is

$$ds_h^2 = \frac{\Sigma_+}{\sin^2\theta} d\theta^2 + \frac{(r_+^2 - 2dr_+ - k^2 + l^2)^2}{\Sigma_+} \sin^4\theta d\varphi^2, \quad (39)$$

where  $\Sigma_+ = r_+^2 - 2dr_+ - k^2 + l^2 \cos^2\theta$ . It is clear that this metric is singular at  $\theta = 0$  and  $\theta = \pi$ . Let us first examine whether the metric is ill-defined at  $\theta = 0$ , and analyze it in the limit  $\theta \rightarrow 0$ . In the small angle case ( $\theta \sim 0$ ), we can introduce a new variable  $\kappa = l(1 - \cos\theta)$ . Using  $\sin^2\theta \simeq 2\kappa/l$ , the two-dimensional cross section (39) for small  $\kappa$  can be written as

$$ds_h^2 = (r_+^2 - 2dr_+ - k^2 + l^2) \left( \frac{d\kappa^2}{4\kappa^2} + \frac{4\kappa^2}{l^2} d\varphi^2 \right). \quad (40)$$

The above metric (40) naturally reduces to what was considered [17] in the ultraspinning Kerr-Sen-AdS<sub>4</sub> black hole case when the magnetic charge parameter vanishes ( $p = 0$ ), and is clearly a metric of constant, negative curvature on a quotient of the hyperbolic space  $\mathbb{H}^2$ . Due to the symmetry, one can perform a similar analysis in the  $\theta \sim \pi$  case and get the same result in the  $\theta \rightarrow \pi$  limit. Apparently, the space is free from pathologies near the north and south poles. Topologically, the event horizon is a sphere with two punctures and sometimes is called the black spindle [16]. This indicates that the above ultraspinning dyonic Kerr-Sen-AdS<sub>4</sub> black hole owns a finite area but noncompact horizon.

Next, we wish to study the conformal boundary of the ultraspinning dyonic Kerr-Sen-AdS<sub>4</sub> black hole. After multiplying the metric (16) with the conformal factor  $l^2/r^2$  and taking the  $r \rightarrow \infty$  limit, the boundary metric reads

$$ds_b^2 = -dt^2 + 2l \sin^2\theta dt d\varphi + l^2 d\theta^2 / \sin^2\theta, \quad (41)$$

which is the same one as those of the superentropic Kerr-Newman-AdS<sub>4</sub> black hole [6] and the ultraspinning Kerr-Sen-AdS<sub>4</sub> black hole [17]. Obviously, the coordinate  $\varphi$  is null on the conformal boundary. In the small  $\kappa = l(1 - \cos\theta)$  limit, the conformal boundary metric (41) can be reexpressed as

$$ds_b^2 = -dt^2 + 4\kappa dt d\varphi + l^2 d\kappa^2 / (4\kappa^2), \quad (42)$$

which is interpreted in [3] as an AdS<sub>3</sub> written as a Hopf-like fibration over  $\mathbb{H}^2$ . It follows that the metric has nothing pathological near two poles:  $\theta = 0$  and  $\theta = \pi$ .

To end this subsection, it needs to point out that the boundary metric (42), which is a negative curvature maximally symmetric spacetime that solves the Einstein- $\Lambda$  in three dimensions, is locally isomorphic to AdS<sub>3</sub>, but

not globally identified as  $\text{AdS}_3$ , since the coordinate  $\varphi$  is a periodic one. In other words, the global identification of this coordinate  $\varphi$  makes the whole spacetime topologically different from  $\text{AdS}_3$ .

### E. Bounds on the mass and horizon radii of extremal ultraspinning dyonic black holes

In Ref. [17], we have discussed the bounds on the mass and horizon radius of the extremal ultraspinning Kerr-Sen- $\text{AdS}_4$  black hole. Here we would like to seek some similar inequalities for its dyonic counterpart. In the extremal dyonic black hole case, two roots of the horizon equation  $\Delta(r) = 0$  coincide with each other, and its location is determined by  $\Delta(r_e) = \Delta'(r_e) = 0$ , whose explicit expressions are given by

$$\begin{aligned} \left(r_e^2 + \frac{q^2 - p^2}{m} r_e - \frac{q^2 p^2}{m^2} + l^2\right)^2 &= 2ml^2 \left(r_e - \frac{p^2}{m}\right), \\ \left(r_e^2 + \frac{q^2 - p^2}{m} r_e - \frac{q^2 p^2}{m^2} + l^2\right) \left(2r_e + \frac{q^2 - p^2}{m}\right) &= ml^2, \end{aligned} \quad (43)$$

from which one can get a quadratic equation and a cubic equation about the radius  $r_e$ ,

$$r_e^2 + \frac{q^2 - 5p^2}{3m} r_e - \frac{m^2 l^2 + p^2(q^2 - 2p^2)}{3m^2} = 0, \quad (44)$$

$$\left(r_e - \frac{p^2}{m}\right) \left(r_e + \frac{q^2 - p^2}{2m}\right)^2 - \frac{ml^2}{8} = 0. \quad (45)$$

Using Eq. (44), one can eliminate the  $r_e^2$  and  $r_e$  terms from Eq. (45) to arrive at the expression for the extremal horizon radius,

$$r_e = \frac{p^2}{m} - 8ml^2 \frac{p^2 + q^2 - 9m^2/16}{(p^2 + q^2)^2 + 12m^2 l^2}, \quad (46)$$

and resubmit it into Eqs. (44) and (45) to obtain an important equality relating the solution parameters,

$$\begin{aligned} \left[l^2 + \frac{27}{256} m^2 - \frac{1}{4m^2} \left(p^2 + q^2 - \frac{9}{8} m^2\right)^2\right]^2 \\ = -\frac{1}{4m^2} \left(p^2 + q^2 - \frac{9}{16} m^2\right)^3, \end{aligned} \quad (47)$$

which will give a stringent restriction on the parameters range allowed by the extremal dyonic solution.

In order to analyze the parameters equation (47), it is convenient to include two new variables: the rescaled mass  $y = m/l$  and the rescaled scalar charge  $x = (p^2 + q^2)/(2ml) = \sqrt{d^2 + k^2}/l$  to rewrite Eq. (47) as a quadratic equation of  $y$ ,

$$\frac{27}{64} y^2 + \frac{1}{4} x(x^2 - 9)y - (x^2 - 1)^2 = 0, \quad (48)$$

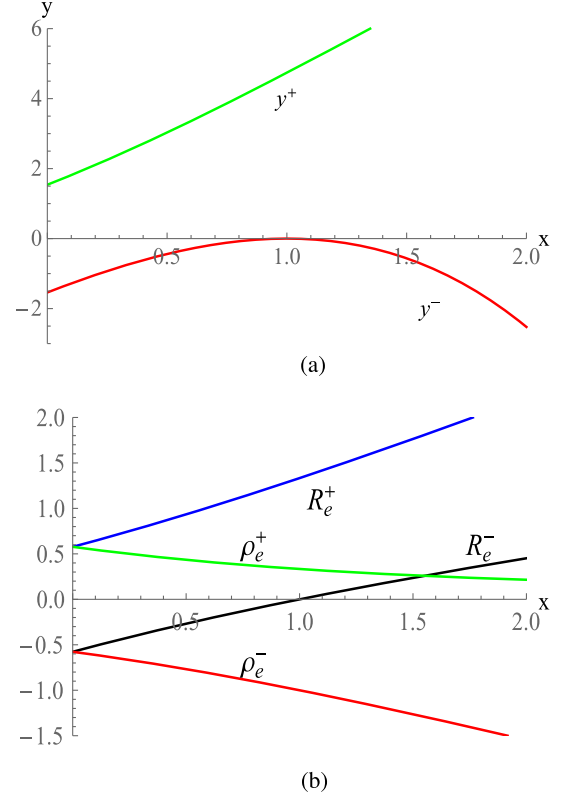


FIG. 1. On the basis of the reasonable range of the positive mass and cosmological scale ( $m > 0$  and  $l > 0$ ), the negative root  $y^-$ , and accordingly  $\rho_e^-$  and  $R_e^-$  should be excluded from our discussions on the ground of physical reason. In the interval  $x \in [0, \infty)$ , the root  $y^+$  is a monotonic increasing function of  $x$ ,  $\rho_e^+$  is monotonic decreasing but  $R_e^+$  is monotonic increasing with increasing  $x$ . The origin  $x = 0$  is equivalent to  $p = q = 0$ , which corresponds to the extremal ultraspinning Kerr- $\text{AdS}_4$  black hole case. (a) Scaled mass vs rescaled scalar charge and (b) Shifted radii vs rescaled scalar charge with  $l = 1$ .

which admits two real roots,

$$y^\pm = \frac{8}{27} [x(9 - x^2) \pm (x^2 + 3)^{3/2}]. \quad (49)$$

Meanwhile, we would like to introduce also two shifted radii<sup>3</sup>

$$\begin{aligned} \rho_e &= r_e - \frac{p^2}{m} = \frac{9y - 32x}{8(x^2 + 3)} l, \\ R_e &= r_e - d = lx + \frac{9y - 32x}{8(x^2 + 3)} l. \end{aligned} \quad (50)$$

In Fig. 1, we plot the rescaled mass and rescaled shifted radii as functions of the rescaled scalar charge in the physical range  $x \in [0, \infty)$ . For the positive mass and

<sup>3</sup>This suggests to adopt the radial coordinate  $\rho = r - p^2/m$  or  $R = r - d$ .

cosmological scale ( $m > 0$  and  $l > 0$ ), only the root  $y^+$  is admissible, and acquires a minimal value:  $8\sqrt{3}/9$  at  $x = 0$  ( $p = q = 0$ ). At the same time,  $\rho_e^+$  and  $R_e^+$  intersect at the Hawking-Page phase transition scale  $r_{\text{HP}} = l/\sqrt{3}$  when  $x = 0$ . This implies that the extremal mass has the lowest bound,

$$m = m_e \geq \frac{8l}{3\sqrt{3}}, \quad (51)$$

but  $r_{\text{HP}} = l/\sqrt{3}$  becomes, respectively, the upper and lower bounds of the shifted horizon radii,

$$\rho_e^+ \leq \frac{l}{\sqrt{3}} \leq R_e^+. \quad (52)$$

The above bounds on the extremal mass and horizon radii are in complete accordance with the results previously obtained in the extremal ultraspinning Kerr-Sen-AdS<sub>4</sub> case [17]. In particular, the detailed discussion made in this subsection further confirms the conclusion in the Note added of our previous paper [17].

### F. Reverse isoperimetric inequality

It has been conjectured [4] that the AdS black holes fulfil the following RII:

$$\mathcal{R} = \left[ \frac{(D-1)V}{\mathcal{A}_{D-2}} \right]^{1/(D-1)} \left( \frac{\mathcal{A}_{D-2}}{A} \right)^{1/(D-2)} \geq 1, \quad (53)$$

with  $\mathcal{A}_{D-2} = 2\pi^{(D-1)/2}/\Gamma[(D-1)/2]$  being the area of the unit  $(D-2)$ -sphere and  $A = 4S$  the horizon area. Equality is attained for the Schwarzschild-AdS black hole, which implies that the Schwarzschild-AdS black hole has the maximum entropy. In other words, for a given entropy, the Schwarzschild-AdS black hole owns the least volume.

Now, we would like to directly check whether or not the ultraspinning dyonic Kerr-Sen-AdS<sub>4</sub> black hole obeys this RII. We have already known that the area of the unit two-dimensional sphere, the thermodynamic volume, and the horizon area are  $\mathcal{A}_2 = 2\mu$ ,  $V = 4(r_+ - d)S/3$ , and  $A = 4S = 2\mu(r_+^2 - 2dr_+ - k^2 + l^2)$ , respectively. Therefore, the isoperimetric ratio is

$$\begin{aligned} \mathcal{R} &= \left( \frac{r_+ - d}{2\mu} A \right)^{1/3} \left( \frac{2\mu}{A} \right)^{1/2} \\ &= \left[ \frac{(r_+ - d)^2}{(r_+ - d)^2 - d^2 - k^2 + l^2} \right]^{1/6}. \end{aligned} \quad (54)$$

Clearly, the ratio of  $\mathcal{R}$  is uncertain. If  $0 \leq d^2 + k^2 < l^2$  (namely,  $0 \leq p^2 + q^2 < 2ml$ ), then  $\mathcal{R} < 1$ , which implies that the ultraspinning dyonic Kerr-Sen-AdS<sub>4</sub> black hole violates the RII, and is superentropic. Otherwise if  $d^2 + k^2 \geq l^2$  (or  $p^2 + q^2 \geq 2ml$ ), one then obtains  $\mathcal{R} \geq 1$ .

In this case, the ultraspinning dyonic Kerr-Sen-AdS<sub>4</sub> black hole obeys the RII and is subentropic. Because the value range of  $\mathcal{R}$  crucially depends upon the values of the solution parameters ( $p$ ,  $q$ ,  $m$ , and  $l$ ), one can find that the ultraspinning dyonic Kerr-Sen-AdS<sub>4</sub> black hole is not always superentropic, similar to the purely electric-charged case that describes the ultraspinning Kerr-Sen-AdS<sub>4</sub> black hole [17]. Only when the parameters obey the inequality  $p^2 + q^2 < 2ml$  does it violate the RII, while the superentropic dyonic Kerr-Newman-AdS<sub>4</sub> black hole always violates the RII [1]. As far as this point is concerned, these dyonic AdS<sub>4</sub> black holes exhibit one remarkable different property.

## V. CONCLUSIONS

In this paper, we have extended our previous work [17] to a more general dyonic case and investigated some interesting properties of the dyonic Kerr-Sen-AdS<sub>4</sub> black hole and its ultraspinning counterpart in the four-dimensional gauged EMDA theory. To this end, we first presented an exquisite form for the dyonic Kerr-Sen black hole solution and found its generalization by including a nonzero negative cosmological constant, namely, the dyonic Kerr-Sen-AdS<sub>4</sub> black hole. Then by applying a simple  $a \rightarrow l$  limit procedure, we obtained its ultraspinning cousin. All the expressions of these solutions, namely, their metric, the Abelian gauge potential, and its dual potential, as well as the dilaton scalar and axion pseudoscalar fields are very convenient for exploring their thermodynamical properties. We presented all necessary thermodynamic quantities and demonstrated that they obey both the differential and integral mass formulas. Furthermore, we displayed new Christodoulou-Ruffini-like squared-mass formulas for these four-dimensional AdS<sub>4</sub> black holes, from which all expected thermodynamic conjugate partners can be computed by differentiating these squared-mass formulas with respect to their corresponding thermodynamic variables and are demonstrated to constitute the ordinary canonical conjugate pairs in the standard forms of black hole thermodynamics.

In particular, we have utilized the method advocated in Refs. [14,17] to show that all thermodynamical quantities of the ultraspinning dyonic Kerr-Sen-AdS<sub>4</sub> black hole can be obtained via taking the same ultraspinning  $a \rightarrow l$  limit to those of their corresponding predecessor in the rotating frame at infinity. After that, we have discussed in detail about the impact of the chirality condition on the actual thermodynamics of this ultraspinning dyonic black hole. To a certain extent, these aspects resemble those of the superentropic dyonic Kerr-Newman-AdS<sub>4</sub> and the ultraspinning Kerr-Sen-AdS<sub>4</sub> black hole.

Paralleling to the work done in Ref. [17], we have discussed some bounds on the mass and horizon radius of the extremal ultraspinning dyonic Kerr-Sen-AdS<sub>4</sub> black hole. Our results further confirm those established for the

ultraspinning Kerr-Sen-AdS<sub>4</sub> black hole [17] and reproduce its conclusion when the magnetic charge parameter vanishes.

Like the purely electric-charged case of the ultraspinning Kerr-Sen-AdS<sub>4</sub> black hole [17], we have also found that the ultraspinning dyonic Kerr-Sen-AdS<sub>4</sub> black hole is not always superentropic, since the RII is violated only when  $p^2 + q^2 < 2ml$ . Once  $p^2 + q^2 \geq 2ml$ , the ultraspinning dyonic Kerr-Sen-AdS<sub>4</sub> black hole will be subentropic. This black hole resembles the ultraspinning Kerr-Sen-AdS<sub>4</sub> black hole which does not always violate the RII [17], but is in sharp contrast with the superentropic dyonic Kerr-Newman-AdS<sub>4</sub> black hole that always violates the RII [1]. A most related issue is to investigate whether or not the

RII is violated in the reduced form of extended thermodynamic phase space, as did in Ref. [15].

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