

Entanglement entropy at critical points of classical evolution in oscillatory and exotic singularity multiverse models

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Using the third quantization formalism we study the quantum entanglement of universes created in pairs within the framework of standard homogeneous and isotropic cosmology. In particular, we investigate entanglement quantities (entropy, temperature) around the maxima, minima and inflection points of the classical evolution. The novelty from previous works is that we show how the entanglement changes in an extended minisuperspace parametrized by the scale factor and additionally, by the massless scalar field. We study the entanglement quantities for the universes which classically exhibit big bang and other than big bang (exotic) singularities such as big brake, big freeze, big separation, and little rip. While taking into account the scalar field, we find that the entanglement entropy is finite at the big bang singularity and diverges at the maxima or minima of expansion. As for the exotic singularity models we find that the entanglement entropy or the temperature in all the critical points and singularities is either finite or infinite, but it never vanishes. This shows that each of the universes of a pair is entangled to a degree parametrized by the entanglement quantities which measure the quantumness of the system. Apart from the von Neumann entanglement entropy, we also check the behavior of the Tsallis and the Renyi entanglement entropies, and find that they behave similarly to the meters of the quantumness. Finally, we find that the best-fit relation between the entanglement entropy and the Hubble parameter (which classically marks special points of universe evolution) is of the logarithmic shape, and not polynomial, as one could initially expect.

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I. INTRODUCTION

Since its beginning [1], the canonical quantum cosmology has suffered from lack of explanations, like, for example, what happens with the arrow of time or finding a unique solution for the wave function of the Universe or the wave function interpretation (see for example [2,3]).

There is, however, the formalism of the third quantization [4] which is quite natural, if we believe in the existence of a wave function of the Universe Ψ . This formalism interprets universes as particles, i.e., the perturbations around the vacuum of the virtual universes, where the most natural way to create them is in pairs, as is the case in quantum field theory. One universe plays the role of a particle and the other (the antiuniverse) of an antiparticle. In the simplest case, we associate wave functions to those

universes as complex scalar functions, and therefore, each of them is described by complex conjugated wave functions corresponding to particles and antiparticles.

The validity of the formalism is still under debate since there are not any experimental results to confirm it yet. Here, we present a recent formalism to analyze the quantumness of a pair of universes using their entanglement quantities, which could shed light on some aspects of quantum cosmology such as, for instance, the question of if the quantumness is restricted to the initial singularity of the universe or takes place throughout the whole evolution of the universe. There have been some studies during last decades that have tried to test the existence of the multiverse as the collection of the universes (see for example [5–10]). They considered the phenomenon of quantum entanglement between universes within the multiverse as essential for the task. The point was to calculate the entanglement entropy explicitly and to draw attention to some particularities of this approach.

In fact, the underlying Wheeler-DeWitt equation of canonical quantum cosmology is nothing else than the

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Hamiltonian constraint $H\Psi = 0$ of the minisuperspace which we work with. It describes the wave function of each individual universe in a set of universes—the multiverse. However, the application of quantum theory to the Universe as a whole challenges some of its fundamentals and forces us to rethink them. How to interpret the wave function Ψ is clearly one of those fundamentals.

Let us point out that there are two main interpretations of Ψ in quantum theory. The first one, the more “wavelike” interpretation of quantum mechanics, where the wave function represents the state of a system, possibly with an internal structure (let us think on a hydrogen atom), and whose quantum state can be described in terms of the elements of some basis of the corresponding Hilbert space. The second one, the more “particlelike” interpretation of the quantum field theory, which describes a field whose quantum state can be interpreted in terms of particles that can, in principle, be measured separately. They are related, but there are subtle differences in the interpretation. For instance, in the former the basis state $|n\rangle$ does not represent any entity by itself but a specific *level* (e.g., an energy level) in terms of which we can describe the state of the whole system. In the latter, the basis state $|n\rangle$ does represent the number of more or less independent entities called *particles*. In fact, a possible way to characterize this dichotomy is to ask what is exactly the entity: the field Ψ or the particles.

In quantum cosmology the former is the customary interpretation given to the wave function Ψ [11–13], where it represents the state of the whole Universe with its internal structure (spacetime + matter fields). The latter is exactly the interpretation adopted in the third quantization formalism [14,15] (see also Ref. [1]), where the universes are interpreted as particles moving in the superspace of spatial geometries and matter field configurations. Here, we are going to pose and analyze a phenomenon that is naturally contextualized in the third quantization formalism, the creation of a universe-antiuniverse pair, but from the point of view of their wave functions. This is justified because we do not want to study the “many-particle” state of the multiverse but only the state and the quantum correlations of a couple of these newborn pairs of universes.

At the beginning of the evolution, the global state of a pair of universes is taken as the ground state $|00\rangle$ for all quantum representations [16]. Once the evolution takes place, the vacuum state changes differently in different representations. For instance, the diagonal representation of the Hamiltonian H is not invariant under the evolution of the universes which means that any instantaneous eigenstate $|N_0\rangle_d$ at a given moment in time spreads into the infinite set of the basis components, $\{|N\rangle_d\}_{N\in\mathbb{N}}$, of the diagonal representation at any later time. On the other hand, the invariant representation of the Hamiltonian H is invariant under the evolution [17,18]. This means that a particular state $|N_0\rangle_i$ of the invariant representation remains the same state along the entire evolution. It seems therefore

a very plausible boundary condition to be imposed that the universes of a universe-antiuniverse pair are in the ground state of the invariant representation which implicitly assumes that the universe-antiuniverse pair is isolated, i.e., noninteracting with the rest of universe-antiuniverse pairs of the multiverse. As we have said, at the beginning of the evolution these two representations coincide, so the universes are also initially in the ground state of the instantaneous diagonal representation. However, as the universes evolve, their quantum states are represented at any later time by a superposition of excited states of the diagonal representation. It turns out that this state is a mixed state for which the entropy can be computed and is different from zero [16].

As we know from quantum optics and quantum mechanics, the entropy of entanglement is a good measure of the *quantumness* of a physical system, being interpreted in terms of the quantum correlations existing between the states of the components. Thus, we can use the entropy of entanglement between a universe and its corresponding antiuniverse to study the degree of quantumness of the universe. It was shown in Ref. [16] that the entanglement entropy is infinite at the maximum point of classical expansion of any universe in the multiverse. A possible weak point of these calculations could be the usage of the WKB approximation, where the wave function diverges precisely at the maximum. In this paper we are not going to make any such approximations and want to analyze the entanglement at any critical point during the evolution of the universe *in time* that in quantum cosmology is supposed to be directly represented by the scale factor. Our procedure is going to be as exact as possible.

In Sec. II, we deal with mathematical details of the third quantization: the existence of conjugated solutions of the Wheeler-DeWitt equation, their physical meaning and the distribution of modes of the scalar field. In Sec. III, we analyze the exact entropy of entanglement of a universe with a massless scalar field as a second degree of freedom in the minisuperspace, in that way generalizing and checking the results of [16]. In Sec. IV, after confirmation that the entanglement entropy is infinite at the maximum, we study what happens at other critical points of the classical evolution (minima in oscillating models and the inflection points). We analyze the entanglement quantities for the universes with some exotic (non-big-bang) singularities in Sec. V, and in Sec. VI we suggest how the entanglement entropy behaves in the vicinity of singularities using our simple model. Finally, in Sec. VII, we give our conclusions.

II. MATHEMATICAL FORMALISM OF THIRD QUANTIZATION

First of all, let's write the total action for gravity and a scalar field [2]:

$$S = \frac{1}{2} \int dt N \left(-\frac{a\dot{a}^2}{N^2} + aK - \frac{\Lambda a^3}{3} \right) + \frac{1}{2} \int dt N a^3 \left(\frac{\dot{\phi}^2}{N^2} - 2V(\phi) \right), \quad (1)$$

where a is the scale factor, ϕ a scalar field, N the lapse function, K the curvature index, Λ the cosmological constant and $V(\phi)$ the potential of the scalar field. From here we see that the canonical momenta are

$$p_a = -\frac{\dot{a}}{N}, \quad p_\phi = \frac{a^3 \dot{\phi}}{N}, \quad (2)$$

which let us write the associated Hamiltonian as

$$H = \frac{N}{2} \left(-\frac{p_a^2}{a} + \frac{p_\phi^2}{a^3} - aK + \frac{\Lambda a^3}{3} + 2a^3 V(\phi) \right). \quad (3)$$

Taking the quantization of the momenta as

$$p_a^2 := -\frac{1}{a} \frac{\partial}{\partial a} \left(a \frac{\partial}{\partial a} \right), \quad p_\phi^2 := -\frac{\partial^2}{\partial \phi^2}, \quad (4)$$

where $\hbar = 1$, the most general Hamiltonian constraint we are going to consider, using the parametrization $\alpha = \ln(a)$, is written as

$$\frac{\partial^2}{\partial \alpha^2} - \frac{\partial^2}{\partial \phi^2} - \left[e^{4\alpha} K - e^{6\alpha} \left(\frac{\Lambda}{3} + 2V(\phi) \right) \right] = 0. \quad (5)$$

A very interesting case is the case for which the scalar field has no potential $V(\phi)$, which leads to a very simple Hamiltonian constraint, where the function in brackets in (5) is only a function of the scale factor.

Lewis and Riesenfeld in Ref. [17] described how to find an invariant operator for the time-dependent quantum harmonic oscillator such that the number operator is constant [17]. Years after, Kim [18] found how to simplify the method by knowing any two linearly independent solutions on the diagonal representation. Our first task is to find the solutions to the Wheeler-DeWitt equation which for a minisuperspace (a, ϕ) built on the massless scalar field ϕ and the scale factor $a = e^\alpha$ has the hyperbolic form

$$\left[\frac{\partial^2}{\partial \alpha^2} - \frac{\partial^2}{\partial \phi^2} + f(\alpha, \Lambda, \dots) \right] \Psi(\alpha, \phi) = 0. \quad (6)$$

This is analogous to the quantum harmonic oscillator equation, where $f(\alpha, \Lambda, \dots)$ is the function which depends on the scale factor and other parameters of the chosen model but the scalar field, since we consider that $V(\phi) = 0$, such as the cosmological constant Λ , and $\Psi(\alpha, \phi)$ is the wave function of the universe.

We opt for making the choice of $\Psi(\alpha, \phi)$ to be separable by Fourier modes such as

$$\Psi(\alpha, \phi) = \int_k dk A(k) e^{\pm i k \phi} \varphi_k(\alpha), \quad (7)$$

where k refers to the number of a mode of the scalar field ϕ , $A(k)$ is a weight function representing the distribution of the modes k and $\varphi_k(\alpha)$ is the wave function of the universe for each mode k that after inserting (7) into (6) fulfills the differential equation

$$\left[\frac{\partial^2}{\partial \alpha^2} + k^2 + f(\alpha, \Lambda, \dots) \right] \varphi_k(\alpha) = 0. \quad (8)$$

Taking k as constant, the solutions of this equation are usually a combination of Bessel functions.

If we follow the analytic Frobenius method to find the general solution to (8), we find that the solutions always exist, and that they strictly depend on the structure ik , and hence the two linearly independent solutions $\varphi_k^{(1,2)}(\alpha)$ of (8) can be found to be complex conjugated. Furthermore, since the solutions depend on the structure ik , the transformation $k \rightarrow -k$ is equivalent to the complex conjugation $i \rightarrow -i$, and therefore the wave functions of both universes can be interpreted as a particle and an antiparticle due to the complex conjugation or because they share an opposite distribution of modes.

This is an indication that the modes of the scalar field can be distributed, as a very physical example, such that the positive modes are taken by one of the universes and the negative modes for the other, making us take the global solutions in (7) as

$$\Psi^{(1)}(\alpha, \phi) := \int_0^\infty dk A(k) e^{-ik\phi} \varphi_k^{(1)}(\alpha), \quad (9)$$

$$\Psi^{(2)}(\alpha, \phi) := \int_{-\infty}^0 dk A(k) e^{-ik\phi} \varphi_k^{(2)}(\alpha), \quad (10)$$

where we use the symmetry of the solutions and choose the sign for $e^{\pm ik\phi}$ as a plane wave that travels toward positive modes. Besides, we can prove that $\Psi^{(1,2)}(\alpha, \phi)$ are complex conjugated if and only if $A(k)$ is symmetric around $k = 0$, which clearly shows the conservation of the energy of the whole system. Thus, $A(k)$ is going to be taken as a symmetric Gaussian distribution around $k = 0$ with σ as its standard deviation—taken as the constant

$$A(k) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{k^2}{2\sigma^2}}, \quad (11)$$

which is more sophisticated than the choice made in Ref. [19]. The reconstruction of $\Psi(\alpha, \phi)$ from $\varphi_k(\alpha)$ cannot in general be made analytically, so we cannot get analytic

results for even the simplest model we analyze, and the results also depend on σ . In view of that, we perform just numerical calculations. Anyway, this method works generally in order to find solutions and afterwards the entanglement entropy of any model with the Hamiltonian constraint like (6).

The function $A(k)$ was taken to be compared with the distribution of Ref. [19], where it was taken as the simplest example. In our case, it was demonstrated to be the only possible option since $A(k)$ must be symmetric around zero. In our approach there is no need to look for renormalization of the theory, since the term inside the integral where Bessel functions appear gives a finite contribution to the integral over k for any square integrable $A(k)$. That is probably the only constraint to $A(k)$, it must not diverge at infinity, and that is even in the most logical case for the distribution of modes; high probability for the ground state $k = 0$, and null probability for $k = \infty$. Introducing the cutoffs into the integral could perhaps be a solution, but we prefer a more natural way of proceeding. What is more, the explicit form of $A(k)$ can be taken as a boundary condition for the state of the universe, which cannot be known *a priori* in cosmology. The best we can do is to try different, *a priori* plausible, boundary conditions, and to check whether they evolve into the universe we observe and to expect distinguishing features that hopefully might discriminate among them. We cannot speculate on the “initial” quantum state of the universe at the very onset, since for that we should have and should use a full quantum theory of gravity. Instead, we can consider quantum states that represent small fluctuations around known (observable) states of the universe. In that sense, we should notice that the value of k is essentially an expected value (i.e., classical) of the momentum of the scalar field, and thus, the value $k = 0$ describes an ideal inflationary regime of the universe, for which $\dot{\phi} = 0$ and $V(\phi) = V(\phi_0) = \text{constant}$. From that point of view, we can interpret the constructed wave packets as representing a semiclassical state (i.e., a classical state with small fluctuations) of the inflationary universe. Thus, the Gaussian wave packets represent a good starting point to study the expected small departures from the classical state stemming from the existence of the entanglement between states of our Universe and a hypothetical partner antiuniverse. Other boundary states should perhaps be considered in some future investigations.

The algorithm starts with writing the Wheeler-DeWitt equation ([5]) as a quantum harmonic oscillator equation.

$$\ddot{\Psi}(\alpha, \phi) + \omega^2(\alpha, \phi)\Psi(\alpha, \phi) = 0, \quad (12)$$

where $\dot{\Psi} := \partial_\alpha \Psi$, and we recognize the momentum $P_\Psi := \dot{\Psi}$, and the frequency as

$$\omega^2(\alpha, \phi) := -\partial_\phi^2 + f(\alpha, \Lambda, \dots). \quad (13)$$

As soon as we are able to, we split the Hamiltonian constraint into two equations: the first for the scale factor

$$\left[\frac{\partial^2}{\partial \alpha^2} + f(\alpha, \Lambda, \dots) \right] \Psi(\alpha, \phi) = E_\alpha \Psi(\alpha, \phi), \quad (14)$$

and the second for the scalar field

$$-\frac{\partial^2}{\partial \phi^2} \Psi(\alpha, \phi) = E_\phi \Psi(\alpha, \phi), \quad (15)$$

recognizing E_α and E_ϕ as the energies associated for each variable and fulfilling $E_\alpha + E_\phi = 0$; the Wheeler-DeWitt equation is only dependent on the scale factor and the energy E_ϕ , which is fixed. Hence, the frequency in (13) is given by

$$\omega^2(\alpha, \phi) \equiv \omega^2(\alpha) := E_\phi + f(\alpha, \Lambda, \dots). \quad (16)$$

In the following, we are assuming that the universes are created in pairs. The notation $|U_- U_+\rangle \equiv |U_-\rangle |U_+\rangle$ represents a state in which $|U_-\rangle$ has been typically interpreted as describing an expanding universe and $|U_+\rangle$ a contracting one. However, in terms of the physical time experienced by their internal observers, whose physical time variables are reversely related [20,21], they can alternatively be interpreted as two expanding universes with conjugated charge and parity properties of their matter contents. Thus, the state $|U_- U_+\rangle$ can equivalently be interpreted as representing a universe-antiuniverse pair. In the present development we shall assume that interpretation. At the beginning, the pair is in the ground state $|00\rangle$ for all representations. However, such a state does not represent here the “no-particle” state, but the state of minimal excitation of the universes. In the invariant representation of the Hamiltonian that leads to (12) (see Appendix), the state $|00\rangle$ remains being the ground state along the entire evolution of the universes. In other words, the universe-antiuniverse pair stays in the ground state of the invariant representation along the entire evolution of the universes. That is the boundary condition which we impose on the state of the universe-antiuniverse pair, which seems to be quite natural provided that the universe-antiuniverse pair does not interact with other universes of the multiverse. However, as the universes expands, i.e., as the scale factor grows, the ground state $|00\rangle$ splits into a linear combination of excited states of the instantaneous diagonal representation, which we shall assume represents the state of the single universes at a given moment of time (i.e., at a given value of the scale factor).

Following Ref. [16] (cf. Appendix) we can write that the invariant representation is described by the annihilation and the creation operators (or perhaps more appropriately the “ladder operators” c_\pm and c_\pm^\dagger)

$$c_+ = \sqrt{\frac{1}{2}} \left[\frac{1}{R} \Psi(\alpha, \phi) + i(R\dot{\Psi} - \dot{R}\Psi(\alpha, \phi)) \right], \quad (17)$$

$$c_-^\dagger = \sqrt{\frac{1}{2}} \left[\frac{1}{R} \Psi(\alpha, \phi) - i(R\dot{\Psi} - \dot{R}\Psi(\alpha, \phi)) \right], \quad (18)$$

where, in this specific case (see Ref. [18] for further explanations), we can take

$$R := \sqrt{\Psi_{(1)}^2(\alpha, \phi) + \Psi_{(2)}^2(\alpha, \phi)}, \quad (19)$$

with $\Psi_{(1,2)}(\alpha, \phi)$ being any two real solutions of (6). Since the functions in (9) are complex conjugated, satisfying the Wheeler-DeWitt equation (6), the real and the imaginary parts of any of them are also solutions, such that both are real functions and are very nice candidates for the functions in (19).

On the other hand, the diagonal representation of the Hamiltonian is given by

$$b_+(\alpha) := \sqrt{\frac{\omega(\alpha)}{2}} \left[\Psi(\alpha, \phi) + \frac{i}{\omega(\alpha)} \dot{\Psi}(\alpha, \phi) \right], \quad (20)$$

$$b_-^\dagger(\alpha) := \sqrt{\frac{\omega(\alpha)}{2}} \left[\Psi(\alpha, \phi) - \frac{i}{\omega(\alpha)} \dot{\Psi}(\alpha, \phi) \right]. \quad (21)$$

The Bogoliubov transformation between representations (17) and (18) and also (20) and (21) is [22]

$$\hat{c}_- = \alpha_B \hat{b}_- - \beta_B \hat{b}_+^\dagger, \quad (22)$$

$$\hat{c}_-^\dagger = \alpha_B^* \hat{b}_-^\dagger - \beta_B^* \hat{b}_+, \quad (23)$$

where α_B and β_B are the Bogoliubov coefficients, that written in terms of R from (19) and the frequency from (15) are

$$\alpha_B = \frac{1}{2} \left[\frac{1}{R\sqrt{\omega}} + R\sqrt{\omega} - \frac{i\dot{R}}{\sqrt{\omega}} \right], \quad (24)$$

$$\beta_B = -\frac{1}{2} \left[\frac{1}{R\sqrt{\omega}} - R\sqrt{\omega} - \frac{i\dot{R}}{\sqrt{\omega}} \right]. \quad (25)$$

Therefore, the ground state of the invariant representation can be expanded in terms of vectors in the Fock space of the diagonal representation as

$$|00\rangle_i = \frac{1}{|\alpha_B|} \sum_{n=0}^{\infty} \left(\frac{|\beta_B|}{|\alpha_B|} \right)^n |n_- n_+\rangle_d, \quad (26)$$

where the states $|n_-\rangle_d$ and $|n_+\rangle_d$ refer to the excited levels of the universe and the antiuniverse, respectively, in the diagonal representation. The diagonal representation is

denoted by the index d , and the invariant representation by the index i .

The quantum state of each single universe of the universe-antiuniverse pair is described by the reduced density matrix that is obtained by tracing out from the composite quantum state the degrees of freedom of the partner universe. This yields

$$\begin{aligned} \rho_- &= \sum_{n=0}^{\infty} \langle n_+ | \rho | n_+ \rangle_d \\ &\propto \sum_n e^{\frac{\omega(\alpha)}{T(\alpha)}(n+\frac{1}{2})} |n_-\rangle \langle n_-\rangle_d, \end{aligned} \quad (27)$$

where $\rho = |0_- 0_+\rangle \langle 0_- 0_+|$ in any representation and $T(\alpha)$ is the temperature of entanglement of the obtained thermal state

$$T(\alpha) = \frac{\omega(\alpha)}{2 \ln[\coth(r)]}, \quad (28)$$

where (cf. [16])

$$q := \tanh(r) = \frac{|\beta_B|}{|\alpha_B|}, \quad (29)$$

whose values are into the interval $[0, 1]$. For some fixed, finite R , we can see that $q = 1$, if $\omega = 0$.

The state (27) shows that in the diagonal representation the quantum state of each single universe of the universe-antiuniverse pair is given by a nonseparable or mixed state whose associated entropy of entanglement measures the degree of quantum correlations between the two universes. In Ref. [16] it was stated that the temperature of entanglement is a good measure of the quantumness of the state. We shall use the temperature of entanglement (28) as well as the entropy of entanglement associated to the state (27) as the measures of the quantumness of the state of the universes.

The entanglement entropy is then calculated as the von Neumann entropy of the system

$$\begin{aligned} S_{\text{ent}}(\alpha) &:= -\text{Tr}[\rho \ln(\rho)] = \cosh^2(r) \ln[\cosh^2(r)] \\ &\quad - \sinh^2(r) \ln[\sinh^2(r)]. \end{aligned} \quad (30)$$

Next, we fix the model and calculate the entropy of entanglement. If we want to calculate it for a one-dimensional scenario, we just need to set $E_\phi = 0$ in (16), and solve the Wheeler-DeWitt equation.

We notice that the entanglement entropy diverges when the frequency $\omega^2(a)$ is zero, since it plays the role of the potential in the Hamiltonian constraint (12). Therefore r in

(30) and the entropy of entanglement in (29) are directly affected, giving $r = 0$, and hence $S_{\text{ent}} \rightarrow \infty$. Something similar happens with the parameter q in (30), which approaches the unity when $\omega \rightarrow 0$.

$$\cosh^2(r) = \frac{1}{1-q^2} = \begin{cases} \frac{(\frac{1}{R} + R|\omega|)^2 + \dot{R}^2}{4|\omega|}, & \text{if } \omega^2 \geq 0 \quad (\text{classically allowed}) \\ -\frac{\frac{1}{R^2} + (R|\omega| - \dot{R})^2}{4|\omega|R\dot{R}}, & \text{if } \omega^2 < 0 \quad (\text{classically forbidden}) \end{cases}. \quad (31)$$

After substitution into (29), we check that the entropy is real for the classically allowed region, and imaginary for the classically forbidden one. This reveals a relation between the entanglement entropy and the classical availability of the universe for certain regions in the phase space (α, ϕ) . Unless $\omega^2(\alpha, \phi)$ reaches an extremum, a point where the entropy is infinite indicates a wall between the classical region and the quantum region, where the entropy of entanglement can even take complex values.

If the universe shows a maximum where the entanglement entropy diverges, and at the initial singularity we assume that the universe is extremely quantum, it is clear that in the middle of its evolution there is, at least, a minimum of this entropy which seems, *a priori*, unrelated to any specific event of the evolution, though still it can have some meaning in the context of information theory which we have not yet discovered.

Another interesting variable is the temperature defined in (28). It was stated in [16] that the temperature could be a better measurement of the quantumness of a system than the entanglement entropy. Let us analyze this temperature together with the entanglement entropy. Using Eqs. (24), (25), (28) and (30), the temperature when the universe approaches a singularity ($\omega \rightarrow 0$) can be approximated to

$$\lim_{\omega \rightarrow 0} T(\alpha) \approx \frac{1 + R^2 \dot{R}^2}{4R^2}, \quad (32)$$

since $R > 0$ and $\omega^2 > 0$ for the classically allowed region. This temperature is never divergent around singularities if R is finite or R and \dot{R} are not divergent at the same time, which is, in general, the case. Therefore, the temperature is finite in general.

In cases when the universe does not recollapse, by assumption the wave function of the universe is made to vanish at infinity and the parameter R in (19) also vanishes. Then the temperature, in general, diverges for most of the frequencies $\omega(\alpha)$.

Finally, it is worth exposing another alternative to the entanglement entropy calculated as von Neumann entropy. First, we could track the behavior of the parameter q from (30), and other entropic measurements like, for example, Tsallis entropy $S_q^{(T)}$ [23,24] or Renyi entropy $S_q^{(R)}$ [25,26],

Furthermore, as $\omega^2(\alpha, \phi)$ is real, being negative for a classically permitted region and positive for the classically forbidden region, then $\omega(\alpha, \phi)$ can be real or purely imaginary, respectively, and from (30) one can obtain that

which are defined and written in terms of r , as we did for the von Neumann entropy (30). For example,

$$S_q^{(T)}(\rho) := \frac{1}{1-q} [\text{Tr}(\rho^q) - 1] \\ = \frac{1}{1-q} \left[\frac{\text{sech}^q(r)}{1 - \tanh^{2q}(r)} - 1 \right], \quad (33)$$

and

$$S_q^{(R)}(\rho) := \frac{1}{1-q} \ln[\text{Tr}(\rho^q)] \\ = \frac{1}{1-q} \ln \left[\frac{\text{sech}^q(r)}{1 - \tanh^{2q}(r)} \right], \quad (34)$$

respectively. These entropies are related to each other by

$$S_q^{(R)} = \frac{1}{1-q} \ln[1 + (1-q)S_q^{(T)}]. \quad (35)$$

III. EXTENDED MINISUPERSPACE ENTANGLEMENT ENTROPY

In order to extend the discussion of Ref. [16] into the matter of degrees of freedom, we consider massless scalar field as another coordinate in minisuperspace as it was presented in Sec. II.

We consider a closed universe ($K = +1$) without the cosmological constant and a massless scalar field [i.e., no potential term, so $V(\phi) = 0$] with an arbitrary distribution of modes k . In this scenario, we write (8) as (see, e.g., [2])

$$\left[\frac{\partial^2}{\partial \alpha^2} + k^2 - e^{4\alpha} \right] \varphi_k(\alpha) = 0, \quad (36)$$

whose solutions are modified Bessel functions of the first kind with complex order

$$\varphi_k^{(1,2)}(\alpha) \propto I_{\pm ik} \left(\frac{e^{2\alpha}}{2} \right), \quad (37)$$

that are complex conjugated. To reconstruct the wave function of the universe as the first solution of (9), we

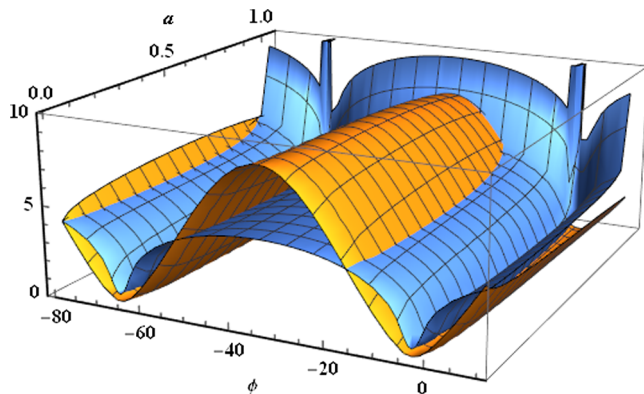


FIG. 1. Entanglement entropy (blue surface) and the temperature of entanglement (orange surface) of a pair of universes in a two-dimensional minisuperspace (a, ϕ) . Here: $\sigma = 1$, $E_\phi = 1$. The entanglement entropy is infinite exactly at the maximum expansion diverging for any ϕ , and finite for any point in the phase space of the classical evolution of the universe, even for $a = 0$; at least for the precision of the numerical method we use. The temperature has been multiplied by a factor of $1/5$ to equalize both surfaces and it shows no divergences at any point of the phase space.

perform a numerical procedure including the distribution function (11). Of the solutions we get for the wave function of the universe, both are complex conjugated as we explained in Sec. II if and only if $A(k)$ is symmetric around $k = 0$. Taking the distribution of the modes as in (11), that condition is fulfilled. Once they are complex conjugated functions, the real and the imaginary parts solve the differential equation (6) by themselves, so we will use them as the input for the calculation of the invariant representation.

Using this numerical wave function, we calculate the parameter R in (19) and its derivative with respect to α , the Bogoliubov coefficients using (24) and (25), and the entropy of entanglement through the parameter r from (29) and (30). The outcome, which is now given over a two-dimensional minisuperspace (a, ϕ) is plotted in Fig. 1.

Here we find a very important result. The entanglement entropy, which is a measurement of the quantumness of the system, is infinite at the maximum of expansion as we expected. However, the entropy at the big bang singularity $a = 0$ is not infinite. As a conclusion, we claim that according to our two-dimensional minisuperspace calculations (the scale factor and the scalar field) the quantumness of the system is larger at the maxima of the evolution of the universe than at the initial singularity, where \dot{a} is nonvanishing, even though it is important to notice that all entanglement entropy is locally decreasing in the neighborhood of the initial singularity. However, still in all the considered classical points (big bang singularity, maxima), both the entropy and temperature of entanglement have nonzero values showing the entanglement of the universes and some degree of quantumness of the system. Another

speculation could be that more degrees of freedom in minisuperspace reduce the entanglement on a similar basis as it happens with the chaotic systems for which adding more particles can stabilize its behavior and reduce disorder.

Furthermore, compared with previous results for the entanglement entropy [16], the main difference is the existence of a quantized scalar field, that is, we did not treat the scalar field classically. It seems now that it may regularize the value of the entropy of entanglement at the origin. However, it does not change the general interpretation. The generalized second principle of quantum thermodynamics [27] states that the entropy of entanglement cannot be increased by means of any local operation and classical communications alone. In the cosmological scenario, that means that the classical processes that happen inside a single universe, which in this context are all local irrespective of the distances between them,¹ cannot increase the entropy of entanglement between the pair of universes. Even more, they are expected to dissipate the quantum correlations between the states of the universes. Thus, the decreasing value of the entropy of entanglement from the very onset of the universes, regardless of whether the entropy of entanglement is infinite or not at the origin, would be associated with a classical evolution of the universes, which follows all the way up to the turning point of maximum expansion where the universe enters again into a highly nonclassical state [16,19].

We have checked that the other measurements of entropy as Tsallis or Renyi entropy in (33) and (34), respectively, are essentially analogous to the von Neumann entropy shown in Fig. 1, and the same happens for the parameter q from (30), but the value at the divergences is in unity.

IV. ENTANGLEMENT ENTROPY AT CRITICAL POINTS OF CLASSICAL EVOLUTION

After our first result where the entanglement entropy diverges at the maximum point of the expansion of the universe, it is necessary to ask if it always happens at any critical point of the oscillating evolution: the maxima, the minima or the inflection points.

In order to figure out how the entropy behaves at these points, we apply a simple one-dimensional model such that the evolution of the universe shows all kinds of points—an oscillatory universe. A simple model explored in Ref. [28] is one with wall-like matter, stringlike matter and a negative cosmological constant $\Lambda < 0$. This model without stringlike matter was later reconsidered and dubbed a simple harmonic universe in the quantum context in Refs. [29–33] based on the early Ref. [34]. Some new aspects of the stability of such models were recently studied in Ref. [35]. In this model the scale factor behaves as

¹In the present context, a nonlocal operation is any that involves the composite state of the two universes of the entangled pair.

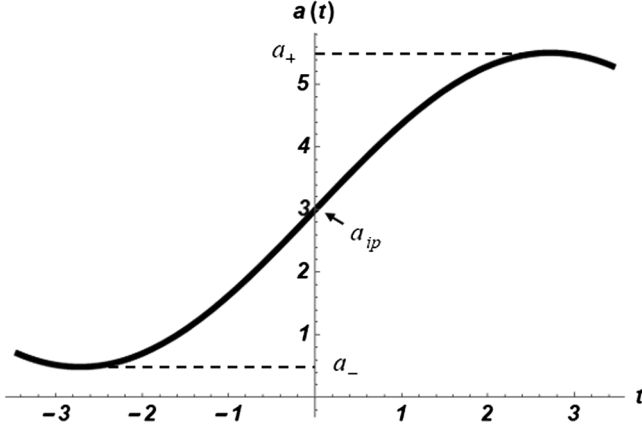


FIG. 2. Oscillation of the universe. Here: $\Lambda = -1$, $K = +1$, $C_s = 0.1$ and $C_w = 2$. The critical points are shown explicitly.

$$a(t) = -\frac{3}{2\Lambda} \left[A \sin \left(\sqrt{-\frac{\Lambda}{3}} t \right) + C_w \right], \quad (38)$$

where C_w is a constant density parameter due to wall-like matter, such that

$$C_w > A := \sqrt{C_w^2 + \frac{4}{3} \Lambda k'}, \quad (39)$$

with $k' = K - C_s$ where K is the curvature index and C_s a constant density parameter due to stringlike matter. The universe is oscillating between the minimum ($-$) and the maximum ($+$) defined by

$$a_{\pm} = -\frac{3}{2\Lambda} (\pm A + C_w). \quad (40)$$

An example of the oscillation in this model is plotted in Fig. 2.

The Friedmann equation for this model is simply

$$H^2 = \frac{\Lambda}{3} + \frac{C_w}{a} - \frac{k'}{a^2}, \quad (41)$$

and the Lagrangian is calculated as

$$L = \frac{1}{2} \int dN a^3 \left[\frac{H^2}{N^2} - \frac{\Lambda}{3} - \frac{C_w}{a} + \frac{k'}{a^2} \right], \quad (42)$$

where N is the lapse function. The corresponding Hamiltonian is

$$H = -\frac{p_a^2}{a} - k'a + C_w a^2 + \frac{\Lambda}{3} a^3, \quad (43)$$

and the Wheeler-DeWitt equation reads as

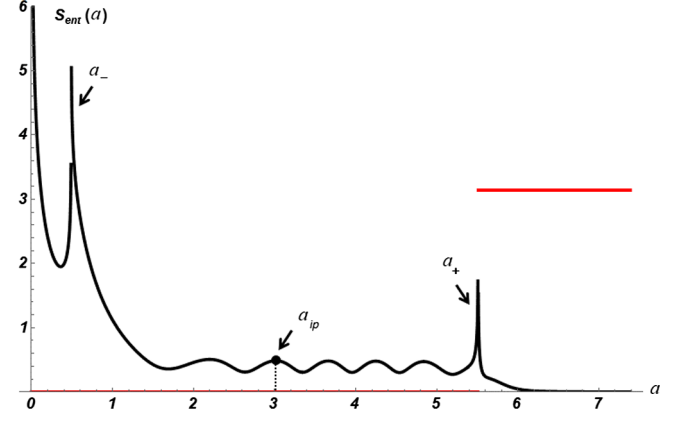


FIG. 3. Entanglement entropy of an oscillating closed universe (38). Here a_{\pm} stands for the maximum and the minimum size of the oscillation as given by (40), and $a_{i.p.}$ stands for the inflection point. We have chosen: $\Lambda = -1$, $C_s = 0.1$ and $C_w = 2$. The black line represents the real part of the entropy while the red line represents its imaginary part. The entropy is finite though nonzero at the inflection point and diverges at those critical points where $\dot{a} = 0$.

$$\left[-\hat{p}_a^2 - k'a^2 + C_w a^3 + \frac{\Lambda}{3} a^4 \right] \Psi(a) = 0, \quad (44)$$

where $\Psi(a)$ is the wave function of the universe. Quantizing the Hamiltonian with the proper factor ordering as described in (4), one rewrites (44) as

$$\left[\frac{1}{a} \frac{d}{da} \left(a \frac{d}{da} \right) - k'a^2 + C_w a^3 + \frac{\Lambda}{3} a^4 \right] \Psi(a) = 0. \quad (45)$$

Using the parametrization $\alpha = \ln(a)$ yields a kind of harmonic oscillator equation

$$\left[\frac{d^2}{d\alpha^2} + \omega^2(\alpha) \right] \Psi(\alpha) = 0, \quad (46)$$

with the frequency

$$\omega^2(\alpha) = -k'e^{4\alpha} + C_w e^{5\alpha} + \frac{\Lambda}{3} e^{6\alpha}. \quad (47)$$

A priori, this equation has no easy analytic solution, but following the same procedure as in Sec. II with $E_\phi = 0$ and the solutions of (46), we get the entropy of entanglement as plotted in Fig. 3. As expected, we can see that the blow-up of the entanglement entropy coincides precisely with the minimum and the maximum of the expansion of the universe. However, the inflection point is shown to be of less importance since the entropy remains smooth (finite) around it. This result is a bit disappointing since the universe we inhabit has passed an inflection point between the matter-dominated and the dark-energy-dominated era not long ago.

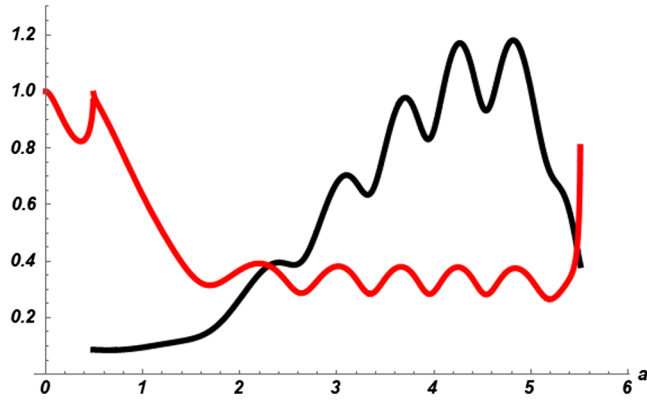


FIG. 4. Representation of the temperature in black and the parameter q in red. The temperature has been multiplied by $1/10$ to make easier the comparison of the oscillations. The temperature is finite, showing a weaker signal of the quantumness of the universe at any point.

For completeness, we calculate the value of the entanglement temperature (28) for this model, and the result is plotted in Fig. 4 together with the parameter q from (30). Both have been plotted qualitatively since their values are not so important as their behavior. Again the temperature shows no divergences, while the parameter q , which is closely related to the entanglement entropy, maintains a similar shape and approaches unity close to the singularities.

V. ENTANGLEMENT OF EXOTIC SINGULARITY UNIVERSES

A. Other than big bang singularities

As one can consult from Table I, the first example of a non-big-bang type (from now on type 0) of a singularity in cosmology which is compatible with observations is a big rip (BR or type I) associated with phantom dark energy [36,37]. Other examples are: a sudden future singularity (SFS or type II) [38] which includes big brake [39], finite scale factor singularities (FSF or type III) which includes

big freeze, big separation singularities (BS or type IV) [40], and w -singularities (type V) [41]. There is also a version of big rip known as little rip [42]. It seems fascinating that some of these singularities are “weaker” and some are “stronger” than a big bang in the sense of leading to infinities of some specific type and not to the other type [43–45].

In order to understand what we mean by “weaker” and “stronger” we have to refer to some mathematical tools to investigate the problem. According to the definition of Tipler [46] a singularity is strong if an extended object represented by three linearly independent, vorticity-free, geodesic deviation vectors at p parallelly transported along causal geodesic l is crushed to zero volume at the singularity by infinite tidal forces. In mathematical terms it means that at least one component of the tensor $I_j^i(\tau) = \int_0^\tau d\tau' \int_0^{\tau'} d\tau'' |R^i_{ajb} u^a u^b|$ (R^i_{ajb} is the Riemann tensor, u^a is the four-velocity vector, $a, b, i, j = 0, 1, 2, 3$ and τ is proper time) diverges on the approach to a singularity at $\tau = \tau_s$. On the other hand, according to the definition of Królak [47], a singularity is strong if the expansion of every future-directed congruence of null (timelike) geodesics emanating from the point p and containing l becomes negative somewhere on l or, in mathematical terms, if at least one component of the tensor $I_j^i(\tau) = \int_0^\tau d\tau' |R^i_{ajb} u^a u^b|$ diverges on the approach to a singularity at $\tau = \tau_s$. For the null geodesics one replaces the Riemann tensor with the Ricci tensor components.

B. Type II singularity universes entanglement

Physically, it is a singularity for which the tidal forces manifest here as the (infinite) impulse which reverses for SFS (or stops to zero for big brake) the increase of separation of geodesics and the geodesics themselves can evolve further—the universe can continue its evolution through a singularity. This behavior is like the turning point of a harmonic oscillator. In a specific example of type II singularity—big brake—the effect of a scalar field which fulfills the equation of state of a generalized Chaplygin gas is

TABLE I. Classification of basic singularities in Friedmann cosmology. Here t_s is the time when a singularity appears, $w = p/\rho$ is the barotropic index, T is Tipler’s definition and K is the Królak definition. In this paper we mainly concentrate on types 0, I_l , II_a , III_a and IV.

| Type | Name | t | $a(t_s)$ | $\rho(t_s)$ | $p(t_s)$ | $\dot{p}(t_s)$ | $w(t_s)$ | T | K |
|---------|---------------------------|----------|----------|-------------|----------|----------------|----------|--------|--------|
| 0 | big bang (BB) | 0 | 0 | ∞ | ∞ | ∞ | finite | strong | strong |
| I | big rip (BR) | t_s | ∞ | ∞ | ∞ | ∞ | finite | strong | strong |
| I_l | little rip (LR) | ∞ | ∞ | ∞ | ∞ | ∞ | finite | strong | strong |
| II | sudden future (SFS) | t_s | a_s | ρ_s | ∞ | ∞ | finite | weak | weak |
| II_a | big brake (BBr) | t_s | a_s | 0 | ∞ | ∞ | finite | weak | weak |
| III | finite scale factor (FSF) | t_s | a_s | ∞ | ∞ | ∞ | finite | weak | strong |
| III_a | big freeze (BF) | t_s | 0 | ∞ | ∞ | ∞ | finite | weak | strong |
| IV | big separation (BS) | t_s | a_s | 0 | 0 | ∞ | ∞ | weak | weak |
| V | w -singularity (w) | t_s | a_s | 0 | 0 | 0 | ∞ | weak | weak |

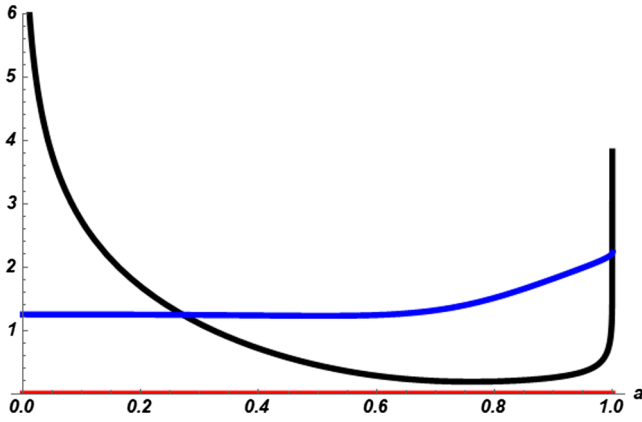


FIG. 5. Entanglement entropy and temperature for a model whose scalar field follows the equation of state (48). We have chosen: $B = -A = 1$. There is a divergence at the big brake singularity for $a = a_s = 1$, as well as for the initial big bang singularity where $a = 0$. The black line is the real part of the entropy, and the red line is the imaginary part. The temperature (blue) has been multiplied by a factor of 5 to make the comparison easier.

$$p = -\frac{A}{\rho^\beta}, \quad (48)$$

where p is the pressure, ρ the density, $A < 0$ a constant and, throughout this section, $\beta = 1$. The dependence of the scalar field with the scale factor is found through the continuity equation which takes the form

$$\rho = \sqrt{\frac{B}{a^6} - A}, \quad (49)$$

where $B > 0$, and where at the singularity time t_s one has $a_s = a(t_s) = (B/A)^{1/6}$. The evolution of the scale factor begins with the big bang singularity at time $t = 0$ where $a(0) = 0$ and stops at the big brake singularity at time t_s where $a(t_s) = a_s > 0$. The exotic big brake singularity that appears in this model has the properties

$$\dot{a}(t_s) = 0, \quad \ddot{a}(t_s) \rightarrow -\infty, \quad \frac{A}{p} = \rho(t_s) \rightarrow 0. \quad (50)$$

The scalar field which leads to such a singularity is treated as classical, so we are going to proceed with quantization using (3) with $N = 1$, i.e.,

$$H = \frac{1}{2} \left(-\frac{p_a^2}{a} - aK + \frac{\Lambda a^3}{3} + 2a^3 \rho \right). \quad (51)$$

After quantization, the Wheeler-DeWitt equation is

$$\left[\frac{\partial^2}{\partial \alpha^2} - e^{4\alpha} K + e^{6\alpha} \left(\frac{\Lambda}{3} + 2\rho(\alpha) \right) \right] \Psi(\alpha) = 0, \quad (52)$$

where, in the case described in [39], there is no cosmological constant ($\Lambda = 0$), the density of the scalar field $\rho(\alpha)$ is as in (49) and the flat case ($K = 0$) is considered.

The entanglement entropy for $B = -A = 1$ is plotted in Fig. 5. There we see the divergence at the initial big bang singularity ($a = 0$) as well as the big brake singularity at ($a = a_s = 1$). Again the temperature does not show anything relevant, but its value is nonzero.

C. Type III singularity universes entanglement

Another exotic singularity—the big freeze—is studied in [48], and appears when there is a scalar field with polytropic equation of state (48), where β is any constant. It is a special case of type III singularity. If the model is considered for a flat universe without a cosmological constant, then the density for which the null, strong and weak energy conditions are fulfilled can be written in terms of the scale factor as

$$\rho = |A|^{1/(1+\beta)} \left[\left(\frac{a_s}{a} \right)^{3(1+\beta)} - 1 \right]^{1/(1+\beta)}, \quad (53)$$

where

$$a_s = a(t_s) = \left| \frac{B}{A} \right|^{1/3(1+\beta)}, \quad (54)$$

and is the minimum size of the universe, $A < 0$, $1 + \beta < 0$, and $B > 0$ is a constant of integration. At a_s the initial big freeze singularity appears, and its properties are

$$\rho(a_s), p(a_s) \rightarrow \infty, \quad \dot{a}(a_s) \rightarrow \infty. \quad (55)$$

Besides, there is no maximum size, the minimum size is not zero in general and it occurs at a finite time t_s .

Again, the scalar field is treated classically, so we stick to the Wheeler-DeWitt equation (52) to proceed. The entanglement entropy in Fig. 6 shows nothing special, even a divergence at a_s that was already expected since $\rho(a_s) \rightarrow \infty$. Therefore the frequency $\omega^2(\alpha)$ in (52) also diverges, the parameter q in (30) approaches unity and finally the entanglement entropy in (29) diverges. Since there is no maximum size for this universe, the envelope of the oscillations of the entanglement entropy decreases as the universe grows.

As the universe expands, the temperature, also showed in Fig. 6, increases since R in (19) goes to zero due to the vanishing behavior of the wave function, which is imposed as a boundary condition for the wave function of the universe. *A priori*, it is said that the quantumness of the universe vanishes when the universe expands. Hence, the temperature cannot be a good indicator of the quantumness, at least in this scenario.

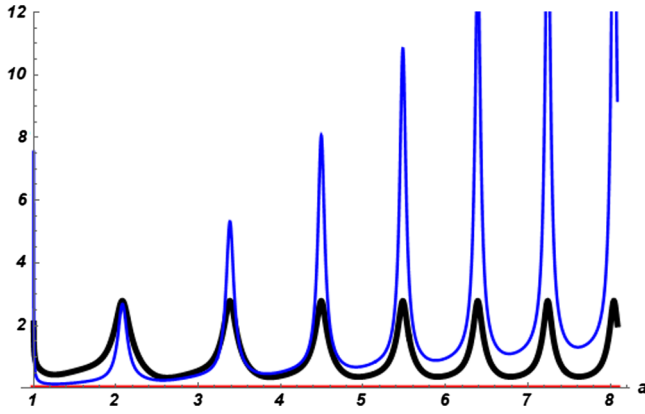


FIG. 6. Entanglement entropy and temperature for a model in which the scalar field follows the equation of state (53). We have chosen: $A = -1$, $B = 1$ and $\beta = -2$. There is a divergence for the Big-Freeze singularity, here at $a = a_s = 1$. The black line is the real part of the entropy and the red line is the imaginary part. The blue line shows the temperature multiplied by $1/10$, which increases as the universe expands.

D. Big separation (type IV) singularity universes entanglement

The model for which the next exotic singularity arises also takes into account a flat universe without a cosmological constant, and a scalar field in which equation of state is (48) and where $A < 0$ and β is any constant. The density in terms of the scale factor which fulfills all the energy conditions is (53), where $B > 0$ and $\beta \in (-1/2, 0)$, but a_s is now the maximum of the expansion of the universe defined by (54).

The type IV singularity appears at $a = a_s$ for $t = t_s$, and its characteristics are

$$p(t_s), \rho(t_s) = 0, \quad \dot{a}(t_s) = 0. \quad (56)$$

The entanglement entropy is calculated from the Hamiltonian constraint (52) and represented in Fig. 7 together with the temperature, which shows a divergence at the initial singularity and another at the type IV singularity. It was expected since $\dot{a}(t_s) = 0$, because the universe reaches a maximum at a_s . The temperature shows no divergences at any point, remaining approximately constant during the evolution. The parameter q behaves as expected; it is the unity at $a = 0$ and $a = a_s = 1$, and it follows the same shape as the entanglement entropy.

E. Little rip singularity entanglement

The little rip event described in [49] is an event for which the scale factor, the Hubble parameter H and its derivative, with respect to the cosmic time, diverge, which is not precisely a singularity. This can be produced by a scalar field with an equation of state like

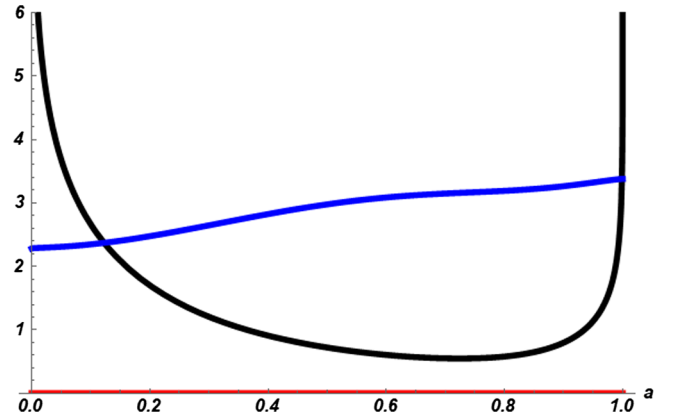


FIG. 7. Entanglement entropy and temperature for a model in which the scalar field follows the equation of state (48). We have chosen: $A = -1$, $B = 1$ and $\beta = -1/2$. There is a divergence in the type IV singularity, here at $a = a_s = 1$. The black line is the real part of the entropy and the red line is the imaginary part. The blue line is the temperature, which has been multiplied by a factor of 10. It shows no special behavior at any singularity.

$$p = -\rho - A\sqrt{\rho} < 0, \quad (57)$$

where $A > 0$ is a constant. If the universe contains this scalar field, its curvature is flat and has no cosmological constant, and the density can be written in terms of the scale factor as

$$\rho = \rho_o \left[\frac{3A}{2\sqrt{\rho_o}} \ln\left(\frac{a}{a_o}\right) + 1 \right]^2, \quad (58)$$

where ρ_o and a_o are constants of integration which correspond to the value of the density and the scale factor, respectively. The density in (58) presents an interesting point at

$$a_s = a_o e^{\frac{2\sqrt{\rho_o}}{3A}}, \quad (59)$$

where the density is zero. Looking at the Wheeler-DeWitt equation (52), with $K = 0$ and $\Lambda = 0$, and comparing it with (12), we see that the frequency $\omega^2(a)$ is directly proportional to the density (58), and therefore the frequency will also be zero so a divergence of the entanglement entropy is expected there. Figure 8 shows the temperature and the entanglement entropy of this model, for which this singularity appears due to the vanishing of the density of the scalar field. There is the initial singularity at $a = 0$, but there is no divergence as the universe is getting closer to the little rip event, where the entanglement entropy is infinitely oscillating and globally decreasing.

The temperature and the parameter q are also shown in Fig. 8. The temperature is increasing for high values of the scale factor. This is again meaningful, because the universe is expected to lose its quantumness when it expands, and thus, the temperature should decrease if it is a good indicator of the quantumness of the universe. That is not

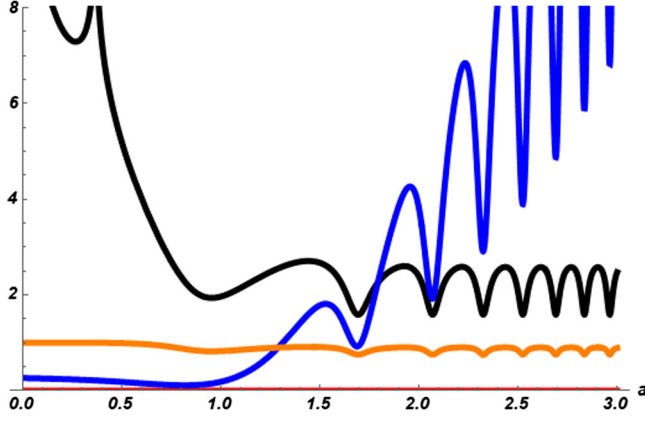


FIG. 8. Entanglement entropy and temperature for a model in which the scalar field follows the equation of state (57). We have chosen: $A = 2/3$, $B = 1$, $\rho_o = 1$ and $a_o = 1$. There are two divergences: the initial singularity at $a = 0$ and another at $a = a_s$. As the universe approaches the little rip event at $a \rightarrow \infty$, nothing special happens. The black line is the real part of the entropy, and the red line is the imaginary part. The temperature, in blue, increases and oscillates as the universe expands. It has been multiplied by a factor $1/20$. The orange curve is the parameter q , which is around the unity in the beginning of the evolution, the unity at the divergences and is oscillating around the value 0.8 as the universe expands.

the case and the temperature increases as the universe evolves. The parameter q is one at the singularities, and for the rest, it moves slightly oscillating around 0.8.

VI. ENTANGLEMENT ENTROPY VERSUS THE HUBBLE PARAMETER

Since the entanglement entropy diverges at the maxima and minima, the question is if it is related somehow with the Hubble parameter, since $H := \partial_a \ln(a)$ is null at those points. In order to analyze this problem, we take the simplest minisuperspace model (a, ϕ) , described by (36), that is the universe with only a scalar field and a unique mode $k = 1$. The wave functions for this model are those of (37) with mode $k = 1$.

The Lagrangian of the model is

$$L = \frac{1}{2} \int dN a^3 \left[\frac{H^2}{N^2} + \frac{1}{a^2} - \frac{\dot{\phi}^2}{N^2} \right], \quad (60)$$

and from this we get

$$p_\phi = \dot{\phi} a = k \equiv \text{const}, \quad (61)$$

which is the mode of the scalar field. Therefore, the Friedmann equation is simply

$$H^2 - \frac{k^2}{a^6} + \frac{1}{a^2} = 0, \quad (62)$$

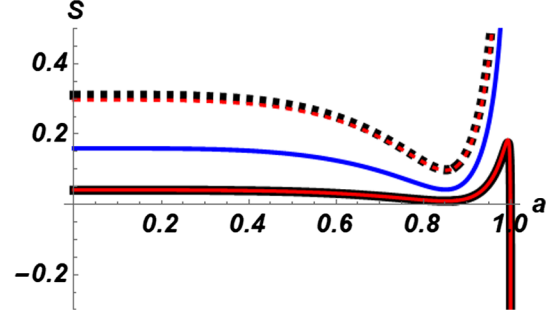


FIG. 9. Different entropies vs the scale factor. The blue line shows the von Neumann entropy, the solid lines show the Renyi (red) and Tsallis (black) entropies for $q = 1.5$, and the dashed lines for $q = 0.7$, respectively. For values $q \in (0, 1)$, it diverges to plus infinity.

and we obtain that, classically, the maximum size of the universe is at $a = 1$ if $k = 1$ (note that k here means the number of the mode, while the curvature index $K = +1$ is “hidden” in the term $1/a^2$). The entanglement entropy is shown in Fig. 9 with a blue line for this model. Again we found a divergence at the maximum of the evolution of the universe, and as in Sec. III, a finite initial entanglement entropy.

As our task is to look for the relation with the Hubble parameter, we use (62) to rewrite the entropy of entanglement numerically and perform a fit as we wish. Our objective is to find a relation like

$$S_{\text{ent}}(H) \sim \frac{1}{H^2}, \quad (63)$$

since it will indicate a relation to the entropy of horizons, such as the Hubble horizon or the black hole horizon. In order to check if this is the case, we approximate the entropy very close to a singularity by the function

$$S_{\text{ent}}(H) \sim c_o + \frac{c_1}{H} + \frac{c_2}{H^2} + \frac{c_3}{H^3}, \quad (64)$$

where we expect the condition $c_2 \gg c_o, c_1, c_3$ to be valid. The result of the fit is shown in Fig. 10 with an unexpected result. The entropy of entanglement seems to fit perfectly to a different function,

$$S_{\text{ent}}(H) \sim c_o - c_1 \ln(H), \quad (65)$$

where $c_1 \approx 1$, and we can see that this entropy has the shape of the Shannon entropy [50] for an event with probability H . The fit can be found in Fig. 10.

We wanted to check if this logarithmic shape holds for alternatives of the von Neumann entropy, as Renyi or Tsallis entropies. Using (33) and (34), we calculated both entropies and drew them in Fig. 9, together with the von Neumann entropy. The blue line shows the von Neumann entropy, the solid lines show the Renyi (red) and Tsallis

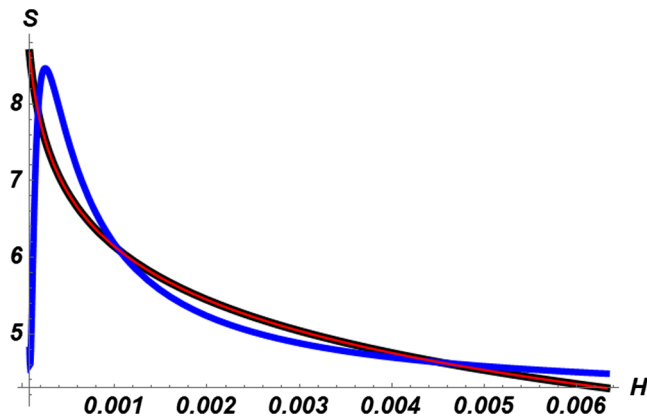


FIG. 10. Entanglement entropy vs the Hubble parameter. The black line is the exact entanglement entropy, while the blue line is the fit for the entropy at (64) and the red line is the fit for the entropy given by the relation (65).

(black) entropies for $q = 1.5$, and the dashed line for $q = 0.7$, respectively. The fits of the form (65) of these alternative entropies are in Fig. 11. The solid lines show the Renyi (black) and Tsallis (blue) entropies in terms of H for $q = 0.7$, and their fits in red and green, respectively. The dashed lines show the absolute values of the Renyi (black) and Tsallis (blue) entropies for $q = 1.5$, with their fits in red and green, respectively.

The fits of Renyi and Tsallis entropies to a function like (64) or (65) are not as appropriate as for von Neumann entropy. Nevertheless, as shown in Fig. 9, all of them behave similarly if $q > 1$, and therefore they could be taken as good indicators of the quantumness of the universe. They even yield an imaginary part of the classical region as the von Neumann entropy. If $q \in (0, 1)$, the entropy diverges to $-\infty$ at the maximum expansion of the universe and a

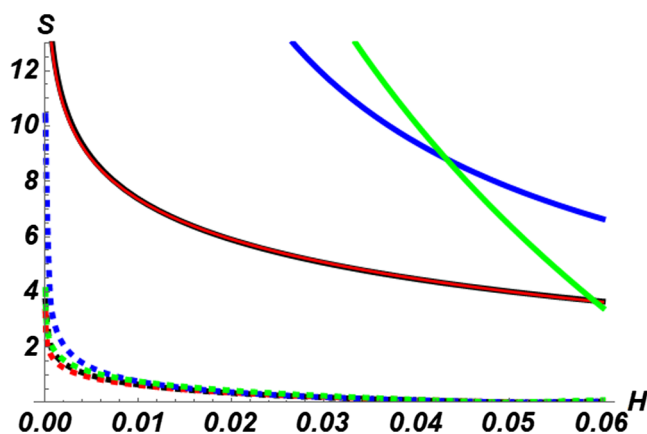


FIG. 11. Different entropies vs the Hubble parameter. The solid lines show the Renyi (black) and Tsallis (blue) entropies in terms of H for $q = 0.7$, and their fits in red and green, respectively. The dashed lines show absolute values of the Renyi (black) and Tsallis (blue) entropies for $q = 1.5$, with their fits in red and green, respectively.

maximum of entropy appears, which is also unrelated to any special event.

What is important to discuss is the possible physical meaning of the infinite values of the entanglement entropy which appears not only in our quantum cosmology considerations, but also in holographic theories and for black holes, where some cutoffs of entropy are applied. In our opinion, a divergent entropy of entanglement indicates that the modes of the quantum states of the universes are highly correlated in a composite state in which a wide range of modes would have a relevant contribution (not only those entropically closed to the classical one). At those points one should expect therefore a *strong departure from the classical behavior*, delving into a regime that is beyond the range of the present work. Let us point out that we consider semiclassical quantum states in the sense that we assume small departures from the classical state (and besides assume homogeneity and isotropy, which are essentially quite simple conditions for the spacetime geometry). Precisely, what the divergent value of the entropy of entanglement seems to say is that at those regimes the semiclassical approximation fails because the universes enter into a very quantum region of their evolution domains. One would expect that a complete quantum theory of gravity would smooth out the divergences, but our aim is to look for some small imprints of the entanglement between the universe-antiuniverse pairs, so that the semiclassical approach is fully justified.

It is worth saying that the Shannon entropy $S = -\log(H)$ can be interpreted as the information (or more precisely, information is negentropy) and so the divergence of entropy would indicate infinite information. However, as it was mentioned already, in our opinion the entropy of entanglement (or information) is a measure of the degree of correlations between the states of the universes. In fact, our formulas (63)–(65) are only approximated expressions to have a glimpse at the problem.

VII. CONCLUSIONS

We have used the third quantization formalism to calculate the entanglement entropy of two universes created in pairs within the framework of the Friedmann cosmology. Our main concern has been to calculate the entanglement entropy and the entanglement temperature around the maxima, minima and inflection points of the classical evolution. This has been done in an extended two-dimensional minisuperspace, parametrized by the scale factor a and the massless scalar field ϕ .

We have found that after the Fourier expansion of the wave function of the universe $\Psi(\alpha, \phi)$ [cf. ([6])], the only way to have two complex conjugated functions $\Psi^{(1,2)}(\alpha, \phi)$ representing both the universe and the antiuniverse is to have symmetric distribution $A(k)$ of the modes of the scalar field around $k = 0$. This is quite a strong, though necessary, constraint on the distribution of the scalar field into a pair of

universes. This has proven the symmetry of the system, i.e., that the energy is conserved.

We have found that, while taking a scalar field ϕ into consideration, the entanglement entropy is finite at the classical place of the big bang singularity, and diverges at the maxima and minima of expansion. We cannot give, *a priori*, any reason to the finite value of the entanglement entropy at the initial singularity, but it seems to be related to the inclusion of another degree of freedom such as the quantized scalar field. Indeed, when we consider the scalar field as classical and give some relation between its energy and the scale factor as we did in Secs. IV and V, the scalar field is no longer an independent variable, and the entanglement entropy blows up at that singular point.

It is worth mentioning that every statement about a critical point or a singularity which one infers from the canonical quantum cosmology is dependent on the construction of the theory far from the semiclassical point of view. The problem is, the factor ordering which affects small scales and/or high curvatures makes the quantization as we took it in (4) to be one of the choices one can make, but there are infinitely many other possibilities which could lead to completely different results (see, e.g., [51]).

We have also studied the entanglement quantities for the universes which classically exhibit singularities other than big bang singularities such as big brake, big freeze, big separation and little rip. We have found that the entanglement entropy or the entanglement temperature in all the critical points and singularities is either finite or infinite but never vanish. This has proven that the universes within each pair are entangled and so subject to some quantumness within the system. Furthermore, we have found in Sec. V, when we analyzed the big freeze singularity, that an earlier statement made in Ref. [16] about the temperature as a proper measure of the quantumness is not always true, since in this specific example, the temperature diverges when the universe expands infinitely, while it was expected that the entanglement of a pair of universes should vanish asymptotically as the universe expands.

Apart from the von Neumann entanglement entropy, we have also calculated the Tsallis and Renyi entanglement entropies and found that they exhibit similar behavior to the measures of quantumness as the von Neumann entropy. This seems to show the robustness of our entanglement calculations and the appropriate conclusions.

Finally, we have been looking for the relation between the entanglement entropy and the Hubble parameter which classically marks special points of the evolution of the universe. In the example analyzed in Sec. VI, we have found the logarithmic relation (65) to be a very good approximation to the behavior of the entanglement entropy in terms of the Hubble parameter around critical points, where the Hubble parameter vanishes. Further investigations should be aimed to find explanations and consistency for this result.

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APPENDIX: INVARIANT REPRESENTATION OF EQUATION (12)

The invariant representation of the Hamiltonian that in the third quantization formalism leads to the Wheeler-DeWitt equation (12) [with the frequency given by (16)] can easily be obtained in the following way. Let us first notice that if we make the change in the wave function of the universe, $\Psi \rightarrow \xi = \frac{1}{R}\Psi$, with R being a function that satisfies

$$\ddot{R} + \omega^2 R = \frac{1}{R^3}, \quad (\text{A1})$$

and a change in the scale factor, $\alpha \rightarrow \tilde{\alpha}$, defined by

$$\tilde{\alpha}(\alpha) = \int^\alpha \frac{d\alpha'}{R^2(\alpha')}, \quad (\text{A2})$$

then, the Wheeler-DeWitt equation (12) transforms into

$$\xi'' + \xi = 0, \quad (\text{A3})$$

where, $\xi' \equiv \frac{\partial \xi}{\partial \tilde{\alpha}}$. Equation (A3) is the equation of a harmonic oscillator with constant unit frequency. The associated creation and annihilation operators, defined as usual by

$$c_{\pm} = \sqrt{\frac{1}{2}}(\xi + i\xi'), \quad (\text{A4})$$

$$c_{\pm}^{\dagger} = \sqrt{\frac{1}{2}}(\xi - i\xi'), \quad (\text{A5})$$

determine a representation $|N\rangle$, given by the eigenstates of the number operator $N = c_{\pm}^{\dagger}c_{\pm}$, that is invariant under the evolution of the scale factor. Now, taking into account that

$$\xi = \frac{1}{R}\Psi, \quad (\text{A6})$$

and

$$\xi' \equiv \frac{\partial \xi}{\partial \tilde{\alpha}} = R\dot{\Psi} - \dot{R}\Psi, \quad (\text{A7})$$

one can write the operators (A4) and (A5) of the invariant representation as (17) and (18).

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